Adaptive time-stepping for Navier-Stokes

Advanced Topics in Scientific Computing Course Final Project
SDIC Master Degree, University of Trieste (UniTS) and SISSA

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Presentation Outline

Adaptive time-stepping for solving the incompressible Navier-Stokes equations with a finite element method using the deal.II library

1. **Introduction**
2. **Adaptive Time-Stepping Algorithms**
   - First Approach
   - Second Approach (*chosen one*)
3. **Results**
4. **Conclusions**

Introduction

Adaptive Time-Stepping

- **Purpose:** control *accuracy* and reduce *computational costs*.
- **Key Idea:** dynamically adjust time step size based on error estimators:
  - **Increase** time step size when error is small (*efficiency*).
  - **Decrease** time step size when error is large (*accuracy*).
- **Challenges with Fixed Time-Stepping:**
  - Too small time steps $\rightarrow$ *waste of computational resources*.
  - Too large time steps $\rightarrow$ *leads to inaccurate solutions*.
- **Goal:** find the *optimal time step size* at each time step, following the time evolution of the *physics* of the flow.

Project Development Overview

- **Foundation:** based on **Step-35 tutorial** of the deal.II library.
  - Solves *incompressible, time-dependent Navier-Stokes equations* using a **projection method** (decouples velocity and pressure).

- **Enhancement:**
  - Original program uses **fixed time steps**.
  - Extended to implement **adaptive time-stepping**:
    - **Dynamic adjustment** of time step size via error estimators.
    - Improves simulation **efficiency** and **accuracy**.

Adaptive Time-Stepping Algorithms

First Approach - Main Concepts

- **Overview**:
  - Based on **two solutions** computed with schemes of **different orders of accuracy**.
  - **Key Idea**: use the difference between the solutions to estimate the next time step size.

Error Estimation and Optimal Time Step Size

- **Error Estimation**:
  - u: less accurate solution, error $\mathcal{O}(\Delta t^p)$.
  - $\bar{u}$: more accurate solution, error $\mathcal{O}(\Delta t^{p+1})$.
  - Difference between solutions:

$$e = ||\bar{u} - u||_{L^2} \approx C\Delta t^p$$

- **Optimal Time Step Size** (Δt_{opt}):
  - Given tolerance $\epsilon > 0$.
  - ρ: safety factor ($0 < \rho \leq 1$).
  - Compute:

$$\Delta t_{\text{opt}} = \rho \Delta t \left(\frac{\epsilon}{e}\right)^{1/p}$$

Drawbacks of the First Approach

- **High computational cost**:
  - Requires **two solutions** with different schemes.
  - **Doubles the cost of the simulation**.

⚠️ Inefficiency may nullify the benefits of adaptive time-stepping.

Second Approach - Main Concepts

- **Overview**:
  - More **efficient** than the first approach: only **one solution** is used (*projection method*).
  - Adjusts time step size based on the **relative change in the solution over two consecutive time steps**.
  - Dynamically adapts to the **physics** of the flow.
  - The time step size is reduced when the system is evolving rapidly, it is increased when the system is evolving slowly.

This approach is the one implemented in the project.

Time Error Indicator

Error Quantification:

- Measures relative change in the solution using:

$$(\eta^n)^2 = \sum_{K \in T_h} \left(\int_{t^{n-1}}^{t^n} \left(||\nabla \left(u_h^n - u_h^{n-1} \right) ||_{L^2(K)}^2 \right) dt \right)$$

  - T_h: mesh.
  - u_h^n, u_h^{n-1}: solutions at t^n, t^{n-1}.

Normalization term:

$$(\sigma^n)^2 = \int_{t^{n-1}}^{t^n} |u_h^n|_1^2 dt$$

- $|u_h^n|_1$: H^1 semi-norm of the solution.

Optimal Time Step Size

Bound on Time Error Indicator:

$$(1-\alpha)\mathsf{TOL} \leq \frac{\eta^n}{\sigma^n} \leq (1+\alpha)\mathsf{TOL}$$

- $\mathsf{TOL} = 0.1$: tolerance.
- $\alpha = 0.5$.

Optimal Time Step Size

Adjustment Rules:

- If bounds are **satisfied**:

$$\Delta t_{\text{opt}} = \Delta t^n$$

- If $(\eta^n)^2 > (\sigma^n)^2(1+\alpha)^2\mathsf{TOL}^2$:

$$\Delta t_{\text{opt}} = \frac{\Delta t^n}{\rho_1} \quad \text{with} \quad \rho_1 = \min\left(\frac{(\eta^n)^2}{(\sigma^n)(1+\frac{\alpha}{2})^2\mathsf{TOL}^2}, 2\right)$$

- If $(\eta^n)^2 < (\sigma^n)^2(1-\alpha)^2\mathsf{TOL}^2$:

$$\Delta t_{\text{opt}} = \frac{\Delta t^n}{\rho_2} \quad \text{with} \quad \rho_2 = \max\left(\frac{(\eta^n)^2}{(\sigma^n)(1-\frac{\alpha}{2})^2\mathsf{TOL}^2}, 0.5\right)$$

Optimal Time Step Size

Stability Constraints:

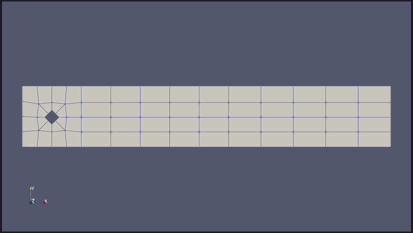
- Choice of ρ_1 and ρ_2 ensures no **large variations** in the time step size, preventing **instabilities** in simulations.
- **Bounded time step size**:

$$\Delta t_{\text{min}} = 10^{-4} \quad \text{and} \quad \Delta t_{\text{max}} = 0.1$$

Results

Test Case: Flow Around a Square Obstacle

- **Geometry**:



- **Boundary Conditions**:
  - **No-slip**: top/bottom walls and the obstacle.
  - **Inflow** (left wall):

$$u = \begin{cases} 4U_m y \frac{(H-y)}{H^2} \\ 0 \end{cases}$$

    - $U_m = 1.5$, $H = 4.1$.
  - **Outflow** (right wall):
    - Vertical velocity component is set to zero.
    - Pressure is set to zero.

Mesh Refinement Levels

- **Meshes Used**:
  - *Level 1* mesh (*coarser*), 960 cells. *2 refinement levels*.
  - *Level 2* mesh (*finer*), 3840 cells. *3 refinement levels*.

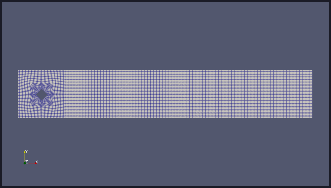


Figure: Level 1 mesh

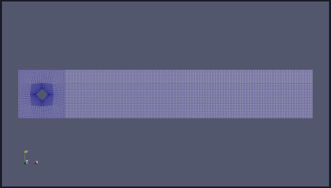


Figure: Level 2 mesh

Parameter Settings

Reynolds Numbers:

- Simulations performed for:

$$\mathrm{Re} = 50, 100, 200$$

Time and Output Settings:

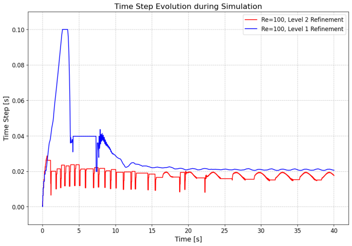
- Final simulation time: $T = 40$.
- Initial time step size: $\Delta t_0 = 0.001$.
- Output interval: every 20 time steps.

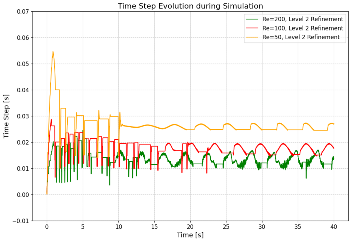
Simulation Results

Case: *level 2* mesh, $Re = 100$.

Animations show the development and extension of a vortex chain behind the obstacle, with formation and shedding of vortices downstream.

Adaptive Time-Stepping Analysis

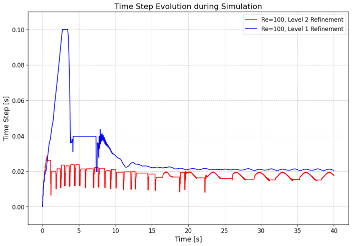


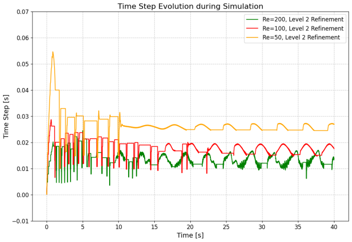


General Trends:

1. **Initialization phase**: time step size decreases.
2. **Stabilization phase**: time step size increases.
3. **Vortex Shedding phase**: time step size oscillates.
4. **Periodic phase**: time step size stabilizes achieving a periodic trend.

Adaptive Time-Stepping Analysis

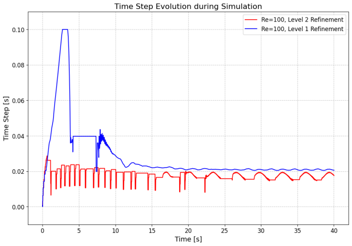


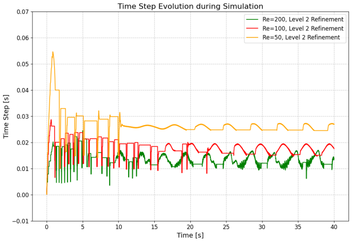


Effect of the Grid Refinement:

1. *Level 2 Mesh (Finer Grid)*:
   - Smaller time step sizes.
   - Increased sensitivity to the flow dynamics.
   - More oscillatory behavior.
2. *Level 1 Mesh (Coarser Grid)*:
   - Larger time step sizes.
   - Reduced sensitivity to the flow dynamics.
   - Smoother trend.

Adaptive Time-Stepping Analysis





Effect of the Reynolds Number:

1. *Higher Reynolds Numbers*:
   - Smaller time step sizes.
   - Increased oscillatory behavior
2. *Lower Reynolds Numbers*:
   - Larger time step sizes.
   - Reduced oscillations.

Effectiveness of the Adaptive Time-Stepping Algorithm

1. **Dynamic Time Step Adjustment**:
   - Time step size is **reduced** during:
     - Rapid solution evolution.
     - Formation of flow structures and vortices.
   - Time step size is **increased** during:
     - Stabilization of the flow.
     - Reduced solution variability.
2. **Optimized Computational Efficiency**:
   - Balances **accuracy** and **efficiency** dynamically.

Effectiveness of the Adaptive Time-Stepping Algorithm

Alignment with Expected Trends:

- *Finer Grid Refinements*:
    - Smaller time step sizes required to resolve detailed flow structures.
- *Higher Reynolds Numbers*:
    - Smaller and more oscillatory time steps to capture faster dynamics.

Conclusions

Conclusions

The adaptive time-stepping algorithm performs as expected, adjusting time step sizes effectively to:

- Reflect the **time evolution** of the physical flow.
- Optimize **efficiency** without compromising **accuracy**.

The analysis of the computed time step size evolution can help to better identify different phases of the flow.

Possible Future Extension:

- Add *adaptive mesh refinement* to further enhance the simulation accuracy.

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