Advanced Numerical Analysis
Proofs left for exercise

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1 Exercises

1.1 Exercise 1

Consider the Poisson problem:

$$\begin{cases} A(u,v) = (u,v)_1 = \int_0^1 u'v'\mathrm{d}x \\ F(v) = \int_0^1 fv\mathrm{d}x \\ V = H_0^1(0,1) \end{cases}$$

Prove that A and F satisfy the hypothesis of the Lax-Milgram lemma.

1.1.1 Proof

Continuity of A:

$$|A(u,v)| = |\int_0^1 u'v'\mathrm{d}x| = |(u,v)_1| \leq \|u\|_1\|v\|_1 = \left(\int_0^1 (u')^2\right)^{\frac{1}{2}}\left(\int_0^1 (v')^2\right)^{\frac{1}{2}}$$

where we applied the Cauchy-Schwarz inequality.
Coercivity of A:

$$A(u,u) = (u,u)_1 = \int_0^1 (u')^2\mathrm{d}x \geq \left(\left(\int_0^1 (u')^2\mathrm{d}x\right)^{\frac{1}{2}}\right)^2 = \|u\|_1^2$$

Continuity of F (assuming f is in $L^2(0,1)$):

$$\begin{aligned} |F(v)| = |\int_0^1 fv\mathrm{d}x| = |(f,v)| \leq \|f\|_{L^2}\|v\|_{L^2} = \|f\|_{L^2}\left(\int_0^1 v^2\mathrm{d}x\right)^{\frac{1}{2}} \leq \\ \leq \|f\|_{L^2}\left(\int_0^1 v^2\mathrm{d}x + \int_0^1 (v')^2\mathrm{d}x\right)^{\frac{1}{2}} = \|f\|_{L^2}\|v\|_{H^1} \end{aligned} \tag{1}$$

where we applied again the Cauchy-Schwarz inequality.

1.2 Exercise 2

The function u minimizes the energy functional $J(v) = \dfrac{1}{2}A(v,v) - F(v) \implies u$ solves the weak problem.

1.2.1 Proof

If u minimizes J, then its first variation must be equal to 0; therefore we have:

$$\begin{aligned} 0 = \frac{d}{d\epsilon} J(u+\epsilon v)\Big|_{\epsilon=0} &= \frac{d}{d\epsilon}\left(\frac{1}{2}\int_0^1 (u+\epsilon v)'(u+\epsilon v)'\mathrm{d}x - \int_0^1 f(u+\epsilon v)\mathrm{d}x\right)\Big|_{\epsilon=0} = \\ &= \left(\int_0^1 (u'v' + \epsilon v'v')\mathrm{d}x - \int_0^1 fv\mathrm{d}x\right)\Big|_{\epsilon=0} = \int_0^1 u'v'\mathrm{d}x - \int_0^1 fv\mathrm{d}x \end{aligned} \tag{2}$$

From which we get:

$$\int_0^1 u'v'\mathrm{d}x = A(u,v) = \int_0^1 fv\mathrm{d}x = F(v)$$

1.3 Exercise 3

First assertion of the *strong maximum principle*: let $u \in C^2(\Omega) \cap C(\bar{\Omega})$, with u harmonic in Ω, Ω open and bounded, then:

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$$

1.3.1 Proof

Having proved the second assertion (given that Ω is connected and $\exists x_0 \in \Omega : u(x_0) = \max_{\bar{\Omega}} u$, then u is constant in Ω), the first assertion follows. In fact, either the maximum is achieved at an interior point $x_0 \in \Omega$ or at the boundary: in the first case, as $u(x_0) = \max_{\bar{\Omega}} u$, by the second assertion u is constant and therefore, in particular, we must have $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$. In the second case, the equality is also true as $\partial\Omega \subseteq \bar{\Omega}$.

1.4 Exercise 4

Uniqueness for Poisson: let $g \in C(\partial\Omega)$, $f \in C(\Omega)$, then there exists at most one solution of the problem:

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega \end{cases}$$

1.4.1 Proof

For absurd, let's assume there are two solutions u and v. Consider the function $w = u - v$; w is harmonic: in fact, as both u and v are solutions of the problem, we can write:

$$\begin{cases} -\Delta w = -\Delta(u-v) = -(\Delta u - \Delta v) = -(f-f) = 0 & \text{in} \quad \Omega \\ w = u - v = g - g = 0 & \text{on} \quad \partial\Omega \end{cases}$$

We can therefore apply the *strong maximum principle* to w:

$$\max_{\bar{\Omega}} w = \max_{\partial\Omega} w = 0 \tag{3}$$

Moreover, w is constant in Ω; specifically, given (3) (w achieves its maximum on the boundary $\partial\Omega$ and $w = 0$ on $\partial\Omega$), $w = u - v = 0$ in Ω. This implies that $u = v$, which means the solution of the problem is unique.

1.5 Exercise 5

Analysis of the FD method - general elliptic 2-points b.v.p: given the following scheme:

$$\frac{-(a_i - \frac{h}{2}b_i)u_{i+1} + 2a_iu_i - (a_i + \frac{h}{2}b_i)u_{i-1}}{h^2} = \mathcal{L}_h u_i$$

The truncation error $T_i = \mathcal{L}u(x_i) - \mathcal{L}_h u(x_i)$ is such that:

$$|T_i| \le \left(\frac{h^2}{12} \|a\|_C \left\|u^{IV}\right\|_C + \frac{h^2}{6} \|b\|_C \left\|u^{III}\right\|_C \right)$$

1.5.1 Proof

Using Taylor's formula:

$$\begin{aligned}
h^2\mathcal{L}_h u(x_i) &= -(a_i - \frac{h}{2}b_i)u_{i+1} + 2a_iu_i - (a_i + \frac{h}{2}b_i)u_{i-1} = \\
&= -(a_i - \frac{h}{2}b_i)(u(x_i) + hu^I(x_i) + \frac{h^2}{2}u^{II}(x_i) + \frac{h^3}{6}u^{III}(x_i) + \frac{h^4}{24}u^{IV}(\xi)) + \\
&+ 2a_iu(x_i) - \\
&- (a_i + \frac{h}{2}b_i)(u(x_i) - hu^I(x_i) + \frac{h^2}{2}u^{II}(x_i) - \frac{h^3}{6}u^{III}(x_i) + \frac{h^4}{24}u^{IV}(\eta)) = \\
&= -a_ih^2u^{II}(x_i) - a_i\frac{h^4}{24}(u^{IV}(\xi) + u^{IV}(\eta)) + h^2b_iu^I(x_i) + b_i\frac{h^4}{6}u^{III}(x_i)
\end{aligned} \tag{4}$$

where $\xi \in [x_i, x_{i+1}]$ and $\eta \in [x_{i-1}, x_i]$.
Considering $\mathcal{L}u = -au^{II} + bu^I$, this implies:

$$\begin{aligned}
T_i &= \mathcal{L}u(x_i) - \mathcal{L}_h u(x_i) = \\
&= -a_iu^{II} + b_iu^I + a_iu^{II}(x_i) + a_i\frac{h^2}{24}(u^{IV}(\xi) + u^{IV}(\eta)) - b_iu^I(x_i) - b_i\frac{h^2}{6}u^{III}(x_i)
\end{aligned} \tag{5}$$

After canceling some terms, assuming $u \in C^{IV}(0,1)$, we finally get:

$$|T_i| \le \frac{h^2}{12} \|a\|_C \left\|u^{IV}\right\|_C + \frac{h^2}{6} \|b\|_C \left\|u^{III}\right\|_C$$

1.6 Exercise 6

Analysis of the FD method - general elliptic 2-points b.v.p: **convergence**. Let $b = 0$, then: $|u(x_i) - u_i| \le ch^2 \left\|u^{IV}\right\|_C \quad \forall i$.

1.6.1 Proof

Consider $|E|_\infty = \max_i |u(x_i) - u_i|$, with $E = \{e_i\}_0^N$, $e_i = u(x_i) - u_i$. Using the stability estimate, we can write:

$$|E|_\infty \le \max\{|e_0|, |e_N|\} + c|\mathcal{L}_h E|_\infty \le c|\mathcal{L}_h E|_\infty$$

where we considered the fact that $|e_0| = |e_N| = 0$ due to the boundary conditions.
Using the consistency (truncation error) with $b = 0$:

$$|\mathcal{L}_h e_i| = |\mathcal{L}_h u(x_i) - \mathcal{L}_h u_i| = |T_i| \leq ch^2 \left\| u^{IV} \right\|_C$$

So overall we proved that $|u(x_i) - u_i| \leq ch^2 \left\| u^{IV} \right\|_C \quad \forall i.$

1.7 Exercise 7

Method of the undetermined coefficients: solve the following linear system to obtain the coefficients.

$$\begin{cases} \alpha + \beta + \gamma = 0 \\ \alpha h_{i+1} - \gamma h_i = 0 \\ \alpha \frac{h_{i+1}^2}{2} + \gamma \frac{h_i^2}{2} = 1 \end{cases}$$

1.7.1 Proof

$$\begin{cases} \beta = -\alpha - \gamma \\ \alpha = \frac{\gamma h_i}{h_{i+1}} \\ \frac{\gamma h_i}{h_{i+1}} \frac{h_{i+1}^2}{2} + \gamma \frac{h_i^2}{2} = 1 \end{cases} \Rightarrow \begin{cases} \beta = -\alpha - \gamma \\ \alpha = \frac{\gamma h_i}{h_{i+1}} \\ \gamma (h_i h_{i+1} + h_i^2) = 2 \end{cases} \Rightarrow \begin{cases} \beta = -\alpha - \frac{2}{h_i(h_{i+1}+h_i)} \\ \alpha = \frac{h_i}{h_{i+1}} \frac{2}{h_i(h_{i+1}+h_i)} \\ \gamma = \frac{2}{h_i(h_{i+1}+h_i)} \end{cases} \Rightarrow$$

$$\begin{cases} \beta = -\frac{2}{h_{i+1}(h_{i+1}+h_i)} - \frac{2}{h_i(h_{i+1}+h_i)} \\ \alpha = \frac{2}{h_{i+1}(h_{i+1}+h_i)} \\ \gamma = \frac{2}{h_i(h_{i+1}+h_i)} \end{cases} \Rightarrow \begin{cases} \beta = -\frac{2}{(h_{i+1}+h_i)} (\frac{1}{h_{i+1}} + \frac{1}{h_i}) \\ \alpha = \frac{2}{h_{i+1}(h_{i+1}+h_i)} \\ \gamma = \frac{2}{h_i(h_{i+1}+h_i)} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \beta = -\frac{2}{(h_{i+1} h_i)} \\ \alpha = \frac{2}{h_{i+1}(h_{i+1}+h_i)} \\ \gamma = \frac{2}{h_i(h_{i+1}+h_i)} \end{cases}$$

1.8 Exercise 8

(*Analysis of the 2D Poisson boundary value problem*) Given V such that $\mathcal{L}_h v_{ij} \leq 0 \; \forall (x_i, x_j)$ internal, then:

$$\max_{ij} v_{ij} = \max_{(x_i, x_j) \in \partial\Omega} v_{ij}$$

1.8.1 Proof

This lemma is a consequence of the general discrete maximum principle.
Suppose $\exists v_{ij} \in \Omega : v_{ij} = M = \max_\Omega u_{ij}$, then:

$$0 \geq \mathcal{L}_h v_{ij} = -\left(\frac{v_{i+1,j} + v_{i,j+1} - 4v_{i,j} + v_{i-1,j} + v_{i,j-1}}{h^2} \right)$$

Then:

$$M = v_{ij} \leq \frac{1}{4} (v_{i+1,j} + v_{i,j+1} + v_{i-1,j} + v_{i,j-1}) \leq M$$

It follows that all the considered v are equal; proceeding similarly up to the boundary, we get that v is constant.

1.9 Exercise 9

(*Analysis of the 2D Poisson boundary value problem*) Let u be smooth enough, then the truncation error T_{ij} is such that:

$$|T_{ij}| \leq \frac{h^2}{12}(M_{xxxx} + M_{yyyy})$$

where $M_{xxxx} = \max_{x,y} |u_{xxxx}(x,y)|$ and $M_{yyyy} = \max_{x,y} |u_{yyyy}(x,y)|$.

1.9.1 Proof

Using Taylor's formula:

$$\begin{aligned}
T_{ij} = \mathcal{L}u(x_i, y_j) - \mathcal{L}_h u(x_i, y_j) = -u_{xx}(x_i, y_j) - u_{yy}(x_i, y_j) + \left(\frac{u_{i+1,j} + u_{i,j+1} - 4u_{i,j} + u_{i-1,j} + u_{i,j-1}}{h^2}\right) = \\
= -u_{xx}(x_i, y_j) - u_{yy}(x_i, y_j) + \left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}\right) = \\
= -u_{xx}(x_i, y_j) - u_{yy}(x_i, y_j) + u_{xx}(x_i, y_j) + \frac{h^2}{24}(u_{xxxx}(\xi_1, y_j) + u_{xxxx}(\eta_1, y_j)) + \\
+ u_{yy}(x_i, y_j) + \frac{h^2}{24}(u_{yyyy}(x_i, \xi_2) + u_{xxxx}(x_i, \eta_2)) = \\
= \frac{h^2}{24}(u_{xxxx}(\xi_1, y_j) + u_{xxxx}(\eta_1, y_j)) + \frac{h^2}{24}(u_{yyyy}(x_i, \xi_2) + u_{xxxx}(x_i, \eta_2))
\end{aligned} \tag{6}$$

where $\xi_1, \eta_1 \in [x_{i-1}, x_{i+1}]$ and $\xi_2, \eta_2 \in [y_{i-1}, y_{i+1}]$.
Finally we get:

$$|T_{ij}| \leq \frac{h^2}{12}(M_{xxxx} + M_{yyyy})$$

1.10 Exercise 10

Show that $\delta^y_{2h}\delta^x_{2h}u = C(u_{i+1,j+1} + u_{i-1,j-1} - u_{i+1,j-1} - u_{i-1,j+1})$.

1.10.1 Proof

$$\begin{aligned}
\delta^y_{2h}\delta^x_{2h}u = \delta^y_{2h}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2h}\right) = \frac{1}{2h}(\delta^y_{2h}u_{i+1,j} - \delta^y_{2h}u_{i-1,j}) = \\
= \frac{1}{2h}\left(\frac{u_{i+1,j+1} - u_{i+1,j-1}}{2h} - \frac{u_{i-1,j+1} - u_{i-1,j-1}}{2h}\right) = \\
= \frac{1}{4h^2}(u_{i+1,j+1} + u_{i-1,j-1} - u_{i+1,j-1} - u_{i-1,j+1})
\end{aligned} \tag{7}$$

1.11 Exercise 11

$\|.\|_1$ is a norm and not just a seminorm.

1.11.1 Proof

Recall that we have:

$$\|v\|_1 = \|v\|_{H^1(\Omega)} = \sqrt{\int_\Omega v^2 \mathrm{d}\Omega + \int_\Omega v'^2 \mathrm{d}\Omega} = \sqrt{\|v\|^2_{L^2(\Omega)} + \|v'\|^2_{L^2(\Omega)}}$$

$$|v|_1 = |v|_{H^1(\Omega)} = \sqrt{\int_\Omega v'^2 \mathrm{d}\Omega} = \left\|v'\right\|_{L^2(\Omega)}$$

Two norms are equivalent if there exist two positive constants c_1 and c_2 such that:

$$c_1\|v\|_A \leq \|v\|_B \leq c_2\|v\|_A \quad \forall v \in V$$

In particular, as $\|v\|_1 = \sqrt{\|v\|^2_{L^2(\Omega)} + |v|_1^2}$, it is evident that $|v_1| \leq \|v\|_1$. On the other hand, using the Poincarè inequality, we have the following:

$$\|v\|_1 = \sqrt{\|v\|^2_{L^2(\Omega)} + |v|_1^2} \leq \sqrt{c_\Omega |v|_1^2 + |v|_1^2} \leq c'_\Omega |v|_1$$

Therefore the two norms are equivalent.

1.12 Exercise 12

Prove the continuity of $\mathcal{A}$, with:

$$\mathcal{A}(u,v) = \int_\Omega A\nabla u \nabla v - \int_\Omega u\vec{b}\cdot\nabla v + \int_\Omega cuv$$

1.12.1 Proof

$$\begin{aligned}
|\mathcal{A}(u,v)| &= \left|\int_\Omega A\nabla u\nabla v - \int_\Omega u\vec{b}\cdot\nabla v + \int_\Omega cuv\right| = \\
&= \left|\int_\Omega A\nabla u\nabla v + \frac{1}{2}\boldsymbol{\nabla}\cdot\vec{b}uv + cuv\right| \leq \int_\Omega |A\nabla u\nabla v + uv(\frac{1}{2}\boldsymbol{\nabla}\cdot\vec{b} + c)| \leq \\
&\leq \int_\Omega |A\nabla u\nabla v| + \int_\Omega |uv(\frac{1}{2}\boldsymbol{\nabla}\cdot\vec{b} + c)| \leq \\
&\leq \|A\|\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \left\|\frac{1}{2}\boldsymbol{\nabla}\cdot\vec{b} + c\right\|\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq \\
&\leq \gamma\|u\|_V\|v\|_V
\end{aligned} \tag{8}$$

where we applied the Cauchy-Schwarz inequality.

1.13 Exercise 13

Céa's lemma: when $\mathcal{A}$ is symmetric, we have:

$$\|u - u_N\|_V \leq \sqrt{\frac{\gamma}{\alpha_0}} \min_{v_N \in V_N} \|u - v_N\|_V$$

1.13.1 Proof

As $\mathcal{A}$ is symmetric and $\mathcal{A}(u-u_N, v_N) = 0\ \forall v_N \in V_N$, we can write:

$$\mathcal{A}(u-u_N, u-u_N) = \min_{v_N \in V_N} \mathcal{A}(u-v_N, u-v_N)$$

Using the coercivity and the continuity we finally obtain:

$$\sqrt{\alpha_0}\|u-u_N\|_V \leq \sqrt{\mathcal{A}(u-u_N, u-u_N)} = \sqrt{\min_{v_N \in V_N} \mathcal{A}(u-v_N, u-v_N)} \leq \sqrt{\gamma} \min_{v_N \in V_N} \|u-v_N\|_V$$
$$\implies \|u-u_N\|_V \leq \sqrt{\frac{\gamma}{\alpha_0}} \min_{v_N \in V_N} \|u-v_N\|_V \tag{9}$$

1.14 Exercise 14

Compute the truncation error for the θ-method.

1.14.1 Proof

Consider:

$$T_i^{n+\frac{1}{2}} = \delta_K^t u(t_{n+\frac{1}{2}}, x_i) - \theta(\delta_h^x)^2 u(t_{n+1}, x_i) - (1-\theta)(\delta_h^x)^2 u(t_n, x_i)$$

Using Taylor expansions:

$$u(t_{n+1}, x_i) = u(t_{n+\frac{1}{2}}, x_i) + \frac{K}{2} u_t(t_{n+\frac{1}{2}}, x_i) + \frac{K^2}{8} u_{tt}(t_{n+\frac{1}{2}}, x_i) + \frac{K^3}{24} u_{ttt}(t_{n+\frac{1}{2}}, x_i)$$

$$u(t_n, x_i) = u(t_{n+\frac{1}{2}}, x_i) - \frac{K}{2} u_t(t_{n+\frac{1}{2}}, x_i) + \frac{K^2}{8} u_{tt}(t_{n+\frac{1}{2}}, x_i) - \frac{K^3}{24} u_{ttt}(t_{n+\frac{1}{2}}, x_i)$$

Obtaining:

$$\delta_K^t u(t_{n+\frac{1}{2}}, x_i) = \frac{u(t_{n+1}, x_i) - u(t_n, x_i)}{K} = u_t(t_{n+\frac{1}{2}}, x_i) + \frac{K^2}{24} u_{ttt}(t_{n+\frac{1}{2}}, x_i)$$

On the other hand we also have:

$$\begin{aligned} &(\delta_h^x)^2 u(t_{n+1}, x_i) = u_{xx}(t_{n+1}, x_i) + \frac{h^2}{12} u_{xxxx}(t_{n+1}, x_i) = \\ &= u_{xx}(t_{n+\frac{1}{2}}, x_i) + \frac{h^2}{12} u_{xxxx}(t_{n+\frac{1}{2}}, x_i) + \frac{K}{2}\left(u_{xxt}(t_{n+\frac{1}{2}}, x_i) + \frac{h^2}{12} u_{xxxxt}(t_{n+\frac{1}{2}}, x_i)\right) + \\ &+ \frac{K^2}{8}\left(u_{xxtt}(t_{n+\frac{1}{2}}, x_i) + \frac{h^2}{12} u_{xxxxtt}(t_{n+\frac{1}{2}}, x_i)\right) \end{aligned} \tag{10}$$

$$\begin{aligned} &(\delta_h^x)^2 u(t_n, x_i) = u_{xx}(t_n, x_i) + \frac{h^2}{12} u_{xxxx}(t_n, x_i) = \\ &= u_{xx}(t_{n+\frac{1}{2}}, x_i) + \frac{h^2}{12} u_{xxxx}(t_{n+\frac{1}{2}}, x_i) - \frac{K}{2}\left(u_{xxt}(t_{n+\frac{1}{2}}, x_i) + \frac{h^2}{12} u_{xxxxt}(t_{n+\frac{1}{2}}, x_i)\right) + \\ &+ \frac{K^2}{8}\left(u_{xxtt}(t_{n+\frac{1}{2}}, x_i) + \frac{h^2}{12} u_{xxxxtt}(t_{n+\frac{1}{2}}, x_i)\right) \end{aligned} \tag{11}$$

Considering the whole expression $\theta(\delta_h^x)^2 u(t_{n+1}, x_i) + (1-\theta)(\delta_h^x)^2 u(t_n, x_i)$ we get:

$$
\begin{aligned}
&\theta(\delta_h^x)^2 u(t_{n+1}, x_i) + (1-\theta)(\delta_h^x)^2 u(t_n, x_i)) = u_{xx}(t_{n+\frac{1}{2}}, x_i) + \frac{h^2}{12} u_{xxxx}(t_{n+\frac{1}{2}}, x_i) + \\
&+ \left(\theta - \frac{1}{2}\right) K \left(u_{xxt}(t_{n+\frac{1}{2}}, x_i) + \frac{h^2}{12} u_{xxxxt}(t_{n+\frac{1}{2}}, x_i) \right) + \\
&+ \frac{K^2}{8} \left(u_{xxtt}(t_{n+\frac{1}{2}}, x_i) + \frac{h^2}{12} u_{xxxxtt}(t_{n+\frac{1}{2}}, x_i) \right)
\end{aligned}
\tag{12}
$$

Putting everything together (stopping to most relevant terms):

$$
\begin{aligned}
|T_i^{n+\frac{1}{2}}| = &\Bigg| u_t(t_{n+\frac{1}{2}}, x_i) - u_{xx}(t_{n+\frac{1}{2}}, x_i) + \left(\frac{1}{2} - \theta\right) K u_{xxt}(t_{n+\frac{1}{2}}, x_i) - \\
&- \frac{h^2}{12} u_{xxxx}(t_{n+\frac{1}{2}}, x_i) + \frac{K^2}{24} u_{ttt}(t_{n+\frac{1}{2}}, x_i) \Bigg|
\end{aligned}
\tag{13}
$$

As the first terms cancel out, we finally get:

$$
|T_i^{n+\frac{1}{2}}| = \left| \left(\frac{1}{2} - \theta\right) K u_{xxt}(t_{n+\frac{1}{2}}, x_i) - \frac{h^2}{12} u_{xxxx}(t_{n+\frac{1}{2}}, x_i) + \frac{K^2}{24} u_{ttt}(t_{n+\frac{1}{2}}, x_i) \right|
$$

Therefore, the truncation error T is $O(K^2, h^2)$ when $\theta = \frac{1}{2}$ (Crank-Nicolson method) and $O(K, h^2)$ when $\theta \neq \frac{1}{2}$.

1.15 Exercise 15

Prove that the θ-method is stable in the l_∞ norm if $\mu(1-\theta) \leq \frac{1}{2}$.

1.15.1 Proof

Recalling the scheme we have:

$$
|(1 + 2\mu\theta) U_i^{n+1}| \leq |\mu\theta(U_{i+1}^{n+1} + U_{i-1}^{n+1})| + |\mu(1-\theta)(U_{i+1}^n + U_{i-1}^n)| + |(1 - 2\mu(1-\theta)) U_i^n|
$$

If $\mu(1-\theta) \leq \frac{1}{2}$ all the coefficients on the right hand side are non-negative and sum to $(1 + 2\mu\theta)$. We then get:

$$
|U_i^{n+1}| \leq \max_{j, t \in (n, n+1)} |U_j^t| = \left\| U^t \right\|_\infty
$$

Applying this process many times we obtain:

$$
|U_i^{n+1}| \leq \left\| U^t \right\|_\infty \leq \cdots \leq \left\| U^0 \right\|_\infty
$$

1.16 Exercise 16

Von-Neumann analysis for the θ-method: compute $\lambda(j)$, where $U_i^n = \lambda(j)^n e^{\iota j (ih)}$.

1.16.1 Proof

Consider the scheme:

$$
(1 + 2\mu\theta) U_i^{n+1} = \mu\theta(U_{i+1}^{n+1} + U_{i-1}^{n+1}) + \mu(1-\theta)(U_{i+1}^n + U_{i-1}^n) + (1 - 2\mu(1-\theta)) U_i^n
$$

If we substitute the mode $U_i^n = \lambda(j)^n e^{\iota j(ih)}$ into the scheme we obtain:

$$\lambda - 1 = \mu(\theta\lambda + (1-\theta))(e^{\iota jh} - 2 + e^{-\iota jh}) = \mu(\theta\lambda + (1-\theta))(-4\sin^2\frac{jh}{2})$$

where we used the fact that $\frac{e^{\iota jh}-2+e^{-\iota jh}}{(2\iota)^2} = \frac{(e^{\frac{\iota jh}{2}}-e^{\frac{-\iota jh}{2}})^2}{(2\iota)^2} = \sin^2\frac{jh}{2}$. From this we get:

$$\begin{aligned}\lambda &= 1 + \mu(\theta\lambda + (1-\theta))(-4\sin^2\frac{jh}{2}) = \\ &= 1 - 4\mu(1-\theta)\sin^2\frac{jh}{2} - 4\mu\theta\lambda\sin^2\frac{jh}{2} \implies \\ &\implies \lambda(1 + 4\mu\theta\sin^2\frac{jh}{2}) = 1 - 4\mu(1-\theta)\sin^2\frac{jh}{2} \implies \\ &\implies \lambda = \frac{1-4\mu(1-\theta)\sin^2\frac{jh}{2}}{1+4\mu\theta\sin^2\frac{jh}{2}}\end{aligned} \tag{14}$$

1.17 Exercise 17

Given the following system of transport equations:

$$\begin{cases} u_t + v_x = 0 \\ u_x + v_t = 0 \end{cases}$$

Prove that v satisfies the wave equation $v_{tt} - v_{xx} = 0$.

1.17.1 Proof

Differentiating the second equation with respect to t and using the first equation, we get:

$$0 = u_{xt} + v_{tt} = (u_t)_x + v_{tt} = -(v_x)_x + v_{tt} = v_{tt} - v_{xx} = 0$$

1.18 Exercise 18

For the wave equation $u_{tt} - u_{xx} = 0$, the energy ϵ is conserved, where:

$$\epsilon = \frac{1}{2}\int_{\mathcal{R}}((u_t)^2 + (u_x)^2)dx$$

1.18.1 Proof

We test with u_t: we multiply the wave equation by u_t and integrate in x:

$$\begin{aligned}u_{tt}u_t - u_{xx}u_t = 0 &\implies \int_{\mathcal{R}} u_{tt}u_t dx - \int_{\mathcal{R}} u_{xx}u_t dx = 0 \\ &\implies \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{R}} u_t^2 dx + \int_{\mathcal{R}} u_x u_{tx} dx = 0 \\ &\implies \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{R}} u_t^2 dx + \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{R}} u_x^2 dx = 0 \\ &\implies \frac{\partial}{\partial t}\left(\frac{1}{2}\int_{\mathcal{R}}((u_t)^2 + (u_x)^2)dx\right) = \frac{\partial\epsilon}{\partial t} = 0 \implies \\ &\implies \epsilon = const = \epsilon(0)\end{aligned} \tag{15}$$

Therefore, the energy ϵ is conserved.

1.19 Exercise 19

Compute the truncation error T_i^n for the *Lax-Wendroff* method.

1.19.1 Proof

Consider the *Lax-Wendroff* method scheme:

$$\frac{u_i^{n+1}-u_i^n}{K}+a\frac{u_{i+1}^n-u_{i-1}^n}{2h}-\frac{Ka^2}{2}\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{h^2}$$

Using Taylor's expansions we can write:

$$u(t_{n+1},x_i)=u(t_n,x_i)+Ku_t(t_n,x_i)+\frac{K^2}{2}u_{tt}(t_n,x_i)+\frac{K^3}{6}u_{ttt}(\psi_n,x_i)$$

with $\psi_n \in (t_n,t_{n+1})$.

$$u(t_n,x_{i+1})=u(t_n,x_i)+hu_x(t_n,x_i)+\frac{h^2}{2}u_{xx}(t_n,x_i)+\frac{h^3}{6}u_{xxx}(t_n,\xi_i)$$

$$u(t_n,x_{i-1})=u(t_n,x_i)-hu_x(t_n,x_i)+\frac{h^2}{2}u_{xx}(t_n,x_i)-\frac{h^3}{6}u_{xxx}(t_n,\eta_i)$$

with $\xi_i \in (x_i,x_{i+1})$, $\eta_i \in (x_{i-1},x_i)$.
We then compute the truncation error T_i^n for the *Lax-Wendroff* scheme using these expansions and get:

$$\begin{aligned}
T_i^n &= \frac{u(t_{n+1},x_i)-u(t_n,x_i)}{K}+a\frac{u(t_n,x_{i+1})-u(t_n,x_{i-1})}{2h}-\frac{Ka^2}{2}\frac{u(t_n,x_{i+1})-2u(t_n,x_i)+u(t_n,x_{i-1})}{h^2}= \\
&= u_t(t_n,x_i)+\frac{K}{2}u_{tt}(t_n,x_i)+\frac{K^2}{6}u_{ttt}(\psi_n,x_i)+au_x(t_n,x_i)+\frac{ah^2}{12}(u_{xxx}(t_n,\xi_i)+u_{xxx}(t_n,\eta_i))- \\
&- \frac{a^2K}{2}(u_{xx}(t_n,x_i))-\frac{a^2Kh}{12}u_{xxx}(t_n,\xi_i)+\frac{a^2Kh}{12}u_{xxx}(t_n,\eta_i)
\end{aligned} \tag{16}$$

Considering the fact that $u_t+au_x=0$ and $u_{tt}=(u_t)_t=(-au_x)_t=-a(u_t)_x=a^2u_{xx}$ we can simplify the expression above and obtain:

$$T_i^n=\frac{K^2}{6}u_{ttt}(\psi_n,x_i)+\frac{ah}{12}(h-aK)u_{xxx}(t_n,\xi_i)+\frac{ah}{12}(h+aK)u_{xxx}(t_n,\eta_i)$$

From which we get:

$$|T_i^n|=\frac{K^2}{6}|u_{ttt}(\psi_n,x_i)|+\frac{|a|h}{12}|h-aK||u_{xxx}(t_n,\xi_i)|+\frac{|a|h}{12}|h+aK||u_{xxx}(t_n,\eta_i)|$$

Finally, considering $|a|\nu \leq 1 \implies |h \pm aK| \leq 2h$, $M_{ttt}=\max|u_{ttt}(t,x)|$ and $M_{xxx}=\max|u_{xxx}(t,x)|$, we finally obtain:

$$|T_i^n| \leq \frac{K^2}{6}M_{ttt}+\frac{|a|h^2}{3}M_{xxx}$$

1.20 Exercise 20

Apply the *leap-frog* method to the wave equation $u_{tt}-u_{xx}=0$.

1.20.1 Proof

We first write the wave equation as a first-order system in U and V, as already done in exercise 17:

$$\begin{cases} U_t + V_x = 0 \\ U_x + V_t = 0 \end{cases}$$

We then discretize U and V by *leap-frog* using a staggered grid:

$$\frac{U_i^{n+\frac{1}{2}} - U_i^{n-\frac{1}{2}}}{K} + \frac{V_{i+\frac{1}{2}}^n - V_{i-\frac{1}{2}}^n}{h} = 0$$

$$\frac{V_{i+\frac{1}{2}}^{n+1} - V_{i+\frac{1}{2}}^n}{K} + \frac{U_{i+1}^{n+\frac{1}{2}} - U_i^{n+\frac{1}{2}}}{h} = 0$$

To obtain the resulting scheme for u we impose $U_i^{n+\frac{1}{2}} = \dfrac{(u_i^{n+1} - u_i^n)}{K}$, $V_{i+\frac{1}{2}}^n = \dfrac{-(u_{i+1}^n - u_i^n)}{h}$ and get:

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{K^2} - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} = 0$$