Investments: Assignment 8

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Problem 1

(a) Let's consider the evolution of the Net asset value of the closed-end fund during a period of time dt. We write NAV(t) the net asset value of the fund at any time t.

Between t and $t+dt$, the net asset value of the fund **increases** of $NAV(t)\cdot(1+R_{t+dt})$ which corresponds on the return on this period. In the same time, the fund pays out investors and management fees which **decrease** the value of the Net Asset Value of : $NAV(t)\cdot(1+R_{t+dt})\cdot(\delta+f)$.

This means that the value of the Nat Asset Value at tiome $t+dt$ can be written : $NAV(t+dt) = NAV(t)\cdot(1+R_{t+dt}) - NAV(t)\cdot(1+R_{t+dt})\cdot(\delta+f)$

Now, the dynamic of the Net Asset Value NAV(t) of the fund is driven by the following equation :

$$\boxed{NAV(t+dt) = NAV(t)\cdot(1+R_{t+dt}) - NAV(t)\cdot(1+R_{t+dt})\cdot(\delta+f)}$$

Equivalent to :

$$\boxed{NAV(t+dt) = NAV(t)\cdot(1+R_{t+dt})\cdot(1-\delta-f)}$$

Which gives in discrete time :

$$\boxed{NAV(t+1) = NAV(t)\cdot(1+R_{t+1})\cdot(1-\delta-f)}$$

(b) We consider that the return of the closed-end fund is given by :

$R_t = R_f + \beta\cdot(R_{M,t} - R_f)$

Considering the previous question written in discrete time, we have the following equations :

$$\begin{aligned}
NAV(t+n) &= NAV(t+n-1)\cdot(1+R_{t+n})\cdot(1-\delta-f)\\
&= NAV(t+n-2)\cdot(1+R_{t+n})\cdot(1+R_{t+n-1})\cdot(1-\delta-f)^2\\
&= ...\\
&= \text{repeat n-2 times}\\
&= ...\\
&= NAV(t)\cdot(1+R_{t+1})\cdot ... \cdot(1+R_{t+n})\cdot(1-\delta-f)^n
\end{aligned} \tag{1}$$

Finally,

$$\boxed{NAV(t+n) = NAV(t)\cdot(1-\delta-f)^n\cdot\prod_{i=1}^{n}(1+R_{t+i})}$$

All returns are independent and identically distributed which means the expectancy of the product is the product of the expectancy, we obtain regarding to the natural filtration $\mathcal{F}_t$:

$$\begin{aligned}
\mathbb{E}_t[NAV(t+n)] &= \mathbb{E}_t\left[NAV(t)\cdot(1-\delta-f)^n\cdot\prod_{i=1}^{n}(1+R_{t+i})\right]\\
&= NAV(t)\cdot(1-\delta-f)^n\cdot\prod_{i=1}^{n}\mathbb{E}_t(1+R_{t+i})\\
&= NAV(t)\cdot(1-\delta-f)^n\cdot\prod_{i=1}^{n}(1+k) \text{ By hypothesis}\\
&= NAV(t)\cdot(1-\delta-f)^n\cdot(1+k)^n
\end{aligned} \tag{2}$$

Finally,

$$\boxed{\mathbb{E}_t[NAV(t+n)] = NAV(t) \cdot (1-\delta-f)^n \cdot (1+k)^n}$$

The funs pays a fraction δ of the net asset value of the fund to investors, let's write CF_{inf} all the cash flows paid to investors now and forever. The dividend received is a fraction of the net asset value of the closed end fund. We must find a relationship between the net asset value at time $t+n$ and at time $t+n-1$ using the formula that have been established above :
$NAV(t+n) = NAV(t) \cdot (1-\delta-f)^n \cdot \prod_{i=1}^{n}(1+R_{t+i})$

$$\begin{aligned} NAV(t+n) &= NAV(t) \cdot (1-\delta-f)^n \cdot \prod_{i=1}^{n}(1+R_{t+i}) \\ &= NAV(t) \cdot (1-\delta-f)^{n-1} \cdot (1-\delta-f) \cdot (1+R_{t+n}) \cdot \prod_{i=1}^{n-1}(1+R_{t+i}) \\ &= (1-\delta-f) \cdot (1+R_{t+n}) \cdot NAV(t+n-1) \end{aligned} \tag{3}$$

The dividend paid out is a fraction δ of the net asset value as explained above compounded with the interest rate which means we obtain :
$Div_{inv} = \delta \cdot NAV(t+n-1) \cdot (1+R_{t+n}) = \frac{NAV(t+n)\cdot\delta}{1-\delta-f}$
Now we are able to calculate the present value of all the cash flows paid to investors which means we sum all the present value of all the cash flows paid out to investors :

$$\begin{aligned} PV(CF_{inv}) &= \mathbb{E}_t\left[\sum_{j=1}^{\infty} \frac{NAV(t+j)\cdot\delta}{(1-\delta-f)\cdot(1+k)^j}\right] \\ &= \frac{\delta}{1-\delta-f} \cdot \sum_{j=1}^{\infty} \frac{\mathbb{E}_t[NAV(t+j)]}{\cdot(1+k)^j} \\ &= \frac{\delta(t)}{1-\delta-f} \cdot \sum_{j=1}^{\infty} \frac{NAV(t)\cdot(1-\delta-f)^i\cdot(1+k)^i}{(1-\delta-f)\cdot(1+k)^j} \\ &= \frac{\delta\cdot NAV(t)}{1-\delta-f} \cdot \sum_{j=1}^{\infty}(1-\delta-f)^i \\ &= \frac{\delta\cdot NAV(t)}{1-\delta-f} \cdot \frac{1}{1-(1-f-\delta)} \cdot (1-f-\delta) \\ &= \frac{\delta\cdot NAV(t)}{\delta+f} \end{aligned} \tag{4}$$

Present value of cash flows paid out to all investors is :

$$\boxed{PV(CF_{inv}) = \frac{\delta\cdot NAV(t)}{\delta+f}}$$

The same calculus would give by analogy - as the fraction of the Net Asset value paid out for managment fees is f - for management fees :

$$\boxed{PV(CF_{fees}) = \frac{f\cdot NAV(t)}{\delta+f}}$$

Detail :

$$
\begin{aligned}
PV(CF_{fees}) &= \mathbb{E}_t\left[\sum_{j=1}^{\infty}\frac{NAV(t+j)\cdot f}{(1-\delta-f)\cdot(1+k)^j}\right] \\
&= \frac{f}{1-\delta-f}\cdot\sum_{j=1}^{\infty}\frac{\mathbb{E}_t[NAV(t+j)]}{\cdot(1+k)^j} \\
&= \frac{f\cdot NAV(t)}{\delta+f}
\end{aligned}
\tag{5}
$$

None of those two values depend on the systematic or the idiosyncratic risk. This might be explained by the fact that the return expected and the discount rate cancel on the present value of cash flows, this means that the return is a convenient one and is what must be expected from the fund. The discount rate which discounts cash flows is steady compared to the expected return.

(c) The average closed end found discount which we write $\mathcal{D}$ is given by the following formula according to what has been written above :

$\mathcal{D} = \frac{NAV(t)-NAV(t)\cdot\frac{\delta}{\delta+f}}{NAV(t)} = 1 - \frac{\delta}{\delta+f} = \frac{f}{\delta+f}$

We plug in the numerical values given in the assignment :

$\mathcal{D} = \frac{0.0044}{0.0044+0.0227} = 16.236\%$

$$\boxed{\text{The average closed-end fund discount implied is : } 16.236\%}$$

(d) The management fees might explain partly the discount puzzle. Indeed, the value we obtain is not absurd and differs only from about 2 points of the observed value. This illustrates how important it is to take those fees into consideration. However, our model is extremely simple and doesn't take into account lots of parameters and it would be a mistake to assert that only management fees are a good explanation for the closed-end fund discount puzzle.