Investments: Report 4

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Problem 1

1. The annualized idiosyincratik risk or PM is

$$\textbf{annualized idiosyncratic risk} = 252 \times Var(\epsilon) = 252 \times std(\epsilon)^2 = 252 \times 0.0140^2 = 0.05081252$$

$$\boxed{\text{Idiosyncratic risk in variance is equal to } 0.05081}$$

2. To compute the systematic risk we use the following formula :

$$R^2 = \frac{\beta_i^2 \cdot \sigma_M^2}{\beta_i^2 \cdot \sigma_M^2 + \nu_i^2}$$

This gives :

$$\beta_i^2 \sigma_M^2 = \frac{R^2 \nu_i^2}{1 - R^2} = \frac{0.416 \cdot 7.35986 \cdot 10^{-2}}{1 - 0.416} = 0.0362$$

$$\boxed{\text{Systematic risk in variance is equal to } 0.0362}$$

3. Total risk is the sum of idiosyncratic and systematic risk :

Finally, using

$$\sigma_i^2 = \beta_i^2 \cdot \sigma_M^2 + \sigma(\epsilon_i)^2$$

$$\boxed{\text{Total Risk in variance} = \text{Systematic} + \text{Idiosyncratic} \quad = 0.087}$$

4. We have

$$\beta_i^2 \cdot \sigma_M^2 = 0.0362 \Leftrightarrow \sigma_M^2 = \frac{0.0362}{0.757^2} = 0.0632$$

$$\boxed{\text{Market Risk in variance} = 0.0632}$$

We take the square root of the volatility portfolio :

$$\boxed{\text{The volatility of the Market portfolio} = 25.13\%}$$

Problem 2

Python part

Problem 3

1. Optimal portfolio is given by :

$$w_o = \frac{1}{a} \cdot \Sigma^{-1} \cdot (\mu - R_0 \cdot \mathbf{1})$$

In matrix form this gives :

$$\begin{aligned} w_{oX} &= \frac{1}{4} \cdot \begin{bmatrix} 0.0144 & 0.0015 & 0.002 \\ 0.0015 & 0.0225 & 0.003 \\ 0.002 & 0.003 & 0.04 \end{bmatrix}^{-1} \cdot \left(\begin{bmatrix} 0.06 \\ 0.08 \\ 0.1 \end{bmatrix} - \begin{bmatrix} 0.03 \\ 0.03 \\ 0.03 \end{bmatrix} \right) \\ &= \frac{1}{4} \cdot \begin{bmatrix} 70.33 & -4.26 & -3.20 \\ -4.26 & 45.15 & -3.17 \\ -3.20 & -3.17 & 25.40 \end{bmatrix} \cdot \begin{bmatrix} 0.03 \\ 0.05 \\ 0.07 \end{bmatrix} \\ &= \begin{bmatrix} 0.418 \\ 0.477 \\ 0.380 \end{bmatrix} \end{aligned}$$

The weight of the portfolio are : $\omega_1 = 0.418, \omega_2 = 0.477$ and $\omega_3 = 0.380$ We observe that the sum of the weight in the optimal portfolio is $\sum_i \omega_i = 1.275$. Because the sum is higher than one, it means that the agent needs to lend money. He needs to lend $\omega_{r_f} = -0.275$ which means he borrows 0.275 at the risk free rate.

2. We assume now that the economy is populated only with mean-variance investors. At equilibrium we can say that the risk-free asset is in zero net supply. We can assume this assumption because some of the agents will not be risk averse and they will short the risk-free asset and other agents are going to be really risk averse and they will invest a lot in the risk-free asset. Hence everything will be compensate. Hence, because this assumption hold at equilibrium, we will obtain that at equilibrium the optimal portfolio will be equal to the the tangency portfolio. The composition of the market portfolio is going to be equal to the composition of the tangency portfolio:

$$w_{tan} = \frac{\Sigma^{-1}(\mu - R_0 \mathbf{1})}{B - AR_0}$$

where the numerator is just equal to $4\omega_0$. A and B are defined as follows:

$$\begin{aligned} A &= \mathbf{1}^{'} \Sigma^{-1} \mathbf{1} \\ &= \mathbf{1}^{'} \begin{bmatrix} 70.33 & -4.26 & -3.20 \\ -4.26 & 45.15 & -3.17 \\ -3.20 & -3.17 & 25.40 \end{bmatrix} \mathbf{1} \\ &= 119.616526 \\ A \cdot R_0 &= 3.58849578 \end{aligned}$$

$$
\begin{aligned}
B &= \mathbf{1}^{'}\Sigma^{-1}\mu \\
&= \mathbf{1}^{'} \begin{bmatrix} 70.33 & -4.26 & -3.20 \\ -4.26 & 45.15 & -3.17 \\ -3.20 & -3.17 & 25.40 \end{bmatrix} \begin{bmatrix} 0.06 \\ 0.08 \\ 0.1 \end{bmatrix} \\
&= 8.69241577
\end{aligned}
$$

Hence we finally obtain that:

$$
\boxed{\omega_{tan} = \frac{\omega_0}{4 \times (8.69241577 - 3.58849578)} = \begin{bmatrix} 0.3278 \\ 0.3737 \\ 0.2985 \end{bmatrix}}
$$

As we can see, the sum of all the weight is now equal to one which make sense since we assumed that the risk-free asset is in zero net supply.

Then the expected return is given by:

$$
\boxed{\mathbf{E}[R] = w_{tan}^{'}\mu = \begin{bmatrix} 0.3278 & 0.3737 & 0.2985 \end{bmatrix} \begin{bmatrix} 0.06 \\ 0.08 \\ 0.1 \end{bmatrix} = 0.079414 \approx 7.94\%}
$$

And the standard deviation is given by:

$$
\boxed{\sigma = \sqrt{\omega_{tan}^{'}\Sigma\omega_{tan}} = \sqrt{\begin{bmatrix} 0.3278 \\ 0.3737 \\ 0.2985 \end{bmatrix}^{'} \begin{bmatrix} 0.0144 & 0.0015 & 0.002 \\ 0.0015 & 0.0225 & 0.003 \\ 0.002 & 0.003 & 0.04 \end{bmatrix} \begin{bmatrix} 0.3278 \\ 0.3737 \\ 0.2985 \end{bmatrix}} = 0.0983959 \approx 9.84\%}
$$

3. We know form the lectures note that in general the optimal portfolio of an agent who has a risk aversion equal to a is proportional to the tangency portfolio as follows:

$$
\omega_o = \frac{B - AR_0}{a}\omega_{tan}
$$

We have seen in the previous question that at equilibrium the optimal portfolio is equal to the tangency portfolio. Hence the economy-wide aggregate risk aversion implicit in the market portfolio is

$$
\boxed{a = B - AR_0 = 8.69241577 - 3.58849578 \approx 5.104}
$$

Note that A and B have been computed in the previous question.

This value is above the risk aversion value used in the first question (4) which means that people are more risk averse which is easy to interpret as in the first question the investor lends at the risk free rate which implies more risk for him.

4. a We Consider now a second mean-variance investor (call her Y) who has the same initial wealth as the investor X. Let's suppose that there are only two investors in this market and that the risk-free asset is in zero net supply. Because investor X is lending $\omega_{r_f X} = -0.275$ of his wealth and because investor Y has the same wealth as investor X. It means that investor Y lends $\boxed{\omega_{r_f Y} = 0.275}$. This is the case because the risk-free asset is indeed in zero net supply.

   b The optimal portfolio is proportional to the tangency portfolio and we know that investor Y is investing $1 - 0.275$ of her wealth in risky assets. Hence:

$$\boxed{\omega_{oY} = \frac{1 - 0.275}{1.275}\omega_{oX} = \begin{bmatrix} 0.2377 \\ 0.2712 \\ 0.2161 \end{bmatrix}}$$

c We know want to compute the risk aversion of investor Y. In order to do this we used the relation between optimal portfolio and tangency portfolio. We use this formula for each risky assets, we should obtain the same result for each risky asset in theory. Because we rounded some results we are not going to obtain exactly the same result for each risky asset but only close result. Hence we are going to take the mean on all the risky assets. For each risky assets we have that:

$$a_Y = \frac{(B - AR_0)\omega_{tan}[i])}{\omega_{oY}[i]}$$

All the components of this equation have been computed in a previous question. We finally get that:

$$\boxed{a_Y \approx 7.04}$$