Assignment 6

1. **APT (50 points)**

   Consider the vector of risky stock returns $R = [R_1 \dots R_N]$, the risk-free rate R_0, and assume the vector or excess returns $R^e = R - R_0\mathbf{1}$ satisfies:

   $$R^e = \alpha + BF + \epsilon$$

   where B is an N, K matrix of factor exposures and F is a K-dimensional vector of random factor realization with $E[F] = \mu_F$ and $V[F] = \Omega_F$ and ϵ is an N-dimensional vector of idiosyncratic risks that satifies $E[\epsilon \mid F] = 0$ and $V[\epsilon] = \Omega_\epsilon$.

   (a) Assume $\Omega_\epsilon = 0$. Show that absence of arbitrage implies that there exists some λ such that $\alpha = B\lambda$.

   (b) Assume that factors are traded assets portfolio excess returns, in the sense that $F_k = R^e_{F_k} = W_k^\top R^e \ \forall k$, for some vector of weights W_k. Show that absence of arbitrage implies that $\alpha = 0$.

   (c) Assume for the following that $\Omega_\epsilon \neq 0$ and $\alpha \neq 0$, and let's define $V[R] = \Sigma = B\Omega_F B^\top + \Omega_\epsilon$. Additionally, define the vector of pure alpha bet excess returns $r^e = R^e - BR^e_F$. Explain what trading strategies (i.e., portfolio weights in various assets) replicate the pure alpha-bet returns.

   Show that the mean-variance efficient portfolio (i.e., that solves $w_{mve} = \text{argmax}_w\{R_0 + w^\top \mu^e - \frac{a}{2} w^\top \Sigma w\}$) can be decomposed into a position in the mean-variance efficient factor portfolio (that invests only in the K factor portfolios and the risk-free rate) and a portfolio of pure alpha bets. Compute the Sharpe ratio of the mean-variance efficient portfolio. Conclude that if $\alpha = 0$ then there is K-fund separation and it is optimal to invest only in the K-factor portfolios.

   Hint: Consider the (mean and variance of a) portfolio that invests in the K factor portfolios and the N pure-alpha bets.

   (d) Assuming that Ω_ϵ is diagonal, what happens to the Sharpe Ratio of the mean-variance portfolio as $n \to \infty$? Is that consistent with the APT?

   (e) Suppose further that factors are traded assets portfolio excess returns, in the sense that $F_k = R^e_{F_k} = W_k^\top R^e \ \forall k$, for some vector of weights W_k. Without loss of generality assume that factors are uncorrelated (note that otherwise you could always "rotate" factors appropriately).

Prove that it is actually not possible for Ω_ϵ to be diagonal in a finite economy (finite number of assets N).

Hint: Use the definition $R_F^e = W^\top R^e$ to decompose factor return total variance into systematic factor risk and idiosyncratic risk components. Use the condition $E[\epsilon \,|\, F] = 0$ to prove that $B = Cov(R^e, F)Var(F)^{-1}$. Use that to derive a contradiction on the factor portfolios' residual risk.

Give an economic interpretation of this result.

2. **Beta and expected returns (50 points).** In this exercise we will test the CAPM using portfolio sorted based on beta.

   Next week, we will continue with the same setup and test momentum and size-sorted portfolios. This two-part exercise will highlight (i) the difference between equal-weighting and value weighting returns, and (ii) the importance of avoiding forward-looking data in testing strategy performances.

   (a) Download monthly stock returns from CRSP for all common stocks (share codes (shcd) 10 and 11) traded on NYSE (exchange codes (excd) 1 and 2) from 1980 to December 31,2021. Also download a risk-free rate and the value-weighted CRSP market return. For the risk-free rate, use the same data that you also used in previous problem sets. Use `select date, vwretd from crsp.msi` to obtain data on the CRSP value-weighted index return. In order to get the exchange codes you can for instance access CRSP's stock event file. You can do this by using a query of the following form:
   `select a.permno, a.date, b.shrcd, b.exchcd, ... (etc.)`
   `from crsp.msf as a left join crsp.msenames as b`
   `on a.permno=b.permno and b.namedt<=a.date ... (etc.)` Delete data of all stocks for which you have less than 504 observations on returns.

   (b) Using the full sample, estimate the market beta for each stock. One way of doing this is to use `df.groupby()` to calculate the relevant moments for each stock,[1] merge these data with the original dataset (left join) and then create a new column

[1] With the .goupby() method you can perform operations on subsets of your data separately, e.g. calculate covariances for each PERMNO.

in the original dataset that contains the market betas (you are however free to choose any procedure that works). For each month, sort stocks by beta into 10 decile portfolios. For each portfolio compute the equal-weighted average average return and compute the beta of the portfolio excess returns with respect to the market excess return for the full sample. Plot the 10 average portfolio returns for the full sample versus the portfolios' betas. If you fit a line through these points, how does the slope of that line compare to the average market excess return for the sample? Are these findings consistent with the CAPM?

(c) Notice that the previous results are forward looking in the sense that the strategy could not have been implemented in real time, since we used the full-sample to estimate the betas. Instead, we would like to have a test that does not suffer from look ahead bias. To that end, compute market betas using the period from 1980 to December 31, 1999. Then, starting in 2000, form 10 portfolios as in point b), but using the betas based on the period from 1980 to 1999. Compute the average returns (for the sample starting in 2000) and betas (for both samples: the first one, starting in 1980 and the second one, starting in 2000) for those portfolios.

Plot the average return to these 'out-of-sample-beta' decile portfolios in the second sample period versus their average beta in the first sample period. Also plot the portfolios' betas in the second sample period against the portfolios' betas in the first sample period. How stable are the betas across periods? Are the out-of-sample findings consistent with the CAPM?

(d) In their Betting-against-Beta paper Frazzini and Pedersen argue that one can make abnormal profits by going long low-beta stocks and shorting high beta stocks, with sufficient leverage in the low-beta portfolio so as to have zero market exposure. To investigate their findings compute the mean, standard deviation, and sharpe ratio in the second period (post 2000) of a strategy that goes long $\frac{1}{\beta_L}$ of the lowest decile beta portfolio financed at the risk-free rate, and goes short $\frac{1}{\beta_H}$ of the highest decile beta portfolio financed at the risk-free rate, where β_L and β_H are the betas of the respective portfolios from the first period (before 2000). Are your findings consistent with the existence of a betting against beta factor? What might explain your findings?