Investments: Report 9

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Problem 1

(a) We want to maximize the following function according to the variable X_1:

$$g(X_1) = R_f + X_1^T(\mu - R_f) - \frac{\gamma}{2} X_1^T \Sigma X_1 - |X_1 - X_0|^T \cdot b$$

We differentiate with respect to X_1:

$$\frac{\partial g}{\partial X_1}(X_1) = (\mu - R_f) - \frac{\gamma}{2} X_1 \Sigma - sign(X_1 - X_0) \cdot b = 0 \tag{1}$$

This gives :

$$X_1 = \frac{\Sigma^{-1}}{\gamma} \cdot [\mu - R_f - b \cdot sign(X_1 - X_0)]$$

We obtain thus two different values for X_1 according to the sign of $X_1 - X_0$:

$$\begin{cases} X_L &= (\gamma \cdot \Sigma)^{-1} \cdot (\mu - R_f - b) \\ X_H &= (\gamma \cdot \Sigma)^{-1} \cdot (\mu - R_f + b) \end{cases}$$

The first case corresponds to the action of **buying** asset 1 which is optimal until X_0 is below the lower bound X_L. On the other side, it is optimal to **sell** the position as long as X_0 is above X_H.

Optimal position is now given by :

$$\text{X}^* = \begin{cases} (\gamma \cdot \Sigma)^{-1} \cdot (\mu - R_f - b) \; if \; X_0 < X_L \\ (\gamma \cdot \Sigma)^{-1} \cdot (\mu - R_f + b) \; if \; X_0 > X_H \\ X_0 \; else \end{cases}$$

By consequence we can conclude that there is no trade if :

$$\boxed{(\gamma \cdot \Sigma)^{-1} \cdot (\mu - R_f - b) \leq X_0 \leq (\gamma \cdot \Sigma)^{-1} \cdot (\mu - R_f + b) \Longleftrightarrow X_L \leq X_0 \leq X_H}$$

(b) We now consider the case where we there are two assets in addition to the risky asset. Where ρ designs the correlation coefficient between the two stocks. The function we want to optimize is the following :

$$h(x_{11}, x_{21}) = R_f + x_{11}(\mu_1 - R_f) - |x_{11} - x_{10}| \cdot b_1 + x_{21}(\mu_2 - R_f) - |x_{21} - x_{20}| \cdot b_2 - \frac{\gamma}{2} \cdot [x_{11}^2 \sigma_1^2 + x_{12}^2 \sigma_2^2 + 2\rho\sigma_1\sigma_2 x_{11} x_{12}]$$

We differentiate with respect to x_{12} and x_{11}

$$\frac{\partial g}{\partial x_{11}}(x_{11}, x_{21}) = (\mu_1 - R_f) - sign(x_{11} - x_{10}) \cdot b_1 - \gamma \cdot (x_{11} \cdot \sigma_1^2 + x_{21} \cdot \sigma_1 \cdot \sigma_2 \cdot \rho) = 0$$

$$\frac{\partial g}{\partial x_{21}}(x_{11}, x_{21}) = (\mu_2 - R_f) - sign(x_{21} - x_{20}) \cdot b_2 - \gamma \cdot (x_{21} \cdot \sigma_2^2 + x_{11} \cdot \sigma_1 \cdot \sigma_2 \cdot \rho) = 0$$

Solving for the optimal condition gives :

$$\boxed{x_{11} = \frac{\gamma^{-1} \cdot [\mu_1 - R_f - sign(x_{11} - x_{10}) \cdot b_1] - x_{21} \cdot \sigma_1 \cdot \sigma_2 \cdot \rho}{\sigma_1^2}}$$

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$$\boxed{x_{21} = \frac{\gamma^{-1} \cdot [\mu_2 - R_f - sign(x_{21} - x_{10}) \cdot b_2] - x_{11} \cdot \sigma_1 \cdot \sigma_2 \cdot \rho}{\sigma_2^2}}$$

We obtain now two bounds for each value of x_{i1} according to the sign of $(x_{i1} - x_{i0})$.

Values for x_{11}

- If $x_{11} - x_{10} > 0$:

$$x_{11} = \frac{\gamma^{-1} \cdot [\mu_1 - R_f - b_1] - x_{21} \cdot \sigma_1 \cdot \sigma_2 \cdot \rho}{\sigma_1^2} = x_{1L} \tag{2}$$

- If $x_{11} - x_{10} < 0$:

$$x_{11} = \frac{\gamma^{-1} \cdot [\mu_1 - R_f + b_1] - x_{21} \cdot \sigma_1 \cdot \sigma_2 \cdot \rho}{\sigma_1^2} = x_{1H} \tag{3}$$

Values for x_{21}

- If $x_{21} - x_{20} > 0$:

$$x_{21} = \frac{\gamma^{-1} \cdot [\mu_2 - R_f - b_2] - x_{11} \cdot \sigma_1 \cdot \sigma_2 \cdot \rho}{\sigma_2^2} = x_{2L} \tag{4}$$

- If $x_{21} - x_{20} < 0$:

$$x_{21} = \frac{\gamma^{-1} \cdot [\mu_2 - R_f + b_2] - x_{11} \cdot \sigma_1 \cdot \sigma_2 \cdot \rho}{\sigma_2^2} = x_{2L} \tag{5}$$

Now we have four boundaries in 2 dimensions which gives us nine trading regions.

Now we find the four points which are going to separate the different trading regions :

We take the first value for x_{11}n (equation 2) and plug in the result for x_{21} in the first case : equation (4), this gives :

$$x_{11} = \frac{\gamma^{-1} \cdot (\mu_1 - R_f + b_1)}{\sigma_1^2} - \left(\frac{\gamma^{-1} \cdot (\mu_2 - R_f - b_2)}{\sigma_2^2} - \frac{x_{11} \cdot \sigma_1 \cdot \rho}{\sigma_2} \right) \cdot \frac{\sigma_2 \cdot \rho}{\sigma_1}$$

$$x_{11} = \frac{\gamma^{-1} \cdot (\mu_1 - R_f + b_1)}{\sigma_1^2} - \frac{\gamma^{-1} \cdot \rho \cdot (\mu_2 - R_f - b_2)}{\sigma_1 \cdot \sigma_2} + x_{11} \cdot \rho^2$$

Now,

$$x_{11} = (\gamma \cdot (1 - \rho^2) \cdot \sigma_1)^{-1} \cdot \left(\frac{\mu_1 - R_f + b_1}{\sigma_1} - \frac{\rho \cdot (\mu_2 - R_f + b_2)}{\sigma_2} \right)$$

We call this value x_{1L1} for the first high value of x_1 :

$$\boxed{x_{1H1} = (\gamma \cdot (1 - \rho^2) \cdot \sigma_1)^{-1} \cdot \left(\frac{\mu_1 - R_f + b_1}{\sigma_1} - \frac{\rho \cdot (\mu_2 - R_f + b_2)}{\sigma_2} \right)}$$

Now, the second low value of x_1 can be found by plug in the other value of x_{21}, the calculation are exactly the same as before with $+b_2$ instead of $-b_2$. This gives us the following answer (the other lower bound for x_1):

$$\boxed{x_{1H2} = (\gamma \cdot (1-\rho^2) \cdot \sigma_1)^{-1} \cdot \left(\frac{\mu_1 - R_f + b_1}{\sigma_1} - \frac{\rho \cdot (\mu_2 - R_f - b_2)}{\sigma_2} \right)}$$

Now we find the high value of x_1 by plugging in the value of the first $x_2 1$ in the second value of $x_1 1$, the difference with the previous question will lie in the b_1 of the first part of the equation, we call those value x_{1L1} and x_{1L2} :

$$\boxed{x_{1L1} = (\gamma \cdot (1-\rho^2) \cdot \sigma_1)^{-1} \cdot \left(\frac{\mu_1 - R_f - b_1}{\sigma_1} - \frac{\rho \cdot (\mu_2 - R_f + b_2)}{\sigma_2} \right)}$$

$$\boxed{x_{1L2} = (\gamma \cdot (1-\rho^2) \cdot \sigma_1)^{-1} \cdot \left(\frac{\mu_1 - R_f - b_1}{\sigma_1} - \frac{\rho \cdot (\mu_2 - R_f - b_2)}{\sigma_2} \right)}$$

For the values of x_2 we obtain symmetrical result and we call them as it follows :

$$\boxed{x_{2L1} = (\gamma \cdot (1-\rho^2) \cdot \sigma_2)^{-1} \cdot \left(\frac{\mu_2 - R_f - b_2}{\sigma_2} - \frac{\rho \cdot (\mu_1 - R_f + b_1)}{\sigma_1} \right)}$$

$$\boxed{x_{2L2} = (\gamma \cdot (1-\rho^2) \cdot \sigma_2)^{-1} \cdot \left(\frac{\mu_2 - R_f + b_2}{\sigma_2} - \frac{\rho \cdot (\mu_1 - R_f + b_1)}{\sigma_1} \right)}$$

$$\boxed{x_{2H1} = (\gamma \cdot (1-\rho^2) \cdot \sigma_2)^{-1} \cdot \left(\frac{\mu_2 - R_f + b_2}{\sigma_2} - \frac{\rho \cdot (\mu_1 - R_f - b_1)}{\sigma_1} \right)}$$

$$\boxed{x_{2H2} = (\gamma \cdot (1-\rho^2) \cdot \sigma_2)^{-1} \cdot \left(\frac{\mu_2 - R_f - b_2}{\sigma_2} - \frac{\rho \cdot (\mu_1 - R_f - b_1)}{\sigma_1} \right)}$$

We can now conclude as in the previous question, we have a lower bound and a higher bound for each asset. If our initial position stands between two boundaries, it will optimal not to trade the asset, if our initial position is above a higher bound it is optimal to **sell** and if we are below a lower bound it is optimal to **buy** the asset.

The value of the intervals have been given in equation (2) to (5):

To sum up :

- If X_{10} and X_{20} are both in intervals $I = [X_{1L}; X_{1H}]$ and $J = [X_{2L}; X_{2H}]$, we are in a no trading region and it is thus not optimal to change the position at time 1 : NO TRADE/NO TRADE
- If X_{10} is above the interval there are three possibilities :
  - X_{20} is in J, the optimal position is to Sell asset 1 and no trade asset 2 : SELL/NO TRADE
  - X_{20} is above J, the optimal position is to Sell asset 1 and asset 2 : SELL/SELL
  - X_{20} is below J, the optimal position is to Sell asset 1 and buy 2 : SELL/BUY
- If X_{10} is below above the interval there are three possibilities again:
  - X_{20} is in J, the optimal position is to Sell asset 1 and no trade asset 2 : BUY/NO TRADE
  - X_{20} is above J, the optimal position is to Sell asset 1 and asset 2 : BUY/SELL
  - X_{20} is below J, the optimal position is to Sell asset 1 and buy 2 : BUY/BUY
- Finally if $X_1 0$ is in I there are two more possibilities :
  - X_{20} is above J, the optimal position is not to trade asset 1 and sell 2 : NO TRADE/SELL
  - X_{20} is below J, the optimal position is not to trade asset 1 and buy 2 : NO TRADE/BUY

(c) The plot are provided in the Jupyter notebook attached, below are the answers to the question (also given in the notebook to explain the graphs).

- As we have four points which delimit our No trade region in a 2D graph, with coordinates : $(X_{1L1}, X_{2L1}), (X_{1H1}, X_{2H1}), (X_{1L2}, X_{2L2}), (X_{1H2}, X_{2H12})$ the shape of the no trade region is going to be a parallelogram. We can notice that when the correlation is equal to 0, $\rho = 0$, we boundaries will be constant : x_H and x_L won't depend on the value of x_21, and thus we obtain a **rectangle**, otherwise a **parallelogram**.
- When we increase correlation between the two assets, the parallelogram will flatten and the no trade region has a smaller area (we can notice that a correlation of 1 is impossible has we divide by $1 - \rho^2$ in or equations).
- When we make asset 2 riskier than asset 1, the no trade region will also flatten. To make this happen, we increase the volatility of asset 2 and make it bigger than asset 1 in percentage.

To conclude, increasing correlation will increase downward sloping. As there is more correlation, when increase a position in an asset, the marginal impact of an increase on the other asset on portfolio variance increase. By consequence, we might decrease the minimal position in the asset where we stop decreasing the position in this asset. This means we have a higher downward when increasing correlation between the two assets and volatility of asset 2 with respect to asset 1.

Problem 2

In this exercise, we consider three type of funds. A passive mutual fund, an active mutual fund and an hedge fund. These funds have the following returns:

$$\begin{aligned} R_t^{\text{Passive before fees}} &= R_t^{\text{stock index}} \\ R_t^{\text{Active before fees}} &= 2.20\% + R_t^{\text{stock index}} + \epsilon_t \\ R_t^{\text{Hedge Fund before fees}} &= 1\% + 4(R_t^{\text{Active before fees}} - R_t^{\text{stock index}}) \end{aligned}$$

We know that ϵ_t has a zero mean and a volatility equal to 3.5%. Moreover, the fees are the following:

Fund	Passive Mutual Fund	Active Mutual Fund
Management Fees	0.1%	1.2%

The risk free rate is $r_f = 1\%$

(a) In this first question, we want top determine the Hedge Fund return's volatility. From the return of all the fund we got that:

$$\begin{aligned} Var(R_t^{\text{Hedge Fund bf}}) &= 4^2(Var(R_t^{\text{Active bf}}) + Var(R_t^{\text{stock index}}) - 2Cov(R_t^{\text{Active bf}}, R_t^{\text{stock index}})) \\ &= 16(Var(R_t^{\text{stock index}}) + Var(\epsilon_t) + Var(R_t^{\text{stock index}}) - 2Var(R_t^{\text{stock index}})) \\ &= 16Var(\epsilon_t) \end{aligned} \tag{6}$$

where we have use the fact that $Cov(R_t^{\text{stock index}}, \epsilon_t) = 0$ in order to compute the variance of the active fund and the covariance terms.

Finally we obtain that the volatility of the Hedge Fund return's is:

$$\boxed{std(R_t^{\text{Hedge Fund bf}}) = 4 \cdot std(\epsilon_t) = 4 \cdot 3.5\% = 14\%} \tag{7}$$

(b) In this second question we are interesting in the beta of the hedge fund. In order to answer this question we remind to the reader that the stock returns must satisfy the following equation:

$$R^e = \alpha + \beta(R_t^{\text{excess premium stock index}}) + \epsilon_t \tag{8}$$

Return of the hedge fund could be rewrite as above:

$$\begin{aligned} R_t^{\text{Excess Hedge Fund bf}} &= 1\% + 4 \cdot (R_t^{\text{Active bf}}) - 4 \cdot (R_t^{\text{stock index}}) - r_f \\ &= 1\% + 4 \cdot (2.20\% + R_t^{\text{stock index}} + \epsilon_t) - 4 \cdot R_t^{\text{stock index}} - r_f \\ &= 9.80\% + 4 \cdot \epsilon_t - r_f \\ &= 8.80\% + 4 \cdot \epsilon_t \end{aligned} \tag{9}$$

This last equality implies that the Hedge Fund has a $\boxed{\beta = 0}$

(c) We are now interesting in the α of the Hedge Fund before any fees.

We can answer this question by using the result from the previous question. The last equation in the last question gives us that $\boxed{\alpha = 8.80\%}$ and this is what we expect since the active fund has an $\alpha = 2.20\%$ and the Hedge fund strategy is to long the active mutual fund and use a leverage of 4.

(d) Suppose that an investor has \$40 invested in the active mutual fund and \$60 in cash (measured in thousands, say). We want to find portfolio using only cash, hedge fund and passive mutual fund that

Problem 2 continued on next page...

will have the same market exposure, same alpha, same volatility and same exposure to ϵ_t. First of all we rewrite the investor portfolio:

$$\begin{aligned} R_{investor_t} &= 0.4 \cdot R_t^{\text{Active bf}} + 0.6 \cdot r_f \\ R^e_{investor_t} &= 0.4 \cdot (2.20\% + R_t^{\text{stock index}} + \epsilon_t) - 0.4 \cdot r_f \\ R^e_{investor_t} &= 0.88\% + 0.4 \cdot R_t^{\text{excess premium stock index}} + 0.4 \cdot \epsilon_t \end{aligned} \tag{10}$$

It implies that $\alpha_{investor} = 0.88\%$, $\beta_{investor} = 0.4$ and $\varepsilon_{investor} = 0.4$.

If we construct a portfolio with the passive mutual fund, the hedge fund and the risk-free rate, we get the following portfolio:

$$\begin{aligned} R^e_{portfolio} &= x_1 \cdot R_t^{\text{stock index}} + x_2 \cdot R_t^{\text{Hedge Fund bf}} + x_3 r_f - r_f \\ &= x_1 \cdot R_t^{\text{stock index}} + x_2 \cdot (9.80\% + 4 \cdot \epsilon_t) + x_3 r_f - r_f + x_1 \cdot r_f - x_1 \cdot r_f \\ &= (x_2 \cdot 9.80\% - r_f \cdot (1 - x_3 - x_1)) + x_1 \cdot R_t^{\text{excess premium stock index}} + 4 \cdot x_2 \cdot \epsilon_t \end{aligned} \tag{11}$$

Hence in order to answer the question asked, we have to impose:

x_1	0.4
x_2	0.1
x_3	$1 - 0.4 - 0.1 = 0.5$

With this weights we get that the alpha of the replicating portfolio is equal to:
$0.1 \cdot 9.80\% - r_f \cdot (1 - 0.5 - 0.4) = 0.980\% - 0.1\% = 0.88\%$.

By investing in the replicating portfolio and have the same market exposure, same volatility, same alpha and same exposure to ϵ_t, the investor should invest:

Passive Mutual Fund	\$40
Hedge Fund	\$10
Risk-free Rate	\$50

By keeping its strategy the investor will paid fees that comes from management of the Active mutual fund. He will pay : $\$40 \cdot 1.20\% = \$0.48$.

By choosing the replicating portfolio, the investor will pay the fees that comes from the Passive mutual fund and the fees that comes from the Hedge Fund.
Then he will pay : $\$40 \cdot 0.10\% + \$10 \cdot$ Hedge Fund's management fees.

Then the fair management fee for the hedge fund in the sense that it would make the investor indifferent between the two allocations is:

$$\boxed{\text{Hedge Fund's management fees} = \frac{\$40 \cdot 1.20\% - \$40 \cdot 0.10\%}{\$10} = 4.40\%} \tag{12}$$

(e) We consider now that there is also performance fee for the Hedge fund and that the hedge fund charges a management fee of 2%. We want to compute the performance fees that the Hedge fund must charge in order to have the same expected fees a above. We remind to the reader that the performance fees are a percentage of the excess return after management fees.

From question b), since ϵ_t has a zero mean, the expected excess return after management fees of the Hedge Fund is $8.80\% - 2\% = 6.80\%$. Then the fees that the investor must pay by putting his/her money in the replicating portfolio must respect the following equation:

$$\$40 \cdot 1.20\% = \$40 \cdot 0.10\% + \$10 \cdot 2\% + \$10 \cdot 6.80\% \cdot \text{Hedge Fund's performance fees} \tag{13}$$

It implies that:

$$\boxed{\text{Hedge Fund's performance fees} = \frac{\$40 \cdot 1.20\% - \$40 \cdot 0.10\% - \$10 \cdot 2\%}{\$10 \cdot 6.80\%} = 35.29\%} \tag{14}$$

(f) From the last question it seems that Hedge fund that charge 2-20 fees are not expensive relative to a typical mutual funds, they are equally expensive. The active management fees must be determine by the performance of the fund. Indeed, agent accept to spend money in these funds because they expect excess return relative to easily replicated factors premia (e.g. market factors).