Investments: Report 3

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Problem 1

a) The optimization problem we want to solve is the following one :

$$max_{\omega_i} \left(\mathbb{E}[R_p] - \frac{\gamma}{2} \cdot \mathbb{V}[R_p] \right)$$

Subject to

$$\sum_i \omega_i = 1$$

We know that $R_p = \sum_i \omega_i \cdot R_i$. Moreover, the risk less asset has zero variance (no volatility). We can by consequence write the function to optimize as :

$$max_{\omega_i} \left(R_f + \mathbb{E}\left[\sum_{i=1}^{N} \omega_i \cdot (R_i - R_f) \right] - \frac{\gamma}{2} \cdot \mathbb{V}\left[\sum_{i=1}^{N} \omega_i \cdot R_i \right] \right)$$

Rewriting the variance and using $\mathbb{E}(R_i) = \mu_i$ we obtain :

$$max_{\omega_i} \left(R_f + \sum_{i=1}^{N} \omega_i \cdot (\mu_i - R_f) - \frac{\gamma}{2} \cdot \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i \cdot \omega_j \cdot Cov(R_i, R_j) \right)$$

We now write the i-th first order condition for the function in the minimum bracket

$$\frac{\partial}{\partial \omega_i} \left(R_f + \sum_{i=1}^{N} \omega_i \cdot (\mu_i - R_f) - \frac{\gamma}{2} \cdot \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i \cdot \omega_j \cdot Cov(R_i, R_j) \right) = 0$$

$$\mu_i - R_f - \gamma \cdot \omega_i \cdot Cov(R_i, R_i) - \frac{\gamma}{2} \cdot \sum_{j \neq i} \omega_j \cdot Cov(R_i, R_j) - \frac{\gamma}{2} \cdot \sum_{j \neq i} \omega_j \cdot Cov(R_j, R_i) = 0$$

Now, as the covariance is symmetric, we can write the last expression as :

$$\mu_i - R_f - \gamma \cdot \omega_i \cdot Cov(R_i, R_i) - \gamma \cdot \sum_{j \neq i} \omega_j \cdot Cov(R_i, R_j) = 0$$

$$\mu_i - R_f - \gamma \cdot \sum_{j} \omega_j \cdot Cov(R_i, R_j) = 0$$

The last expression is equal to $Cov(R_i, R_p)$

Finally,

$$\mu_i - R_f - \gamma \cdot Cov(R_i, R_p) = 0$$

We obtain the required result :

$$\boxed{\mu_i - R_f = \gamma \cdot Cov[R_i, R_p]}$$

b) We call A the variance/covariance matrix, by definition we have : $\omega^T \cdot A \cdot \omega = \sigma_p^2$

Proof :

$$\begin{aligned}
\omega^T \cdot A \cdot \omega &= (\sum_i \omega_i \cdot Cov(R_i, R_1), \sum_i \omega_i \cdot Cov(R_i, R_2), ..., \sum_i \omega_i \cdot Cov(R_i, R_N)) \cdot \omega \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i \cdot \omega_j \cdot Cov(R_i, R_j) \\
&= \sigma_p^2
\end{aligned} \tag{1}$$

We write the expression proven in the previous question in matrix form :

$$\mu - R_f \mathbb{1} = \gamma \cdot A \cdot \omega$$

Multiplying both sides by ω^T we obtain :

$$\mu_p - R_f = \gamma \cdot \sigma_p^2$$

We obtain now :

$$\frac{\mu - R_f \mathbb{1}}{\mu_p - R_f} = \frac{\gamma \cdot A \cdot \omega}{\gamma \cdot \sigma_p^2}$$

And using the fact that the i-th element of $A \cdot \omega$ is $Cov(R_i, r_p)$

Taking the i-th element we obtain the equation asked :

$$\boxed{\mu_i - R_f = \beta_{i,p} \cdot (\mu_p - R_f)}$$

c) We consider a linear regression of R_i on $R_p - R_f$ which gives the following equality :

$$R_i = \beta_0 + \beta_1 \cdot (R_p - R_f) + \epsilon_i$$

Where β_0 is the intercept, β_1 the slope and ϵ_i the errors terms where we suppose $\mathbb{E}(\epsilon_i) = 0$ and that the error term are uncorrelated with other terms : $Cov(\epsilon_i, R_p - R_f) = 0$

$$\begin{aligned}
Cov(\epsilon_i, R_p - R_f) &= Cov(R_i - \beta_0 - \beta_1 \cdot (R_p - R_f), R_p - R_f) \\
&= Cov(R_i, R_p - R_f) - \beta_1 \cdot Cov(R_p - R_f, R_p - R_f) \\
&= Cov(R_i, R_p) - \beta_1 \cdot Cov(R_p, R_p) \text{ (Using } \mathbb{V}(R_f) = 0) \\
&= 0 \text{ (By hypothesis)}
\end{aligned}$$

Now we can write $\beta_1 = \frac{Cov(R_i, R_p)}{\mathbb{V}(R_p)} = \frac{Cov(R_i, R_p)}{\sigma_p^2}$

The linear regression is now :

$$R_i = \beta_0 + \frac{Cov(R_i, R_p)}{\sigma_p^2} \cdot (R_p - R_f) + \epsilon_i$$

Taking expectancy on both side of the equation and substracting by R_f :

$$\mu_i - R_f = \beta_0 - R_f + \frac{Cov(R_i, R_p)}{\sigma_p^2} \cdot (R_p - R_f) + \epsilon_i$$

Now, we use

$$\mu_i - R_f = \beta_{i,p} \cdot (\mu_p - R_f)$$

This gives $\beta_0 = R_f$ and the result for the linear regression.

d) We saw in the lecture that all mean-variance efficient portfolio are a linear combination of the risk free asset and the tangency portfolio. If we consider two mean-variance efficient portfolios called P_1 and P_2 we have :

$$\frac{\mu_{P_1} - R_f}{\sigma_{P_1}} = \frac{\frac{Cov(R_{P_1}, R_{P_2})}{\sigma_{P_2}^2} \cdot (\mu_{P_2} - R_f)}{\sigma_{P_1}} = \frac{\mu_{P_2} - R_f}{\sigma_{P_2}} \cdot \frac{Cov(R_{P_1}, R_{P_2})}{\sigma_{P_2} \cdot \sigma_{P_1}}$$

The term

$$\frac{Cov(R_{P_1}, R_{P_2})}{\sigma_{P_2} \cdot \sigma_{P_1}}$$

is the correlation between the two portfolio which is equal to one as they both lie on the mean variance efficience frontier.

Now, both portoflios which were randomly chosen have the same sharp ratio which means all portfolios on the mean-variance frontier have the same sharp ratio.