Investments: Report 10

Due on May, Tuesday 24^{th}, 2022

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Problem 1

(a) We have :

$$V(k, n_{k-1}) = max_{n_t, t \geq k} \mathbb{E}\left[\sum_{t=k}^{T} \rho^{t-k}\left(n_t \cdot \mu - \frac{\lambda}{2} \cdot (n_t - n_{t-1})^2 - \frac{\gamma}{2} \cdot n_t^2 \cdot \sigma^2\right)\right]$$

Now,

$$V(T, n_{T-1)} = max_{n_T} \mathbb{E}_T[n_T \cdot \mu - \frac{\lambda}{2} \cdot (n_T - n_{T-1})^2 - \frac{\gamma}{2} \cdot n_T^2 \cdot \sigma^2]$$

We write the first order condition :

$$\begin{aligned} \frac{\partial V}{\partial n_T} &= \mu - \lambda \cdot (n_T - n_{T-1}) - \gamma \cdot n_T \cdot \sigma^2 \\ 0 &= \mu - n_T \cdot (\lambda + \gamma \cdot \sigma^2) + \lambda \cdot n_{T-1} \\ n_T &= \frac{\mu + \lambda \cdot n_{T-1}}{\lambda + \gamma \cdot \sigma^2} \end{aligned} \tag{1}$$

Now we obtain the optimal value for the n umber of share :

$$\boxed{n^*(T, n_{T-1}) = \frac{\mu + \lambda \cdot n_{T-1}}{\lambda + \gamma \cdot \sigma^2}}$$

We plug in the value for the optimal number of share we found above in $V(T, n)$ which has been explicitly given at the beginning of the exercise.

$$\begin{aligned} V(T, n) &= \frac{\mu^2 + \lambda \cdot \mu \cdot n_{T-1}}{\lambda + \gamma \cdot \sigma^2} - \frac{\lambda}{2} \cdot \left(\frac{\mu - \gamma \cdot \sigma^2 \cdot n_{T-1}}{\lambda + \gamma \cdot \sigma^2}\right)^2 - \frac{\gamma}{2} \cdot \left(\frac{\mu + \lambda \cdot n_{T-1}}{\lambda + \gamma \cdot \sigma^2}\right)^2 \cdot \sigma^2 \\ &= \frac{\mu^2 + \lambda \cdot \mu \cdot n_{T-1}}{\lambda + \gamma \cdot \sigma^2} - \frac{\frac{\lambda \cdot \mu^2}{2} - \lambda \cdot \gamma \cdot \sigma^2 \cdot n_{T-1} + \frac{\lambda \cdot \gamma^2 \cdot \sigma^4 \cdot n_{T-1}^2}{2}}{(\lambda + \gamma \cdot \sigma^2)^2} - \frac{\frac{\lambda \cdot \mu^2 \cdot \sigma^2}{2} + \lambda \cdot \gamma \cdot \sigma^2 \cdot n_{T-1} + \frac{\lambda^2 \cdot \gamma \cdot \sigma^2 \cdot n_{T-1}^2}{2}}{(\lambda + \gamma \cdot \sigma^2)^2} \\ &= \frac{1}{(\lambda + \gamma \cdot \sigma^2)} \cdot \left[\frac{\mu^2}{2} + \lambda \cdot \mu \cdot n_{T-1} - \frac{1}{2} \frac{\gamma \cdot \lambda^2 \cdot \sigma^2 \cdot (1 + \sigma^2)}{\lambda + \gamma \cdot \sigma^2} \cdot n_{T-1}^2\right] \end{aligned}$$

Now we can identify the values for the constant required :

$$\boxed{Q_T = \frac{\lambda \cdot \gamma \cdot \sigma^2}{\lambda + \gamma \cdot \sigma^2}}$$

$$\boxed{q_T = \frac{\lambda \cdot \mu}{\lambda + \gamma \cdot \sigma^2}}$$

$$\boxed{C_T = \frac{\mu^2}{2 \cdot (\lambda + \gamma \cdot \sigma^2)}}$$

(b) We replace the value of $V(t+1, n)$ in the expression of $V(t, n_{t-1})$. This yields to :

$$V(t, n) = max_{n_t}(n_t \cdot \mu - \frac{\lambda}{2} \cdot (n_t - n_{t-1})^2 - \frac{\gamma}{2} \cdot n_t^2 + \rho \cdot (-\frac{1}{2} \cdot Q_{t+1} \cdot n_t^2 + q_{t+1} \cdot n_t + C_{t+1}))$$

The first order condition gives :

$$0 = \mu - \lambda \cdot (n_t - n_{t-1}) - \gamma \cdot n_t \cdot \sigma^2 - \rho \cdot Q_{t+1} \cdot n_t + \rho \cdot q_{t+1}$$

Solving for n_t gives :

$$\boxed{n_t^* = \frac{\mu + \rho \cdot q_{t+1} + \lambda \cdot n_{t-1}}{\lambda + \gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}}}$$

Now we use the Bellman equation provided to show the recursive form required : Let $\alpha = \lambda + \gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}$

$$\begin{aligned}
V(t, n-1) &= n_t \cdot \mu - \frac{\lambda}{2} \cdot n_t^2 + \lambda \cdot n_t \cdot n_{t-1} - \frac{\lambda}{2} \cdot n_{t-1}^2 - \frac{\gamma}{2} \cdot n_t^2 \sigma^2 - \frac{1}{2} \cdot Q_{t+1} \rho \cdot n_t^2 + \rho \cdot q_{t+1} \cdot n_t + \rho \cdot c_{t+1} \\
&= \frac{\mu^2 + \rho \cdot q_{t+1}}{\alpha} + \frac{\lambda \cdot \mu \cdot n_{t-1}}{\alpha} - \frac{\lambda}{2} \cdot \frac{\left[(\mu + \rho \cdot q_{t+1})^2 + 2 \cdot \lambda \cdot n_{t-1} \cdot (\mu + \rho \cdot q_{t+1}) + \lambda^2 \cdot n_{t-1}^2\right]}{\alpha^2} + \\
&\lambda \cdot n_{t-1} \cdot \frac{\mu + \rho \cdot q_{t+1}}{\alpha} + \frac{(\lambda \cdot n_{t-1})^2}{\alpha} - \frac{\lambda}{2} \cdot n_{t-1}^2 - \\
&\frac{\gamma}{2} \cdot \sigma^2 \cdot \frac{(\mu + \rho \cdot q_{t+1})^2 + 2 \cdot \lambda \cdot n_{t-1} + (\mu + \rho \cdot q_{t+1}) + \lambda^2 \cdot n_{t-1}^2}{\alpha^2} \\
&- \frac{1}{2} \cdot Q_{t+1} \cdot \rho \cdot \frac{(\mu + \rho \cdot q_{t+1})^2 + 2 \cdot \lambda \cdot n_{t-1} + (\mu + \rho \cdot q_{t+1}) + \lambda^2 \cdot n_{t-1}^2}{\alpha^2} \\
&+ \rho \cdot q_{t+1} \cdot \frac{\mu + \rho \cdot q_{t+1} + \lambda \cdot n_{t-1}}{\alpha} + \rho \cdot C_{t+1}
\end{aligned}$$

We now identify the different coefficients :

$$\begin{aligned}
C_t &= \frac{\mu^2 + \rho \cdot q_{t+1} \cdot \mu}{\alpha} - \frac{\lambda \cdot \mu^2 + 2 \cdot \lambda \cdot \rho \cdot q_{t+1} + \lambda \cdot \rho^2 \cdot q_{t+1}^2}{2 \cdot \alpha^2} - \frac{\gamma \cdot \sigma^2 \cdot \mu^2 + 2 \cdot \gamma \cdot \sigma^2 \cdot \mu \cdot \rho \cdot q_{t+1} + \gamma \cdot \sigma^2 \cdot \rho^2 \cdot q_{t+1}^2}{2 \cdot \alpha^2} \\
&- \frac{Q_{t+1} \cdot \rho \cdot \mu^2 + 2 \cdot Q_{t+1} \cdot \rho^2 \cdot \mu \cdot q_{t+1} + Q_{t+1} \cdot \rho^3 \cdot q_{t+1}^1}{2 \cdot \alpha^2} + \frac{\rho \cdot q_{t+1} \cdot \mu + \rho^2 \cdot q_{t+1}^2}{\alpha} + \rho \cdot C_{t+1} \\
&= \rho \cdot C_{t+1} + \frac{1}{2} \cdot \frac{(\mu + \rho \cdot q_{t+1})^2}{\lambda + \gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}}
\end{aligned}$$

$$\begin{aligned}
q_t &= \frac{\lambda \cdot \mu + \lambda \cdot \mu + \lambda \cdot \rho \cdot q_{t+1} + \lambda \cdot \rho \cdot q_{t+1}}{\alpha} - \frac{\lambda \cdot (\mu + \rho \cdot q_{t+1})}{\alpha} \\
&= \frac{2 \cdot \lambda \cdot (\mu + \rho \cdot q_{t+1})}{\alpha} - \frac{\lambda}{\cdot} (\mu + \rho \cdot q_{t+1}) \alpha \\
&= \frac{\lambda \cdot (\mu + \rho \cdot q_{t+1})}{\lambda + \gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}}
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{2} \cdot Q_t &= -\frac{\lambda^2}{2 \cdot \alpha} - \frac{\lambda}{2} + \frac{\lambda^2}{\alpha} \\
Q_t &= \lambda - \frac{\lambda^2}{\lambda + \gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}}
\end{aligned}$$

Now, recursively we can solve for all time t from T to 1 by considering the following values : $Q_{T+1} = q_{T+1} = C_{T+1} = 0$

(c) We now that the aim_t portfolio maximizes the value function at any time t which means at any time the derivative of the value function taking at aim_t is equal to 0. We can thus write :

$$\begin{aligned}
\frac{\partial V(t+1,n)}{\partial aim_t} &= \frac{\partial V}{\partial aim_t}(-\frac{1}{2}\cdot aim_t^2 \cdot Q_{t+1} + aim_t \cdot q_{t+1} + c_{t+1}) \\
0 &= -aim_t \cdot Q_{t+1} + q_{t+1} \\
aim_t &= \frac{q_{t+1}}{Q_{t+1}} = \frac{\mu + \rho \cdot q_{t+2}}{\gamma \cdot \sigma^2 + \rho \cdot Q_{t+2}}
\end{aligned}$$

We use the expression of the optimal number of shares :

$$\begin{aligned}
n_t^* &= \frac{\mu + \rho \cdot q_{t+1} + \lambda \cdot n_{t-1}}{\lambda + \gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}} \\
&= \frac{\gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}}{\lambda + \gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}} \cdot \frac{\mu + \rho \cdot q_{t+1}}{\rho \cdot Q_{t+1} + \gamma \cdot \sigma^2} + \frac{\lambda}{\lambda + \gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}} \cdot n_{t-1} \\
&= \tau_{t-1} \cdot aim_{t-1} + (1 - \tau_{t-1}) \cdot n_{t-1}
\end{aligned}$$

With the following value for τ_t :

$$\boxed{\tau_t = \frac{\gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}}{\lambda + \gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}}}$$

(d) see python code

(e) Since we are only considering the quadratic cost on the number of share sells from t to $t+1$, it implies that the optimal trading strategy does not depends of the price shocks. Moreover, a price shock does not affect the expected future return and neither the future return variance, then it does not change the optimal strategy which only depends of future expected return and future variance return. From this aspects, it is optimal to trade without analysing the price shocks. However, this assumption is not reasonable in the real life. In such a situation, a proportional cost in addition to the quadratic cost must be a a reasonable way to compute the total cost of the trade because the bid-ask spread will increase. An other explanation of why it is not reasonable is the momentum strategy. We saw with the momentum strategy that there is an auto-correlation with return of a small time interval.

(f) Taking into account lambda varying through time we get:

$$V(t, n_{t-1}) = max_{n_t}\left\{n_t \cdot \mu - \frac{\lambda_t}{2} \cdot (n_t - n_{t-1})^2 - \frac{\gamma}{2} \cdot n_t^2 \cdot \sigma^2 + \rho \mathbb{E}_T[V(t+1, n_t)]\right\}$$

$$\boxed{n_t^* = \frac{\mu + \rho \cdot q_{t+1} + \lambda_t \cdot n_{t-1}}{\lambda_t + \gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}}}$$

$$C_t = \rho \cdot C_{t+1} + \frac{1}{2} \cdot \frac{(\mu + \rho \cdot q_{t+1}^2)}{\lambda_t + \gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}}$$

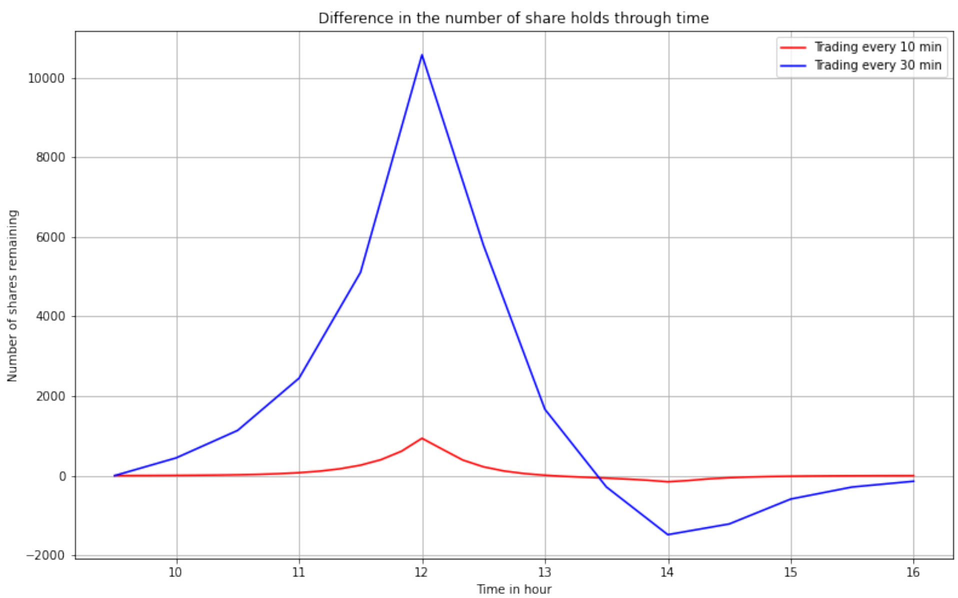
$$q_t = \frac{\lambda_t \cdot (\mu + \rho \cdot q_{t+1})}{\lambda_t + \gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}}$$

$$Q_t = \lambda_t - \frac{\lambda_t^2}{\lambda_t + \gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}}$$

The equation of aim_t do not change.

$$\boxed{\tau_t = \frac{\gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}}{\lambda_t + \gamma \cdot \sigma^2 + \rho \cdot Q_{t+1}}}$$

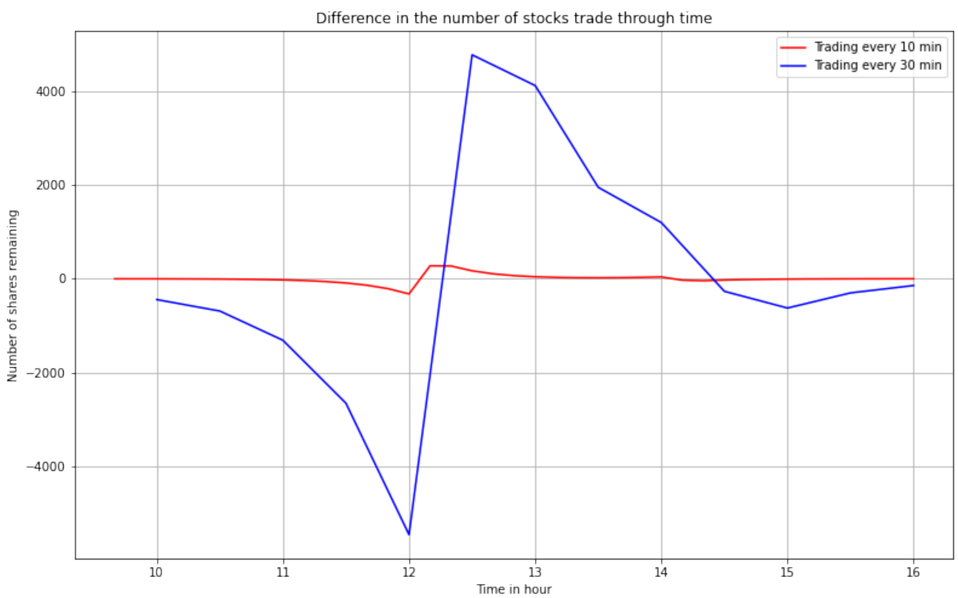
Then we would solve the model just like before but using thcose new equation that take into account the λ_t varying throught time.



We remark that when it is more expensive to trade, he trade a lot before (more than before) the mid day, as cost are lower at this point he is benefiting from it. Then from 12:00 to 14:00, he trades less than in the constant lambda, indeed because trading cost are higher he don't want to trade much for this time. After this he trade again faster than the original, as fast as before 12:00 at the trading cost are similar.

Problem 2

See Jupyter notebook attached


Difference in the number of stocks trade through time
Trading every 10 min
Trading every 30 min
Number of shares remaining
Time in hour
4000
2000
0
−2000
−4000
10
11
12
13
14
15
16