$\mathscr{N}$onLocal $\mathscr{B}$oxes & $\mathscr{C}$ommunication $\mathscr{C}$omplexity

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Ion Nechita, Clément Pellegrini.
(Toulouse, Tuesday 14th of February, 2023.)

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Bob

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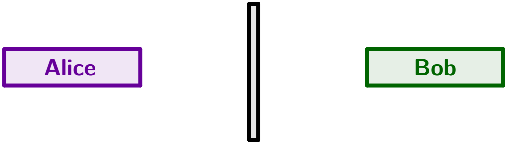

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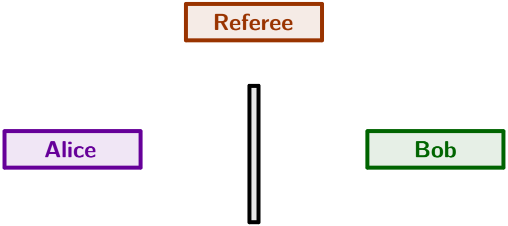

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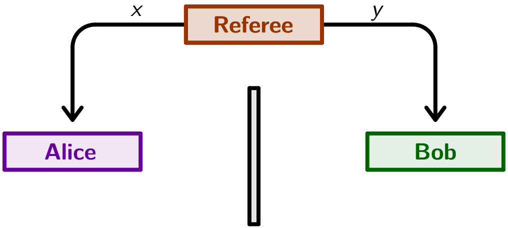

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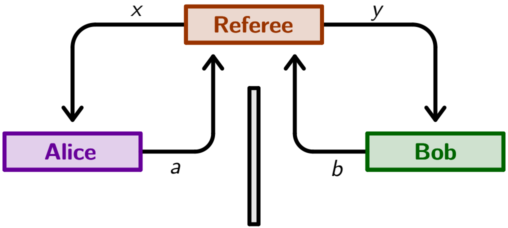

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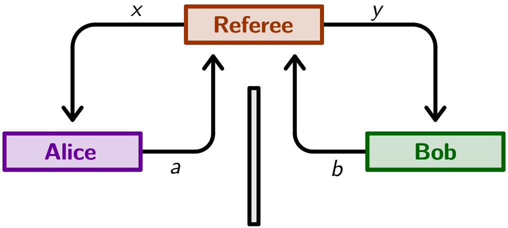

Win at CHSH. $a \oplus b = x\,y$.

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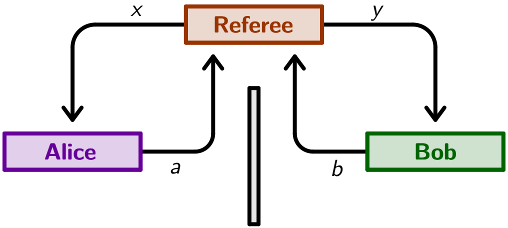

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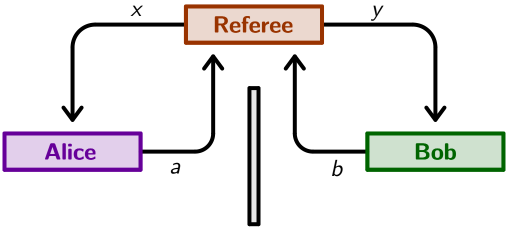

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  $\rightsquigarrow \max \mathbb{P}(\text{win}) = 75\%$.

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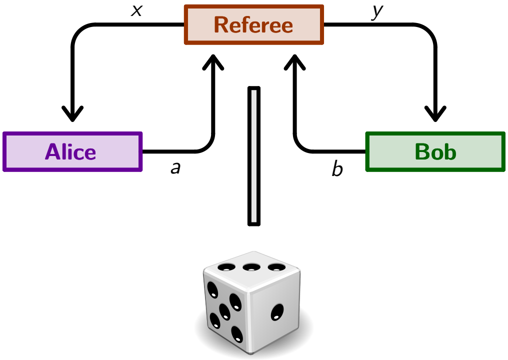

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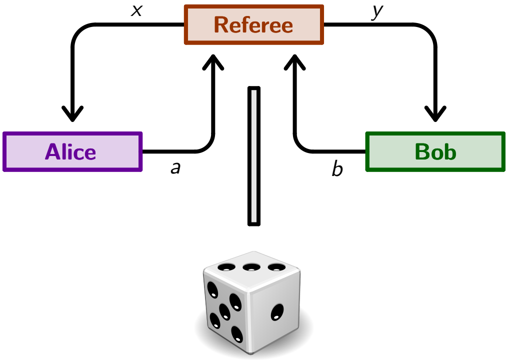

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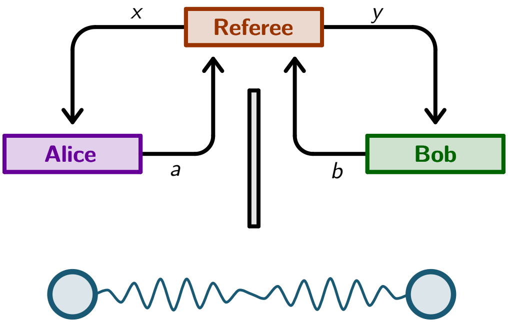

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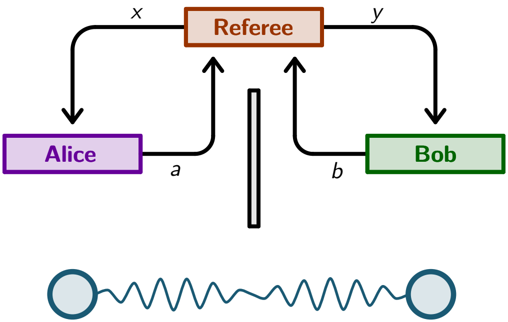

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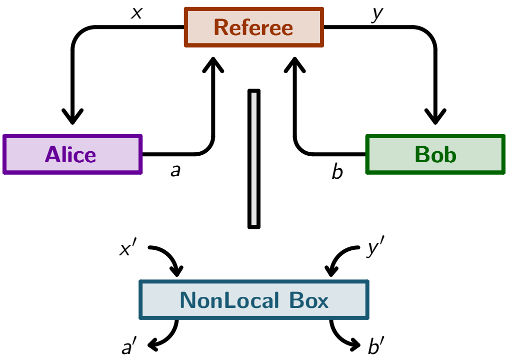

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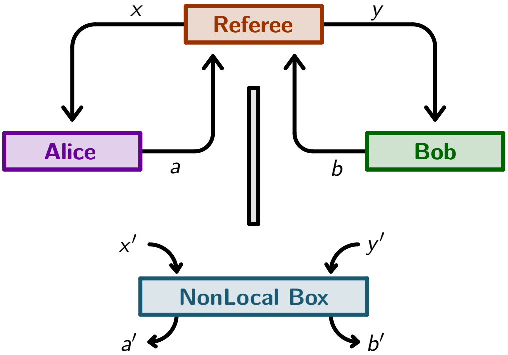

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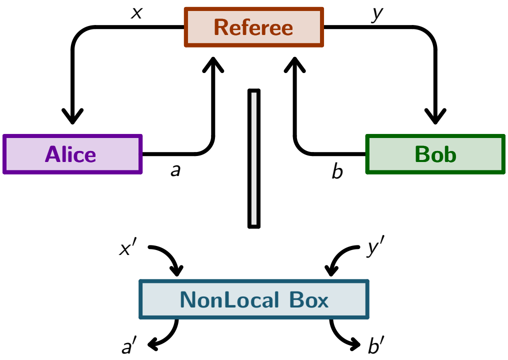

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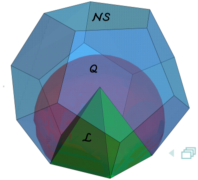

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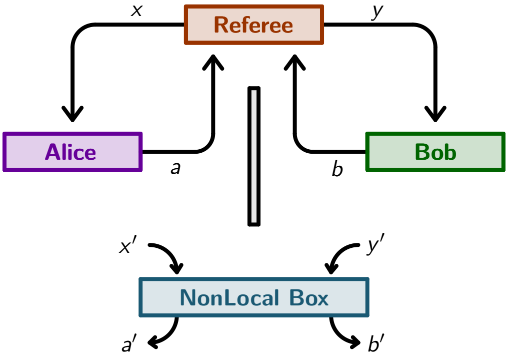

Win at CHSH. $a \oplus b = x\,y$.
Win at CHSH'. $a \oplus b = (x \oplus 1)\,(y \oplus 1)$.

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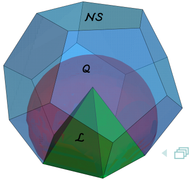

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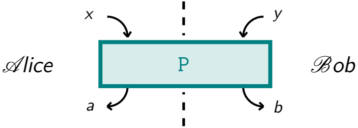

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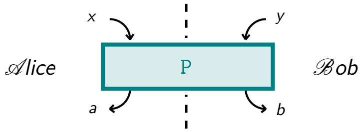

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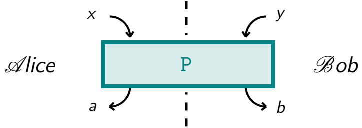

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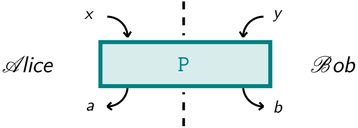

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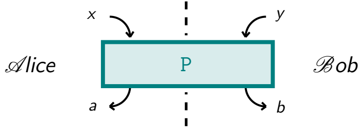

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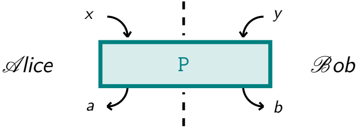

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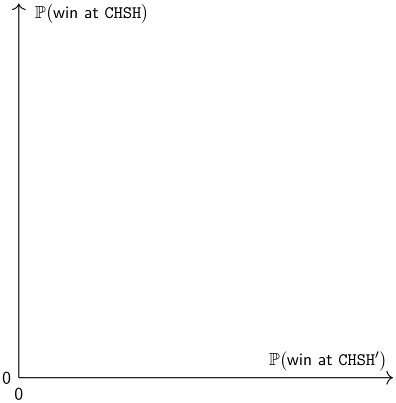

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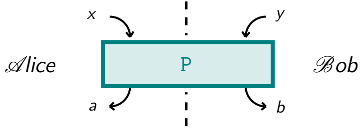

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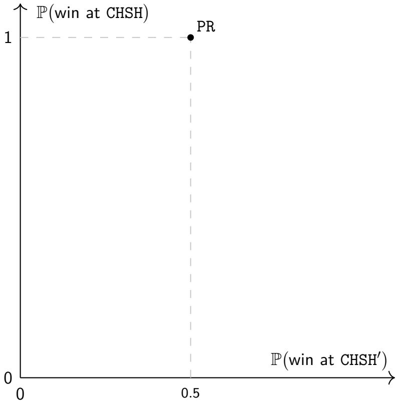

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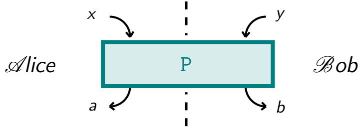

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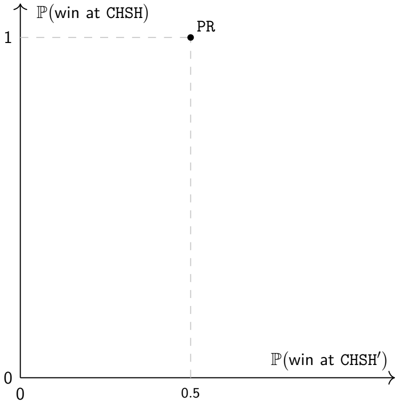

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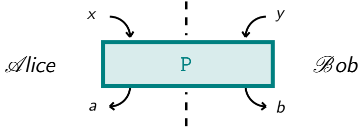

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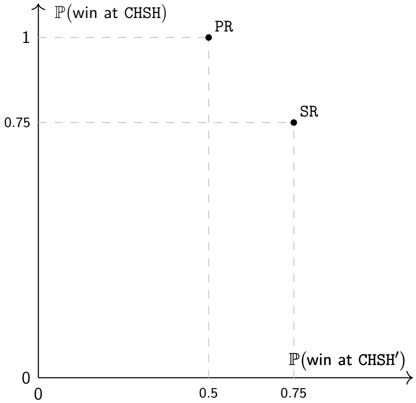

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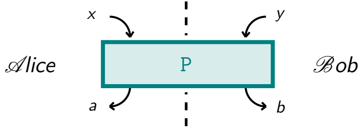

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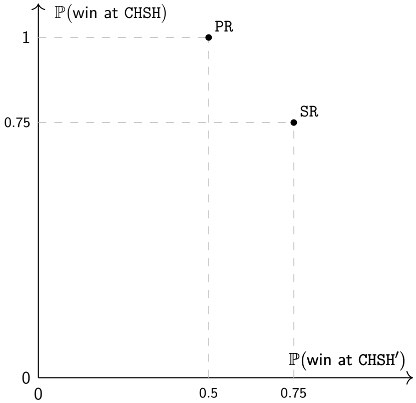

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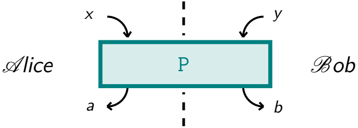

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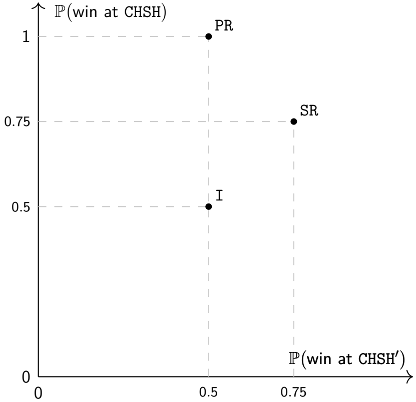

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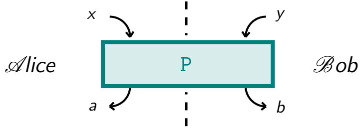

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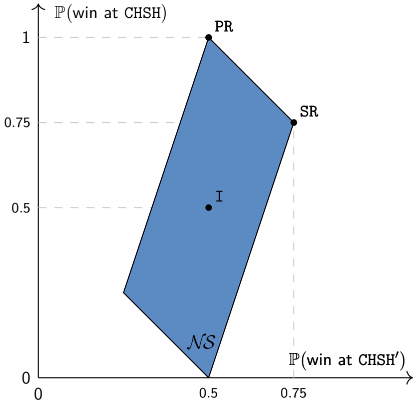

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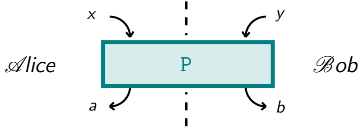

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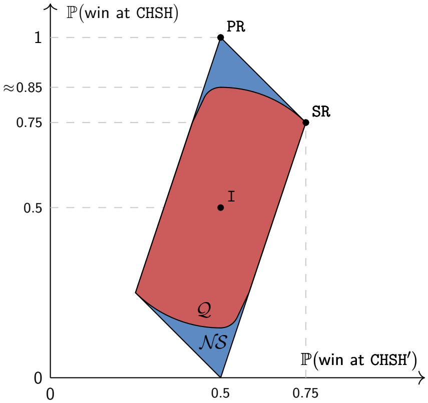

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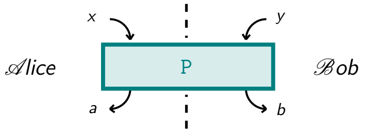

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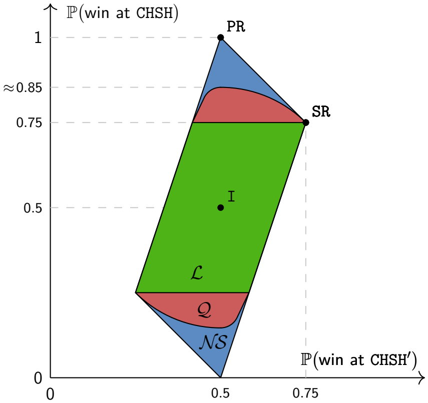

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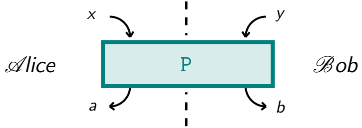

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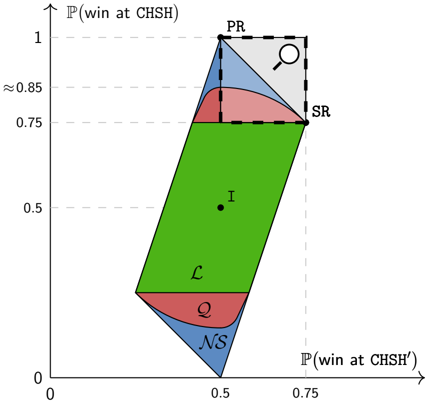
- $\mathtt{I}\big(a, b \,|\, x, y\big) := \frac{1}{4}$.



Communication Complexity

Communication Complexity

Alice

Bob

Communication Complexity

$f: \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$

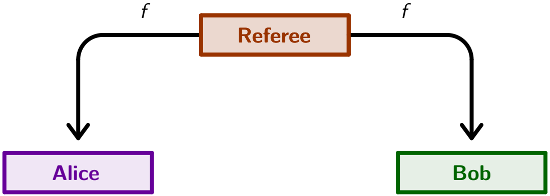
Referee

Alice

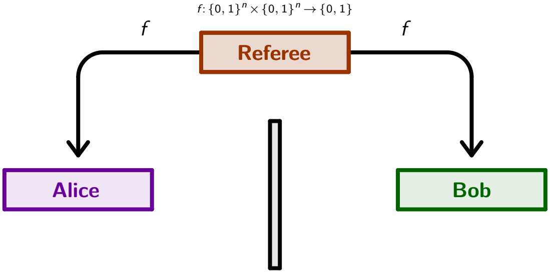
Bob

Communication Complexity

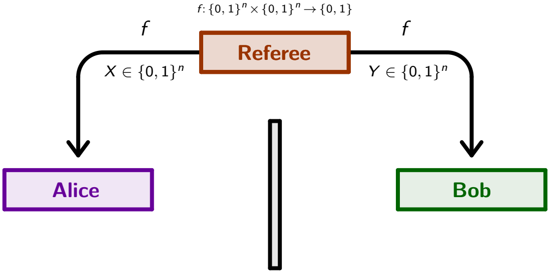
$f\colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$



Communication Complexity



Communication Complexity



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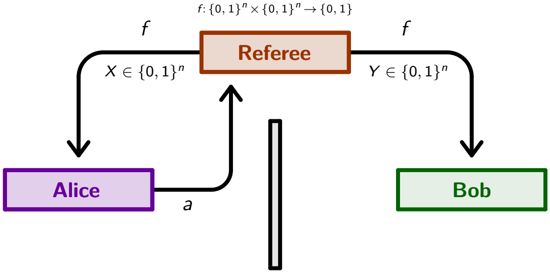

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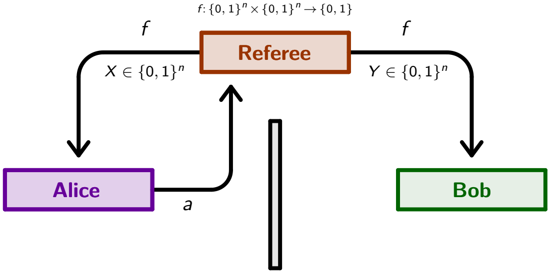
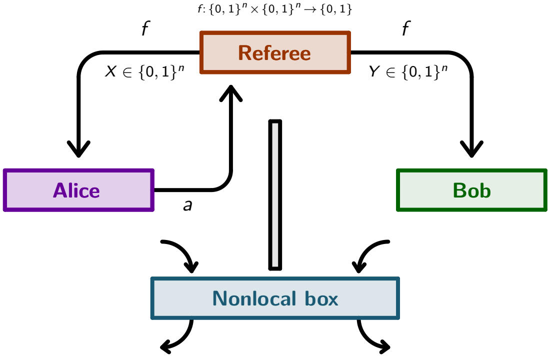


$$\text{Win} \iff a = f(X, Y).$$

Communication Complexity



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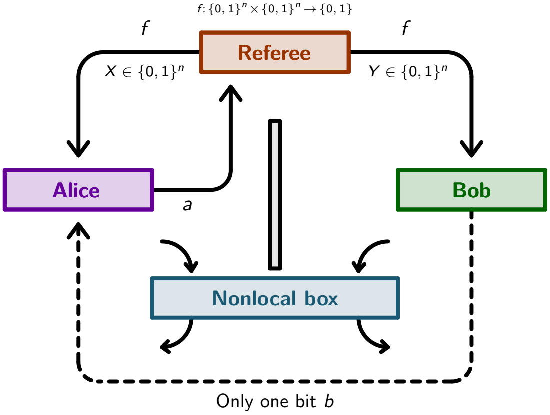

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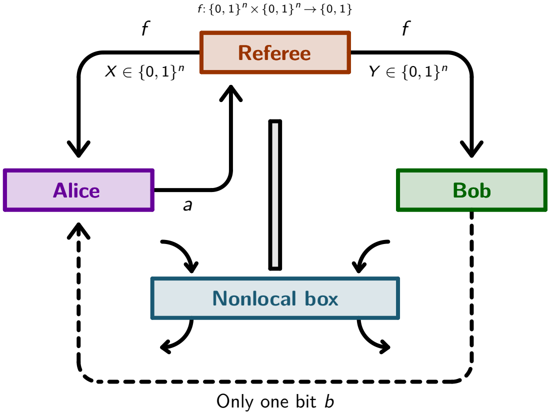

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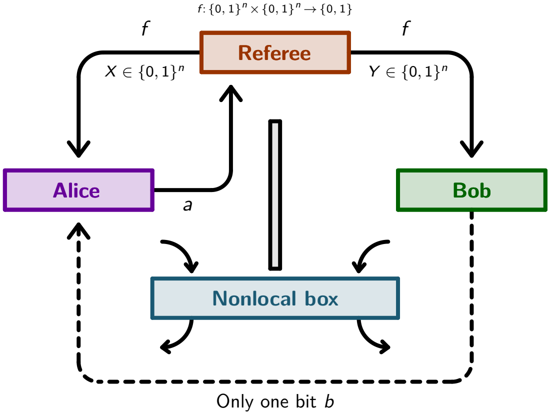

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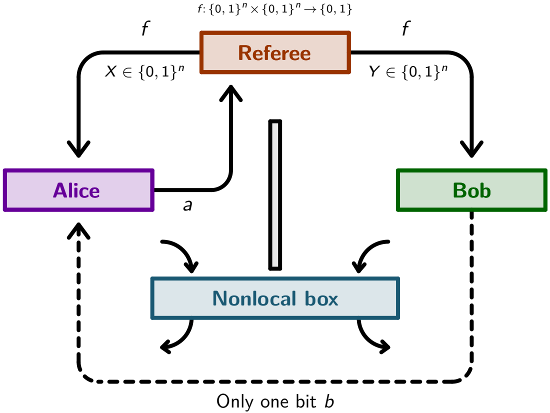

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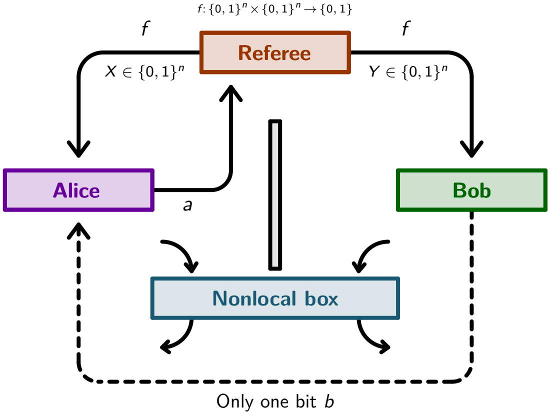
- $f := x_1 \oplus y_1 \oplus x_2 \oplus y_2 \oplus 1$ is trivial.

Communication Complexity



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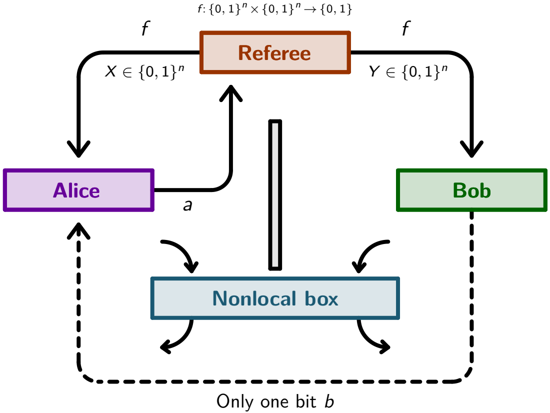
Communication Complexity



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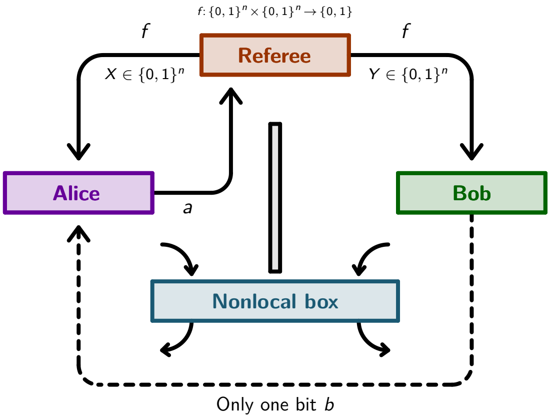
Communication Complexity



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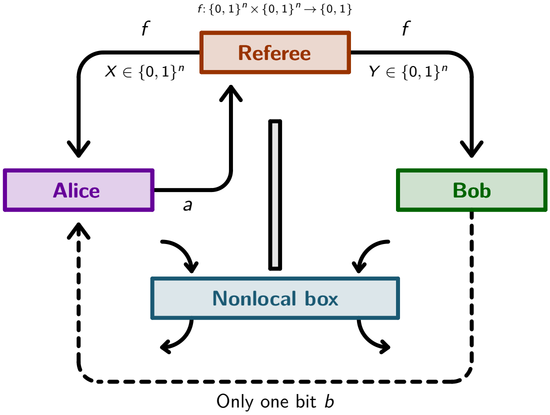
Def. *A box* $\mathtt{P}$ *is said to be* **collapsing** *(or* trivial*) if using copies of this box* $\mathtt{P}$ *any Boolean function f is trivial, with probability $\geq q > \frac{1}{2}$.*

Communication Complexity



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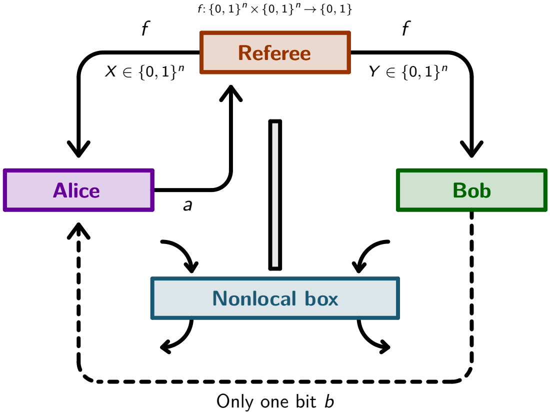
Ex. Link with previous boxes:

Communication Complexity



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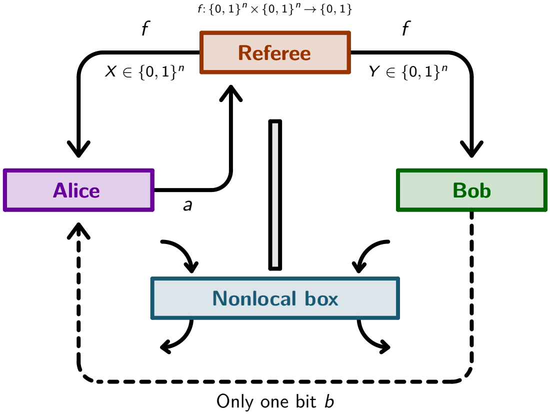
Ex. Link with previous boxes:

- The $\mathtt{PR}$ box is collapsing.

Communication Complexity



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Def. *A box* P *is said to be* **collapsing** *(or trivial) if using copies of this box* P *any Boolean function f is trivial, with probability $\geq q > \frac{1}{2}$.*

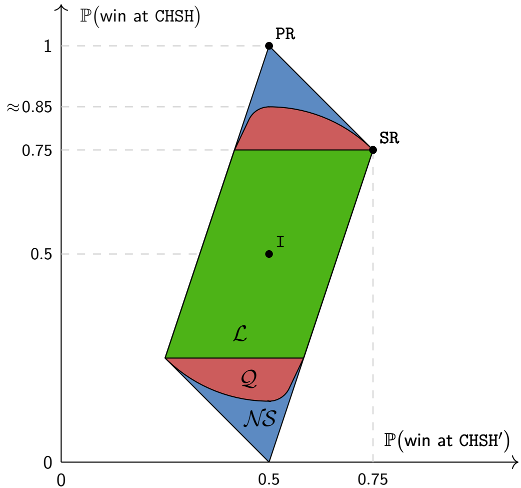
Ex. Link with previous boxes:

- The PR box is collapsing.
- The boxes SR and I are NOT collapsing.

— *Part* 2 —

Historical Overview

Goal. Show that quantum boxes are **non-collapsing** but that post-quantum boxes are **collapsing**.



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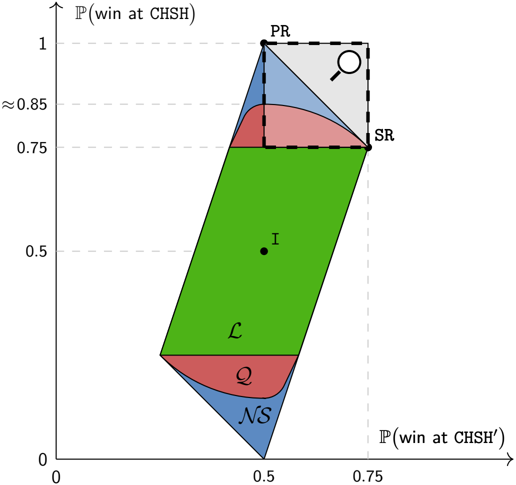
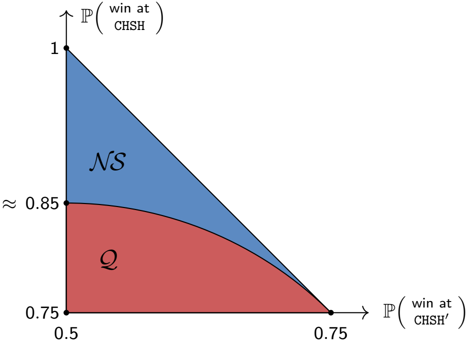


1999: Quantum Boxes are Non-Collapsing



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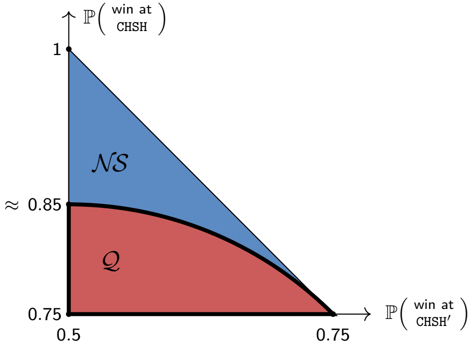

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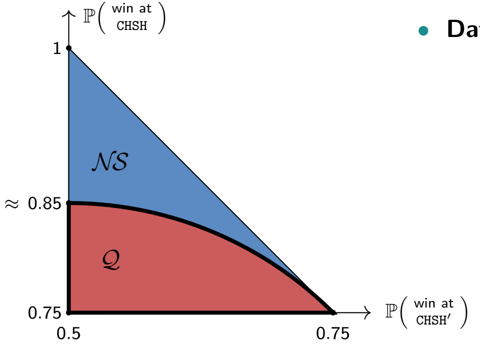

- **Date.** 1999 [1].

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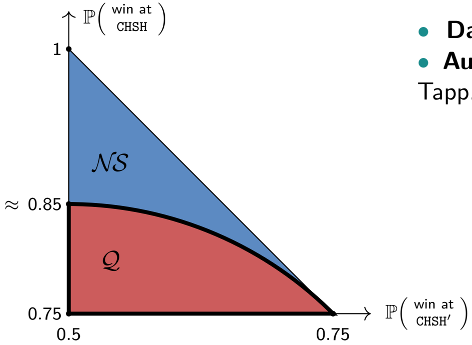

- **Date.** 1999 [1].
- **Authors.** Cleve, van Dam, Nielson, Tapp.

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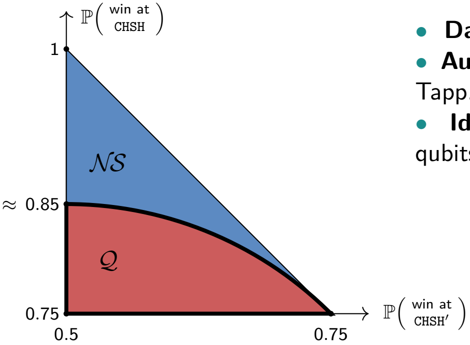

- **Date.** 1999 [1].
- **Authors.** Cleve, van Dam, Nielson, Tapp.
- **Ideas.** (1) Prove the result with qubits,

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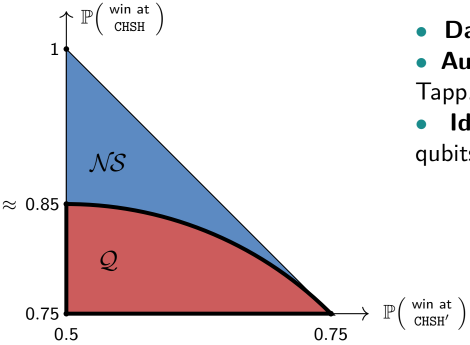
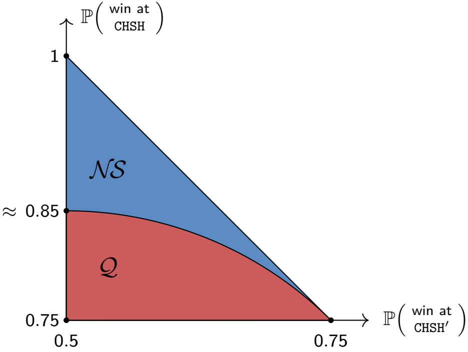


- **Date.** 1999 [1].
- **Authors.** Cleve, van Dam, Nielson, Tapp.
- **Ideas.** (1) Prove the result with qubits, (2) Go back to bits.

1999: The PR Box is Collapsing



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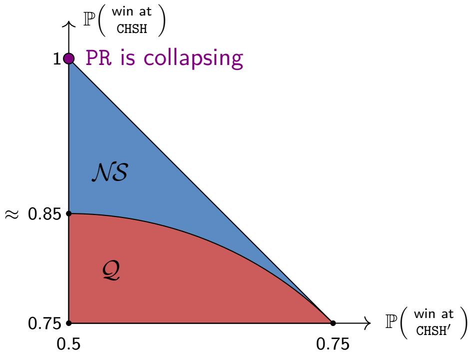

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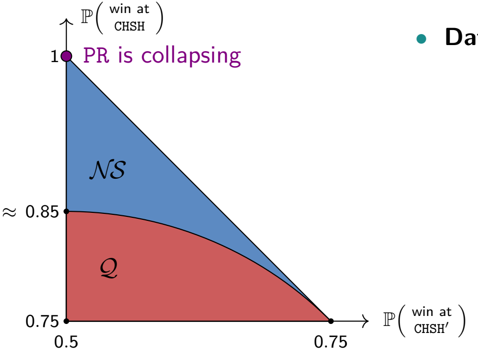
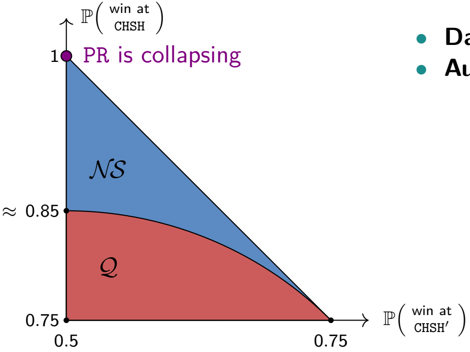


- **Date.** 1999 [2].

1999: The PR Box is Collapsing



- **Date.** 1999 [2].
- **Author.** van Dam.

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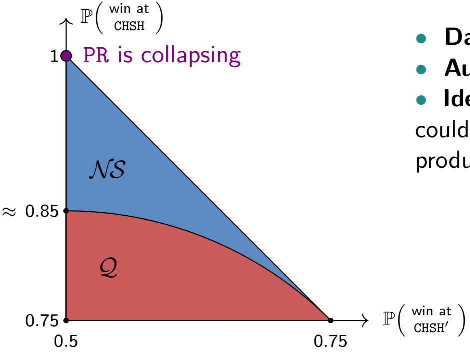

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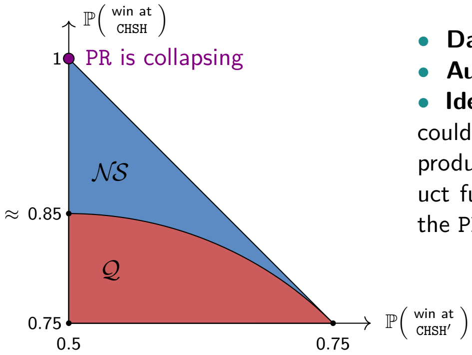

- **Date.** 1999 [2].
- **Author.** van Dam.
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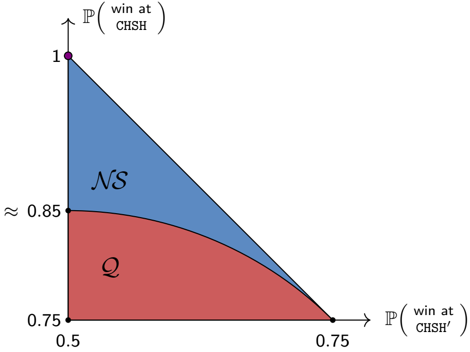
2006: Boxes Above $\approx 91\%$ are Collapsing



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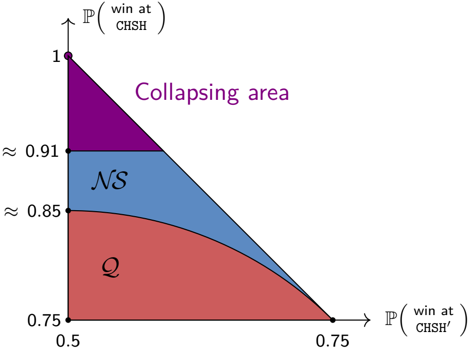

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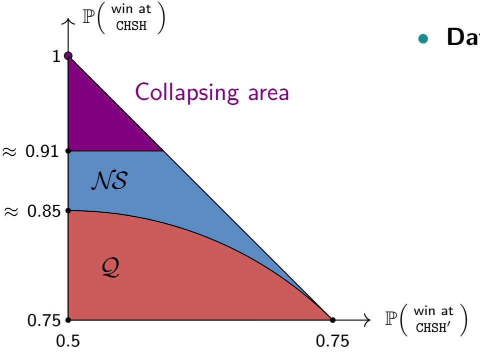

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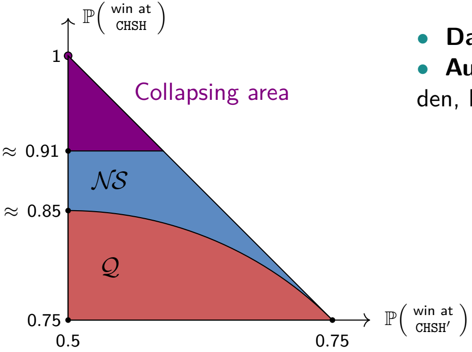

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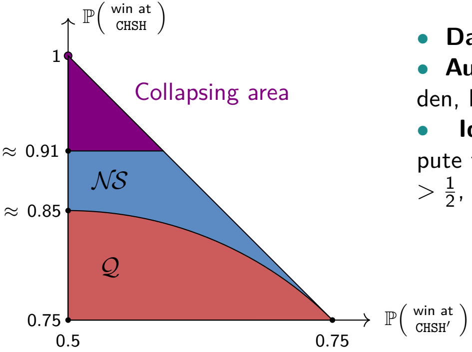

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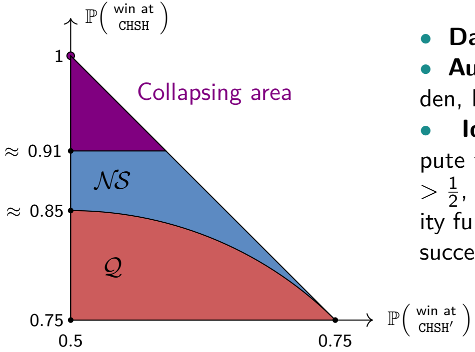

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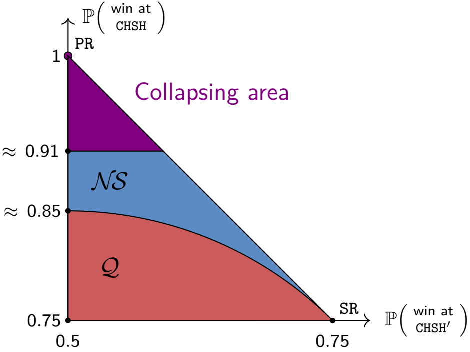
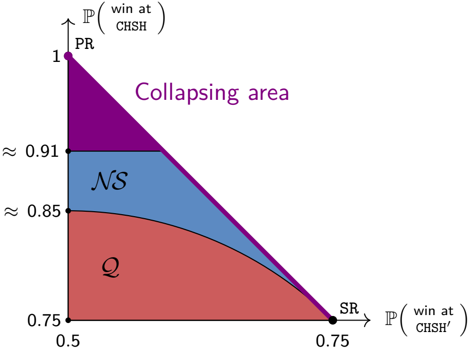
- **Authors.** Brassard, Buhrman, Linden, Méthot, Tapp, Unger.
- **Ideas.** (1) Distributively compute the given function f with proba $> \frac{1}{2}$, (2) Inductively apply the majority function $\texttt{Maj}$ in order to boost the success probability.

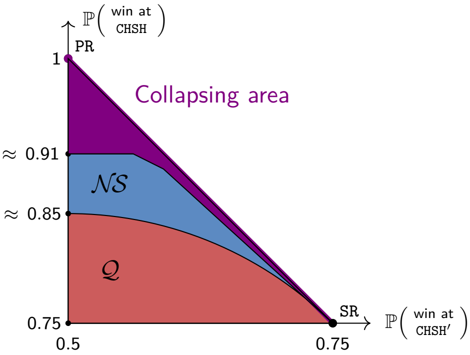
2009: Correlated Boxes are Collapsing



2009: Correlated Boxes are Collapsing



2009: Correlated Boxes are Collapsing



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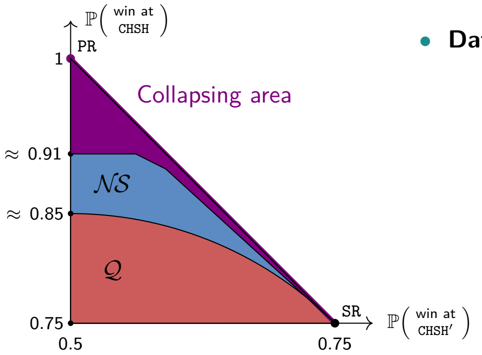
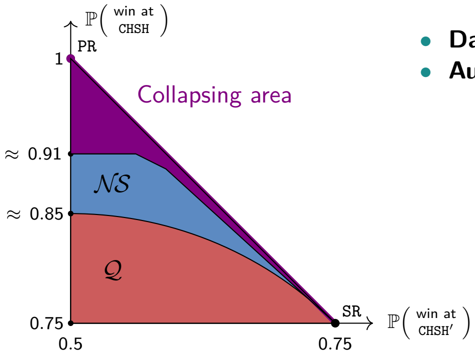


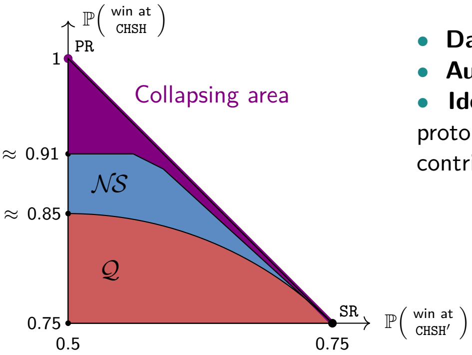
- **Date.** 2009 [4].

2009: Correlated Boxes are Collapsing



- **Date.** 2009 [4].
- **Authors.** Brunner, Skrzypczyk.

2009: Correlated Boxes are Collapsing



- **Date.** 2009 [4].
- **Authors.** Brunner, Skrzypczyk.
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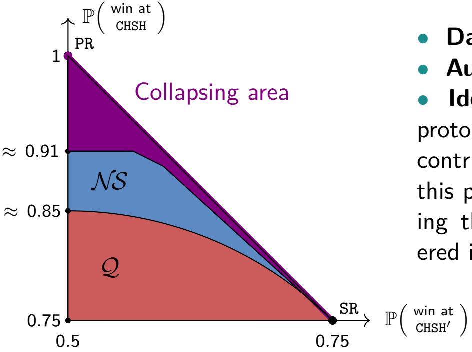
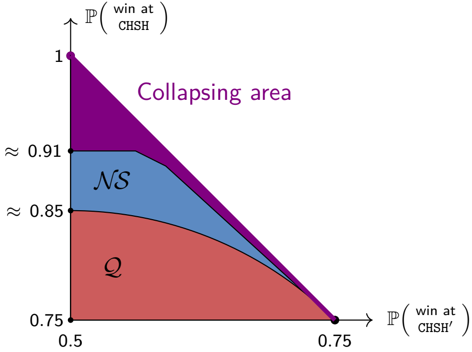
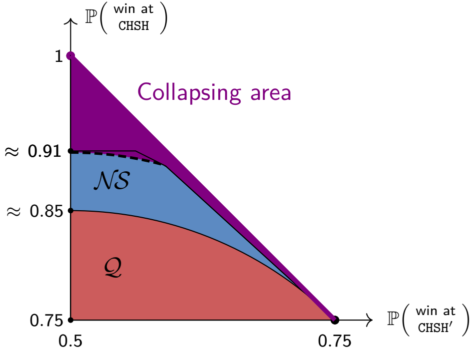


- **Date.** 2009 [4].
- **Authors.** Brunner, Skrzypczyk.
- **Ideas.** (1) Introduce a distillation protocol, cf. generalization in "Our contribution", (2) Inductively apply this protocol many times until reaching the "collapsing triangle" discovered in 2006.

2023: Boxes above a certain Ellipse are Collapsing



2023: Boxes above a certain Ellipse are Collapsing



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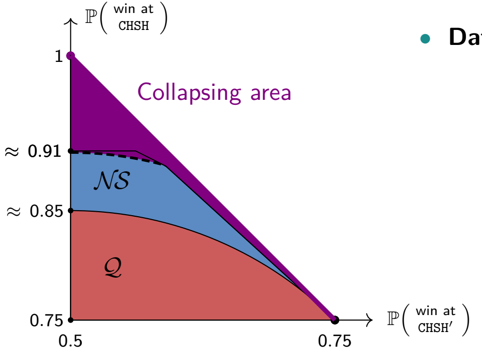
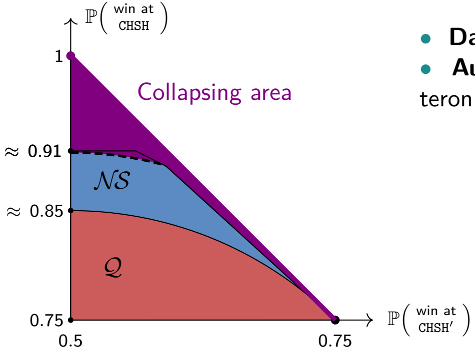


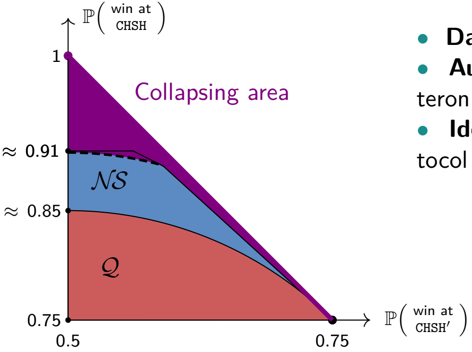
- **Date.** 2023 [5].

2023: Boxes above a certain Ellipse are Collapsing



- **Date.** 2023 [5].
- **Author.** Proulx, Broadbent, Botteron.

2023: Boxes above a certain Ellipse are Collapsing



- **Date.** 2023 [5].
- **Author.** Proulx, Broadbent, Botteron.
- **Idea.** Generalize BBLMTU's protocol (cf. 2006).

— *Part* 3 —

Our Contribution: Algebra of Boxes

Our Contribution [6]

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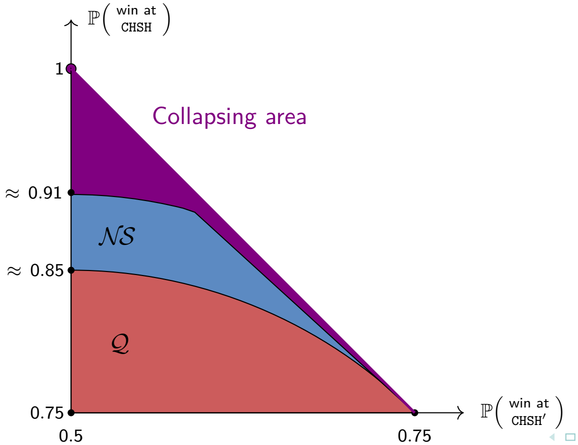

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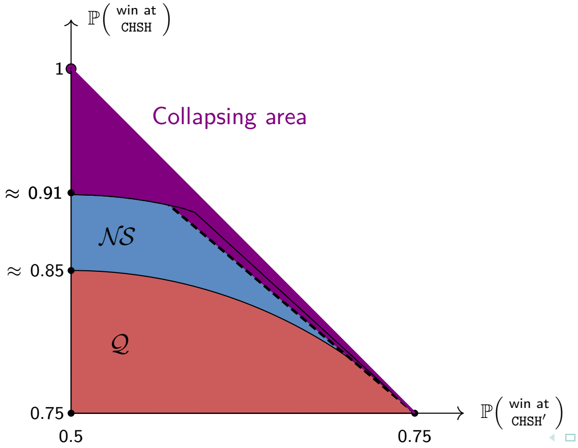

Algebra of Boxes

Algebra of Boxes

Recall. *A* nonlocal box $\mathtt{P}$ *is a conditional probability distribution* $(a, b, x, y) \in \{0, 1\}^4 \mapsto \mathtt{P}(a, b \,|\, x, y) \in [0, 1]$ *such that* $\mathtt{P} \in \mathcal{NS} \backslash \mathcal{L}$*.*

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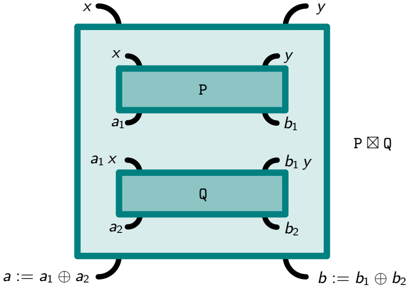

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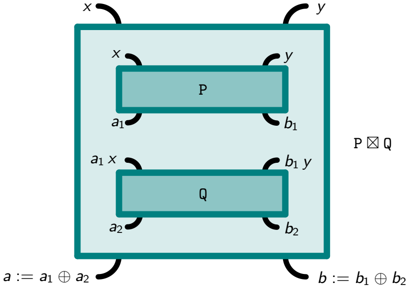

$$\mathtt{P}\boxtimes\mathtt{Q}\left(a,\ b\,\middle|\,x,\ y\right) := \sum_{a_1,b_1\in\{0,1\}} \mathtt{P}\left(a_1,\ b_1\,\middle|\,x,\ y\right)\times\mathtt{Q}\left(a\oplus a_1,\ b\oplus b_1\,\middle|\,a_1x,\ b_1y\right)$$

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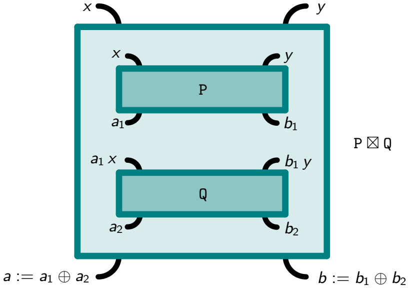

$$\mathtt{P}\boxtimes\mathtt{Q}\left(a,\,b\,\middle|\,x,\,y\right) := \sum_{a_1,b_1\in\{0,1\}} \mathtt{P}\left(a_1,\,b_1\,\middle|\,x,\,y\right)\times\mathtt{Q}\left(a\oplus a_1,\,b\oplus b_1\,\middle|\,a_1x,\,b_1y\right)$$

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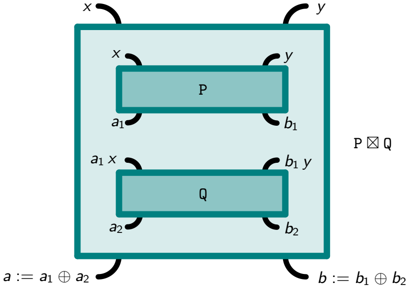

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Algebra of Boxes. The vector space $\mathcal{B} := \mathcal{F}\left(\{0,1\}^4, \mathbb{R}\right)$ endowed with the operations $\{+, \cdot, \boxtimes\}$ defines a non-commutative and non-associative algebra.

Orbit of a Box

Orbit of a Box

Orbit of order k.

Orbit of a Box

Orbit of order k**.** $\mathtt{Orbit}_k(\mathtt{P}) :=$
$\{$products of exactly k times the term $\mathtt{P}\}$.

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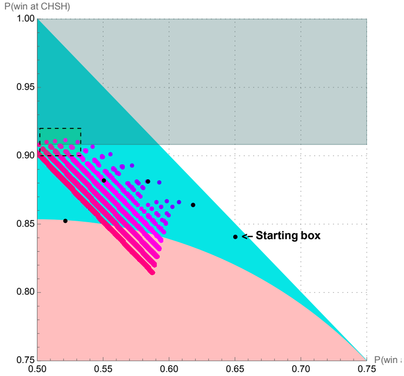
Trivial Area

Post–Quantum Correlations

Quantum Correlations

This orbit intersects the trivial area, so this starting box is trivial!



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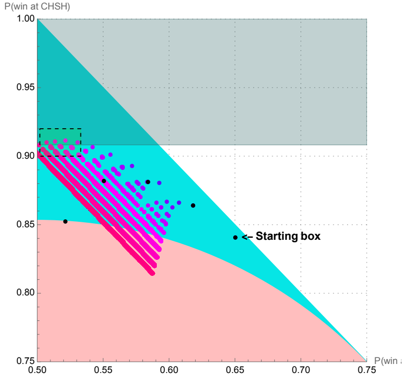

The "highest" box in each orbit.

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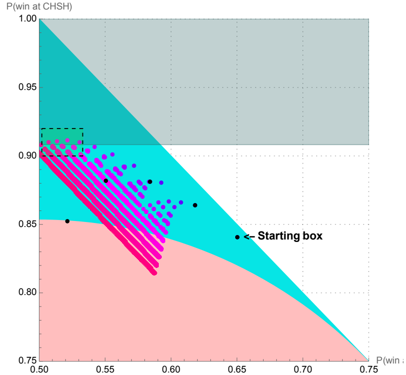

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The "highest" box in each orbit.

$$\mathtt{P}_{\max,\,k} = (((\mathtt{P} \boxtimes \mathtt{P}) \boxtimes \mathtt{P}) \cdots) \boxtimes \mathtt{P} =: \mathtt{P}^{\boxtimes k}.$$

New Collapsing Boxes: Numerical Proof

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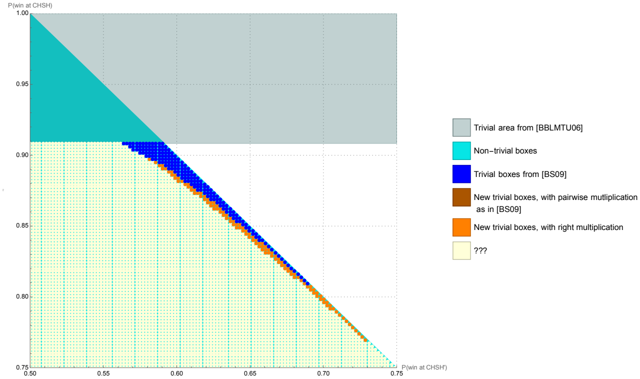

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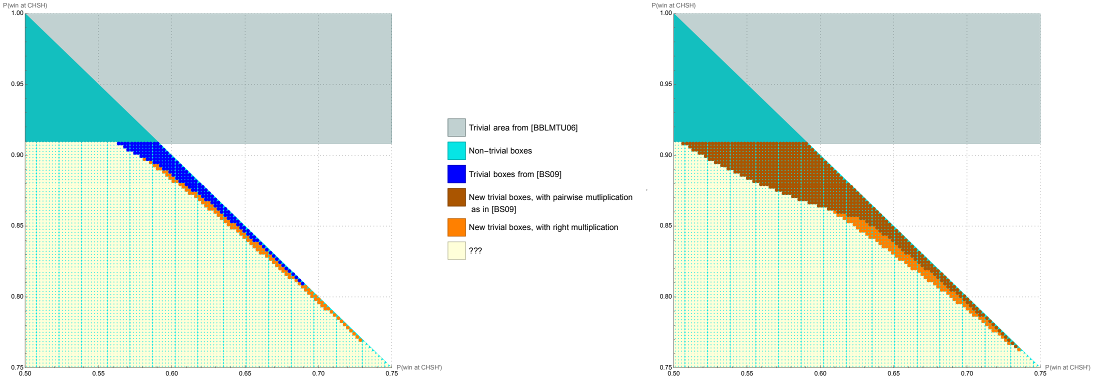

New Collapsing Boxes: Analytical Proof

Theorem 1 (New collapsing boxes)

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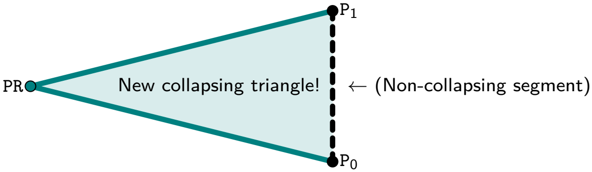
In the triangle whose vertices are $\{\mathrm{PR},\ \mathrm{P}_0 := \mathbf{1}_{a=b=0},\ \mathrm{P}_1 := \mathbf{1}_{a=b=1}\}$, *all the points are collapsing boxes, except points in the segment* P_0-P_1.



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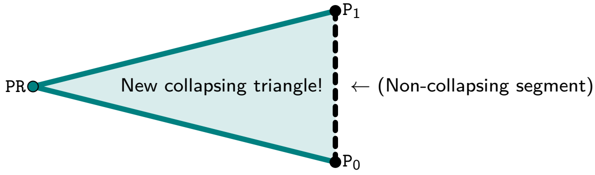

Proof (idea).

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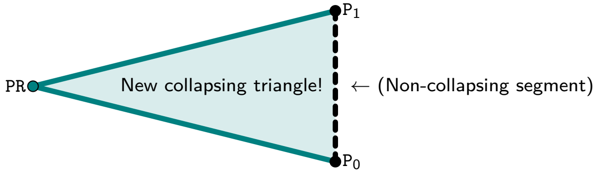

Proof (idea). **(1)** The triangle is stable under ⊠.

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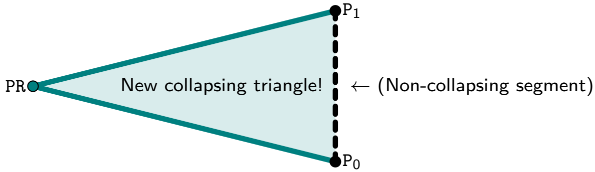

Proof (idea). **(1)** The triangle is stable under $\boxtimes$. **(2)** Define a sequence:

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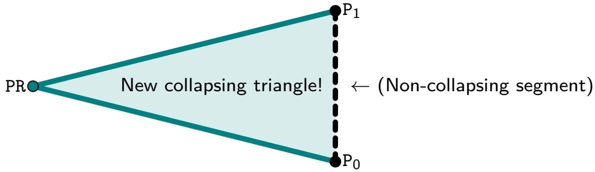

Proof (idea). **(1)** The triangle is stable under ⊠. **(2)** Define a sequence: initialize at an arbitrary point of the triangle (except in the vertical segment),

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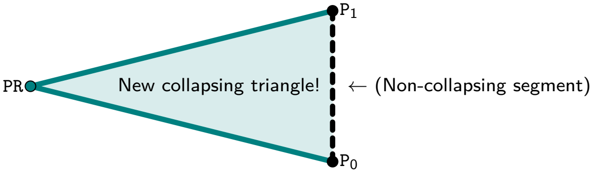

Proof (idea). **(1)** The triangle is stable under $\boxtimes$. **(2)** Define a sequence: initialize at an arbitrary point of the triangle (except in the vertical segment), and inductively apply the multiplication $\boxtimes$.

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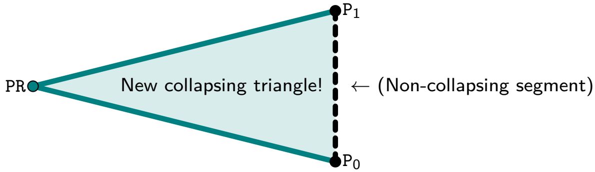

Proof (idea). **(1)** The triangle is stable under $\boxtimes$. **(2)** Define a sequence: initialize at an arbitrary point of the triangle (except in the vertical segment), and inductively apply the multiplication $\boxtimes$. **(3)** This sequence converges to PR.

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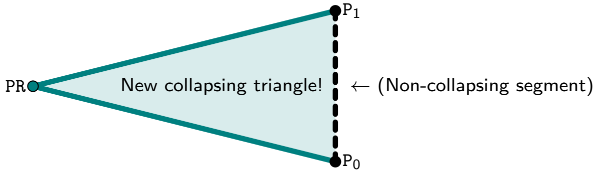

Proof (idea). **(1)** The triangle is stable under ⊠. **(2)** Define a sequence: initialize at an arbitrary point of the triangle (except in the vertical segment), and inductively apply the multiplication ⊠. **(3)** This sequence converges to PR. **(4)** But, near PR, all boxes are collapsing (cf. 2006).

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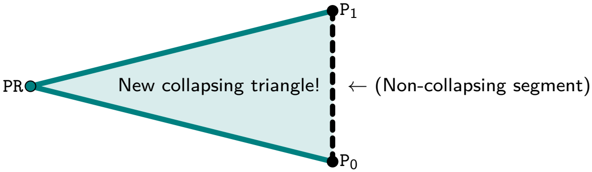

Proof (idea). **(1)** The triangle is stable under $\boxtimes$. **(2)** Define a sequence: initialize at an arbitrary point of the triangle (except in the vertical segment), and inductively apply the multiplication $\boxtimes$. **(3)** This sequence converges to $\mathtt{PR}$. **(4)** But, near $\mathtt{PR}$, all boxes are collapsing (cf. 2006). **(5)** Hence, the orbit intersects the collapsing area and the starting box must be collapsing as well. □

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Under the supervision of Anne Broadbent, Ion Nechita and Clément Pellegrini.