# Open Question: link between NonLocal Boxes and Communication Complexity?

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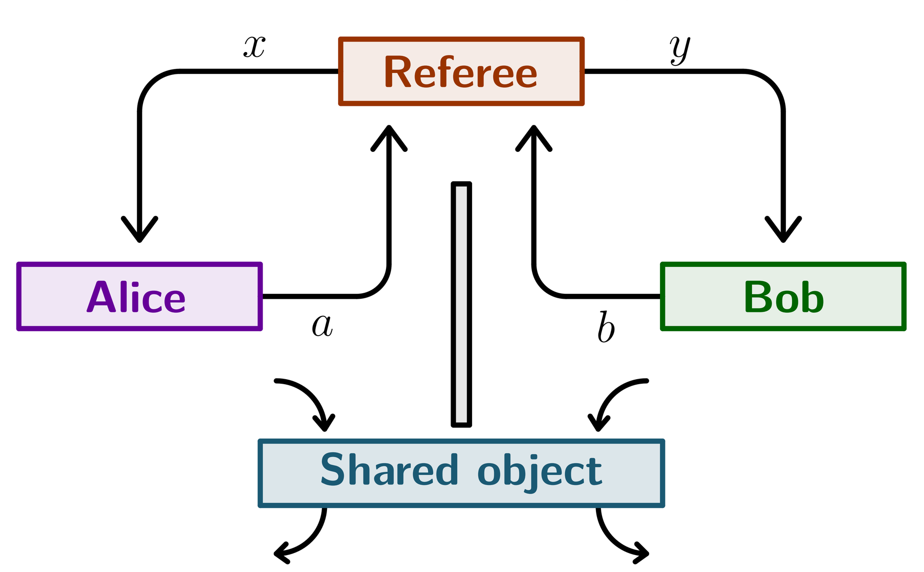
## Goal

Prove that post-quantum boxes collapse communication complexity, and deduce that they are unlikely to exist in Nature.

## 1. CHSH game

Alice and Bob receive some bits $x, y \in \{0,1\}$, and they answer some bits $a, b \in \{0,1\}$ to the referee.




- **Win at CHSH** *iff* $a \oplus b = x \times y$.
- **Win at CHSH′** *iff* $a \oplus b = (x \oplus 1) \times (y \oplus 1)$.

Depending on the type of the shared object, Alice and Bob can reach different wining probabilities:

- **Classical Strategy.** $\max \mathrm{P}\left(\substack{\text{win}\\ \text{CHSH}}\right) = 75\%$.
  ⇝ Shared object: *shared randomness.*
- **Quantum Strategy.** $\max \mathrm{P}\left(\substack{\text{win}\\ \text{CHSH}}\right) = \frac{2+\sqrt{2}}{4} \approx 85\%$.
  ⇝ Shared object: *quantum states.*
- **Non-Signaling Strategy.** $\max \mathrm{P}\left(\substack{\text{win}\\ \text{CHSH}}\right) = 100\,\%$.
  ⇝ Shared object: *nonlocal boxes.*

## References

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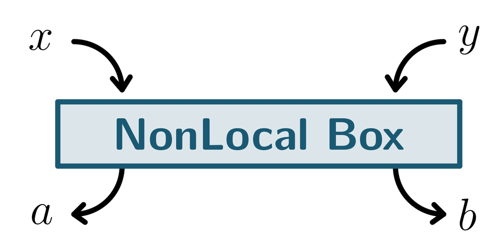
## 2. NonLocal Boxes

**Def.** A *nonlocal box* is formalized by a conditional probability distribution $\mathrm{P}(a, b \,|\, x, y)$.

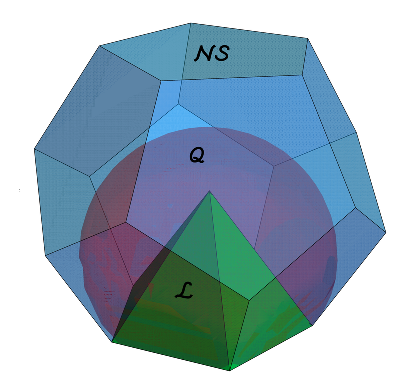



**Examples.** • $\mathtt{PR}(a,b\,|\,x,y) := \begin{cases} 1/2 & \text{if } a \oplus b = x \times y, \\ 0 & \text{otherwise.} \end{cases}$

• Shared Randomness: $\mathtt{SR}(a,b\,|\,x,y) := \begin{cases} 1/2 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$

• Fully mixed box: $\mathtt{I}(a,b\,|\,x,y) := 1/4$.

**Non-signalling boxes.** The set $\mathcal{NS} := \{\text{non-signaling boxes}\}$ is an 8-dimensional convex set, containing $\mathcal{Q} := \{\text{quantum boxes}\}$.




## 3. Communication Complexity

Let $f : \{0,1\}^n \times \{0,1\}^m \to \{0,1\}$. Assume Alice knows $f$ and $X \in \{0,1\}^n$, and Bob knows $f$ and $Y \in \{0,1\}^m$.

**Def.** The *communication complexity* of $f$ at $(X,Y)$, denoted $\mathbf{CC}_p(f, X, Y)$, is the minimal number of communication bits between Alice and Bob so that Alice knows the value $f(X,Y)$ with probability $> p$.

**Def.** A box $\mathrm{P}$ *collapses communication complexity* if it allows to compute any Boolean function with only one bit of communication and bounded error:

$$\exists p > \frac{1}{2}, \quad \forall f, \forall X, \forall Y, \quad \mathbf{CC}_p(f, X, Y) \le 1\,.$$

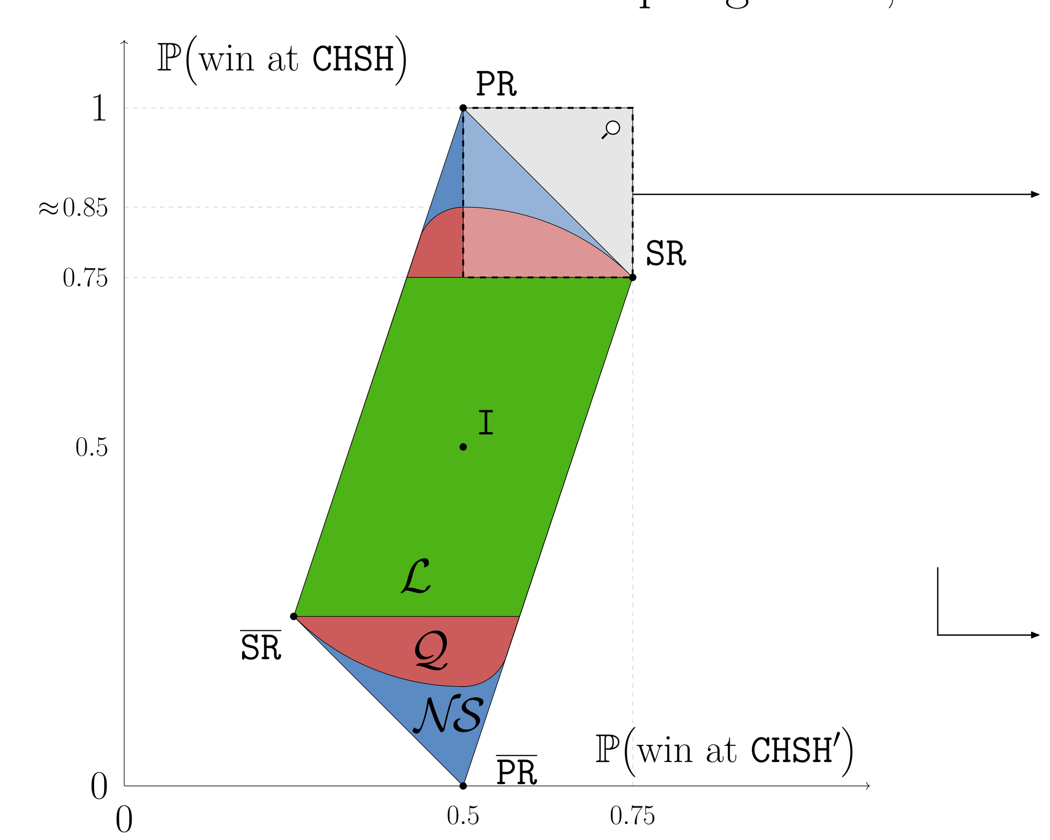
**Intuition.** It is strongly believed that such a collapsing box could not exist in Nature (it would be too powerful) [7, 3, 4, 1].

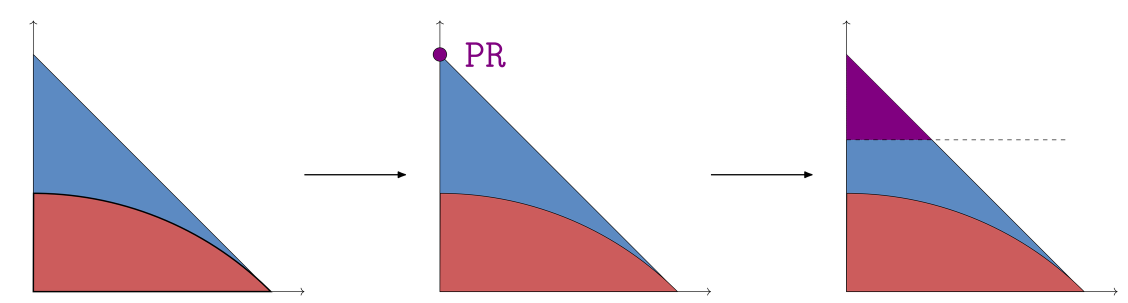
## 4. Open Question

Which nonlocal boxes collapse communication complexity?

## 5. Partial Answers

**Historical Overview of Partial Answers.** This overview is presented in the slice of $\mathcal{NS}$ passing through the boxes $\mathtt{PR}$, $\mathtt{SR}$ and $\mathtt{I}$, and we zoom in the top-right corner of the diagram. The open question consists in determining what portion of the **blue** area (the "post-quantum boxes") is collapsing, and what portion is not collapsing. In **purple** are drawn the known collapsing boxes, whereas in **red** are represented the known non-collapsing boxes.

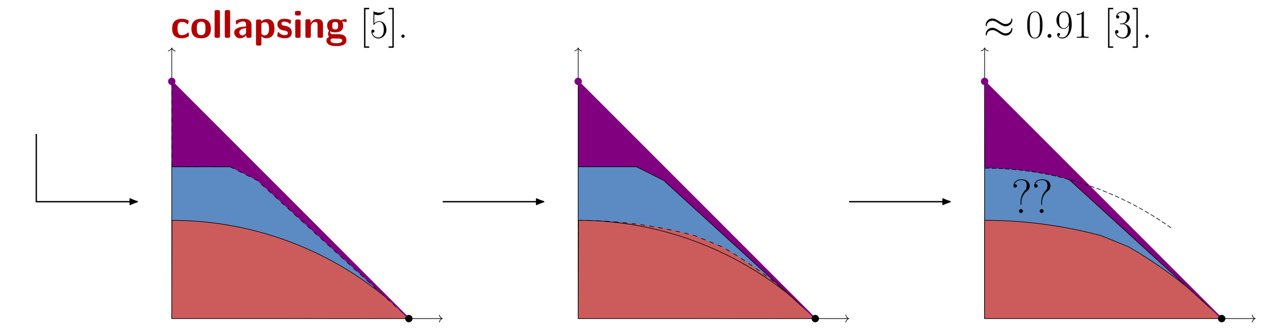



**1999**: Quantum boxes are **non-collapsing** [5].

**1999**: The $\mathtt{PR}$ box is **collapsing** [7].

**2006**: **Collapsing** region above $\approx 0.91$ [3].




**2009**: The "thickened" diagonal is **collapsing** (part. numerical result) [4].

**2015**: "Almost quantum" boxes are **non-collapsing** [6].

**2023**: **Collapsing** region above an ellipse (analytical result) [2].

The question is still open today: there is still a **blue** gap to be filled!

## 6. Ideas of our proof [2] (2023)

The proof is a generalization of [3] (2006).

**Notations.** Let $\mathrm{P} \in \mathcal{NS}$ and consider:

$$\begin{aligned} \eta_{xy} &:= -1 + 2 \textstyle\sum_c \mathrm{P}(c, c \oplus xy \,|\, x, y)\,; \\ A &:= \big(\eta_{00} + \eta_{01} + \eta_{10} + \eta_{11}\big)^2\,; \\ B &:= 2\,\eta_{00}^2 + 4\eta_{01}\eta_{10} + 2\eta_{11}^2\,. \end{aligned}$$

**Theorem** (Sufficient conditon). If the box $\mathrm{P}$ satisfies $A + B > 16$, then $\mathrm{P}$ is collapsing.

*Idea of the proof.* Let $f : \{0,1\}^n \times \{0,1\}^m \to \{0,1\}$ a Boolean function known by both Alice and Bob, and let two strings $X \in \{0,1\}^n$ and $Y \in \{0,1\}^m$ known by Alice and Bob respectively. Alice and Bob share infinitely many copies of a certain nonlocal box $\mathrm{P}$ and infinitely many shared random bits.

If the condition $A + B > 16$ is valid, then we exhibit a sequence of protocols $(\mathcal{P}_k)_k$ such that for each $k$, Alice is able to produce a bit $a$ that equals $f(X,Y)$ with some probability $p_k > 1/2$ using only 1 bit of communication. Moreover, we show that the sequence $(p_k)_k$ converges to some $p_* > 1/2$:

$$p_k \underset{k\to\infty}{\longrightarrow} p_* > 1/2\,,$$

and that $p_*$ does not depend on $f$ nor $X$ nor $Y$ (it only depends on $\mathrm{P}$).

Hence, for any $f$, there exists a $k$ large enough such that the protocol $\mathcal{P}_k$ correctly computes $f(X,Y)$ with probability $p_k > (1 + p_*)/2 > 1/2$ and only 1 bit of communication, and as the constant $p := (1 + p_*)/2$ is independent of $f$, $X$, $Y$, we indeed obtain that $\mathrm{P}$ collapses communication complexity by definition. □

**Examples of new collapsing regions (in black).**