$\mathscr{A}$lgebra of $\mathscr{N}$on$\mathscr{L}$ocal $\mathscr{B}$oxes & the $\mathscr{C}$ollapse of $\mathscr{C}$ommunication $\mathscr{C}$omplexity

Pierre Botteron. Reference: arXiv:2312.00725 [1] (to appear in Quantum, 2024).

Together with:

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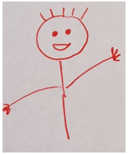

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— Part 1 —

Motivation

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Goal. Combine several theoretical principles to rule out the quantum theory ($\mathcal{Q}$) from the non-signalling theory ($\mathcal{NS}$).

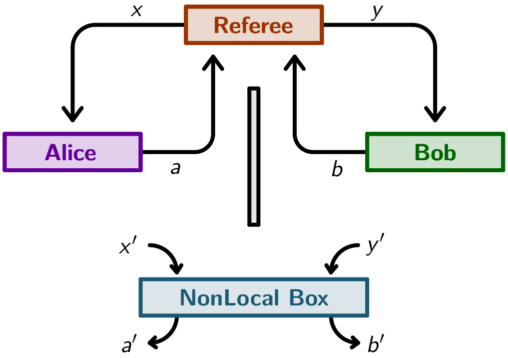
Here. We will study the principle of no-collapse of **communication complexity** (CC). Intuitively, a violation of this principle seems impossible in Nature [2, 3, 4]. Quantum theory satisfies this principle, but some non-signalling correlations violate it.

Open Question. What are all non-signalling correlations that violate this principle?

— *Part 2* —

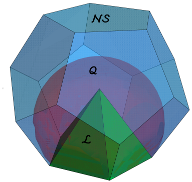
Setup

2.1. CHSH Game & Nonlocal Boxes

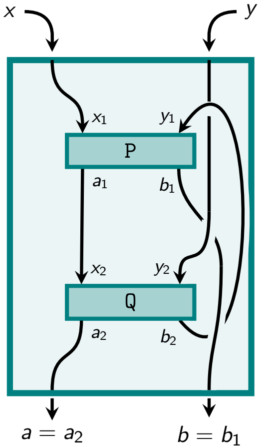


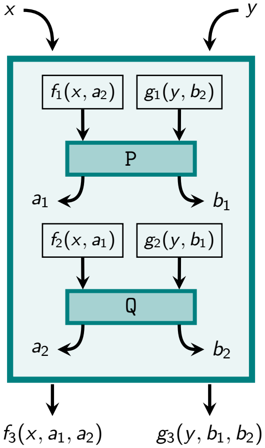
Win at CHSH $\iff$ $a \oplus b = x\,y$.

- **Deterministic Strategies.**
  $\rightsquigarrow$ $\max \mathbb{P}(\text{win}) = 75\%$.
- **Classical Strategies ($\mathcal{L}$).**
  $\rightsquigarrow$ $\max \mathbb{P}(\text{win}) = 75\%$.
- **Quantum Strategies ($\mathcal{Q}$).**
  $\rightsquigarrow$ $\max \mathbb{P}(\text{win}) = \cos^2\left(\frac{\pi}{8}\right) \approx 85\%$.
- **Non-signalling Strategies ($\mathcal{NS}$).**
  $\rightsquigarrow$ $\max \mathbb{P}(\text{win}) = 100\%$.



2.2. Wiring of Nonlocal Boxes



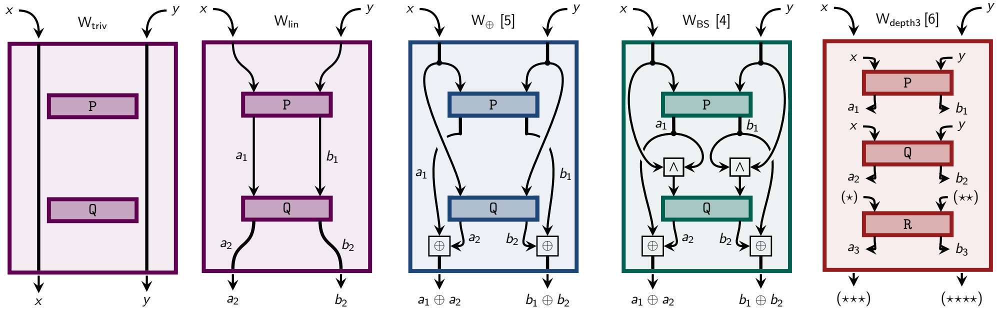


Definition. A **wiring** W between two boxes $\mathtt{P}, \mathtt{Q} \in \mathcal{NS}$ consists in six functions $f_1, f_2, g_1, g_2 : \{0,1\}^2 \to [0,1]$ and $f_3, g_3 : \{0,1\}^3 \to [0,1]$ satisfying the *non-cyclicity conditions* for all x, y:

$$f_1(x,0) \neq f_1(x,1) \Rightarrow f_2(x,0) = f_2(x,1),$$
$$f_2(x,0) \neq f_2(x,1) \Rightarrow f_1(x,0) = f_1(x,1),$$

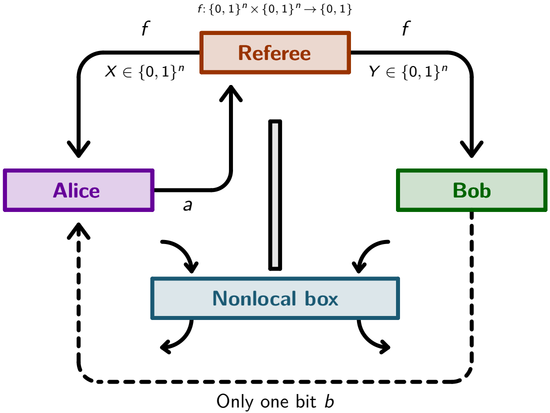
and similarly for g_1, g_2. The new box is denoted $\mathtt{P} \boxtimes_{\mathrm{W}} \mathtt{Q} \in \mathcal{NS}$.

Examples of Wirings in the Litterature



where the overline bar is the NOT gate: $\overline{x} = x \oplus 1$, the symbol $(\star)$ stands for $xa_2 \vee x\overline{a_1} \vee \overline{x\,a_2}a_1$, and $(\star\star)$ for $yb_2 \vee y\overline{b_1}$, and $(\star\star\star)$ for $a_3a_2 \vee a_3\overline{a_1} \vee \overline{a_3\,a_2}a_1$, and $(\star\star\star\star)$ for $b_3b_2 \vee b_3\overline{b_1} \vee \overline{b_3\,b_2}b_1$.

2.3. Collapse of Communication Complexity



Win $\iff a = f(X, Y)$.

Def. *A function f is said to be* **trivial** *(in the sense of communication complexity) if Alice knows any value $f(X, Y)$ with only one bit transmitted between Alice and Bob.*

Ex. For $n = 2$, $X = (x_1, x_2)$, $Y = (y_1, y_2)$:

- $f := x_1 \oplus y_1 \oplus x_2 \oplus y_2 \oplus 1$ is trivial.
- $g := (x_1\, x_2) \oplus (y_1\, y_2)$ is trivial.
- $h := (x_1\, y_1) \oplus (x_2\, y_2)$ is NOT trivial.

Def. *A box* $\mathtt{P}$ *is said to be* **collapsing** *(or* trivial*) if using copies of this box* $\mathtt{P}$ *any Boolean function f is trivial, with probability $\geq q > \frac{1}{2}$ for some q independent of n, f, X, Y.*

Ex.
- The famous PR box is collapsing [2].
- Local ($\mathcal{L}$) and quantum ($\mathcal{Q}$) boxes are NOT collapsing [7].

— *Part* 3 —

Results

3.1. Algebra of Boxes

Fact. Given a wiring W, the new box $\mathtt{P} \boxtimes_{\mathsf{W}} \mathtt{Q}$ is bilinear in the boxes $(\mathtt{P}, \mathtt{Q})$.
So $\mathcal{B}_{\mathsf{W}} := (\{boxes\}, \boxtimes_{\mathsf{W}})$ is an algebra, that we call the **algebra of boxes**.

Proposition (Characterization of commutativity and associativity)

Assume W is a wiring such that $f_1 = f_2 = f(x)$ and $g_1 = g_2 = g(y)$. Then:

1. $\mathcal{B}_{\mathsf{W}}$ is commutative $\iff$ $f_3(x, a_1, a_2) = f_3(x, a_2, a_1)$ and $g_3(y, b_1, b_2) = g_3(y, b_2, b_1)$.

If in addition $f(x) = x$ and $g(y) = y$:

2. $\mathcal{B}_{\mathsf{W}}$ is associative $\iff$ $f_3(x, a_1, f_3(x, a_2, a_3)) = f_3(x, f_3(x, a_1, a_2), a_3)$ and $g_3(y, b_1, g_3(y, b_2, b_3)) = g_3(y, g_3(y, b_1, b_2), b_3)$.

3.2. Orbit of a Box

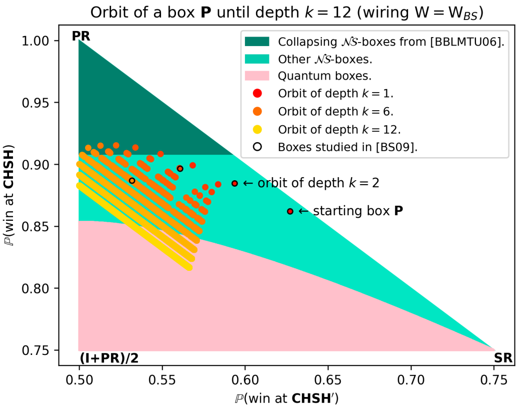
$$\mathtt{Orbit}^{(3)}(\mathrm{P}) = \big\{(\mathrm{P}\boxtimes\mathrm{P})\boxtimes\mathrm{P},\ \mathrm{P}\boxtimes(\mathrm{P}\boxtimes\mathrm{P})\big\}\,,$$

$$\mathtt{Orbit}^{(4)}(\mathrm{P}) = \Big\{\big((\mathrm{P}\boxtimes\mathrm{P})\boxtimes\mathrm{P}\big)\boxtimes\mathrm{P},\ \big(\mathrm{P}\boxtimes(\mathrm{P}\boxtimes\mathrm{P})\big)\boxtimes\mathrm{P},\ (\mathrm{P}\boxtimes\mathrm{P})\boxtimes(\mathrm{P}\boxtimes\mathrm{P}),\ \mathrm{P}\boxtimes\big((\mathrm{P}\boxtimes\mathrm{P})\boxtimes\mathrm{P}\big),\ \mathrm{P}\boxtimes\big(\mathrm{P}\boxtimes(\mathrm{P}\boxtimes\mathrm{P})\big)\Big\}\,,$$

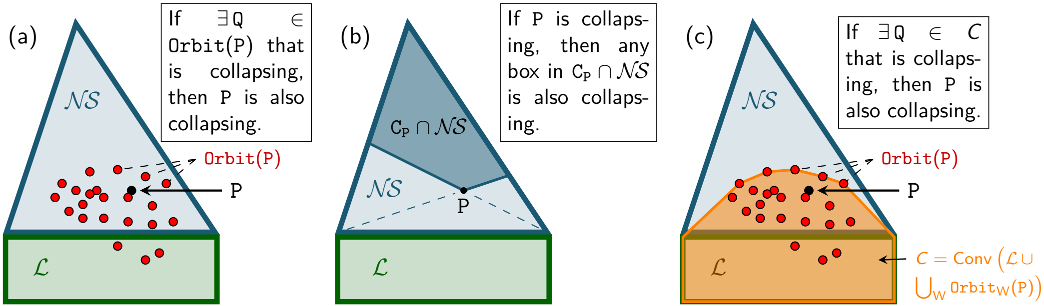
$$\mathtt{Orbit}^{(k)}_{\mathrm{W}}(\mathrm{P}) := \Big\{ \text{all possible products with } k \text{ times the term } \mathrm{P}, \text{ using the multiplication } \boxtimes_{\mathrm{W}} \Big\}.$$

Proposition. *For fixed k, the points of the orbit are aligned, and the highest CHSH-value is achieved by the parenthesization with only multiplication on the right:*

$$\mathrm{P}^{\boxtimes k} := \Big(\big((\mathrm{P}\boxtimes\mathrm{P})\boxtimes\mathrm{P}\big)\cdots\Big)\boxtimes\mathrm{P}\,.$$

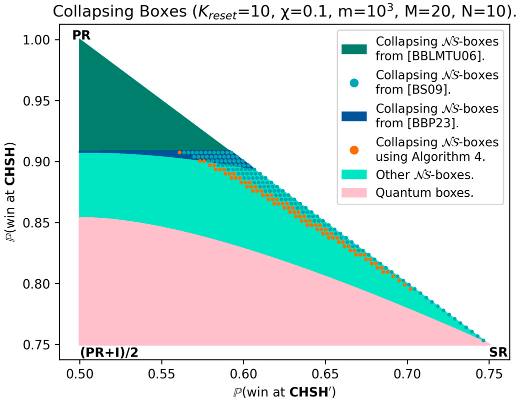


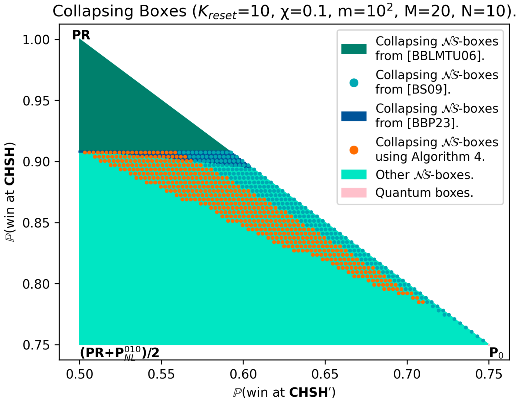
Here is the consequence to Communication Complexity:



3.3. Numerical Results

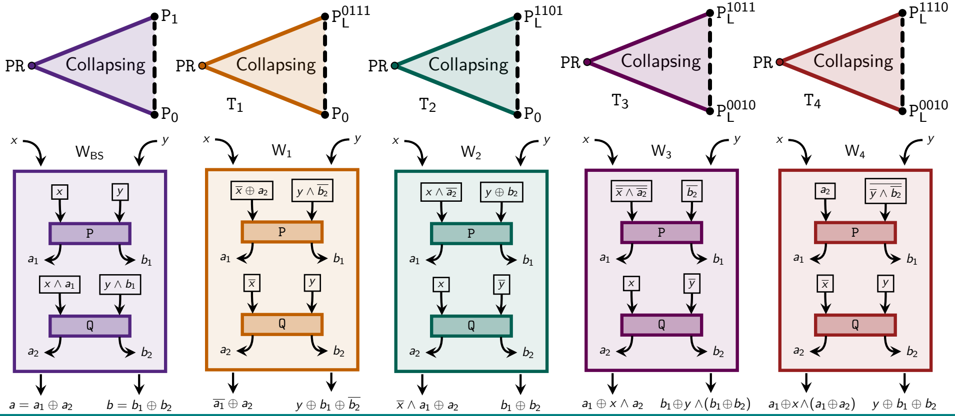
Using a gradient descent algorithm, we obtain in **orange** new collapsing boxes (this result is similar to the independent and concurrent work of [6]):





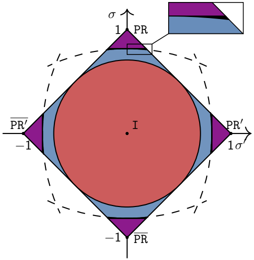
3.4. Analytical Results

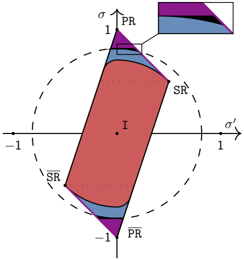
Based on the algebra of boxes and fixed-point theorems, we recover from [8] the following collapsing triangles of nonlocal boxes, with their respective wiring:



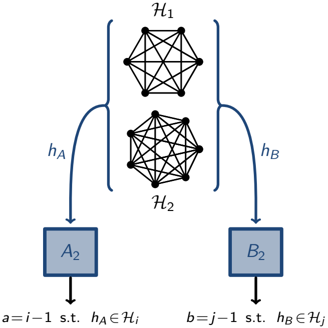
Our Other Related Results

B.–Broadbent–Proulx, PRL:132 (7 2024) [9].
We find that boxes above a certain ellipse collapse CC, using bias amplification by majority function:





B.–Weber, arXiv:2406.02199 [10] (online yesterday!)
We show that certain correlations for the graph isomorphism game, the graph coloring game, and the vertex distance game collapse CC:



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