Research questions
Tree-based methods: application to economics

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Master in economics (2^{nd} year)

What is Machine Learning?

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- Development, analysis and implementation of **computational methods**
- Allows a machine to evoluates in a given environment through a **training process**

Applications of Machine Learning?

Use examples:

- Pattern recognition (objects, hand-written characters)
- Clustering of phone-calls
- Medical and clinical decision making
- Web mining, search engines
- Image indexation

The impact of machine learning on economics

"I believe that ML will have a dramatic impact on the field of economics within a short time frame."

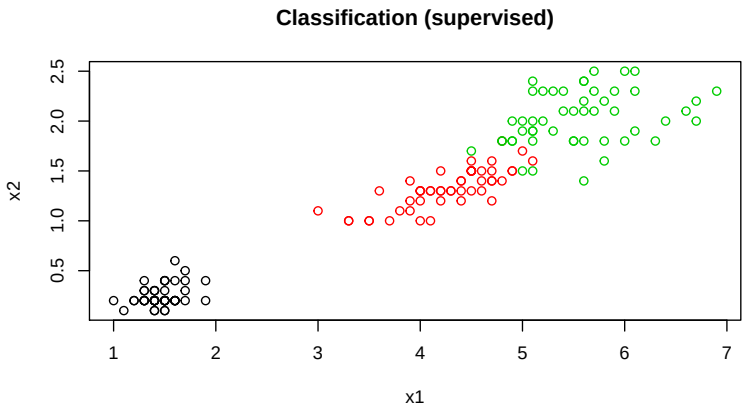
— Susan Athey, The Impact of Machine Learning on Economics, 2018.

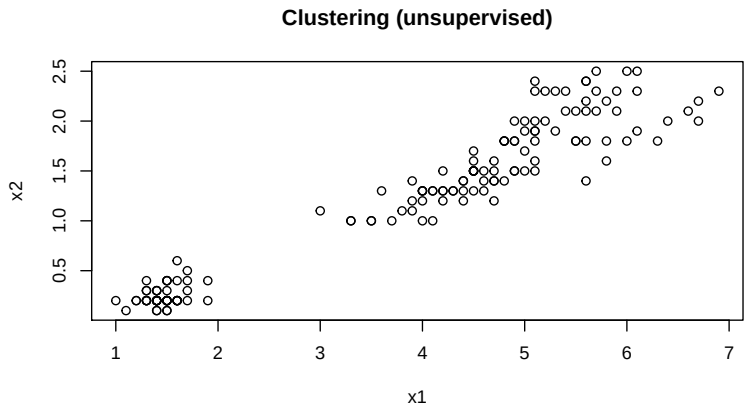
The impact of machine learning on economics

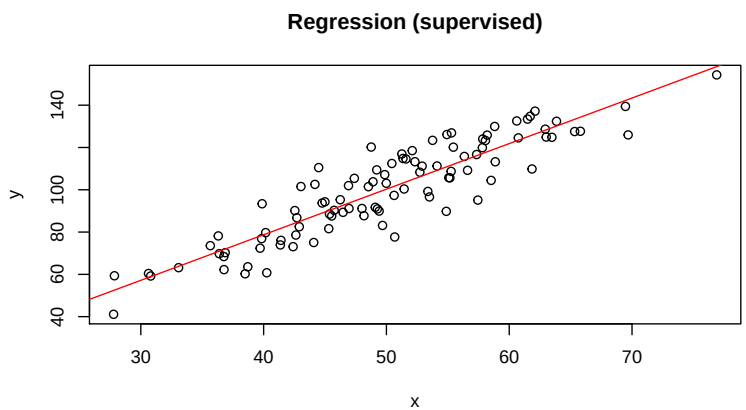
Applications of ML in many fields of economics:

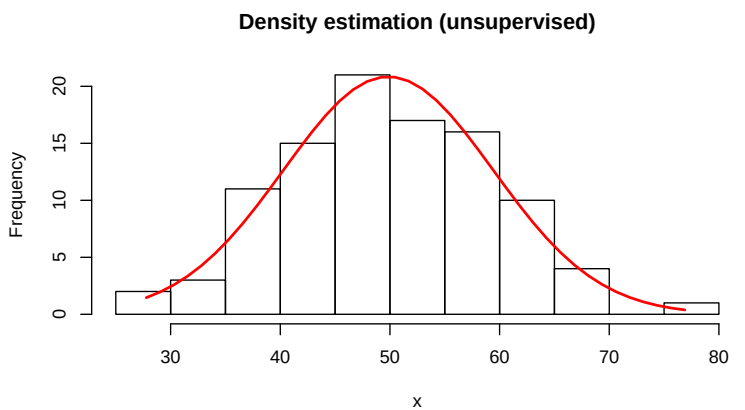
- Econometrics
- Health economics
- Labour economics
- Housing economics
- Economics of the internet

Supervised versus Unsupervised learning









2 Tree-based methods
- Binary trees
- Regression trees
- Classification trees

Decision trees

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Most popular method: Classification and Regression Trees (CART), Breiman, 1984.

Building a regression tree

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There are two steps for building a tree:

- Divide the feature space in J **non-overlapping regions** $R_1, R_2, ..., R_J$.
- For each observation in region R_j, compute the mean of the response values:

$$\hat{y}_{R_j} = \frac{1}{\#R_j} \sum_{i \in R_j} y_i$$

Building a regression tree

Q: How the regions are constructed ?

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Goal: find regions $R_1, ..., R_J$ that **minimize the residual sum of squares**, defined by:

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Note: RSS is also called the **training error**

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$$R_l(j,s) = \{X | X_j < s\} \text{ and } R_r(j,s) = \{X | X_j \geq s\}$$

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Finally we keep the pair (j,s) that **minimize**

$$\sum_{i:x_i \in R_l(j,s)} (y_i - \hat{y}_{R_l})^2 + \sum_{i:x_i \in R_r(j,s)} (y_i - \hat{y}_{R_r})^2$$

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Continues until a **stopping criterion** (e.g minimum size of a region) is verified

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The resulting tree is call the **maximal tree**

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Pruning consists in growing a very large tree T_0 and reduce it back to obtain a **subtree**

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Goal: For a given value α (**complexity parameter**), find a subtree $T \subset T_0$ that minimizes

$$\sum_{m=1}^{|T|} \sum_{x_i \in R_m} (y_i - \hat{y}_{R_m})^2 + \alpha |T|$$

Example: Predicting Baseball Players' Salaries

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Goal: Predict a baseball player's `Salary` based on `Years` (number of years played in major leagues) and `Hits` (number of hits made in the previous season).

Name	Years	Hits	Salary
Alan Ashby	14	81	475.0
Alvin Davis	3	130	480.0
Andre Dawson	11	141	500.0
Andres Galarraga	2	87	91.5
Alfredo Griffin	11	169	750.0

Figure: A sample of baseball players

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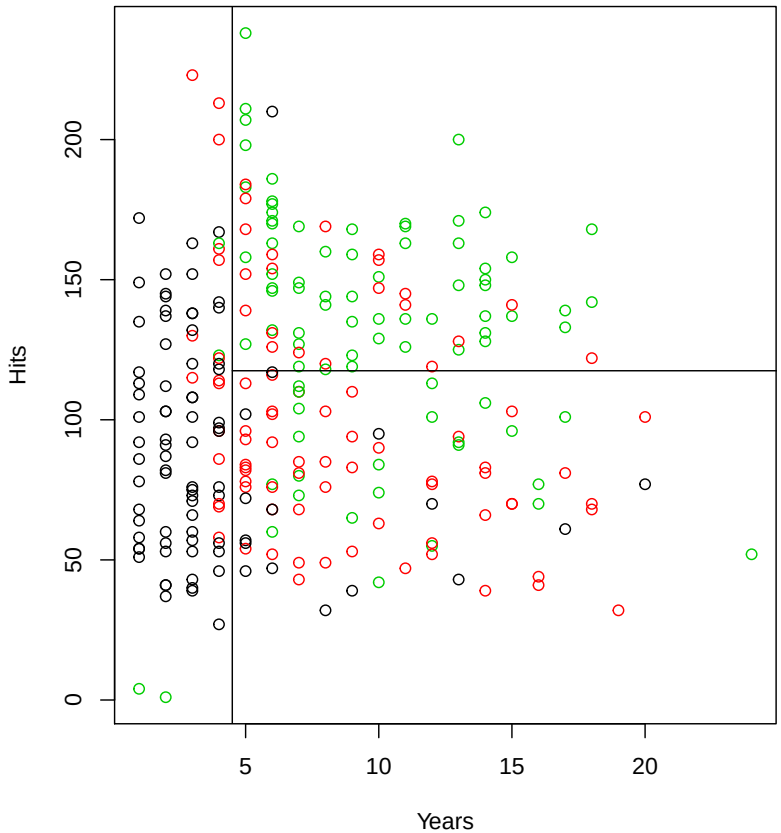
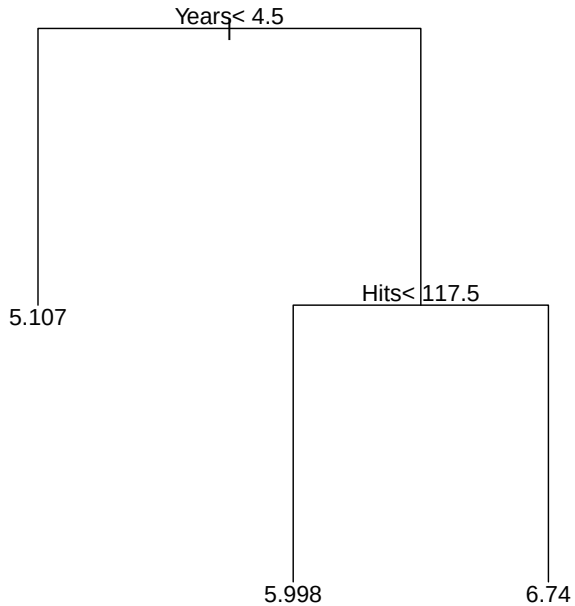
We denote:

- X_1: `Years` (exogeneous)
- X_2: `Hits` (exogeneous)
- Y: log(`Salary`) (endogeneous)


Hits
0
50
100
150
200
5
10
15
20
Years


Years< 4.5
5.107
Hits< 117.5
5.998
6.74

Classification tree

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Used to predict a **qualitative** response variable:
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$Y \in \{0, ..., K\}, K \in \mathbb{N}^*$

Assign to each region R_j the **most commonly occuring class** of observations in R_j

Problem: RSS cannot be used in classification: an alternative to RSS is the **classification error rate**

Classification error rate

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Note: in practice, two other measures are used

Gini index

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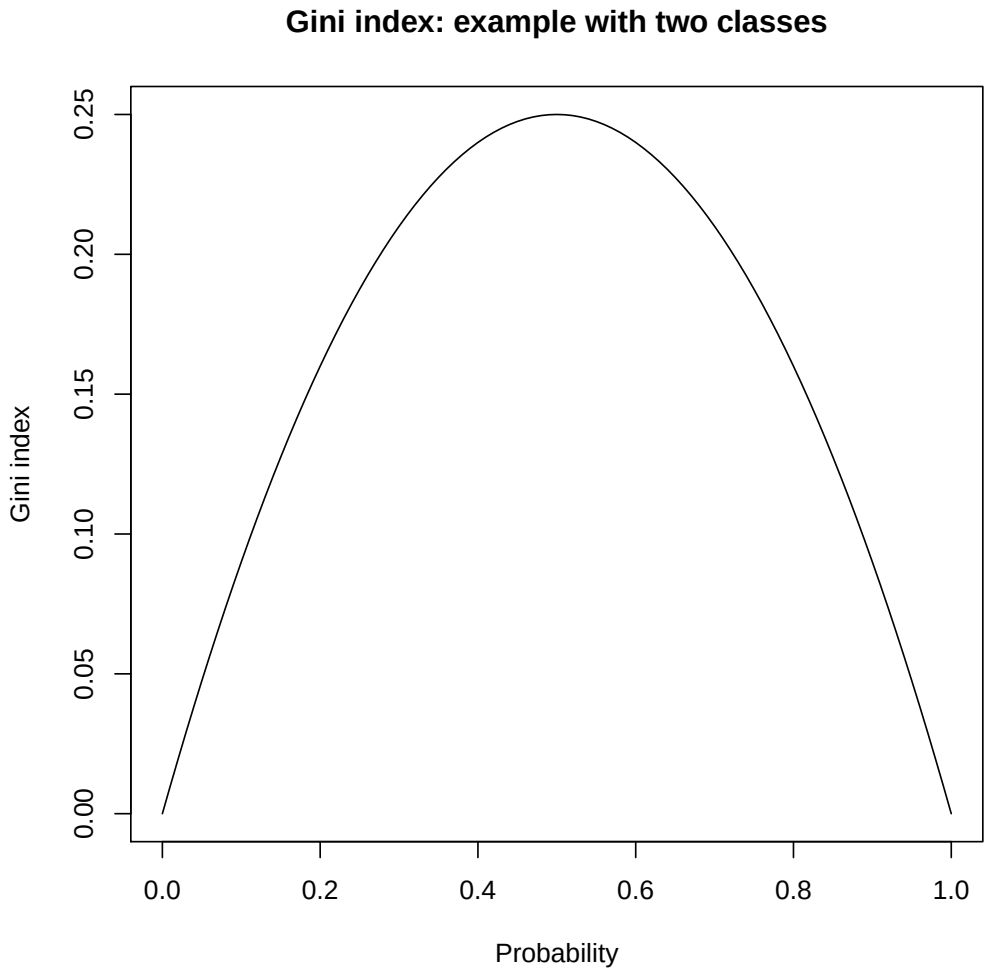
The Gini index is a measure of **purity**: small values indicate that a node contains a predominant class

Gini index


Gini index: example with two classes
Gini index
0.00
0.05
0.10
0.15
0.20
0.25
0.0
0.2
0.4
0.6
0.8
1.0
Probability

Cross-entropy

An alternative to the Gini index

$$D = -\sum_{k=1}^{K} \hat{p}_{mk} \log \hat{p}_{mk}$$

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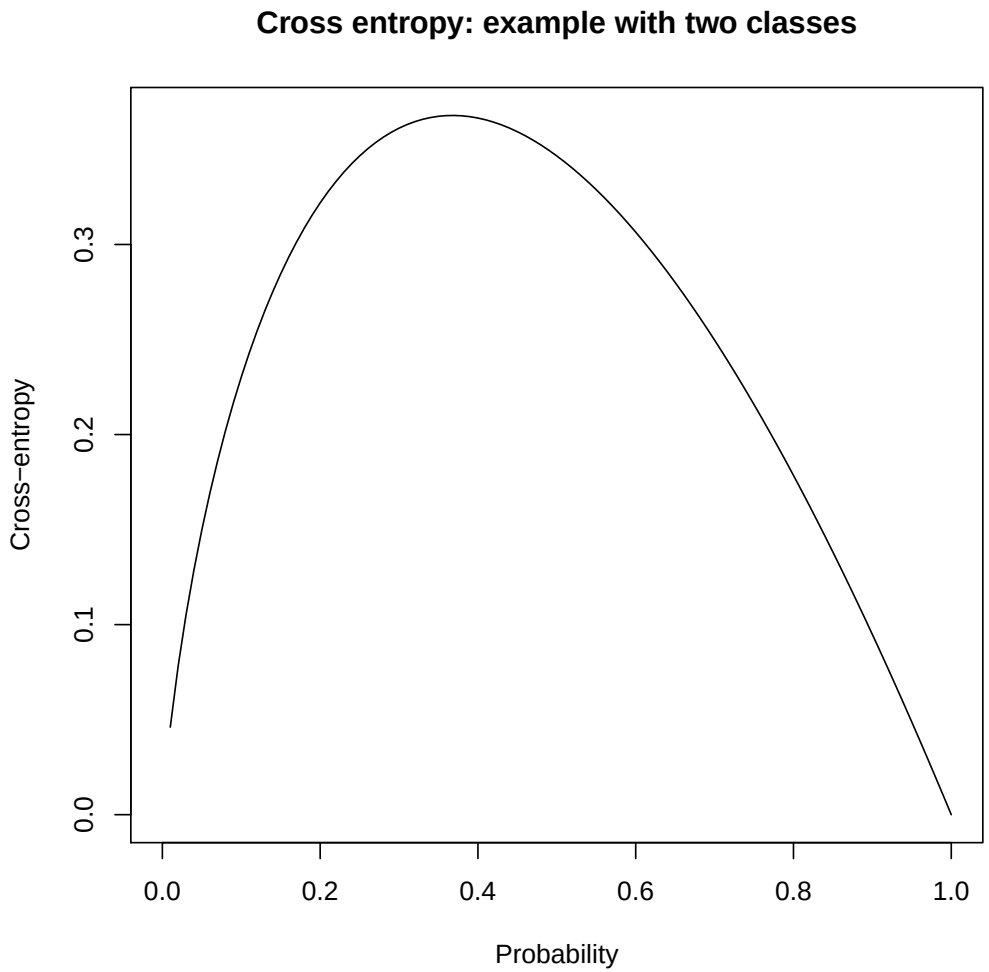
Note: Gini index and cross-entropy are similar numerically

Cross-entropy



Example: predicting heart disease

Data: **Heart** dataset containing 303 observations of 14 features.

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Goal: Predict the presence of heart disease `HD` based on different predictors such as `Age`, `Sex`, `Chol` (a cholesterol measurement)...

Age	Sex	Chestpain	...	HD
63	1	typical	...	No
67	1	asymptomatic	...	Yes
67	1	asymptomatic	...	Yes
37	1	nonanginal	...	No
41	0	nontypical	...	No

Figure: A sample of patients

Age	Sex	Chestpain	...	HD
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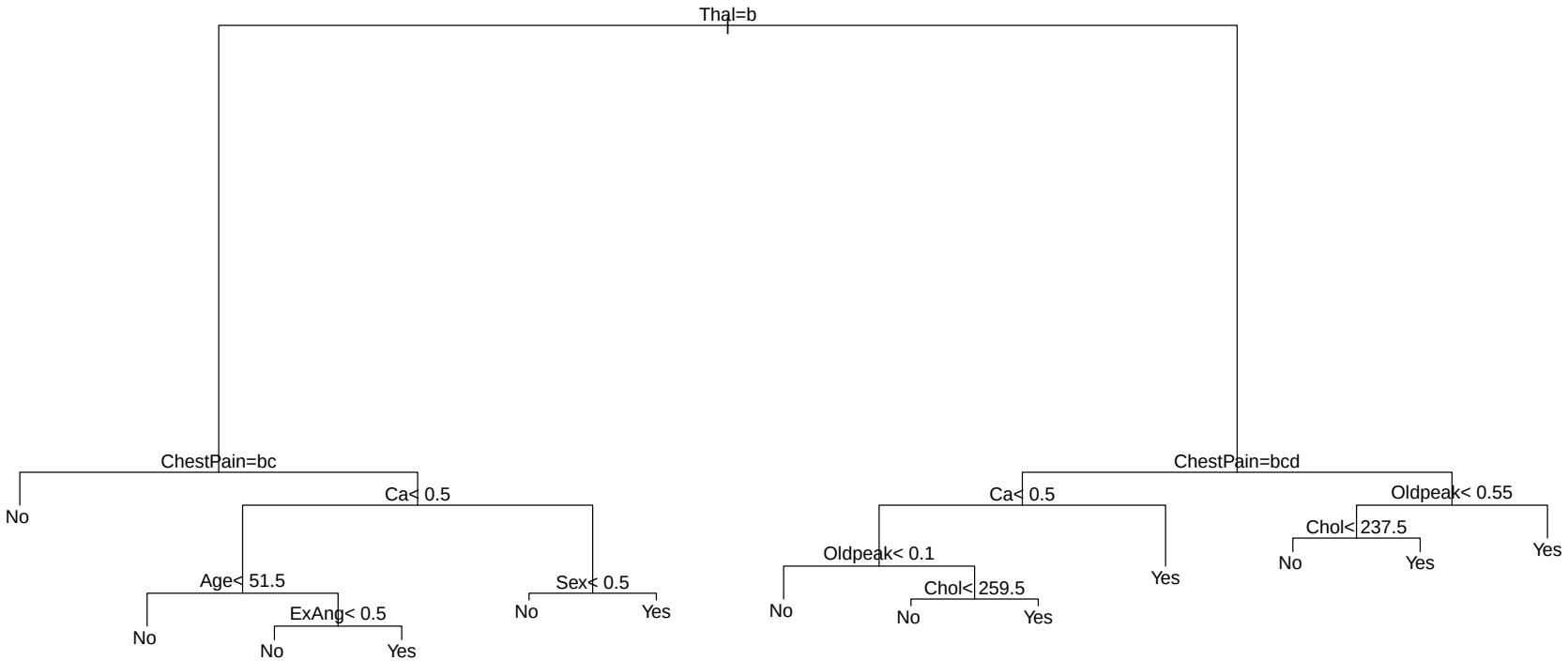
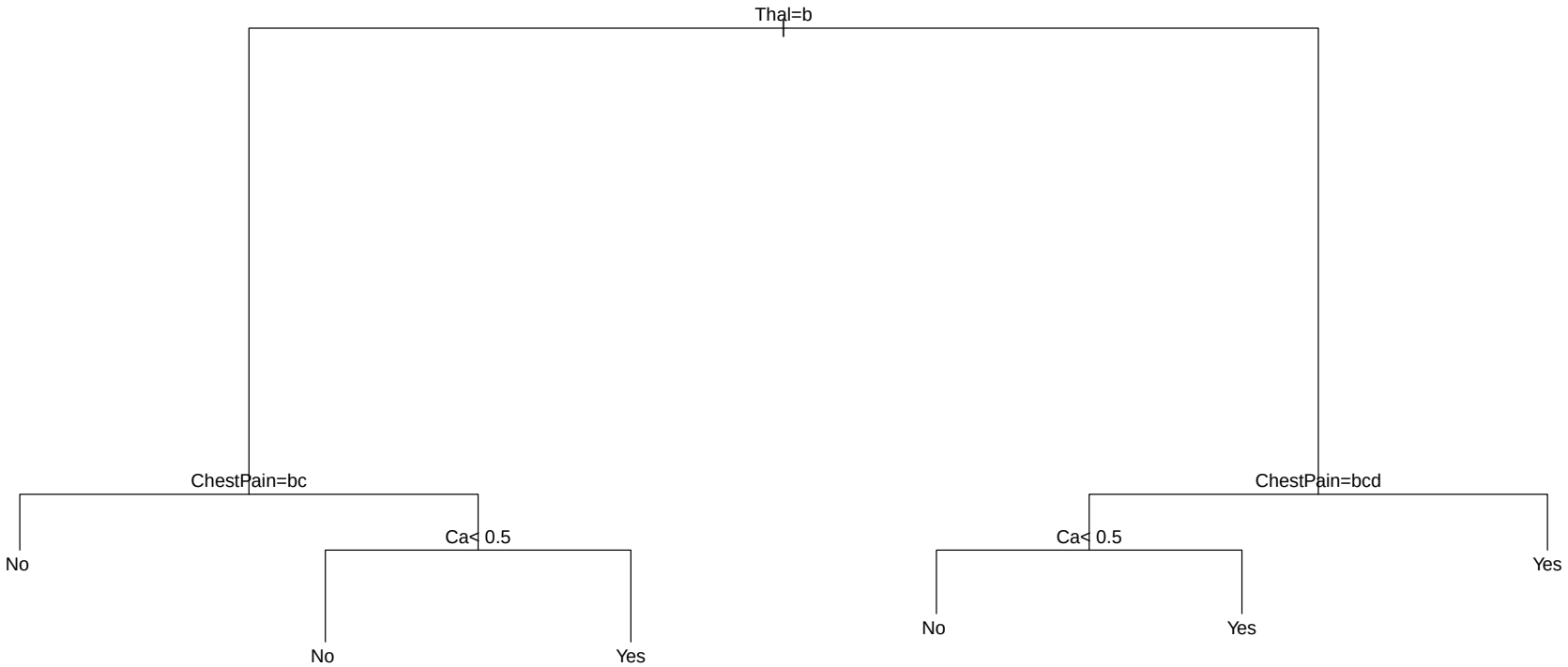
37	1	nonanginal	...	No
41	0	nontypical	...	No

Figure: A sample of patients

We denote:

- X_1: `Age` (exogeneous)
- X_2: `Sex` (exogeneous)
- ...
- Y: `HD` (endogeneous)


Thal=b
ChestPain=bc
Ca< 0.5
No
Age< 51.5
Sex< 0.5
No
ExAng< 0.5
No
No
Yes
Yes
ChestPain=bcd
Ca< 0.5
Oldpeak< 0.55
Oldpeak< 0.1
Yes
Chol< 237.5
No
Chol< 259.5
Yes
No
No
Yes
Yes


Thal=b
ChestPain=bc
ChestPain=bcd
Ca< 0.5
Ca< 0.5
No
No
Yes
No
Yes
Yes

Trees: pros and cons

Pros:

- Easy to use
- Help decision making
- Graphical, interpretable
- Deal with both quantitative and qualitative features

Cons:

- Not the best method in terms of prediction accuracy
- Trees are unstable

3 Advanced tree-based methods
- Bagging and Random Forest
- Clustering using binary trees
- Variable importance

Bagging

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Averaging a set of obervations reduces variance

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Idea:

- Calculate B predictions denoted $\hat{f}^1(x), \hat{f}^2(x), ..., \hat{f}^B(x)$ using B separate training sets

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- Problem: In practice, we do not have access to multiple training sets

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- Train a model (a tree) on the b^{th} bootstrapped training dataset in ordre to get $\hat{f}*b(x)$
- Average all the predictions:

$$\hat{f}_{bag}(x) = \frac{1}{B}\sum_{b=1}^{B}\hat{f}*b(x)$$

Random forest

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Random forest

- RF is an improvement over bagged trees, using a "trick" that **decorrelated** the bootstrapped trees
- A random sample of m predictors is chosen among the full set of p predictors at each split of the tree
- Typically, we choose $m = \sqrt{p}$, useful when we have a high number of correlated predictors

Unsupervised learning

- Two issues in unsupervised learning: **Clustering** and **Density estimation**

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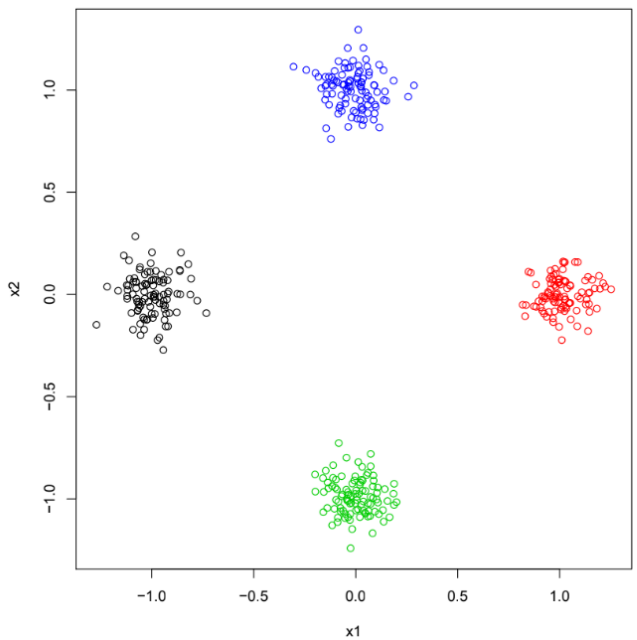

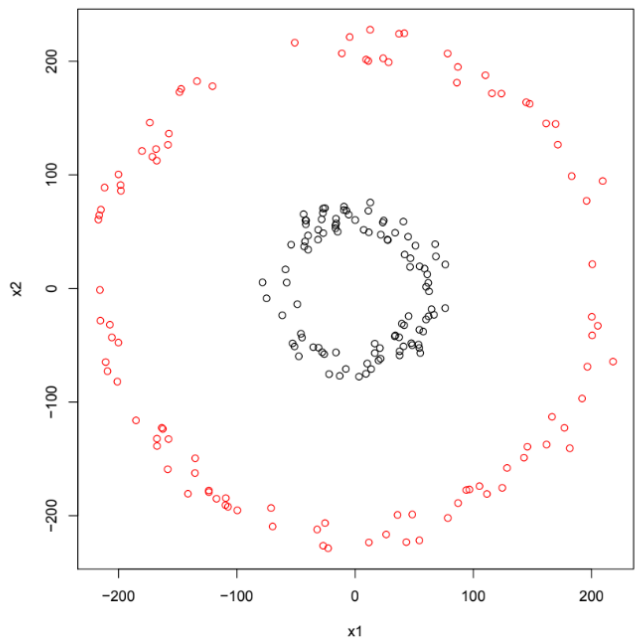

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Classical problems:

- Measure the **performance** of a partition
- Choose the optimal value of K, the **number of clusters**

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- **joining** the leaves of the tree (alternative pruning)

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- Good convergence properties

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Let $X_t = \{X | X \in t\}$, $\alpha_t = P(X \in t)$ and $R(t)$ a heterogeneity measure (**deviance**) of t, defined as:

Step 1: Growing the maximal tree

Let t be a tree node containing a set of observations in $\mathbb{R}^p$. The **child nodes** of t are denoted t_L and t_R, defined as follows:

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The **best split** of t is defined by the pair $(j, a) \in \{1, ...p\} \times \mathbb{R}$ maximizing

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The clustering **tree** represents the **partition**. Each **leaf** represents a **cluster**.

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$$\bar{d}_L^\delta = \frac{1}{\delta n_L} \sum_{i=1}^{\delta n_L} d_i \text{ and } \bar{d}_R^\delta = \frac{1}{\delta n_R} \sum_{j=1}^{\delta n_R} d_j$$

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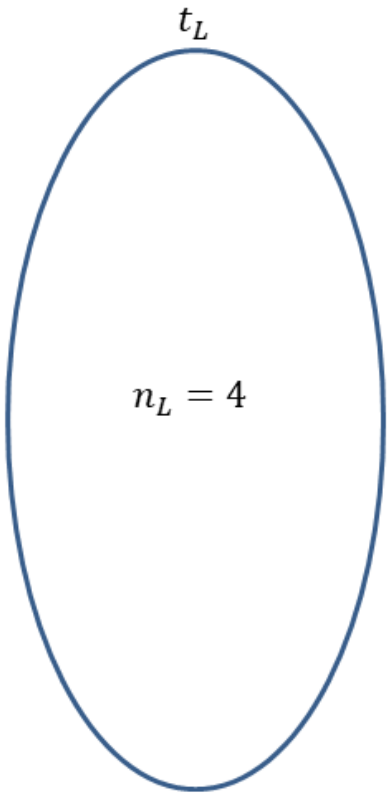
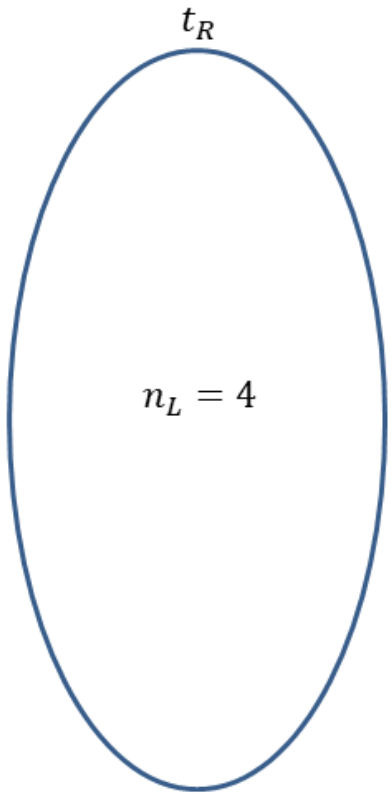
and their ordered versions $\{d_i\}$ and $\{d_j\}$, where d(X,y) is the Euclidean distance

Dissimilarity measure: illustration





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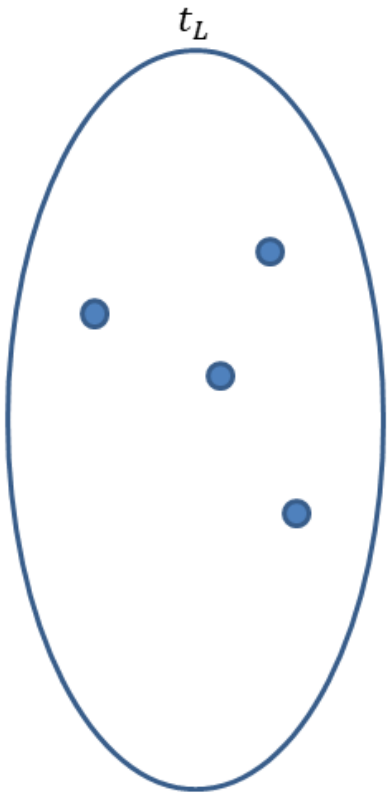

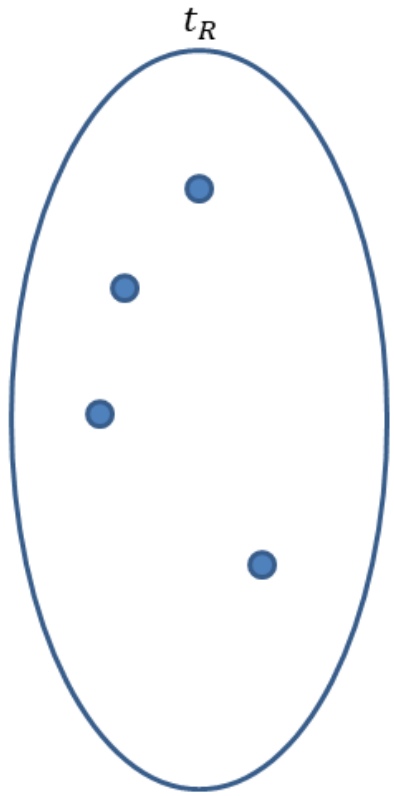

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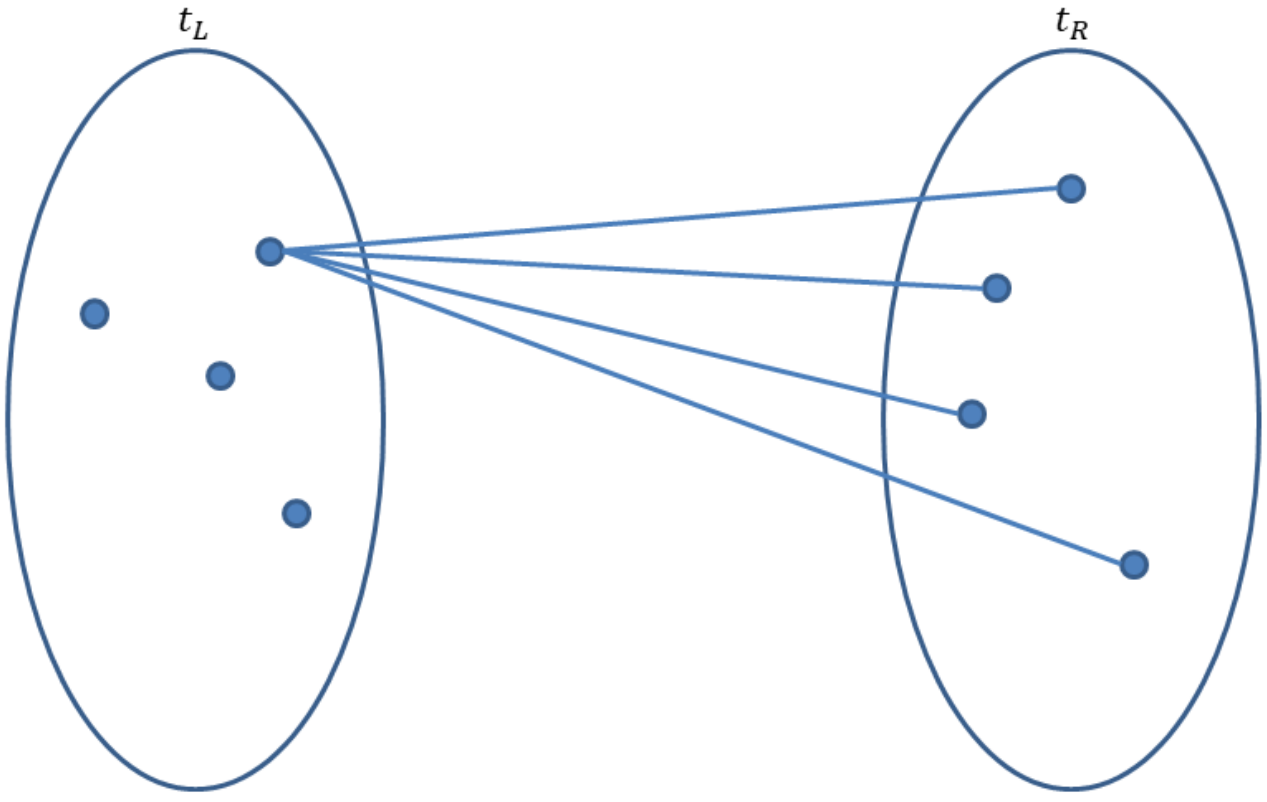

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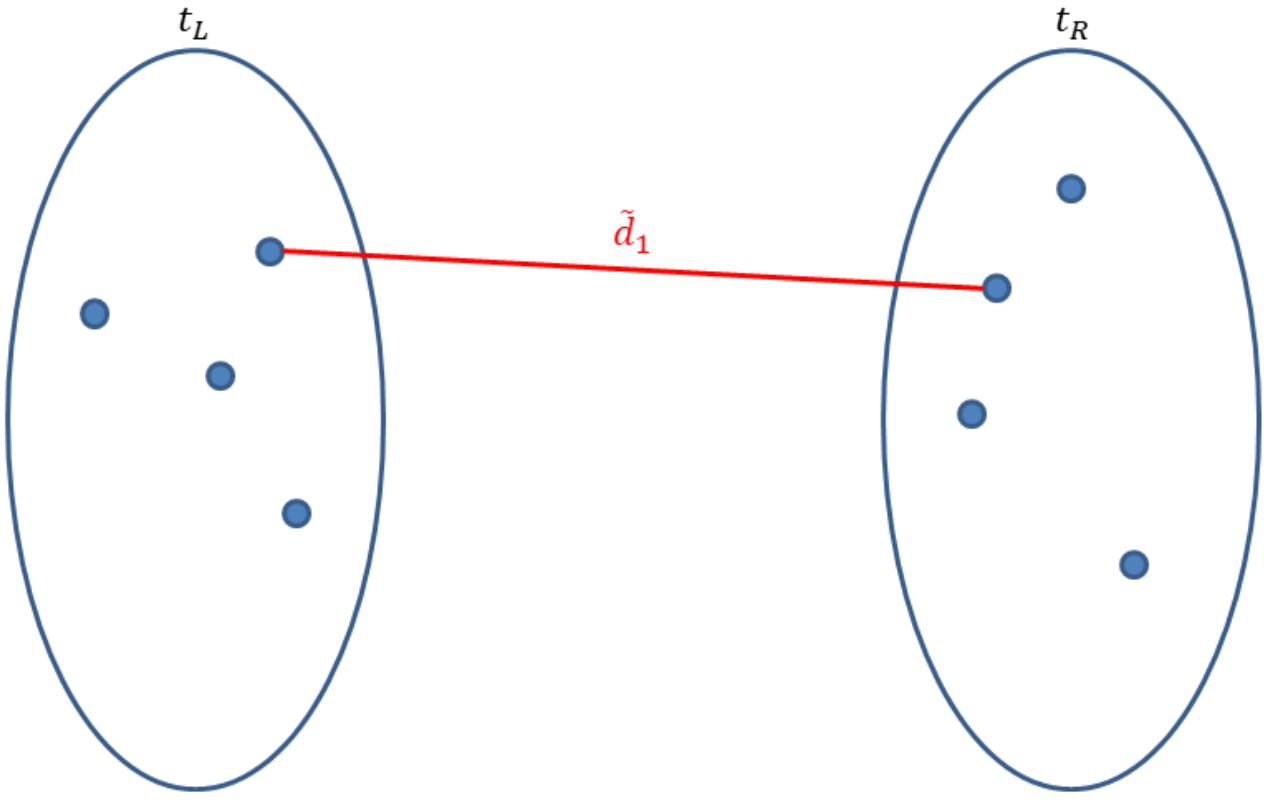

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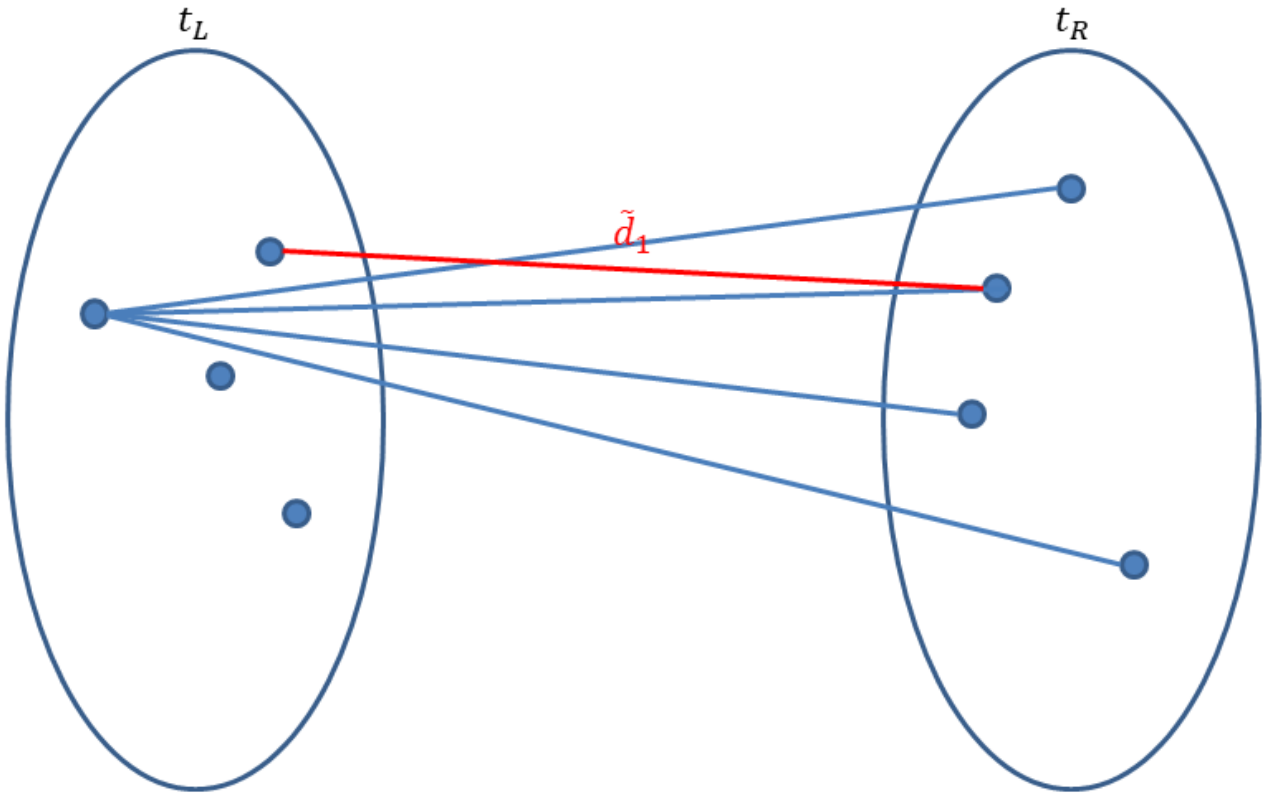

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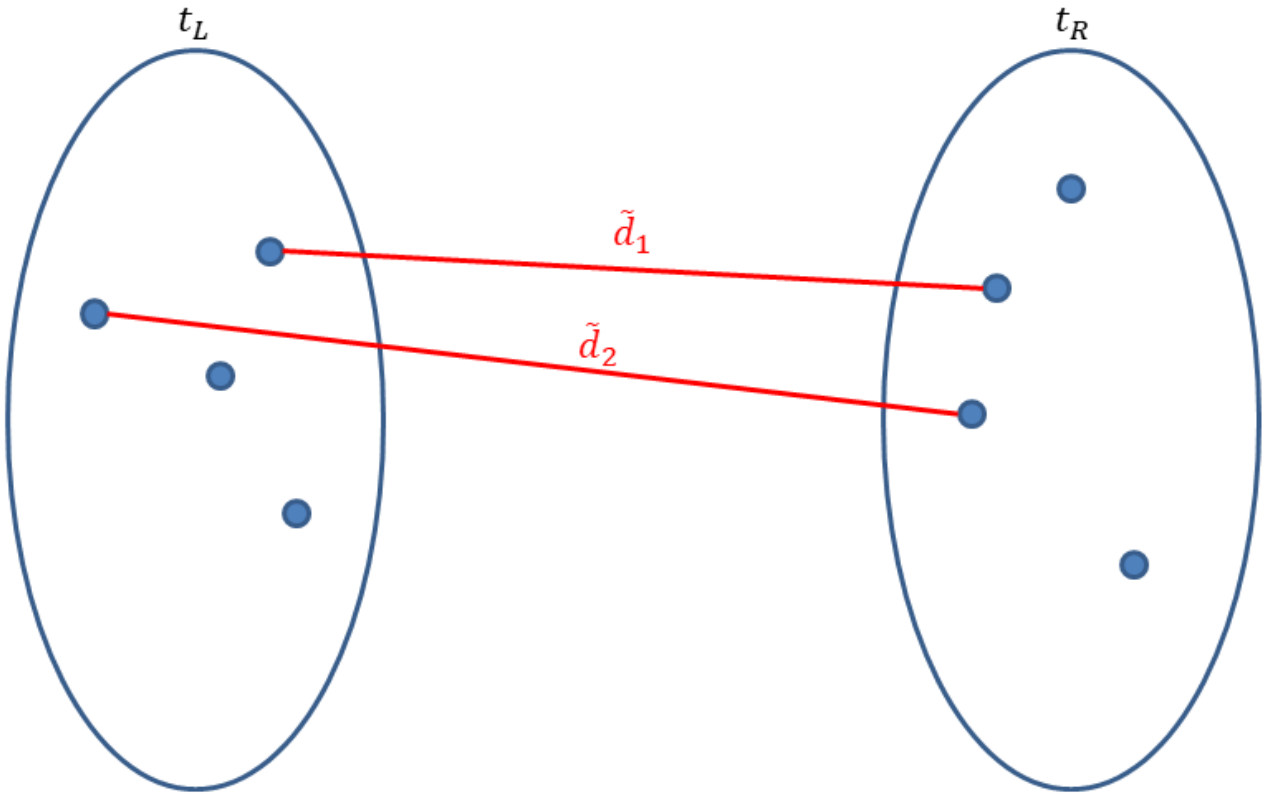

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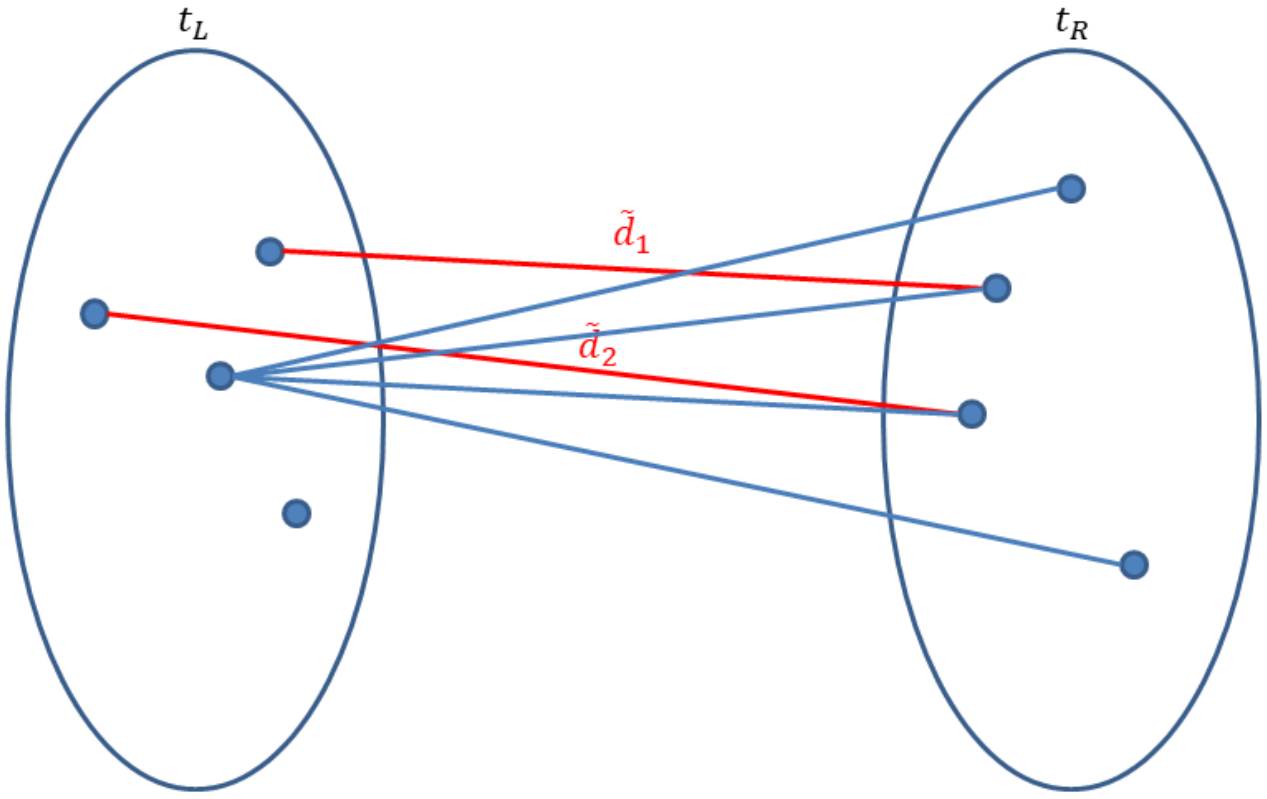

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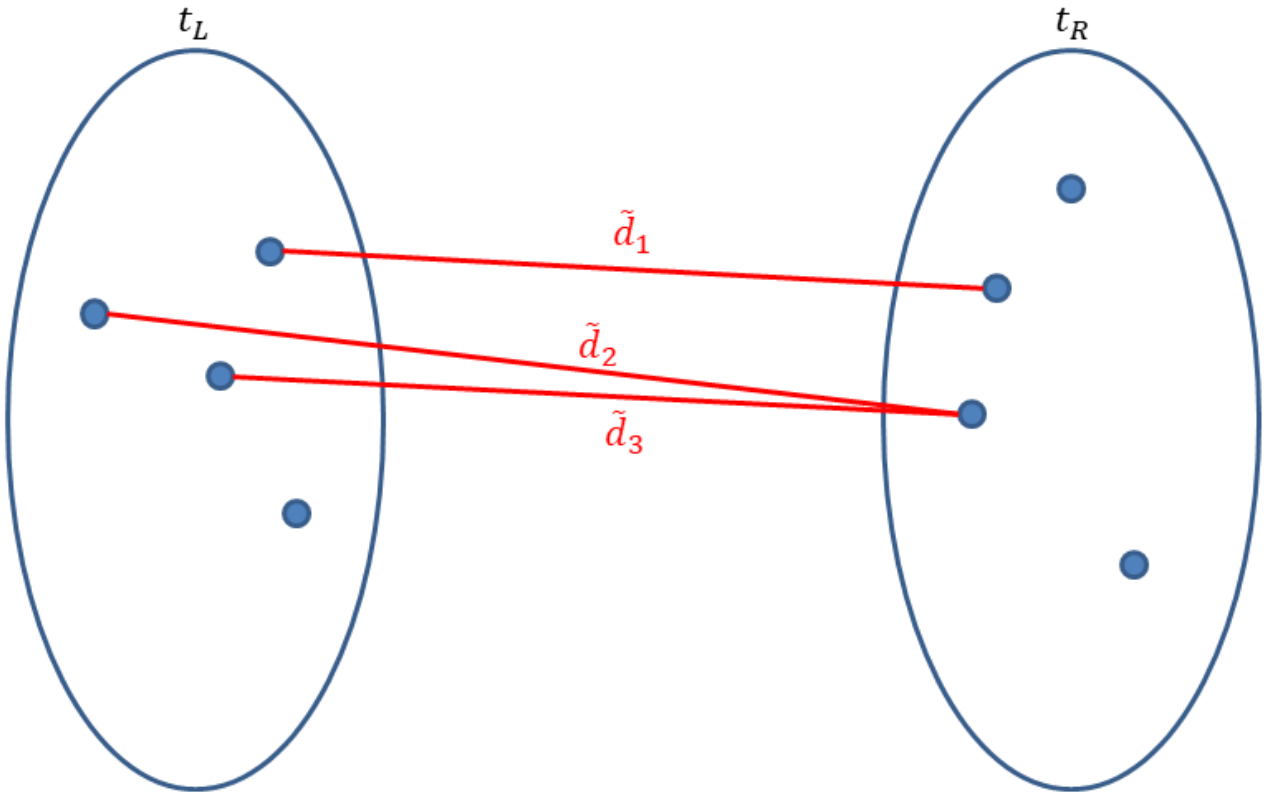

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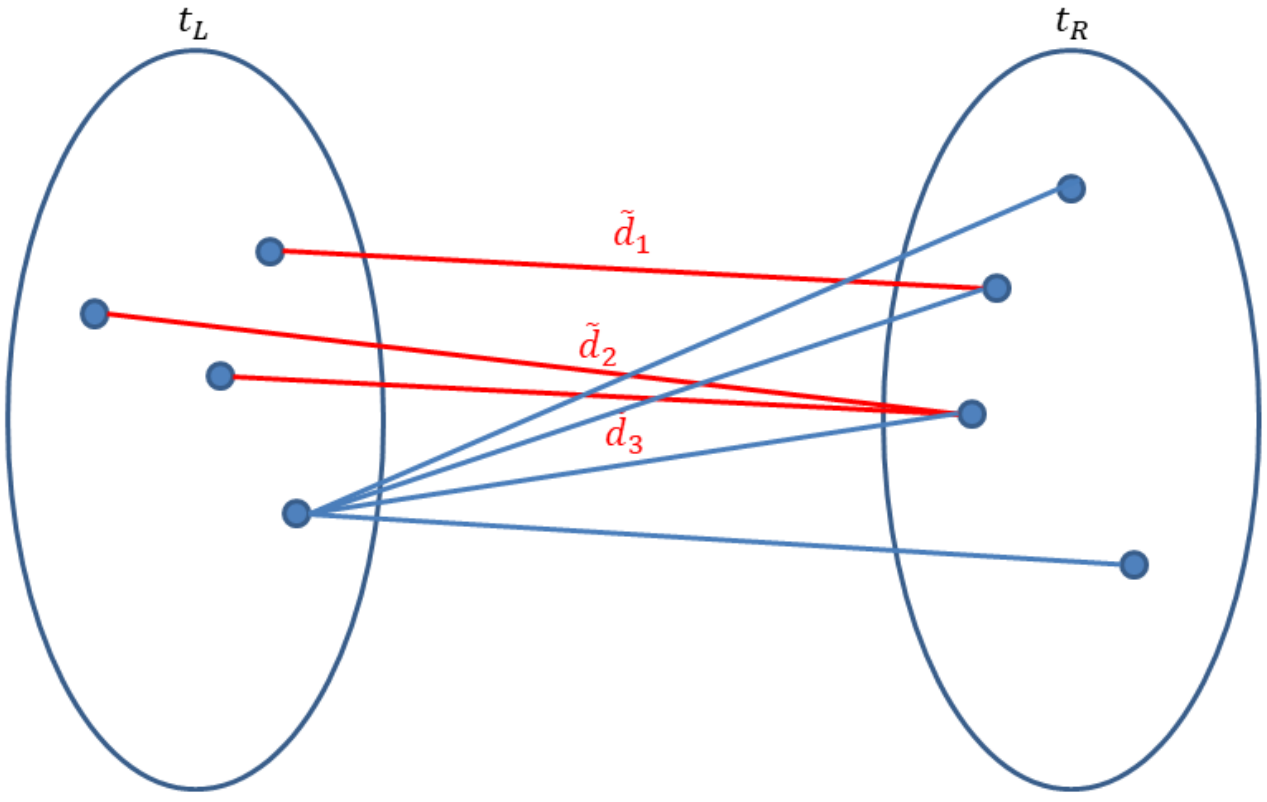

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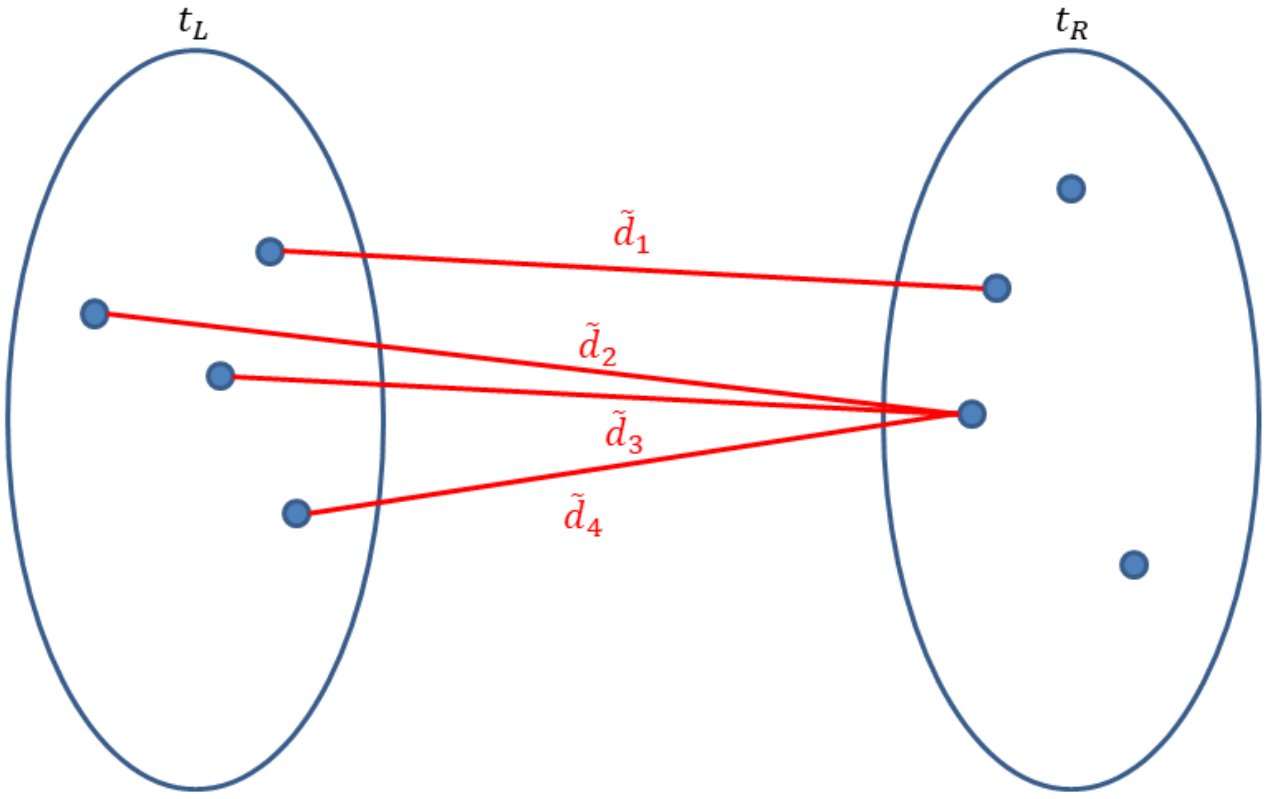

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Joining the leaves

$t_{\tilde{L}}$ and $t_{\tilde{R}}$ are replaced by their union $t_{\tilde{L}} \cup t_{\tilde{R}}$ and $N_L = N_L - 1$.
Stop when $N_L = K$.

CUBT: pros and cons

Pros:

- Decisional method
- Interpretable
- Extensions to other types of data
- Adapted to parallel computing
- Partition of the feature space, not only the training dataset

Cons:

- Same as CART
- Trees are unstable

Motivation and objectives

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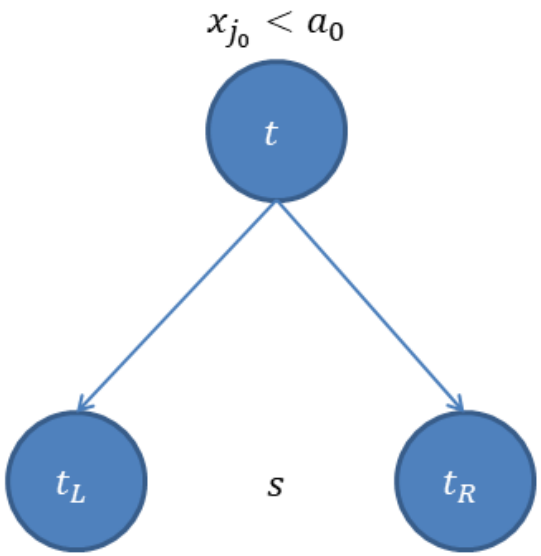
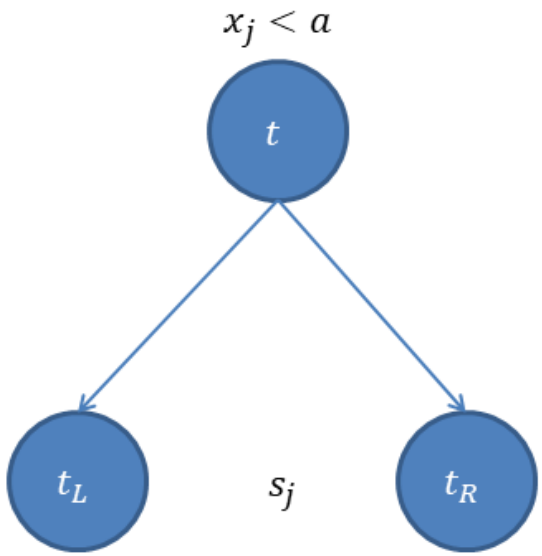
- Define variable importance in CUBT
- Analyze its stability
- Compare to other methods

Competitive splits

To compute the importance of a feature j, we define the **competitive split** of a feature j_0 in a node t.





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p_{RR} can be defined equivalently

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which is the **loss of deviance** induced if each node is replaced by the surrogate split defined on X_j

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- Stability of variable importance

4 Applications to Economics: Causal Trees

Causal trees

Recursive partitioning for heterogeneous causal effect . Susan Athey and Guido Imbens (2016). In PNAS.

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This approach is based on regression trees, modified to take into account the treatment effects in causal models

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- We have N **units**, indexed by $i = 1, ..., N$

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- $W_i \in \{0, 1\}$, binary indicator of the treatment (0 for control, 1 for treatment)
- The outcome is

$$Y_i^{obs} = Y_i(W_i) = \begin{cases} Y_i(0) & \text{if } W_i = 0 \\ Y_i(1) & \text{if } W_i = 1 \end{cases}$$

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We try to estimate the **conditional average treatment effect** (CATE)

$$\tau(x) = \mathbb{E}[Y_i(1) - Y_i(0)|X_i = x]$$

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- Estimation CATE rather than predicting outcomes
- "Honest" estimation: use separate datasets for constructing the partition (the regions) and estimating the effects within the leaves
- In CART, the estimation is "adaptive" (same dataset)

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$$\pi(S) = \begin{cases} \{\{L, R\}\} & \text{if } \bar{Y}_L - \bar{Y}_R \leq c \\ \{\{L\}, \{R\}\} & \text{if } \bar{Y}_L - \bar{Y}_R > c \end{cases}$$

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$$\hat{\mu}(x, S, \Pi) = \frac{1}{\#(i \in S : X_i \in l(x, \Pi)} \sum_{i \in S : X_i \in l(x, \Pi)} Y_i$$

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In the same way, the authors also define "honest" splitting and cross-valdation.

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- From trees to forests: *Generalized Random Forests.* Susan Athey, Julie Tibshirani and Stefan Wager(2018). In arxiv.

End of lecture.