# A few comments about "Topology" by Munkres

Pierre-Yves Gaillard[1]

As the title indicates, we make a comments about the book **Topology** by James R. Munkres. This is a work in progress.

---

• **Definition of $\mathbb{R}$ p. 31.** The object $\mathbb{R}$ is defined by assuming that there exists a set having certain properties. We take this assumption for granted. Then it is easy to see that there are several sets having these properties. So, let $\mathbb{R}'$ be a set having the same properties as $\mathbb{R}$. Let $\mathbb{Z}'_+, \mathbb{Z}'$ and $\mathbb{Q}'$ be to $\mathbb{R}'$ what $\mathbb{Z}_+, \mathbb{Z}$ and $\mathbb{Q}$ are to $\mathbb{R}$.

**Theorem 1.** *There is a unique morphism of fields from $f : \mathbb{R} \to \mathbb{R}'$. This morphism is an isomorphism of ordered fields, and it induces isomorphisms $\mathbb{Z}_+ \to \mathbb{Z}'_+, \mathbb{Z} \to \mathbb{Z}'$ and $\mathbb{Q} \to \mathbb{Q}'$.*

**Lemma 2.** *There is a unique map $g : \mathbb{Z}_+ \to \mathbb{Z}'_+$ such that $g(0) = 0$ and $g(n+1) = g(n)+1$ for all $n$ in $\mathbb{Z}_+$. Similarly, there is a unique map $h : \mathbb{Z}'_+ \to \mathbb{Z}_+$ such that $h(0) = 0$ and $h(n+1) = h(n)+1$ for all $n$ in $\mathbb{Z}'_+$.*

*Proof.* For $i \in \mathbb{Z}_+$ and $\varphi : \{1, \dots, i\} \to \mathbb{Z}'_+$ define $\rho(\varphi) \in \mathbb{Z}'_+$ by $\rho(\varphi) := \varphi(i) + 1$. Then the first statement follows from the Principle of Recursive Definition (Theorem 3 p. 3). The proof of the second statement is similar. □

*Proof of Theorem 1.* In the notation of Lemma 2, set $u := h \circ g$. Then $u : \mathbb{Z}_+ \to \mathbb{Z}_+$ satisfies $u(0) = 0$ and $u(n + 1) = u(n) + 1$ for all $n$ in $\mathbb{Z}_+$. One can easily prove that $u(n) = n$ by induction. The same argument works for $g \circ h$. This shows that $g : \mathbb{Z}_+ \to \mathbb{Z}'_+$ and $h : \mathbb{Z}'_+ \to \mathbb{Z}_+$ are inverse isomorphisms. Then we extend $g$ to morphisms $\mathbb{Z} \to \mathbb{Z}'$ and $\mathbb{Q} \to \mathbb{Q}'$, and similarly for $h$, and, arguing as before, we show that these morphisms are isomorphisms. More precisely, we see that there is a unique morphism $\mathbb{Z} \to \mathbb{Z}'$ extending $g$, and that this morphism is an isomorphism, and similarly for the morphism $\mathbb{Q} \to \mathbb{Q}'$. So we can make the identifications $\mathbb{Z}_+ = \mathbb{Z}'_+, \mathbb{Z} = \mathbb{Z}', \mathbb{Q} = \mathbb{Q}'$. To show that there is a unique morphism of fields $\mathbb{R} \to \mathbb{R}'$, and that this morphism is an isomorphism (inducing the identity of $\mathbb{Q}$), we argue as in Section *Appendix to Chapter 1* in *A few comments about "Principles of Mathematical Analysis" by Rudin*, available at `https://zenodo.org/records/13955297`. □

---

• **Exercise 7.6. p. 51.** We say that two sets $A$ and $B$ **have the same cardinality** if there is a bijection of $A$ with $B$.

[1]ORCID https://orcid.org/0000-0002-7960-1698

(a) Show that if $B \subset A$ and if there is an injection

$$f : A \to B,$$

then $A$ and $B$ have the same cardinality. [Hint: Define $A_1 = A, B_1 = B$, and for $n > 1$, $A_n = f(A_{n-1})$ and $B_n = f(B_{n-1})$. (Recursive definition again!) Note that $A_1 \supset B_1 \supset A_2 \supset B_2 \supset A_3 \supset \cdots$ Define a bijection $h : A \to B$ by the rule

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_n \setminus B_n \text{ for some } n, \\ x & \text{otherwise.]} \end{cases}$$

(b) *Theorem (Schroeder-Bernstein theorem).* If there are injections $f : A \to C$ and $g : C \to A$, then $A$ and $C$ have the same cardinality.

**Solution.** (a) We will freely use the following two obvious facts:

(F1) For $x \in A$ and $n \in \mathbb{Z}_+$ we have

$$x \in A_n \iff f(x) \in A_{n+1} \text{ and } x \in B_n \iff f(x) \in B_{n+1}.$$

(F2) We have $\bigcap_{n\geq 1} A_n = \bigcap_{\geq 1} B_n =: I$.

Setting $A'_n := A_n \setminus B_n, B'_n := B_n \setminus A_{n+1}$, we get

$$A = \left(\bigcup_{n\geq 1} A'_n\right) \cup \left(\bigcup_{n\geq 1} B'_n\right) \cup I,$$

and this union is disjoint. We also have

$$B = \left(\bigcup_{n\geq 2} A'_n\right) \cup \left(\bigcup_{n\geq 1} B'_n\right) \cup I.$$

The injection $f$ induces bijections $f_n : A'_n \to A'_{n+1}$ (here we are using (F1)). To define a bijection $h : A \to B$, it suffices to define three bijections

$$u : \bigcup_{n\geq 1} A'_n \to \bigcup_{n\geq 2} A'_n, \quad v : \bigcup_{n\geq 1} B'_n \to \bigcup_{n\geq 1} B'_n, \quad w : I \to I.$$

We define $u$ by $u(x) = f_n(x)$ if $x \in A'_n$, and take $v$ and $w$ to be the identity maps.

(b) We set $B := g(C) \subset A$ and define $f' : A \to B$ by $f'(a) := g(f(a))$. Then $f' : A \to B$ satisfies the assumptions for $f : A \to B$ in (a).

---

• **Exercise 8.7. p. 56.** Prove Theorem 8.4 p. 54.

**Solution.** We follow closely Section 8 of the book. Recall the statement of Theorem 8.4.

**Theorem 3** (Principle of Recursive Definition)**.** *Let $A$ be a set; let $a_0$ be an element of $A$. Suppose $\rho$ is a function that assigns, to each function $f$ mapping a nonempty section of the positive integers into $A$, an element of $A$. Then there exists a unique function*

$$h : \mathbb{Z}^+ \to A$$

*such that*

$$\begin{aligned} h(1) &= a_0, \\ h(i) &= \rho(h|\{1,\dots,i-1\}) \text{ for } i > 1. \end{aligned} \tag{$\star$}$$

The formula ($\star$) is called a recursion formula for $h$. It specifies $h(1)$, and it expresses the value of $h$ at $i > 1$ in terms of the values of $h$ for positive integers less than $i$.

The first step is to prove that there exist functions defined on sections $\{1,\dots,n\}$ of $\mathbb{Z}^+$ that satisfy ($\star$):

**Lemma 4.** *Given $n \in \mathbb{Z}^+$, there exists a function*

$$f : \{1,\dots,n\} \to C$$

*that satisfies ($\star$) for all $i$ in its domain.*

*Proof.* The point of this lemma is that it is a statement that depends on $n$; therefore, it is capable of being proved by induction. Let $A$ be the set of all $n$ for which the lemma holds. We show that $A$ is inductive. It then follows that $A = \mathbb{Z}^+$.

The lemma is true for $n = 1$, since the function $f : \{1\} \to C$ defined by the equation

$$f(1) = a_0$$

satisfies ($\star$).

Supposing the lemma to be true for $n-1$, we prove it true for $n$. By hypothesis, there is a function $f' : \{1,\dots,n-1\} \to C$ satisfying ($\star$) for all $i$ in its domain. Define $f : \{1,\dots,n\} \to C$ by the equations

$$\begin{aligned} f(i) &= f'(i) \text{ for } i \in \{1,\dots,n-1\}, \\ f(n) &= \rho(f'|\{1,\dots,n-1\}). \end{aligned}$$

Note that this definition is an acceptable one; it does not define $f$ in terms of itself but in terms of the given function $f'$.

It is easy to check that $f$ satisfies ($\star$) for all $i$ in its domain. The function $f$ satisfies ($\star$) for $i \le n-1$ because it equals $f'$ there. And $f$ satisfies ($\star$) for $i = n$ because, by definition,

$$f(n) = \rho(f'|\{1,\dots,n-1\})$$

and $f'|\{1,\dots,n-1\} = f|\{1,\dots,n-1\}$. $\square$



munkres-a-b, 2024/10/25, 19:54:45 -04'00', 1729900478

**Lemma 5.** *Suppose that $f : \{1, \dots, n\} \to C$ and $g : \{1, \dots, m\} \to C$ both satisfy $(\star)$ for all $i$ in their respective domains. Then $f(i) = g(i)$ for all $i$ in both domains.*

*Proof.* Suppose not. Let $i$ be the smallest integer for which $f(i) \neq g(i)$. The integer $i$ is not 1, because

$$f(1) = a_0 = g(1),$$

by $(\star)$. Now for all $j < i$, we have $f(j) = g(j)$. Because $f$ and $g$ satisfy $(\star)$,

$$f(i) = \rho(f|\{1, \dots, n-1\}),$$
$$g(i) = \rho(g|\{1, \dots, n-1\}).$$

Since $f|\{1, \dots, i-1\} = g|\{1, \dots, i-1\}$, we have $f(i) = g(i)$, contrary to the choice of $i$. □

**Theorem 6.** *There exists a unique function $h : \mathbb{Z}^+ \to C$ satisfying $(\star)$ for all $i \in \mathbb{Z}^+$.*

*Proof.* By Lemma 4, there exists for each $n$ a function that maps $\{1, \dots, n\}$ into $C$ and satisfies $(\star)$ for all $i$ in its domain. Given $n$, Lemma 5 shows that this function is unique; two such functions having the same domain must be equal. Let $f_n : \{1, \dots, n\} \to C$ denote this unique function.

Now comes the crucial step. We define a function $h : \mathbb{Z}^+ \to C$ by defining its rule to be the union $U$ of the rules of the functions $f_n$. The rule for $f_n$ is a subset of $\{1, \dots, n\} \times C$; therefore, $U$ is a subset of $\mathbb{Z}^+ \times C$. We must show that $U$ is the rule for a function $h : \mathbb{Z}^+ \to C$.

That is, we must show that each element $i$ of $\mathbb{Z}^+$ appears as the first coordinate of exactly one element of $U$. This is easy. The integer $i$ lies in the domain of $f_n$ if and only if $n > i$. Therefore, the set of elements of $U$ of which $i$ is the first coordinate is precisely the set of all pairs of the form $(i, f_n(i))$, for $n \geq i$. Now Lemma 5 tells us that $f_n(i) = f_m(i)$ if $n, m \geq i$. Therefore, all these elements of $U$ are equal; that is, there is only one element of $U$ that has $i$ as its first coordinate.

To show that $h$ satisfies $(\star)$ is also easy; it is a consequence of the following facts:

$$h(i) = f_n(i) \text{ for } i \leq n,$$
$$f_n \text{ satisfies } (\star) \text{ for all } i \text{ in its domain.}$$

The proof of uniqueness is a copy of the proof of Lemma 5. □

---

• **Solution to Exercise 13.6 p. 83.** We must show that the topologies $\mathcal{T}_\ell$ and $\mathcal{T}_K$ are incomparable.

Claim: $[2, 3) \notin \mathcal{T}_K$. Proof. If not we would have $2 \in (a, b) \setminus K \subset [2, 3)$ for some $a$ and $b$, hence $a < 2$ and $a \leq 2$, contradiction.

Claim: $(-1, 1) \setminus K \notin \mathcal{T}_\ell$. Proof. If not we would have $0 \in [a, b) \subset (-1, 1) \setminus K \subset [2, 3)$ for some $a$ and $b$, hence $a \leq 0 < b$, hence $a < \frac{1}{n} < b$ for some $n$, contradiction.

---

• **Solution to Exercise 13.7 p. 83.** Let us use the following notation:

$\mathcal{T}_s :=$ standard topology,

$\mathcal{T}_K :=$ topology of $\mathbb{R}_K$,

$\mathcal{T}_{fc} :=$ finite complement topology,

$\mathcal{T}_u :=$ upper limit topology (having the sets $(a,b]$ as basis),

$\mathcal{T}_\infty :=$ topology having the sets $(-\infty,a)$ as basis.

We denote the corresponding topological spaces by $\mathbb{R}_s, \mathbb{R}_K, \mathbb{R}_{fc}, \mathbb{R}_u$ and $\mathbb{R}_\infty$. Finally we write $\mathcal{B}_s, \mathcal{B}_K, \mathcal{B}_u$ and $\mathcal{B}_\infty$ for the obvious bases.

The inclusions between these five topologies on $\mathbb{R}$ can be summarized by the diagram

$$\begin{array}{ccc} & u & \\ & K & \\ & s & \\ fc & & \infty, \end{array}$$

where "$i$ below $j$" means "$\mathcal{T}_i \subsetneq \mathcal{T}_j$"[2], and "$i$ and $j$ on the same level" means "$\mathcal{T}_i$ and $\mathcal{T}_j$ are incomparable".

Preliminary comments: It is easy to see that the elements of $\mathcal{T}_\infty$ are $\varnothing$, the intervals $(-\infty,a)$, and $\mathbb{R}$, and to observe that $\mathcal{T}_\infty \cap \mathcal{T}_{fc} = \{\varnothing, \mathbb{R}\}$. It is also easy to compare the standard topology $\mathcal{T}_s$ to the others: the elements of $\mathcal{T}_{fc}$ and $\mathcal{T}_\infty$ are clearly open in $\mathbb{R}_s$, and it is plain that the intervals $(a,b)$ (which are the elements on $\mathcal{B}_s$) are open in $\mathbb{R}_K$ and in $\mathbb{R}_\infty$ (note that $(a,b) = \bigcup_{d<b}(a,d]$). Clearly, $(-1,1) \setminus K \in \mathcal{T}_K$ and $(a,b] \in \mathcal{T}_u$ are not open in $\mathbb{R}_s$. Moreover $(2,3]$ is in $\mathcal{T}_u$ but not in $\mathcal{T}_K$. So, it only remains to prove $\mathcal{T}_K \subset \mathcal{T}_u$.

Let $x$ be in $(a,b) \setminus K$. It suffices to show that there is a $c$ such that $x \in (c,x] \subset (a,b) \setminus K$. If $x \le 0$ we set $c := a$. If $\frac{1}{n+1} < x < \frac{1}{n}$ we set $c := \frac{1}{n+1}$. If $x > 1$ we set $c := \max(1,a)$.

[2]I denote inclusion by $\subset$ and proper inclusion by $\subsetneq$. I know that, in some sense, it would be more coherent to use $\subseteq$ for inclusion, but I prefer to do it that way, and I hope the reader will not be confused.