A few comments about "Linear Algebra Done Right" by Axler

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As the title indicates, we make a comments about the book **Linear Algebra Done Right** by Sheldon Axler. This is a work in progress. This is a complement to Jon U jubnoske08's excellent solutions in

[1] `https://github.com/jubnoske08/linear_algebra/tree/main`

• **Exercise 1C9.** A function $f : \mathbb{R} \to \mathbb{R}$ is called **periodic** if there exists a positive number p such that $f(x) = f(x + p)$ for all $x \in \mathbb{R}$. Is the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$? Explain.

Solution. The answer is no. To prove this, consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Z} \\ 0 \text{ otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} 1 \text{ if } x \in \sqrt{2}\,\mathbb{Z} \\ 0 \text{ otherwise.} \end{cases}$$

These functions are periodic (of period 1 and $\sqrt{2}$ respectively). Claim: $f + g$ is not periodic. The claim will imply that the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ is not a subspace of $\mathbb{R}^{\mathbb{R}}$. Proof of the claim: We have $(f + g)(0) = 2$. If $f + g$ admitted the period $p > 0$, we would have $(f + g)(p) = 2$, and there would be nonzero integers m and n such that $m = p = n\sqrt{2}$, and thus $\sqrt{2} = m/n$, contradiction.

• **Exercises 1C12 and 1C13.** Exercises 1C12 and 1C13 are particular cases of the following statement.

Statement. Let n be a positive integer, let K be a field containing at least n distinct elements, let V be a K-vector space, and let $V_1, \dots, V_n$ be n *proper* subspaces of V. Then $V \neq V_1 \cup \cdots \cup V_n$.

Proof. Assume by contradiction that we have $V = V_1 \cup \cdots \cup V_n$. Given $v, w \in V$ with $v \neq w$ say that the **line** through v and w is the subset

$$L(v, w) = v + K\,(w - v) = \{v + a\,(w - v) \mid a \in K\}$$

of V. Note that $L(w, v) = L(v, w)$. The lines $L(v, w)$ are easily seen to have the following properties:

(a) $v, w \in L(v, w)$,

(b) $L(v, w)$ contains at least n distinct points[2],

(c) if a subspace W of V has at least two points in common with $L(v, w)$, then $L(v, w) \subset W$ and $w - v \in W$.

We prove the statement. We can assume that n is the least positive integer for which we have $V = V_1 \cup \cdots \cup V_n$ for some family (V_i) of proper subspaces. Clearly the integer n is ≥ 2. We claim $V_n \subset V_1 \cup \cdots \cup V_{n-1}$. This will imply $V = V_1 \cup \cdots \cup V_{n-1}$, contradiction. To prove the claim, let v_n be in V_n and v in $V \setminus V_n$, and set $L := L(v, v + v_n)$. We have $L \cap V_n = \varnothing$ because $v + av_n \in V_n$ would imply $v \in V_n$, contradiction. As a result, L is contained in $V_1 \cup \cdots \cup V_{n-1}$. By (b) and the pigeonhole principle, L intersects some V_i with $i < n$ in at least two points, and (c) yields $v_n \in V_i$, proving the claim.

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[2]In this proof, the vectors of V are called "points".