A few comments about "Linear Algebra Done Right" by Axler

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As the title indicates, we make a comments about the book **Linear Algebra Done Right** by Sheldon Axler. This is a work in progress. This is a complement to Jon U jubnoske08's excellent solutions in

- [1] https://github.com/jubnoske08/linear_algebra/tree/main
- Exercise 1C9. A function $f : \mathbb{R} \to \mathbb{R}$ is called **periodic** if there exists a positive number p such that f(x) = f(x+p) for all $x \in \mathbb{R}$. Is the set of periodic functions from \mathbb{R} to \mathbb{R} a subspace of $\mathbb{R}^{\mathbb{R}}$? Explain.

Solution. The answer is no. To prove this, consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} 1 \text{ if } x \in \sqrt{2} \ \mathbb{Z} \\ 0 \text{ otherwise.} \end{cases}$$

These functions are periodic (of period 1 and $\sqrt{2}$ respectively). Claim: f+g is not periodic. The claim will imply that the set of periodic functions from \mathbb{R} to \mathbb{R} is not a subspace of $\mathbb{R}^{\mathbb{R}}$. Proof of the claim: We have (f+g)(0)=2. If f+g admitted the period p>0, we would have (f+g)(p)=2, and there would be nonzero integers m and n such that $m=p=n\sqrt{2}$, and thus $\sqrt{2}=m/n$, contradiction.

• Exercises 1C12 and 1C13. Exercises 1C12 and 1C13 are particular cases of the following statement.

Statement. Let n be a positive integer, let K be a field containing at least n distinct elements, let V be a K-vector space, and let V_1, \ldots, V_n be n proper subspaces of V. Then $V \neq V_1 \cup \cdots \cup V_n$.

Proof. Assume by contradiction that we have $V = V_1 \cup \cdots \cup V_n$. Given $v, w \in V$ with $v \neq w$ say that the line through v and w is the subset

$$L(v, w) = v + K(w - v) = \{v + a(w - v) \mid a \in K\}$$

of V. Note that L(w,v) = L(v,w). The lines L(v,w) are easily seen to have the following properties:

- (a) $v, w \in L(v, w)$,
- (b) L(v, w) contains at least n distinct points²,
- (c) if a subspace W of V has at least two points in common with L(v,w), then $L(v,w) \subset W$ and $w-v \in W$.

We prove the statement. We can assume that n is the least positive integer for which we have $V = V_1 \cup \cdots \cup V_n$ for some family (V_i) of proper subspaces. Clearly the integer n is ≥ 2 . We claim $V_n \subset V_1 \cup \cdots \cup V_{n-1}$. This will imply $V = V_1 \cup \cdots \cup V_{n-1}$, contradiction. To prove the claim, let v_n be in V_n and v in $V \setminus V_n$, and set $L := L(v, v + v_n)$. We have $L \cap V_n = \emptyset$ because $v + av_n \in V_n$ would imply $v \in V_n$, contradiction. As a result, L is contained in $V_1 \cup \cdots \cup V_{n-1}$. By (b) and the pigeonhole principle, L intersects some V_i with i < n in at least two points, and (c) yields $v_n \in V_i$, proving the claim.

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 $^{^{2}}$ In this proof, the vectors of V are called "points".