# A few comments about "Topology" by Munkres

Pierre-Yves Gaillard[1]

As the title indicates, we make a comments about the book **Topology** by James R. Munkres. This is a work in progress. The last version of this text is available here: `https://www.overleaf.com/read/kdwwjvqjrzwb#9fe3a6`

---

• **Definition of $\mathbb{R}$ p. 31.** The object $\mathbb{R}$ is defined by assuming that there exists a set $\mathbb{R}$ having certain properties. We take this assumption for granted. Then it is easy to see that there are several sets having these properties. So, let $\mathbb{R}'$ be a set having the same properties as $\mathbb{R}$. Let $\mathbb{Z}'_+, \mathbb{Z}'$ and $\mathbb{Q}'$ be to $\mathbb{R}'$ what $\mathbb{Z}_+, \mathbb{Z}$ and $\mathbb{Q}$ are to $\mathbb{R}$.

**Theorem 1.** *There is a unique morphism of fields from $f : \mathbb{R} \to \mathbb{R}'$. This morphism is an isomorphism of ordered fields, and it induces isomorphisms $\mathbb{Z}_+ \to \mathbb{Z}'_+, \mathbb{Z} \to \mathbb{Z}'$ and $\mathbb{Q} \to \mathbb{Q}'$.*

**Lemma 2.** *There is a unique map $g : \mathbb{Z}_+ \to \mathbb{Z}'_+$ such that $g(0) = 0$ and $g(n+1) = g(n)+1$ for all $n$ in $\mathbb{Z}_+$. Similarly, there is a unique map $h : \mathbb{Z}'_+ \to \mathbb{Z}_+$ such that $h(0) = 0$ and $h(n+1) = h(n)+1$ for all $n$ in $\mathbb{Z}'_+$.*

*Proof.* For $i \in \mathbb{Z}_+$ and $\varphi : \{1, \dots, i\} \to \mathbb{Z}'_+$ define $\rho(\varphi) \in \mathbb{Z}'_+$ by $\rho(\varphi) := \varphi(i) + 1$. Then the first statement follows from the Principle of Recursive Definition (Theorem 4 p. 3). The proof of the second statement is similar. □

*Proof of Theorem 1.* In the notation of Lemma 2, set $u := h \circ g$. Then $u : \mathbb{Z}_+ \to \mathbb{Z}_+$ satisfies $u(0) = 0$ and $u(n + 1) = u(n) + 1$ for all $n$ in $\mathbb{Z}_+$. One can easily prove that $u(n) = n$ by induction. The same argument works for $g \circ h$. This shows that $g : \mathbb{Z}_+ \to \mathbb{Z}'_+$ and $h : \mathbb{Z}'_+ \to \mathbb{Z}_+$ are inverse isomorphisms. Then we extend $g$ to morphisms $\mathbb{Z} \to \mathbb{Z}'$ and $\mathbb{Q} \to \mathbb{Q}'$, and similarly for $h$, and, arguing as before, we show that these morphisms are isomorphisms. More precisely, we see that there is a unique morphism $\mathbb{Z} \to \mathbb{Z}'$ extending $g$, and that this morphism is an isomorphism, and similarly for the morphism $\mathbb{Q} \to \mathbb{Q}'$. So we can make the identifications $\mathbb{Z}_+ = \mathbb{Z}'_+, \mathbb{Z} = \mathbb{Z}', \mathbb{Q} = \mathbb{Q}'$. To show that there is a unique morphism of fields $\mathbb{R} \to \mathbb{R}'$, and that this morphism is an isomorphism (inducing the identity of $\mathbb{Q}$), we argue as in Section *Appendix to Chapter 1* in *A few comments about "Principles of Mathematical Analysis" by Rudin*, available at `https://zenodo.org/records/13955297`. □

---

**Theorem 7.8. p. 50.** Recall the statement:

**Theorem 3** (Theorem 7.8. p. 50 of the book)**.** *Let $A$ be a set. There is no injective map $f : \mathcal{P}(A) \to A$, and there is no surjective map $g : A \to \mathcal{P}(A)$.*

[1]ORCID https://orcid.org/0000-0002-7960-1698

Here is my favourite way of phrasing the argument showing that there is no surjective map $g : A \to \mathcal{P}(A)$. Let $g : A \to \mathcal{P}(A)$ be a map, and set $B := \{a \mid a \notin g(a)\}$, so that we have, for all $a$ in $A$,

$$a \in B \iff a \notin g(a).$$

Let $a_0$ be in $A$. If we had $g(a_0) = B$, we would get, for all $a$ in $A$,

$$a \in g(a_0) \iff a \notin g(a),$$

and we immediately that setting $a := a_0$ yields a contradiction. This shows that $B$ is not in the range of $g$.

---

• **Exercise 7.6. p. 51.** We say that two sets $A$ and $B$ **have the same cardinality** if there is a bijection of $A$ with $B$.

(a) Show that if $B \subset A$ and if there is an injection

$$f : A \to B,$$

then $A$ and $B$ have the same cardinality. [Hint: Define $A_1 = A, B_1 = B$, and for $n > 1$, $A_n = f(A_{n-1})$ and $B_n = f(B_{n-1})$. (Recursive definition again!) Note that $A_1 \supset B_1 \supset A_2 \supset B_2 \supset A_3 \supset \cdots$ Define a bijection $h : A \to B$ by the rule

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_n \setminus B_n \text{ for some } n, \\ x & \text{otherwise.]} \end{cases}$$

(b) *Theorem (Schroeder-Bernstein theorem).* If there are injections $f : A \to C$ and $g : C \to A$, then $A$ and $C$ have the same cardinality.

**Solution.** (a) We will freely use the following two obvious facts:

(F1) For $x \in A$ and $n \in \mathbb{Z}_+$ we have

$$x \in A_n \iff f(x) \in A_{n+1} \text{ and } x \in B_n \iff f(x) \in B_{n+1}.$$

(F2) We have $\bigcap_{n\geq 1} A_n = \bigcap_{\geq 1} B_n =: I$.

Setting $A'_n := A_n \setminus B_n, B'_n := B_n \setminus A_{n+1}$, we get

$$A = \left(\bigcup_{n\geq 1} A'_n\right) \cup \left(\bigcup_{n\geq 1} B'_n\right) \cup I,$$

and this union is disjoint. We also have

$$B = \left(\bigcup_{n\geq 2} A'_n\right) \cup \left(\bigcup_{n\geq 1} B'_n\right) \cup I.$$

The injection $f$ induces bijections $f_n : A'_n \to A'_{n+1}$ (here we are using (F1)). To define a bijection $h : A \to B$, it suffices to define three bijections

$$u : \bigcup_{n\geq 1} A'_n \to \bigcup_{n\geq 2} A'_n, \quad v : \bigcup_{n\geq 1} B'_n \to \bigcup_{n\geq 1} B'_n, \quad w : I \to I.$$

We define $u$ by $u(x) = f_n(x)$ if $x \in A'_n$, and take $v$ and $w$ to be the identity maps.

(b) We set $B := g(C) \subset A$ and define $f' : A \to B$ by $f'(a) := g(f(a))$. Then $f' : A \to B$ satisfies the assumptions for $f : A \to B$ in (a).

---

• **Exercise 8.7. p. 56.** Prove Theorem 8.4 p. 54.

**Solution.** Recall the statement of Theorem 8.4.

**Theorem 4** (Principle of Recursive Definition, Theorem 8.4 of the book)**.** *Let $A$ be a set; let $a_0$ be an element of $A$. Suppose $\rho$ is a function that assigns, to each function $f$ mapping a nonempty section of the positive integers into $A$, an element of $A$. Then there exists a unique function*

$$h : \mathbb{Z}^+ \to A$$

*such that*

$$\begin{aligned} h(1) &= a_0, \\ h(i) &= \rho(h|\{1, \ldots, i-1\}) \text{ for } i > 1. \end{aligned} \tag{$*$}$$

The formula $(*)$ is called a recursion formula for $h$. It specifies $h(1)$, and it expresses the value of $h$ at $i > 1$ in terms of the values of $h$ for positive integers less than $i$.

The book gives a detailed proof of the particular case when $\rho(h|\{1, \ldots, i-1\})$ is equal to $\min(C \setminus h(\{1, \ldots, i-1\}))$, where "min" means "*minimum*", and $C$ is an infinite set. A close inspection of this proof reveals that the sole property of the element $c$ of $C$ defined by the equality $c := \min(C \setminus h(\{1, \ldots, i-1\}))$ is that it depends only on the restriction $h|\{1, \ldots, i-1\}$. This implies that, if, in the proof given by the book, we replace "$\min(C \setminus h(\{1, \ldots, i-1\}))$" with "$\rho(h|\{1, \ldots, i-1\})$", then we obtain a proof of Theorem 4.

---

• **Exercise 10.7 p. 67.** Let $J$ be a well-ordered set. A subset $J_0$ of $J$ is said to be **inductive** if for every $\alpha \in J$,

$$(S_\alpha \subset J_0) \Rightarrow \alpha \in J_0.$$

**Theorem 5** (The principle of transfinite induction)**.** *If $J$ is a well-ordered set and $J_0$ is an inductive subset of $J$, then $J_0 = J$.*

**Solution.** If $J_0 \neq J$, let $\alpha$ be the least element of $J \setminus J_0$. We get $S_\alpha \subset J_0$, and thus $\alpha \in J_0$, contradiction.

---

• **Exercise 10.10 p. 67. Theorem.** Let $J$ and $C$ be well-ordered sets; assume that there is no surjective function mapping a section of $J$ onto $C$. Then there exists a unique function $h : J \to C$ satisfying the equation

$$h(x) = \min(C \setminus h(S_x)) \tag{$*$}$$

for each $x \in J$, where $S_x$ is the section of $J$ by $x$.

**Proof.**

(a) If $h$ and $k$ map sections of $J$, or all of $J$, into $C$ and satisfy $(*)$ for all $x$ in their respective domains, show that $h(x) = k(x)$ for all $x$ in both domains.

(b) If there exists a function $h : S_\alpha \to C$ satisfying $(*)$, show that there exists a function $k : S_\alpha \cup \{\alpha\} \to C$ satisfying $(*)$.

(c) If $K \subset J$ and for all $\alpha \in K$ there exists a function $h_\alpha : S_\alpha \to C$ satisfying $(*)$, show that there exists a function

$$k : \bigcup_{\alpha \in K} S_\alpha \to C$$

satisfying $(*)$.

(d) Show by transfinite induction that for every $\beta \in J$, there exists a function $h_\beta : S_\beta \to C$ satisfying $(*)$. [Hint: If $\beta$ has an immediate predecessor $\alpha$, then $S_\beta = S_\alpha \cup \{\alpha\}$. If not, $S_\beta$ is the union of all $S_\alpha$ with $\alpha < \beta$.]

(e) Prove the theorem.

**Solution.**

(a) Otherwise there would be a least $x$ such that $h(x) \neq k(x)$, we would get $h(S_x) = k(S_x)$, and $(*)$ would yield a contradiction.

(b) We define $k$ by $k(x) = h(x)$ if $x < \alpha$ and $k(x) = \min(C \setminus h(S_x))$ if $x = \alpha$, and verify that $k$ satisfies $(*)$.

(c) Set $k(x) = h_\alpha(x)$ if $x \in S_\alpha$. To show that $k(x)$ is well defined, we must check that $\beta > \alpha$ implies $h_\beta(x) = h_\alpha(x)$. But this follows from (a).

(d) Let $I$ be the set of all $\beta \in J$ such that there is a map $h_\beta : S_\beta \to C$ satisfying $(*)$. It suffices to show that $I$ is inductive. So, assume that $\beta$ is in $J$ and that $S_\beta \subset I$. We must show $\beta \in I$. To do that, we use (b) if $\beta$ has an immediate predecessor, and we use (c) if not.

(e) We define $h$ by

$$h(x) = \begin{cases} \min(C \setminus h_x(S_x)) & \text{if } x = \max(J) \\ h_{x+1}(x) & \text{if } x \neq \max(J), \end{cases}$$

where "$x \neq \max(J)$" means "$x \neq \max(J)$ if $J$ has a maximum", and $x + 1$ is the least element greater than $x$. Let us show that $h$ satisfies $(*)$, that is, $h(x) = \min(C \setminus h(S_x))$. We can assume $x \neq \max(J)$ (in the above sense). We must show $h_{x+1}(x) = \min(C \setminus h(S_x))$. Since we have $h_{x+1}(x) = \min(C \setminus h_{x+1}(S_x))$ by (d) it suffices to prove $h(S_x) = h_{x+1}(S_x)$. Let $y$ be in $S_x$, that is, $y \in J$ and $y < x$. It is enough to verify $h(y) = h_{x+1}(y)$, that is, $h_{y+1}(y) = h_{x+1}(y)$. We have $y + 1 < x + 1$, and thus $S_{y+1} \subset S_{x+1}$, and (a) implies $h_{x+1}|S_{y+1} = h_{y+1}$. This proves $h_{y+1}(y) = h_{x+1}(y)$, which is what we wanted.

---

## • Supplementary Exercise 11.1 p. 72.

**Theorem 6** (General principle of recursive definition)**.** *Let $J$ be a well-ordered set; let $C$ be a set. Let $\mathcal{F}$ be the set of all functions mapping sections of $J$ into $C$. Given a function $\rho : \mathcal{F} \to C$, there exists a unique function $h : J \to C$ such that $h(\alpha) = \rho(h|S_\alpha)$ for each $\alpha \in J$.*

[Hint: Follow the pattern outlined in Exercise 10 of §10.]

**Solution.** A close inspection of the solution to Exercise 10 of §10 reveals that the sole property of the element $c$ of $C$ defined by the equality $c := \min(C \setminus h(S_x))$ is that it depends only on the restriction $h|S_x$. This implies that, if, in the proof given by the book, we replace "$\min(C \setminus h(S_x))$" with "$\rho(h|S_x)$", then we obtain a proof of Theorem 6.

---

## • Supplementary Exercise 11.2 p. 72.

(a) Let $J$ and $E$ be well-ordered sets; let $h : J \to E$. Show the following two statements are equivalent:

(i) $h$ is order preserving and its image is $E$ or a section of $E$.

(ii) $h(\alpha) = $ smallest $[E - h(S_\alpha)]$ for all $\alpha$.

[Hint: Show that each of these conditions implies that $h(S_\alpha)$ is a section of $E$; conclude that it must be the section by $h(\alpha)$.]

(b) If $E$ is a well-ordered set, show that no section of $E$ has the order type of $E$, nor do two different sections of $E$ have the same order type. [Hint: Given $J$, there is at most one order-preserving map of $J$ into $E$ whose image is $E$ or a section of $E$.]

**Solution.**

(a) For all $X \subset E$ set $X^c := E \setminus X$. For the sake of prudence, we change (ii) to:

(ii') $h(S_x) \neq E$ and $h(x) = \min(h(S_x)^c)$ for all $x$.

We want to show that (i) and (ii') are equivalent.

(i) implies (ii'). We prove $h(S_x) \neq E$ by noting that $h(S_x) = E$ we would entail $h(x) = h(y)$ for some $y < x$, contradiction. To prove $h(x) = \min(h(S_x)^c)$, assume by contradiction that we have $h(x) \neq \min(h(S_x)^c) =: e$. If $h(x) < e$, then $h(x) \notin h(S_x)^c$, that is, $h(x) \in h(S_x)$, and we reach a contradiction as above. If $e < h(x)$, then $e = h(y)$ for some $y < x$, that is, $\min(h(S_x)^c) = e \in h(S_x)$, contradiction.

(ii') implies (i). We assume (ii'), and, in particular, that $h$ is weakly increasing. To show that $h$ is increasing, suppose $x < y$ and $h(x) = h(y)$ (we cannot have $h(x) > h(y)$ because $h$ is weakly increasing). Since $h(x) = h(y) = \min(h(S_y)^c)$, we have $h(x) \in h(S_y)^c$, but $h(x) \in h(S_y)$, contradiction. Finally, $h(J)$ is downward closed because $e < h(x) = \min(h(S_x)^c)$ implies $e \in h(S_x) \subset h(J)$.

In the statement of the Exercise, the condition that $J$ is well-ordered can be changed from an assumption to a conclusion.

(b) Let $a$ be in $E$, and assume there is an isomorphism of well-ordered sets $h : S_a \to E$. It suffices to derive a contradiction. Let $i : S_a \to E$ be the inclusion. By (a) $h$ and $i$ satisfy the same recursion relation. By the Theorem about the General Principle of Definition by Recursion, we have $h = i$, and thus $a \in h(S_a) = i(S_a) = S_a$, contradiction.

---

• **Supplementary Exercise 11.3 p. 73.** Let $J$ and $E$ be well-ordered sets; suppose there is an order-preserving map $k : J \to E$. Using Exercises 1 and 2, show that $J$ has the order type of $E$ or a section of $E$. [Hint: Choose $e_0 \in E$. Define $h : J \to E$ by the recursion formula

$$h(\alpha) = \text{smallest } [E - h(S_\alpha)] \text{ if } h(S_\alpha) \neq E,$$

and $h(\alpha) = e_0$ otherwise. Show that $h(\alpha) \leq k(\alpha)$ for all $\alpha$; conclude that $h(S_\alpha) \neq E$ for all $\alpha$.]

**Solution.** We can assume $E \neq \varnothing$. Let $e_0$ be in $E$. Let $x$ be in $J$. We define $h : J \to E$ as in the hint.

Claim 1: $h(x) \leq k(x)$ for all $x$.

Claim 2: $h(y) \leq k(y)$ for all $y$ in $S_x$ implies $h(S_x) \neq E$.

Proof of Claim 2. For all $y$ in $S_x$ we have $h(y) \leq k(y) < k(x)$, and in particular $k(x) \neq h(y)$. This implies $k(x) \notin h(S_x)$.

Proof of Claim 1. Assume by contradiction $h(x) > k(x)$ for some $x$. We can assume that $x$ is minimum for this condition. For $y < x$ we have $h(y) \leq k(y)$, hence $h(S_x) \neq E$ by Claim 2.

Claims 1 and 2 imply $h(x) = \min(h(S_x)^c)$ for all $x$, hence $h$ is increasing and $h(J)$ is downward closed by Supplementary Exercise 11.2 above, hence $J$ has the order type of $E$ or a section of $E$.

---

• **Supplementary Exercise 11.4 p. 73.** Use Exercises 1–3 to prove the following:

(a) If $A$ and $B$ are well-ordered sets, then exactly one of the following three conditions holds: $A$ and $B$ have the same order type, or $A$ has the order type of a section of $B$, or $B$ has the order type of a section of $A$. [Hint: Form a well-ordered set containing both $A$ and $B$, as in Exercise 8 of §10; then apply the preceding exercise.]

(b) Suppose that $A$ and $B$ are well-ordered sets that are uncountable, such that every section of $A$ and of $B$ is countable. Show $A$ and $B$ have the same order type.

**Solution.** (a) For any element $x$ of any ordered set $X$, let $X_{<x}$ denote the corresponding section, and let us set $X_{<\infty} := X$. Let $C$ be the well-ordered set containing both $A$ and $B$, described in Exercise 8 of §10, and let $k : A \to C$ and $\ell : B \to C$ be the natural increasing maps. By the previous Exercise, we have isomorphisms $A \simeq C_{<x}$ and $B \simeq C_{<y}$ for some $x$ and $y$ in $C \cup \{\infty\}$. We can assume $x \leq y$. Then $x = y$ implies $A \simeq B$. If $x < y$, we get

$$A \simeq C_{<x} = (C_{<y})_{<x} \subsetneq C_{<y} \simeq B.$$

This implies $A \simeq B_{<b}$ for some $b$ in $B$. The fact that the various cases are exclusive follows from Supplementary Exercise 11.2b.

(b) Follows from (a).

Here is an important consequence of (a):

**Theorem 7** (Comparability Theorem)**.** *If $A$ and $B$ are sets, then exactly one of the following three conditions holds:*

(i) *there is a bijection $A \to B$,*

(ii) *there is an injection $A \to B$ and a surjection $B \to A$,*

(iii) *there is an injection $B \to A$ and a surjection $A \to B$.*

---

• **Supplementary Exercise 11.5 p. 73.** Let $X$ be a set; let $\mathcal{A}$ be the collection of all pairs $(A, <)$, where $A$ is a subset of $X$ and $<$ is a well-ordering of $A$. Define

$$(A, <) \prec (A', <')$$

if $(A, <)$ equals a section of $(A', <')$.

(a) Show that $\prec$ is a strict partial order on $A$.

(b) Let $\mathcal{B}$ be a subcollection of $A$ that is simply ordered by $\prec$. Define $B'$ to be the union of the sets $B$, for all $(B, <) \in \mathcal{B}$; and define $<'$ to be the union of the relations $<$, for all $(B, <) \in \mathcal{B}$. Show that $(B', <')$ is a well-ordered set.

**Solution.** Left to the reader.

---

• **Supplementary Exercise 11.6 p. 73.** Use Exercises 1 and 5 to prove the following:

**Theorem.** The maximum principle is equivalent to the well-ordering theorem.

**Solution.** The fact that the well-ordering theorem implies the maximum principle is proved on p. 70 of the book. Let us prove the converse. In the setting of Supplementary Exercise 11.5b, take $\mathcal{B}$ maximal. Then it suffices to show that $B' = X$. If it was not so, we could add to $B'$ a new element $x$ and make it the largest element of $B' \cup \{x\}$, which would then be a well-ordered set larger than $B'$, contradiction.

---

• **Supplementary Exercise 11.7 p. 73.** Use Exercises 1–5 to prove the following:

**Theorem.** The choice axiom is equivalent to the well-ordering theorem.

Proof. Let $X$ be a set; let $c$ be a fixed choice function for the nonempty subsets of $X$. If $T$ is a subset of $X$ and $<$ is a relation on $T$, we say that $(T, <)$ is a tower in $X$ if $<$ is a well-ordering of $T$ and if for each $x \in T$,

$$x = c(X - S_x(T)),$$

where $S_x(T)$ is the section of $T$ by $x$.

(a) Let $(T_1, <_1)$ and $(T_2, <_2)$ be two towers in $X$. Show that either these two ordered sets are the same, or one equals a section of the other. [Hint: Switching indices if necessary, we can assume that $h : T_1 \to T_2$ is order preserving and $h(T_1)$ equals either $T_2$ or a section of $T_2$. Use Exercise 2 to show that $h(x) = x$ for all $x$.]

(b) If $(T, <)$ is a tower in $X$ and $T \neq X$, show there is a tower in $X$ of which $(T, <)$ is a section.

(c) Let $\{(T_k, <_k) | k \in K\}$ be the collection of all towers in $X$. Let

$$T = \bigcup_{k \in K} T_k \text{ and } < = \bigcup_{k \in K} (<_k).$$

Show that $(T, <)$ is a tower in $X$. Conclude that $T = X$.

**Solution.** (a) The map $h$ in the hint exists by Supplementary Exercise 4a. Let us show $h(x) = x$ for all $x$ thanks to Supplementary Exercise 2. Assume by contradiction that we have $h(x) \neq x$ for some $x$ in $T_1$, which we can suppose to be minimum for this condition. The map $h$ induces an isomorphism $T_1 \simeq h(T_1)$, implying $h(T_{1,<y}) = T_{2,h(y)}$ for all $y$ in $T_1$. By the choice of $x$ we get $T_{1,<x} = T_{2,<h(x)}$. Since $T_1$ and $T_2$ are towers, this entails

$$h(x) = c(X \setminus T_{2,<h(x)}) = c(X \setminus T_{1,<x}) = x,$$

contradiction. The fact that $h(x) = x$ for all $x$ in $T_1$ implies, by Supplementary Exercise 11.2a, that $T_1$ is contained and downward closed in $T_2$.

(b) Add $c(X \setminus T)$ to $T$, and make it the largest element.

(c) Left to the reader.

This shows that the choice axiom implies the well-ordering theorem. The converse is clear.

---

• **Solution to Exercise 13.6 p. 83.** We must show that the topologies $\mathcal{T}_\ell$ and $\mathcal{T}_K$ are incomparable.

Claim: $[2,3) \notin \mathcal{T}_K$. Proof. If not we would have $2 \in (a,b) \setminus K \subset [2,3)$ for some $a$ and $b$, hence $a < 2$ and $a \leq 2$, contradiction.

Claim: $(-1,1) \setminus K \notin \mathcal{T}_\ell$. Proof. If not we would have $0 \in [a,b) \subset (-1,1) \setminus K \subset [2,3)$ for some $a$ and $b$, hence $a \leq 0 < b$, hence $a < \frac{1}{n} < b$ for some $n$, contradiction.

---

• **Solution to Exercise 13.7 p. 83.** Let us use the following notation:

$\mathcal{T}_s :=$ standard topology,

$\mathcal{T}_K :=$ topology of $\mathbb{R}_K$,

$\mathcal{T}_{fc} :=$ finite complement topology,

$\mathcal{T}_u :=$ upper limit topology (having the sets $(a,b]$ as basis),

$\mathcal{T}_\infty :=$ topology having the sets $(-\infty, a)$ as basis.

We denote the corresponding topological spaces by $\mathbb{R}_s, \mathbb{R}_K, \mathbb{R}_{fc}, \mathbb{R}_u$ and $\mathbb{R}_\infty$. Finally we write $\mathcal{B}_s, \mathcal{B}_K, \mathcal{B}_u$ and $\mathcal{B}_\infty$ for the obvious bases.

The inclusions between these five topologies on $\mathbb{R}$ can be summarized by the diagram

$$\begin{array}{ccc} & u & \\ & K & \\ & s & \\ fc & & \infty, \end{array}$$

where "$i$ below $j$" means "$\mathcal{T}_i \subsetneq \mathcal{T}_j$"[2], and "$i$ and $j$ on the same level" means "$\mathcal{T}_i$ and $\mathcal{T}_j$ are incomparable".

Preliminary comments: It is easy to see that the elements of $\mathcal{T}_\infty$ are $\varnothing$, the intervals $(-\infty, a)$, and $\mathbb{R}$, and to observe that $\mathcal{T}_\infty \cap \mathcal{T}_{fc} = \{\varnothing, \mathbb{R}\}$. It is also easy to compare the standard topology $\mathcal{T}_s$ to the others: the elements of $\mathcal{T}_{fc}$ and $\mathcal{T}_\infty$ are clearly open in $\mathbb{R}_s$, and it is plain that the intervals $(a,b)$ (which are the elements on $\mathcal{B}_s$) are open in $\mathbb{R}_K$ and in $\mathbb{R}_\infty$ (note that $(a,b) = \bigcup_{d<b}(a,d]$). Clearly, $(-1,1) \setminus K \in \mathcal{T}_K$ and $(a,b] \in \mathcal{T}_u$ are not open in $\mathbb{R}_s$. Moreover $(2,3]$ is in $\mathcal{T}_u$ but not in $\mathcal{T}_K$. So, it only remains to prove $\mathcal{T}_K \subset \mathcal{T}_u$.

Let $x$ be in $(a,b) \setminus K$. It suffices to show that there is a $c$ such that $x \in (c,x] \subset (a,b) \setminus K$. If $x \leq 0$ we set $c := a$. If $\frac{1}{n+1} < x < \frac{1}{n}$ we set $c := \frac{1}{n+1}$. If $x > 1$ we set $c := \max(1,a)$.

[2] I denote inclusion by $\subset$ and proper inclusion by $\subsetneq$. I know that, in some sense, it would be more coherent to use $\subseteq$ for inclusion, but I prefer to do it that way, and hope the reader will not be confused.