

# A few comments about “Topology” by Munkres

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As the title indicates, we make a comments about the book **Topology** by James R. Munkres. This is a work in progress. The last version of this text is available here: <https://www.overleaf.com/read/kdwwjvqjrzb#9fe3a6>. Another version is available here: <https://github.com/Pierre-Yves-Gaillard/About-Topology-by-Munkres>.

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• **Definition of  $\mathbb{R}$  p. 31.** The object  $\mathbb{R}$  is defined by assuming that there exists a set  $\mathbb{R}$  having certain properties. We take this assumption for granted. Then it is easy to see that there are several sets having these properties. So, let  $\mathbb{R}'$  be a set having the same properties as  $\mathbb{R}$ . Let  $\mathbb{Z}'_+, \mathbb{Z}'$  and  $\mathbb{Q}'$  be to  $\mathbb{R}'$  what  $\mathbb{Z}_+, \mathbb{Z}$  and  $\mathbb{Q}$  are to  $\mathbb{R}$ .

**Theorem 1.** *There is a unique morphism of fields from  $f : \mathbb{R} \rightarrow \mathbb{R}'$ . This morphism is an isomorphism of ordered fields, and it induces isomorphisms  $\mathbb{Z}_+ \rightarrow \mathbb{Z}'_+, \mathbb{Z} \rightarrow \mathbb{Z}'$  and  $\mathbb{Q} \rightarrow \mathbb{Q}'$ .*

**Lemma 2.** *There is a unique map  $g : \mathbb{Z}_+ \rightarrow \mathbb{Z}'_+$  such that  $g(0) = 0$  and  $g(n+1) = g(n) + 1$  for all  $n$  in  $\mathbb{Z}_+$ . Similarly, there is a unique map  $h : \mathbb{Z}'_+ \rightarrow \mathbb{Z}_+$  such that  $h(0) = 0$  and  $h(n+1) = h(n) + 1$  for all  $n$  in  $\mathbb{Z}'_+$ .*

*Proof.* For  $i \in \mathbb{Z}_+$  and  $\varphi : \{1, \dots, i\} \rightarrow \mathbb{Z}'_+$  define  $\rho(\varphi) \in \mathbb{Z}'_+$  by  $\rho(\varphi) := \varphi(i) + 1$ . Then the first statement follows from the Principle of Recursive Definition (Theorem 4 p. 3). The proof of the second statement is similar.  $\square$

*Proof of Theorem 1.* In the notation of Lemma 2, set  $u := h \circ g$ . Then  $u : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$  satisfies  $u(0) = 0$  and  $u(n+1) = u(n) + 1$  for all  $n$  in  $\mathbb{Z}_+$ . One can easily prove that  $u(n) = n$  by induction. The same argument works for  $g \circ h$ . This shows that  $g : \mathbb{Z}_+ \rightarrow \mathbb{Z}'_+$  and  $h : \mathbb{Z}'_+ \rightarrow \mathbb{Z}_+$  are inverse isomorphisms. Then we extend  $g$  to morphisms  $\mathbb{Z} \rightarrow \mathbb{Z}'$  and  $\mathbb{Q} \rightarrow \mathbb{Q}'$ , and similarly for  $h$ , and, arguing as before, we show that these morphisms are isomorphisms. More precisely, we see that there is a unique morphism  $\mathbb{Z} \rightarrow \mathbb{Z}'$  extending  $g$ , and that this morphism is an isomorphism, and similarly for the morphism  $\mathbb{Q} \rightarrow \mathbb{Q}'$ . So we can make the identifications  $\mathbb{Z}_+ = \mathbb{Z}'_+, \mathbb{Z} = \mathbb{Z}', \mathbb{Q} = \mathbb{Q}'$ . To show that there is a unique morphism of fields  $\mathbb{R} \rightarrow \mathbb{R}'$ , and that this morphism is an isomorphism (inducing the identity of  $\mathbb{Q}$ ), we argue as in Section *Appendix to Chapter 1* in *A few comments about “Principles of Mathematical Analysis” by Rudin*, available at <https://zenodo.org/records/13955297>.  $\square$

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**Theorem 7.8. p. 50.** Recall the statement:

**Theorem 3** (Theorem 7.8. p. 50 of the book). *Let  $A$  be a set. There is no injective map  $f : \mathcal{P}(A) \rightarrow A$ , and there is no surjective map  $g : A \rightarrow \mathcal{P}(A)$ .*

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Here is my favourite way of phrasing the argument showing that there is no surjective map  $g : A \rightarrow \mathcal{P}(A)$ . Let  $g : A \rightarrow \mathcal{P}(A)$  be a map, and set  $B := \{a \mid a \notin g(a)\}$ , so that we have, for all  $a$  in  $A$ ,

$$a \in B \iff a \notin g(a).$$

Let  $a_0$  be in  $A$ . If we had  $g(a_0) = B$ , we would get, for all  $a$  in  $A$ ,

$$a \in g(a_0) \iff a \notin g(a),$$

and we immediately that setting  $a := a_0$  yields a contradiction. This shows that  $B$  is not in the range of  $g$ .

• **Exercise 7.6. p. 51.** We say that two sets  $A$  and  $B$  **have the same cardinality** if there is a bijection of  $A$  with  $B$ .

(a) Show that if  $B \subset A$  and if there is an injection

$$f : A \rightarrow B,$$

then  $A$  and  $B$  have the same cardinality. [Hint: Define  $A_1 = A, B_1 = B$ , and for  $n > 1$ ,  $A_n = f(A_{n-1})$  and  $B_n = f(B_{n-1})$ . (Recursive definition again!) Note that  $A_1 \supset B_1 \supset A_2 \supset B_2 \supset A_3 \supset \dots$ . Define a bijection  $h : A \rightarrow B$  by the rule

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_n \setminus B_n \text{ for some } n, \\ x & \text{otherwise.} \end{cases}$$

(b) *Theorem (Schröder-Bernstein theorem).* If there are injections  $f : A \rightarrow C$  and  $g : C \rightarrow A$ , then  $A$  and  $C$  have the same cardinality.

**Solution.** (a) We will freely use the following two obvious facts:

(F1) For  $x \in A$  and  $n \in \mathbb{Z}_+$  we have

$$x \in A_n \iff f(x) \in A_{n+1} \text{ and } x \in B_n \iff f(x) \in B_{n+1}.$$

(F2) We have  $\bigcap_{n \geq 1} A_n = \bigcap_{n \geq 1} B_n =: I$ .

Setting  $A'_n := A_n \setminus B_n, B'_n := B_n \setminus A_{n+1}$ , we get

$$A = \left( \bigcup_{n \geq 1} A'_n \right) \cup \left( \bigcup_{n \geq 1} B'_n \right) \cup I,$$

and this union is disjoint. We also have

$$B = \left( \bigcup_{n \geq 2} A'_n \right) \cup \left( \bigcup_{n \geq 1} B'_n \right) \cup I.$$

The injection  $f$  induces bijections  $f_n : A'_n \rightarrow A'_{n+1}$  (here we are using (F1)). To define a bijection  $h : A \rightarrow B$ , it suffices to define three bijections

$$u : \bigcup_{n \geq 1} A'_n \rightarrow \bigcup_{n \geq 2} A'_n, \quad v : \bigcup_{n \geq 1} B'_n \rightarrow \bigcup_{n \geq 1} B'_n, \quad w : I \rightarrow I.$$

We define  $u$  by  $u(x) = f_n(x)$  if  $x \in A'_n$ , and take  $v$  and  $w$  to be the identity maps.

(b) We set  $B := g(C) \subset A$  and define  $f' : A \rightarrow B$  by  $f'(a) := g(f(a))$ . Then  $f' : A \rightarrow B$  satisfies the assumptions for  $f : A \rightarrow B$  in (a).

• **Exercise 8.7. p. 56.** Prove Theorem 8.4 p. 54.

**Solution.** Recall the statement of Theorem 8.4.

**Theorem 4** (Principle of Recursive Definition, Theorem 8.4 of the book). *Let  $A$  be a set; let  $a_0$  be an element of  $A$ . Suppose  $\rho$  is a function that assigns, to each function  $f$  mapping a nonempty section of the positive integers into  $A$ , an element of  $A$ . Then there exists a unique function*

$$h : \mathbb{Z}^+ \rightarrow A$$

such that

$$\begin{aligned} h(1) &= a_0, \\ h(i) &= \rho(h|_{\{1, \dots, i-1\}}) \text{ for } i > 1. \end{aligned} \tag{*}$$

The formula (\*) is called a recursion formula for  $h$ . It specifies  $h(1)$ , and it expresses the value of  $h$  at  $i > 1$  in terms of the values of  $h$  for positive integers less than  $i$ .

The book gives a detailed proof of the particular case when  $\rho(h|_{\{1, \dots, i-1\}})$  is equal to  $\min(C \setminus h(\{1, \dots, i-1\}))$ , where “min” means “*minimum*”, and  $C$  is an infinite set. A close inspection of this proof reveals that the sole property of the element  $c$  of  $C$  defined by the equality  $c := \min(C \setminus h(\{1, \dots, i-1\}))$  is that it depends only on the restriction  $h|_{\{1, \dots, i-1\}}$ . This implies that, if, in the proof given by the book, we replace “ $\min(C \setminus h(\{1, \dots, i-1\}))$ ” with “ $\rho(h|_{\{1, \dots, i-1\}})$ ”, then we obtain a proof of Theorem 4.

• **Exercise 10.7 p. 67.** Let  $J$  be a well-ordered set. A subset  $J_0$  of  $J$  is said to be **inductive** if for every  $\alpha \in J$ ,

$$(S_\alpha \subset J_0) \Rightarrow \alpha \in J_0.$$

**Theorem 5** (The Principle of Transfinite Induction). *If  $J$  is a well-ordered set and  $J_0$  is an inductive subset of  $J$ , then  $J_0 = J$ .*

**Solution.** If  $J_0 \neq J$ , let  $\alpha$  be the least element of  $J \setminus J_0$ . We get  $S_\alpha \subset J_0$ , and thus  $\alpha \in J_0$ , contradiction.

• **Exercise 10.10 p. 67. Theorem.** Let  $J$  and  $C$  be well-ordered sets; assume that there is no surjective function mapping a section of  $J$  onto  $C$ . Then there exists a unique function  $h : J \rightarrow C$  satisfying the equation

$$h(x) = \min(C \setminus h(S_x)) \tag{*}$$

for each  $x \in J$ , where  $S_x$  is the section of  $J$  by  $x$ .

**Proof.**

- (a) If  $h$  and  $k$  map sections of  $J$ , or all of  $J$ , into  $C$  and satisfy  $(*)$  for all  $x$  in their respective domains, show that  $h(x) = k(x)$  for all  $x$  in both domains.
- (b) If there exists a function  $h : S_\alpha \rightarrow C$  satisfying  $(*)$ , show that there exists a function  $k : S_\alpha \cup \{\alpha\} \rightarrow C$  satisfying  $(*)$ .
- (c) If  $K \subset J$  and for all  $\alpha \in K$  there exists a function  $h_\alpha : S_\alpha \rightarrow C$  satisfying  $(*)$ , show that there exists a function

$$k : \bigcup_{\alpha \in K} S_\alpha \rightarrow C$$

satisfying  $(*)$ .

- (d) Show by transfinite induction that for every  $\beta \in J$ , there exists a function  $h_\beta : S_\beta \rightarrow C$  satisfying  $(*)$ . [Hint: If  $\beta$  has an immediate predecessor  $\alpha$ , then  $S_\beta = S_\alpha \cup \{\alpha\}$ . If not,  $S_\beta$  is the union of all  $S_\alpha$  with  $\alpha < \beta$ .]
- (e) Prove the theorem.

**Solution.**

- (a) Otherwise there would be a least  $x$  such that  $h(x) \neq k(x)$ , we would get  $h(S_x) = k(S_x)$ , and  $(*)$  would yield a contradiction.
- (b) We define  $k$  by  $k(x) = h(x)$  if  $x < \alpha$  and  $k(x) = \min(C \setminus h(S_x))$  if  $x = \alpha$ , and verify that  $k$  satisfies  $(*)$ .
- (c) Set  $k(x) = h_\alpha(x)$  if  $x \in S_\alpha$ . To show that  $k(x)$  is well defined, we must check that  $\beta > \alpha$  implies  $h_\beta(x) = h_\alpha(x)$ . But this follows from (a).
- (d) Let  $I$  be the set of all  $\beta \in J$  such that there is a map  $h_\beta : S_\beta \rightarrow C$  satisfying  $(*)$ . It suffices to show that  $I$  is inductive. So, assume that  $\beta$  is in  $J$  and that  $S_\beta \subset I$ . We must show  $\beta \in I$ . To do that, we use (b) if  $\beta$  has an immediate predecessor, and we use (c) if not.
- (e) We define  $h$  by

$$h(x) = \begin{cases} \min(C \setminus h_x(S_x)) & \text{if } x = \max(J) \\ h_{x+1}(x) & \text{if } x \neq \max(J), \end{cases}$$

where “ $x \neq \max(J)$ ” means “ $x \neq \max(J)$  if  $J$  has a maximum”, and  $x+1$  is the least element greater than  $x$ . Let us show that  $h$  satisfies  $(*)$ , that is,  $h(x) = \min(C \setminus h(S_x))$ . We can assume  $x \neq \max(J)$  (in the above sense). We must show  $h_{x+1}(x) = \min(C \setminus h(S_x))$ . Since we have  $h_{x+1}(x) = \min(C \setminus h_{x+1}(S_x))$  by (d) it suffices to prove  $h(S_x) = h_{x+1}(S_x)$ . Let  $y$  be in  $S_x$ , that is,  $y \in J$  and  $y < x$ . It is enough to verify  $h(y) = h_{x+1}(y)$ , that is,  $h_{y+1}(y) = h_{x+1}(y)$ . We have  $y+1 < x+1$ , and thus  $S_{y+1} \subset S_{x+1}$ , and (a) implies  $h_{x+1}|_{S_{y+1}} = h_{y+1}$ . This proves  $h_{y+1}(y) = h_{x+1}(y)$ , which is what we wanted.

• **Supplementary Exercise 11.1 p. 72.**

**Theorem 6** (General principle of recursive definition). *Let  $J$  be a well-ordered set; let  $C$  be a set. Let  $\mathcal{F}$  be the set of all functions mapping sections of  $J$  into  $C$ . Given a function  $\rho : \mathcal{F} \rightarrow C$ , there exists a unique function  $h : J \rightarrow C$  such that  $h(\alpha) = \rho(h|S_\alpha)$  for each  $\alpha \in J$ .*

[Hint: Follow the pattern outlined in Exercise 10 of §10.]

**Solution.** A close inspection of the solution to Exercise 10 of §10 reveals that the sole property of the element  $c$  of  $C$  defined by the equality  $c := \min(C \setminus h(S_x))$  is that it depends only on the restriction  $h|S_x$ . This implies that, if, in the proof given by the book, we replace “ $\min(C \setminus h(S_x))$ ” with “ $\rho(h|S_x)$ ”, then we obtain a proof of Theorem 6.

Here is a slightly different way of proving the General Principle of Recursive Definition. We state and prove Theorem 8 below, which we call General Principle of Transfinite Induction, and which generalizes both the usual Principle of Transfinite Induction (Exercise 10.7 p. 67 of the book and Theorem 5 p. 3. above) and the General Principle of Recursive Definition (Supplementary Exercise 1 p. 72 of the book and Theorem 6 above).

For each ordered set  $X$  and each  $x$  in  $X$  we denote the subset  $\{y \in X \mid y < x\}$  by  $X(x)$ . (This is the so-called *section by  $x$* .) Let  $X$  be a well-ordered set, let  $A$  be a set, let

$$\rho : \bigcup_{x \in X} A^{X(x)} \rightarrow A,$$

where  $A^{X(x)}$  stands for the set of all maps from  $X(x)$  to  $A$ . (Note that the sets  $A^{X(x)}$  with  $x$  in  $X$  are disjoint.)

**Theorem 7** (General Principle of Recursive Definition). *There is a unique map  $f : X \rightarrow A$  such that  $f(x) = \rho(f|X(x))$  for all  $x$  in  $X$ .*

The main ingredient to prove Theorem 7 is

**Theorem 8** (General Principle of Transfinite Induction). *If  $P(X)$  is a property that a well-ordered set  $X$  may or may not have, and if  $P(X)$  holds whenever  $P(X(x))$  holds for all  $x$  in  $X$ , then  $P(X)$  holds for all well-ordered set  $X$ .*

Before proving Theorem 8 recall the usual Principle of Transfinite Induction:

**Theorem 9** (Principle of Transfinite Induction). *Let  $U$  be a well-ordered set. If  $Q(u)$  is a property that an element  $u$  of  $U$  may or may not have, and if  $Q(u)$  holds whenever  $Q(v)$  holds for all  $v < u$ , then  $Q(u)$  holds for all  $u$  in  $U$ .*

This is Exercise 10.7 p. 67 of the book.

*Proof of Theorem 8.* Let  $X$  be a well-ordered set. We assume that  $P(X)$  holds whenever  $P(X(x))$  holds for all  $x$  in  $X$ , and we want to prove  $P(X)$ . Let  $\mathcal{D}$  be the set of all downward closed subsets of  $X$ . Then we have  $\mathcal{D} = \{X(x) \mid x \in X\} \cup \{X\}$ , and  $X(x) \subsetneq X(y)$  if and only if  $x < y$ , and  $X(x) \subsetneq X$  for all  $x$ , and  $\mathcal{D}$  is well-ordered by proper inclusion. We want to apply the Principle of Transfinite Induction (Theorem 9). To this end we set  $U := \mathcal{D}$  and, for  $D \in \mathcal{D}$  we define  $Q(D)$  as being  $P(D)$ . Then Theorem 9 tells us that  $P(D)$  holds for all  $D$  in  $\mathcal{D}$ , and thus in particular for  $D = X \in \mathcal{D}$ .  $\square$

*Proof of Theorem 7.* We denote the statement of Theorem 7 by  $P(X)$ , and we want to apply Theorem 8. So we assume that  $P(X(x))$  holds for all  $x$  in  $X$ . It suffices to prove  $P(X)$ .

Case 1:  $X$  has a largest element  $\infty \in X$ . By assumption, for all  $x < \infty$  there is a unique map  $f_x : X(x) \rightarrow A$  such that  $f(y) = \rho(f|X(y))$  for all  $y < x$ . It is easy to check that each  $f_x$  is the restriction of  $f_\infty$  to  $X(x)$ , and that, if we define  $f : X \rightarrow A$  by  $f(\infty) := \rho(f_\infty)$  and  $f(x) := f_\infty(x)$  if  $x < \infty$ , then  $f$  is the unique solution to our problem.

Case 2:  $X = \bigcup_{x \in X} X(x)$ . We have maps  $f_x : X(x) \rightarrow A$  as above, and it is easy to check that that map  $f : X \rightarrow A$  defined by  $f(x) := f_{x+1}(x)$ , where  $x+1$  is the successor of  $x$ , is the unique solution to our problem.  $\square$

To see why the General Principle of Transfinite Induction (Theorem 8) generalizes the Principle of Transfinite Induction (Theorem 9), note that we can define  $P(X)$  in terms of the  $Q(u)$  by decreeing the  $P(X)$  holds if and only if

$$\left( (\exists u \in U) (X = U(u)) \right) \implies Q(u).$$

• **Supplementary Exercise 11.2 p. 72.**

(a) Let  $J$  and  $E$  be well-ordered sets; let  $h : J \rightarrow E$ . Show the following two statements are equivalent:

- (i)  $h$  is order preserving and its image is  $E$  or a section of  $E$ .
- (ii)  $h(\alpha) = \text{smallest } [E - h(S_\alpha)]$  for all  $\alpha$ .

[Hint: Show that each of these conditions implies that  $h(S_\alpha)$  is a section of  $E$ ; conclude that it must be the section by  $h(\alpha)$ .]

(b) If  $E$  is a well-ordered set, show that no section of  $E$  has the order type of  $E$ , nor do two different sections of  $E$  have the same order type. [Hint: Given  $J$ , there is at most one order-preserving map of  $J$  into  $E$  whose image is  $E$  or a section of  $E$ .]

**Solution.**

(a) For all  $X \subset E$  set  $X^c := E \setminus X$ . For the sake of prudence, we change (ii) to:

(ii')  $h(S_x) \neq E$  and  $h(x) = \min(h(S_x)^c)$  for all  $x$ .

We want to show that (i) and (ii') are equivalent.

(i) implies (ii'). We prove  $h(S_x) \neq E$  by noting that  $h(S_x) = E$  we would entail  $h(x) = h(y)$  for some  $y < x$ , contradiction. To prove  $h(x) = \min(h(S_x)^c)$ , assume by contradiction that we have  $h(x) \neq \min(h(S_x)^c) =: e$ . If  $h(x) < e$ , then  $h(x) \notin h(S_x)^c$ , that is,  $h(x) \in h(S_x)$ , and we reach a contradiction as above. If  $e < h(x)$ , then  $e = h(y)$  for some  $y < x$ , that is,  $\min(h(S_x)^c) = e = h(y) \in h(S_x)$ , contradiction.

(ii') implies (i). We assume (ii'), and, in particular, that  $h$  is weakly increasing. To show that  $h$  is increasing, suppose  $x < y$  and  $h(x) = h(y)$  (we cannot have  $h(x) > h(y)$  because  $h$  is weakly increasing). Since  $h(x) = h(y) = \min(h(S_y)^c)$ , we have  $h(x) \in h(S_y)^c$ , but  $h(x) \in h(S_y)$ ,

contradiction. Finally,  $h(J)$  is downward closed because  $e < h(x) = \min(h(S_x)^c)$  implies  $e \in h(S_x) \subset h(J)$ .

In the statement of the Exercise, the condition that  $J$  is well-ordered can be changed from an assumption to a conclusion.

(b) Let  $a$  be in  $E$ , and assume there is an isomorphism of well-ordered sets  $h : S_a \rightarrow E$ . It suffices to derive a contradiction. Let  $i : S_a \rightarrow E$  be the inclusion. By (a)  $h$  and  $i$  satisfy the same recursion relation. By the Theorem about the General Principle of Definition by Recursion, we have  $h = i$ , and thus  $a \in h(S_a) = i(S_a) = S_a$ , contradiction.

• **Supplementary Exercise 11.3 p. 73.** Let  $J$  and  $E$  be well-ordered sets; suppose there is an order-preserving map  $k : J \rightarrow E$ . Using Exercises 1 and 2, show that  $J$  has the order type of  $E$  or a section of  $E$ . [Hint: Choose  $e_0 \in E$ . Define  $h : J \rightarrow E$  by the recursion formula

$$h(\alpha) = \text{smallest } [E - h(S_\alpha)] \text{ if } h(S_\alpha) \neq E,$$

and  $h(\alpha) = e_0$  otherwise. Show that  $h(\alpha) \leq k(\alpha)$  for all  $\alpha$ ; conclude that  $h(S_\alpha) \neq E$  for all  $\alpha$ .]

**Solution.** We can assume  $E \neq \emptyset$ . Let  $e_0 \in E$ . Let  $x$  be in  $J$ . We define  $h : J \rightarrow E$  as in the hint.

Claim 1:  $h(x) \leq k(x)$  for all  $x$ .

Claim 2:  $h(y) \leq k(y)$  for all  $y$  in  $S_x$  implies  $h(S_x) \neq E$ .

Proof of Claim 2. For all  $y$  in  $S_x$  we have  $h(y) \leq k(y) < k(x)$ , and in particular  $k(x) \neq h(y)$ . This implies  $k(x) \notin h(S_x)$ .

Proof of Claim 1. Assume by contradiction  $h(x) > k(x)$  for some  $x$ . We can assume that  $x$  is minimum for this condition. For  $y < x$  we have  $h(y) \leq k(y)$ , hence  $h(S_x) \neq E$  by Claim 2.

Claims 1 and 2 imply  $h(x) = \min(h(S_x)^c)$  for all  $x$ , hence  $h$  is increasing and  $h(J)$  is downward closed by Supplementary Exercise 11.2 above, hence  $J$  has the order type of  $E$  or a section of  $E$ .

• **Supplementary Exercise 11.4 p. 73.** Use Exercises 1–3 to prove the following:

- (a) If  $A$  and  $B$  are well-ordered sets, then exactly one of the following three conditions holds:  $A$  and  $B$  have the same order type, or  $A$  has the order type of a section of  $B$ , or  $B$  has the order type of a section of  $A$ . [Hint: Form a well-ordered set containing both  $A$  and  $B$ , as in Exercise 8 of §10; then apply the preceding exercise.]
- (b) Suppose that  $A$  and  $B$  are well-ordered sets that are uncountable, such that every section of  $A$  and of  $B$  is countable. Show  $A$  and  $B$  have the same order type.

**Solution.** (a) For any element  $x$  of any ordered set  $X$ , let  $X_{<x}$  denote the corresponding section, and let us set  $X_{<\infty} := X$ . Let  $C$  be the well-ordered set containing both  $A$  and  $B$ , described in Exercise 8 of §10, and let  $k : A \rightarrow C$  and  $\ell : B \rightarrow C$  be the natural increasing maps. By the previous Exercise, we have isomorphisms  $A \simeq C_{<x}$  and  $B \simeq C_{<y}$  for some  $x$  and  $y$  in  $C \cup \{\infty\}$ . We can assume  $x \leq y$ . Then  $x = y$  implies  $A \simeq B$ . If  $x < y$ , we get

$$A \simeq C_{<x} = (C_{<y})_{<x} \subsetneq C_{<y} \simeq B.$$

This implies  $A \simeq B_{<b}$  for some  $b$  in  $B$ . The fact that the various cases are exclusive follows from Supplementary Exercise 11.2b.

(b) Follows from (a).

Here is an important consequence of (a):

**Theorem 10** (Comparability Theorem). *If  $A$  and  $B$  are sets, then exactly one of the following three conditions holds:*

- (i) *there is a bijection  $A \rightarrow B$ ,*
- (ii) *there is an injection  $A \rightarrow B$  and a surjection  $B \rightarrow A$ ,*
- (iii) *there is an injection  $B \rightarrow A$  and a surjection  $A \rightarrow B$ .*

• **Supplementary Exercise 11.5 p. 73.** Let  $X$  be a set; let  $\mathcal{A}$  be the collection of all pairs  $(A, <)$ , where  $A$  is a subset of  $X$  and  $<$  is a well-ordering of  $A$ . Define

$$(A, <) \prec (A', <')$$

if  $(A, <)$  equals a section of  $(A', <')$ .

- (a) Show that  $\prec$  is a strict partial order on  $\mathcal{A}$ .
- (b) Let  $\mathcal{B}$  be a subcollection of  $\mathcal{A}$  that is simply ordered by  $\prec$ . Define  $B'$  to be the union of the sets  $B$ , for all  $(B, <) \in \mathcal{B}$ ; and define  $<'$  to be the union of the relations  $<$ , for all  $(B, <) \in \mathcal{B}$ . Show that  $(B', <')$  is a well-ordered set.

**Solution.** Left to the reader.

• **Supplementary Exercise 11.6 p. 73.** Use Exercises 1 and 5 to prove the following:

**Theorem.** The maximum principle is equivalent to the well-ordering theorem.

**Solution.** The fact that the well-ordering theorem implies the maximum principle is proved on p. 70 of the book. Let us prove the converse. In the setting of Supplementary Exercise 11.5b, take  $\mathcal{B}$  maximal. Then it suffices to show that  $B' = X$ . If it was not so, we could add to  $B'$  a new element  $x$  and make it the largest element of  $B' \cup \{x\}$ , which would then be a well-ordered set larger than  $B'$ , contradiction.

• **Supplementary Exercise 11.7 p. 73.** Use Exercises 1–5 to prove the following:

**Theorem.** The choice axiom is equivalent to the well-ordering theorem.

**Proof.** Let  $X$  be a set; let  $c$  be a fixed choice function for the nonempty subsets of  $X$ . If  $T$  is a subset of  $X$  and  $<$  is a relation on  $T$ , we say that  $(T, <)$  is a tower in  $X$  if  $<$  is a well-ordering of  $T$  and if for each  $x \in T$ ,

$$x = c(X - S_x(T)),$$

where  $S_x(T)$  is the section of  $T$  by  $x$ .



- (a) Let  $(T_1, <_1)$  and  $(T_2, <_2)$  be two towers in  $X$ . Show that either these two ordered sets are the same, or one equals a section of the other. [Hint: Switching indices if necessary, we can assume that  $h : T_1 \rightarrow T_2$  is order preserving and  $h(T_1)$  equals either  $T_2$  or a section of  $T_2$ . Use Exercise 2 to show that  $h(x) = x$  for all  $x$ .]
- (b) If  $(T, <)$  is a tower in  $X$  and  $T \neq X$ , show there is a tower in  $X$  of which  $(T, <)$  is a section.
- (c) Let  $\{(T_k, <_k) | k \in K\}$  be the collection of all towers in  $X$ . Let

$$T = \bigcup_{k \in K} T_k \text{ and } < = \bigcup_{k \in K} (<_k).$$

Show that  $(T, <)$  is a tower in  $X$ . Conclude that  $T = X$ .

**Solution.** (a) The map  $h$  in the hint exists by Supplementary Exercise 4a. Let us show  $h(x) = x$  for all  $x$  thanks to Supplementary Exercise 2. Assume by contradiction that we have  $h(x) \neq x$  for some  $x$  in  $T_1$ , which we can suppose to be minimum for this condition. The map  $h$  induces an isomorphism  $T_1 \simeq h(T_1)$ , implying  $h(T_{1, <_1 y}) = T_{2, h(y)}$  for all  $y$  in  $T_1$ . By the choice of  $x$  we get  $T_{1, <_1 x} = T_{2, <_2 h(x)}$ . Since  $T_1$  and  $T_2$  are towers, this entails

$$h(x) = c(X \setminus T_{2, <_2 h(x)}) = c(X \setminus T_{1, <_1 x}) = x,$$

contradiction. The fact that  $h(x) = x$  for all  $x$  in  $T_1$  implies, by Supplementary Exercise 11.2a, that  $T_1$  is contained and downward closed in  $T_2$ .

- (b) Add  $c(X \setminus T)$  to  $T$ , and make it the largest element.
- (c) Left to the reader.

This shows that the choice axiom implies the well-ordering theorem. The converse is clear.

• **Solution to Exercise 13.6 p. 83.** We must show that the topologies  $\mathcal{T}_\ell$  and  $\mathcal{T}_K$  are incomparable.

Claim:  $[2, 3] \notin \mathcal{T}_K$ . Proof. If not we would have  $2 \in (a, b) \setminus K \subset [2, 3)$  for some  $a$  and  $b$ , hence  $a < 2$  and  $a \leq 2$ , contradiction.

Claim:  $(-1, 1) \setminus K \notin \mathcal{T}_\ell$ . Proof. If not we would have  $0 \in [a, b) \subset (-1, 1) \setminus K \subset [2, 3)$  for some  $a$  and  $b$ , hence  $a \leq 0 < b$ , hence  $a < \frac{1}{n} < b$  for some  $n$ , contradiction.

• **Solution to Exercise 13.7 p. 83.** Let us use the following notation:

$\mathcal{T}_s$  := standard topology,

$\mathcal{T}_K$  := topology of  $\mathbb{R}_K$ ,

$\mathcal{T}_{fc}$  := finite complement topology,

$\mathcal{T}_u$  := upper limit topology (having the sets  $(a, b]$  as basis),

$\mathcal{T}_\infty$  := topology having the sets  $(-\infty, a)$  as basis.

We denote the corresponding topological spaces by  $\mathbb{R}_s, \mathbb{R}_K, \mathbb{R}_{fc}, \mathbb{R}_u$  and  $\mathbb{R}_\infty$ . Finally we write  $\mathcal{B}_s, \mathcal{B}_K, \mathcal{B}_u$  and  $\mathcal{B}_\infty$  for the obvious bases.

The inclusions between these five topologies on  $\mathbb{R}$  can be summarized by the diagram

$$\begin{array}{ccc} & u & \\ & K & \\ & s & \\ fc & & \infty, \end{array}$$

where “ $i$  below  $j$ ” means “ $\mathcal{T}_i \subsetneq \mathcal{T}_j$ ”<sup>2</sup>, and “ $i$  and  $j$  on the same level” means “ $\mathcal{T}_i$  and  $\mathcal{T}_j$  are incomparable”.

Preliminary comments: It is easy to see that the elements of  $\mathcal{T}_\infty$  are  $\emptyset$ , the intervals  $(-\infty, a)$ , and  $\mathbb{R}$ , and to observe that  $\mathcal{T}_\infty \cap \mathcal{T}_{fc} = \{\emptyset, \mathbb{R}\}$ . It is also easy to compare the standard topology  $\mathcal{T}_s$  to the others: the elements of  $\mathcal{T}_{fc}$  and  $\mathcal{T}_\infty$  are clearly open in  $\mathbb{R}_s$ , and it is plain that the intervals  $(a, b)$  (which are the elements on  $\mathcal{B}_s$ ) are open in  $\mathbb{R}_K$  and in  $\mathbb{R}_\infty$  (note that  $(a, b) = \bigcup_{d < b} (a, d]$ ). Clearly,  $(-1, 1) \setminus K \in \mathcal{T}_K$  and  $(a, b] \in \mathcal{T}_u$  are not open in  $\mathbb{R}_s$ . Moreover  $(2, 3]$  is in  $\mathcal{T}_u$  but not in  $\mathcal{T}_K$ . So, it only remains to prove  $\mathcal{T}_K \subset \mathcal{T}_u$ .

Let  $x$  be in  $(a, b) \setminus K$ . It suffices to show that there is a  $c$  such that  $x \in (c, x] \subset (a, b) \setminus K$ . If  $x \leq 0$  we set  $c := a$ . If  $\frac{1}{n+1} < x < \frac{1}{n}$  we set  $c := \frac{1}{n+1}$ . If  $x > 1$  we set  $c := \max(1, a)$ .

We give a slightly different proof of Theorem 17.2 p. 94 of the book. First recall the statement.

**Theorem 11** (Theorem 17.2 p. 94 of the book). *Let  $X$  be a topological space,  $Y$  a subspace, and  $A$  a subset of  $Y$ . Then  $A$  is closed in  $Y$  if and only if  $A = Y \cap C$  for some closed subset  $C$  of  $X$ .*

*Proof.* It suffices to show that  $Y \setminus A = Y \cap U$ , where  $U$  is an open subset of  $X$ , if and only if  $A = Y \cap (X \setminus U)$ . Hence it is enough to prove that we have  $Y \setminus (Y \cap U) = Y \cap (X \setminus U)$ , or equivalently  $Y \setminus U = Y \cap (X \setminus U)$ , for all subset  $U$  of  $X$ . But this is clear.  $\square$

<sup>2</sup>I denote inclusion by  $\subset$  and proper inclusion by  $\subsetneq$ . I know that, in some sense, it would be more coherent to use  $\subseteq$  for inclusion, but I prefer to do it that way, and hope the reader will not be confused.