

A few comments about “Topology” by Munkres

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As the title indicates, we make a comments about the book **Topology** by James R. Munkres. This is a work in progress. The last version of this text is available here: <https://www.overleaf.com/read/kdwwjvqjrzb#9fe3a6>. Another version is available here: <https://github.com/Pierre-Yves-Gaillard/About-Topology-by-Munkres>.

• **Definition of \mathbb{R} p. 31.** The object \mathbb{R} is defined by assuming that there exists a set \mathbb{R} having certain properties. We take this assumption for granted. Then it is easy to see that there are several sets having these properties. So, let \mathbb{R}' be a set having the same properties as \mathbb{R} . Let $\mathbb{Z}'_+, \mathbb{Z}'$ and \mathbb{Q}' be to \mathbb{R}' what \mathbb{Z}_+, \mathbb{Z} and \mathbb{Q} are to \mathbb{R} .

Theorem 1. *There is a unique morphism of fields from $f : \mathbb{R} \rightarrow \mathbb{R}'$. This morphism is an isomorphism of ordered fields, and it induces isomorphisms $\mathbb{Z}_+ \rightarrow \mathbb{Z}'_+, \mathbb{Z} \rightarrow \mathbb{Z}'$ and $\mathbb{Q} \rightarrow \mathbb{Q}'$.*

Lemma 2. *There is a unique map $g : \mathbb{Z}_+ \rightarrow \mathbb{Z}'_+$ such that $g(0) = 0$ and $g(n+1) = g(n) + 1$ for all n in \mathbb{Z}_+ . Similarly, there is a unique map $h : \mathbb{Z}'_+ \rightarrow \mathbb{Z}_+$ such that $h(0) = 0$ and $h(n+1) = h(n) + 1$ for all n in \mathbb{Z}'_+ .*

Proof. For $i \in \mathbb{Z}_+$ and $\varphi : \{1, \dots, i\} \rightarrow \mathbb{Z}'_+$ define $\rho(\varphi) \in \mathbb{Z}'_+$ by $\rho(\varphi) := \varphi(i) + 1$. Then the first statement follows from the Principle of Recursive Definition (Theorem 4 p. 3). The proof of the second statement is similar. \square

Proof of Theorem 1. In the notation of Lemma 2, set $u := h \circ g$. Then $u : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ satisfies $u(0) = 0$ and $u(n+1) = u(n) + 1$ for all n in \mathbb{Z}_+ . One can easily prove that $u(n) = n$ by induction. The same argument works for $g \circ h$. This shows that $g : \mathbb{Z}_+ \rightarrow \mathbb{Z}'_+$ and $h : \mathbb{Z}'_+ \rightarrow \mathbb{Z}_+$ are inverse isomorphisms. Then we extend g to morphisms $\mathbb{Z} \rightarrow \mathbb{Z}'$ and $\mathbb{Q} \rightarrow \mathbb{Q}'$, and similarly for h , and, arguing as before, we show that these morphisms are isomorphisms. More precisely, we see that there is a unique morphism $\mathbb{Z} \rightarrow \mathbb{Z}'$ extending g , and that this morphism is an isomorphism, and similarly for the morphism $\mathbb{Q} \rightarrow \mathbb{Q}'$. So we can make the identifications $\mathbb{Z}_+ = \mathbb{Z}'_+, \mathbb{Z} = \mathbb{Z}', \mathbb{Q} = \mathbb{Q}'$. To show that there is a unique morphism of fields $\mathbb{R} \rightarrow \mathbb{R}'$, and that this morphism is an isomorphism (inducing the identity of \mathbb{Q}), we argue as in Section *Appendix to Chapter 1* in *A few comments about “Principles of Mathematical Analysis” by Rudin*, available at <https://zenodo.org/records/13955297>. \square

Theorem 7.8. p. 50. Recall the statement:

Theorem 3 (Theorem 7.8. p. 50 of the book). *Let A be a set. There is no injective map $f : \mathcal{P}(A) \rightarrow A$, and there is no surjective map $g : A \rightarrow \mathcal{P}(A)$.*

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Here is my favourite way of phrasing the argument showing that there is no surjective map $g : A \rightarrow \mathcal{P}(A)$. Let $g : A \rightarrow \mathcal{P}(A)$ be a map, and set $B := \{a \mid a \notin g(a)\}$, so that we have, for all a in A ,

$$a \in B \iff a \notin g(a).$$

Let a_0 be in A . If we had $g(a_0) = B$, we would get, for all a in A ,

$$a \in g(a_0) \iff a \notin g(a),$$

and we immediately that setting $a := a_0$ yields a contradiction. This shows that B is not in the range of g .

• **Exercise 7.6. p. 51.** We say that two sets A and B **have the same cardinality** if there is a bijection of A with B .

(a) Show that if $B \subset A$ and if there is an injection

$$f : A \rightarrow B,$$

then A and B have the same cardinality. [Hint: Define $A_1 = A, B_1 = B$, and for $n > 1$, $A_n = f(A_{n-1})$ and $B_n = f(B_{n-1})$. (Recursive definition again!) Note that $A_1 \supset B_1 \supset A_2 \supset B_2 \supset A_3 \supset \dots$. Define a bijection $h : A \rightarrow B$ by the rule

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_n \setminus B_n \text{ for some } n, \\ x & \text{otherwise.} \end{cases}$$

(b) *Theorem (Schroeder-Bernstein theorem).* If there are injections $f : A \rightarrow C$ and $g : C \rightarrow A$, then A and C have the same cardinality.

Solution. (a) We will freely use the following two obvious facts:

(F1) For $x \in A$ and $n \in \mathbb{Z}_+$ we have

$$x \in A_n \iff f(x) \in A_{n+1} \text{ and } x \in B_n \iff f(x) \in B_{n+1}.$$

(F2) We have $\bigcap_{n \geq 1} A_n = \bigcap_{n \geq 1} B_n =: I$.

Setting $A'_n := A_n \setminus B_n, B'_n := B_n \setminus A_{n+1}$, we get

$$A = \left(\bigcup_{n \geq 1} A'_n \right) \cup \left(\bigcup_{n \geq 1} B'_n \right) \cup I,$$

and this union is disjoint. We also have

$$B = \left(\bigcup_{n \geq 2} A'_n \right) \cup \left(\bigcup_{n \geq 1} B'_n \right) \cup I.$$

The injection f induces bijections $f_n : A'_n \rightarrow A'_{n+1}$ (here we are using (F1)). To define a bijection $h : A \rightarrow B$, it suffices to define three bijections

$$u : \bigcup_{n \geq 1} A'_n \rightarrow \bigcup_{n \geq 2} A'_n, \quad v : \bigcup_{n \geq 1} B'_n \rightarrow \bigcup_{n \geq 1} B'_n, \quad w : I \rightarrow I.$$

We define u by $u(x) = f_n(x)$ if $x \in A'_n$, and take v and w to be the identity maps.

(b) We set $B := g(C) \subset A$ and define $f' : A \rightarrow B$ by $f'(a) := g(f(a))$. Then $f' : A \rightarrow B$ satisfies the assumptions for $f : A \rightarrow B$ in (a).

• **Exercise 8.7. p. 56.** Prove Theorem 8.4 p. 54.

Solution. Recall the statement of Theorem 8.4.

Theorem 4 (Principle of Recursive Definition, Theorem 8.4 of the book). *Let A be a set; let a_0 be an element of A . Suppose ρ is a function that assigns, to each function f mapping a nonempty section of the positive integers into A , an element of A . Then there exists a unique function*

$$h : \mathbb{Z}^+ \rightarrow A$$

such that

$$\begin{aligned} h(1) &= a_0, \\ h(i) &= \rho(h|_{\{1, \dots, i-1\}}) \text{ for } i > 1. \end{aligned} \tag{*}$$

The formula (*) is called a recursion formula for h . It specifies $h(1)$, and it expresses the value of h at $i > 1$ in terms of the values of h for positive integers less than i .

The book gives a detailed proof of the particular case when $\rho(h|_{\{1, \dots, i-1\}})$ is equal to $\min(C \setminus h(\{1, \dots, i-1\}))$, where “min” means “*minimum*”, and C is an infinite set. A close inspection of this proof reveals that the sole property of the element c of C defined by the equality $c := \min(C \setminus h(\{1, \dots, i-1\}))$ is that it depends only on the restriction $h|_{\{1, \dots, i-1\}}$. This implies that, if, in the proof given by the book, we replace “ $\min(C \setminus h(\{1, \dots, i-1\}))$ ” with “ $\rho(h|_{\{1, \dots, i-1\}})$ ”, then we obtain a proof of Theorem 4.

• **Exercise 10.7 p. 67.** Let J be a well-ordered set. A subset J_0 of J is said to be **inductive** if for every $\alpha \in J$,

$$(S_\alpha \subset J_0) \Rightarrow \alpha \in J_0.$$

Theorem 5 (The Principle of Transfinite Induction). *If J is a well-ordered set and J_0 is an inductive subset of J , then $J_0 = J$.*

Solution. If $J_0 \neq J$, let α be the least element of $J \setminus J_0$. We get $S_\alpha \subset J_0$, and thus $\alpha \in J_0$, contradiction.

• **Exercise 10.10 p. 67. Theorem.** Let J and C be well-ordered sets; assume that there is no surjective function mapping a section of J onto C . Then there exists a unique function $h : J \rightarrow C$ satisfying the equation

$$h(x) = \min(C \setminus h(S_x)) \tag{*}$$

for each $x \in J$, where S_x is the section of J by x .

Proof.

- (a) If h and k map sections of J , or all of J , into C and satisfy $(*)$ for all x in their respective domains, show that $h(x) = k(x)$ for all x in both domains.
- (b) If there exists a function $h : S_\alpha \rightarrow C$ satisfying $(*)$, show that there exists a function $k : S_\alpha \cup \{\alpha\} \rightarrow C$ satisfying $(*)$.
- (c) If $K \subset J$ and for all $\alpha \in K$ there exists a function $h_\alpha : S_\alpha \rightarrow C$ satisfying $(*)$, show that there exists a function

$$k : \bigcup_{\alpha \in K} S_\alpha \rightarrow C$$

satisfying $(*)$.

- (d) Show by transfinite induction that for every $\beta \in J$, there exists a function $h_\beta : S_\beta \rightarrow C$ satisfying $(*)$. [Hint: If β has an immediate predecessor α , then $S_\beta = S_\alpha \cup \{\alpha\}$. If not, S_β is the union of all S_α with $\alpha < \beta$.]
- (e) Prove the theorem.

Solution.

- (a) Otherwise there would be a least x such that $h(x) \neq k(x)$, we would get $h(S_x) = k(S_x)$, and $(*)$ would yield a contradiction.
- (b) We define k by $k(x) = h(x)$ if $x < \alpha$ and $k(x) = \min(C \setminus h(S_x))$ if $x = \alpha$, and verify that k satisfies $(*)$.
- (c) Set $k(x) = h_\alpha(x)$ if $x \in S_\alpha$. To show that $k(x)$ is well defined, we must check that $\beta > \alpha$ implies $h_\beta(x) = h_\alpha(x)$. But this follows from (a).
- (d) Let I be the set of all $\beta \in J$ such that there is a map $h_\beta : S_\beta \rightarrow C$ satisfying $(*)$. It suffices to show that I is inductive. So, assume that β is in J and that $S_\beta \subset I$. We must show $\beta \in I$. To do that, we use (b) if β has an immediate predecessor, and we use (c) if not.
- (e) We define h by

$$h(x) = \begin{cases} \min(C \setminus h_x(S_x)) & \text{if } x = \max(J) \\ h_{x+1}(x) & \text{if } x \neq \max(J), \end{cases}$$

where “ $x \neq \max(J)$ ” means “ $x \neq \max(J)$ if J has a maximum”, and $x+1$ is the least element greater than x . Let us show that h satisfies $(*)$, that is, $h(x) = \min(C \setminus h(S_x))$. We can assume $x \neq \max(J)$ (in the above sense). We must show $h_{x+1}(x) = \min(C \setminus h(S_x))$. Since we have $h_{x+1}(x) = \min(C \setminus h_{x+1}(S_x))$ by (d) it suffices to prove $h(S_x) = h_{x+1}(S_x)$. Let y be in S_x , that is, $y \in J$ and $y < x$. It is enough to verify $h(y) = h_{x+1}(y)$, that is, $h_{y+1}(y) = h_{x+1}(y)$. We have $y+1 < x+1$, and thus $S_{y+1} \subset S_{x+1}$, and (a) implies $h_{x+1}|_{S_{y+1}} = h_{y+1}$. This proves $h_{y+1}(y) = h_{x+1}(y)$, which is what we wanted.

• **Supplementary Exercise 11.1 p. 72.**

Theorem 6 (General principle of recursive definition). *Let J be a well-ordered set; let C be a set. Let \mathcal{F} be the set of all functions mapping sections of J into C . Given a function $\rho : \mathcal{F} \rightarrow C$, there exists a unique function $h : J \rightarrow C$ such that $h(\alpha) = \rho(h|S_\alpha)$ for each $\alpha \in J$.*

[Hint: Follow the pattern outlined in Exercise 10 of §10.]

Solution. A close inspection of the solution to Exercise 10 of §10 reveals that the sole property of the element c of C defined by the equality $c := \min(C \setminus h(S_x))$ is that it depends only on the restriction $h|S_x$. This implies that, if, in the proof given by the book, we replace “ $\min(C \setminus h(S_x))$ ” with “ $\rho(h|S_x)$ ”, then we obtain a proof of Theorem 6.

Here is a slightly different way of proving the General Principle of Recursive Definition. We state and prove Theorem 8 below, which we call General Principle of Transfinite Induction, and which generalizes both the usual Principle of Transfinite Induction (Exercise 10.7 p. 67 of the book and Theorem 5 p. 3. above) and the General Principle of Recursive Definition (Supplementary Exercise 1 p. 72 of the book and Theorem 6 above).

For each ordered set X and each x in X we denote the subset $\{y \in X \mid y < x\}$ by $X(x)$. (This is the so-called *section by x* .) Let X be a well-ordered set, let A be a set, let

$$\rho : \bigcup_{x \in X} A^{X(x)} \rightarrow A,$$

where $A^{X(x)}$ stands for the set of all maps from $X(x)$ to A . (Note that the sets $A^{X(x)}$ with x in X are disjoint.)

Theorem 7 (General Principle of Recursive Definition). *There is a unique map $f : X \rightarrow A$ such that $f(x) = \rho(f|X(x))$ for all x in X .*

The main ingredient to prove Theorem 7 is

Theorem 8 (General Principle of Transfinite Induction). *If $P(X)$ is a property that a well-ordered set X may or may not have, and if $P(X)$ holds whenever $P(X(x))$ holds for all x in X , then $P(X)$ holds for all well-ordered set X .*

Before proving Theorem 8 recall the usual Principle of Transfinite Induction:

Theorem 9 (Principle of Transfinite Induction). *Let U be a well-ordered set. If $Q(u)$ is a property that an element u of U may or may not have, and if $Q(u)$ holds whenever $Q(v)$ holds for all $v < u$, then $Q(u)$ holds for all u in U .*

This is Exercise 10.7 p. 67 of the book.

Proof of Theorem 8. Let X be a well-ordered set. We assume that $P(X)$ holds whenever $P(X(x))$ holds for all x in X , and we want to prove $P(X)$. Let \mathcal{D} be the set of all downward closed subsets of X . Then we have $\mathcal{D} = \{X(x) \mid x \in X\} \cup \{X\}$, and $X(x) \subsetneq X(y)$ if and only if $x < y$, and $X(x) \subsetneq X$ for all x , and \mathcal{D} is well-ordered by proper inclusion. We want to apply the Principle of Transfinite Induction (Theorem 9). To this end we set $U := \mathcal{D}$ and, for $D \in \mathcal{D}$ we define $Q(D)$ as being $P(D)$. Then Theorem 9 tells us that $P(D)$ holds for all D in \mathcal{D} , and thus in particular for $D = X \in \mathcal{D}$. \square

Proof of Theorem 7. We denote the statement of Theorem 7 by $P(X)$, and we want to apply Theorem 8. So we assume that $P(X(x))$ holds for all x in X . It suffices to prove $P(X)$.

Case 1: X has a largest element $\infty \in X$. By assumption, for all $x < \infty$ there is a unique map $f_x : X(x) \rightarrow A$ such that $f(y) = \rho(f|X(y))$ for all $y < x$. It is easy to check that each f_x is the restriction of f_∞ to $X(x)$, and that, if we define $f : X \rightarrow A$ by $f(\infty) := \rho(f_\infty)$ and $f(x) := f_\infty(x)$ if $x < \infty$, then f is the unique solution to our problem.

Case 2: $X = \bigcup_{x \in X} X(x)$. We have maps $f_x : X(x) \rightarrow A$ as above, and it is easy to check that that map $f : X \rightarrow A$ defined by $f(x) := f_{x+1}(x)$, where $x+1$ is the successor of x , is the unique solution to our problem. \square

To see why the General Principle of Transfinite Induction (Theorem 8) generalizes the Principle of Transfinite Induction (Theorem 9), note that we can define $P(X)$ in terms of the $Q(u)$ by decreeing the $P(X)$ holds if and only if

$$\left((\exists u \in U) (X = U(u)) \right) \implies Q(u).$$

• **Supplementary Exercise 11.2 p. 72.**

(a) Let J and E be well-ordered sets; let $h : J \rightarrow E$. Show the following two statements are equivalent:

- (i) h is order preserving and its image is E or a section of E .
- (ii) $h(\alpha) = \text{smallest } [E - h(S_\alpha)]$ for all α .

[Hint: Show that each of these conditions implies that $h(S_\alpha)$ is a section of E ; conclude that it must be the section by $h(\alpha)$.]

(b) If E is a well-ordered set, show that no section of E has the order type of E , nor do two different sections of E have the same order type. [Hint: Given J , there is at most one order-preserving map of J into E whose image is E or a section of E .]

Solution.

(a) For all $X \subset E$ set $X^c := E \setminus X$. For the sake of prudence, we change (ii) to:

(ii') $h(S_x) \neq E$ and $h(x) = \min(h(S_x)^c)$ for all x .

We want to show that (i) and (ii') are equivalent.

(i) implies (ii'). We prove $h(S_x) \neq E$ by noting that $h(S_x) = E$ we would entail $h(x) = h(y)$ for some $y < x$, contradiction. To prove $h(x) = \min(h(S_x)^c)$, assume by contradiction that we have $h(x) \neq \min(h(S_x)^c) =: e$. If $h(x) < e$, then $h(x) \notin h(S_x)^c$, that is, $h(x) \in h(S_x)$, and we reach a contradiction as above. If $e < h(x)$, then $e = h(y)$ for some $y < x$, that is, $\min(h(S_x)^c) = e = h(y) \in h(S_x)$, contradiction.

(ii') implies (i). We assume (ii'), and, in particular, that h is weakly increasing. To show that h is increasing, suppose $x < y$ and $h(x) = h(y)$ (we cannot have $h(x) > h(y)$ because h is weakly increasing). Since $h(x) = h(y) = \min(h(S_y)^c)$, we have $h(x) \in h(S_y)^c$, but $h(x) \in h(S_y)$,

contradiction. Finally, $h(J)$ is downward closed because $e < h(x) = \min(h(S_x)^c)$ implies $e \in h(S_x) \subset h(J)$.

In the statement of the Exercise, the condition that J is well-ordered can be changed from an assumption to a conclusion.

(b) Let a be in E , and assume there is an isomorphism of well-ordered sets $h : S_a \rightarrow E$. It suffices to derive a contradiction. Let $i : S_a \rightarrow E$ be the inclusion. By (a) h and i satisfy the same recursion relation. By the Theorem about the General Principle of Definition by Recursion, we have $h = i$, and thus $a \in h(S_a) = i(S_a) = S_a$, contradiction.

• **Supplementary Exercise 11.3 p. 73.** Let J and E be well-ordered sets; suppose there is an order-preserving map $k : J \rightarrow E$. Using Exercises 1 and 2, show that J has the order type of E or a section of E . [Hint: Choose $e_0 \in E$. Define $h : J \rightarrow E$ by the recursion formula

$$h(\alpha) = \text{smallest } [E - h(S_\alpha)] \text{ if } h(S_\alpha) \neq E,$$

and $h(\alpha) = e_0$ otherwise. Show that $h(\alpha) \leq k(\alpha)$ for all α ; conclude that $h(S_\alpha) \neq E$ for all α .]

Solution. We can assume $E \neq \emptyset$. Let $e_0 \in E$. Let x be in J . We define $h : J \rightarrow E$ as in the hint.

Claim 1: $h(x) \leq k(x)$ for all x .

Claim 2: $h(y) \leq k(y)$ for all y in S_x implies $h(S_x) \neq E$.

Proof of Claim 2. For all y in S_x we have $h(y) \leq k(y) < k(x)$, and in particular $k(x) \neq h(y)$. This implies $k(x) \notin h(S_x)$.

Proof of Claim 1. Assume by contradiction $h(x) > k(x)$ for some x . We can assume that x is minimum for this condition. For $y < x$ we have $h(y) \leq k(y)$, hence $h(S_x) \neq E$ by Claim 2.

Claims 1 and 2 imply $h(x) = \min(h(S_x)^c)$ for all x , hence h is increasing and $h(J)$ is downward closed by Supplementary Exercise 11.2 above, hence J has the order type of E or a section of E .

• **Supplementary Exercise 11.4 p. 73.** Use Exercises 1–3 to prove the following:

- (a) If A and B are well-ordered sets, then exactly one of the following three conditions holds: A and B have the same order type, or A has the order type of a section of B , or B has the order type of a section of A . [Hint: Form a well-ordered set containing both A and B , as in Exercise 8 of §10; then apply the preceding exercise.]
- (b) Suppose that A and B are well-ordered sets that are uncountable, such that every section of A and of B is countable. Show A and B have the same order type.

Solution. (a) For any element x of any ordered set X , let $X_{<x}$ denote the corresponding section, and let us set $X_{<\infty} := X$. Let C be the well-ordered set containing both A and B , described in Exercise 8 of §10, and let $k : A \rightarrow C$ and $\ell : B \rightarrow C$ be the natural increasing maps. By the previous Exercise, we have isomorphisms $A \simeq C_{<x}$ and $B \simeq C_{<y}$ for some x and y in $C \cup \{\infty\}$. We can assume $x \leq y$. Then $x = y$ implies $A \simeq B$. If $x < y$, we get

$$A \simeq C_{<x} = (C_{<y})_{<x} \subsetneq C_{<y} \simeq B.$$

This implies $A \simeq B_{<b}$ for some b in B . The fact that the various cases are exclusive follows from Supplementary Exercise 11.2b.

(b) Follows from (a).

Here is an important consequence of (a):

Theorem 10 (Comparability Theorem). *If A and B are sets, then exactly one of the following three conditions holds:*

- (i) *there is a bijection $A \rightarrow B$,*
- (ii) *there is an injection $A \rightarrow B$ and a surjection $B \rightarrow A$,*
- (iii) *there is an injection $B \rightarrow A$ and a surjection $A \rightarrow B$.*

• **Supplementary Exercise 11.5 p. 73.** Let X be a set; let \mathcal{A} be the collection of all pairs $(A, <)$, where A is a subset of X and $<$ is a well-ordering of A . Define

$$(A, <) \prec (A', <')$$

if $(A, <)$ equals a section of $(A', <')$.

- (a) Show that \prec is a strict partial order on \mathcal{A} .
- (b) Let \mathcal{B} be a subcollection of \mathcal{A} that is simply ordered by \prec . Define B' to be the union of the sets B , for all $(B, <) \in \mathcal{B}$; and define $<'$ to be the union of the relations $<$, for all $(B, <) \in \mathcal{B}$. Show that $(B', <')$ is a well-ordered set.

Solution. Left to the reader.

• **Supplementary Exercise 11.6 p. 73.** Use Exercises 1 and 5 to prove the following:

Theorem. The maximum principle is equivalent to the well-ordering theorem.

Solution. The fact that the well-ordering theorem implies the maximum principle is proved on p. 70 of the book. Let us prove the converse. In the setting of Supplementary Exercise 11.5b, take \mathcal{B} maximal. Then it suffices to show that $B' = X$. If it was not so, we could add to B' a new element x and make it the largest element of $B' \cup \{x\}$, which would then be a well-ordered set larger than B' , contradiction.

• **Supplementary Exercise 11.7 p. 73.** Use Exercises 1–5 to prove the following:

Theorem. The choice axiom is equivalent to the well-ordering theorem.

Proof. Let X be a set; let c be a fixed choice function for the nonempty subsets of X . If T is a subset of X and $<$ is a relation on T , we say that $(T, <)$ is a tower in X if $<$ is a well-ordering of T and if for each $x \in T$,

$$x = c(X - S_x(T)),$$

where $S_x(T)$ is the section of T by x .

- (a) Let $(T_1, <_1)$ and $(T_2, <_2)$ be two towers in X . Show that either these two ordered sets are the same, or one equals a section of the other. [Hint: Switching indices if necessary, we can assume that $h : T_1 \rightarrow T_2$ is order preserving and $h(T_1)$ equals either T_2 or a section of T_2 . Use Exercise 2 to show that $h(x) = x$ for all x .]
- (b) If $(T, <)$ is a tower in X and $T \neq X$, show there is a tower in X of which $(T, <)$ is a section.
- (c) Let $\{(T_k, <_k) | k \in K\}$ be the collection of all towers in X . Let

$$T = \bigcup_{k \in K} T_k \text{ and } < = \bigcup_{k \in K} (<_k).$$

Show that $(T, <)$ is a tower in X . Conclude that $T = X$.

Solution. (a) The map h in the hint exists by Supplementary Exercise 4a. Let us show $h(x) = x$ for all x thanks to Supplementary Exercise 2. Assume by contradiction that we have $h(x) \neq x$ for some x in T_1 , which we can suppose to be minimum for this condition. The map h induces an isomorphism $T_1 \simeq h(T_1)$, implying $h(T_{1, <_1 y}) = T_{2, h(y)}$ for all y in T_1 . By the choice of x we get $T_{1, <_1 x} = T_{2, <_2 h(x)}$. Since T_1 and T_2 are towers, this entails

$$h(x) = c(X \setminus T_{2, <_2 h(x)}) = c(X \setminus T_{1, <_1 x}) = x,$$

contradiction. The fact that $h(x) = x$ for all x in T_1 implies, by Supplementary Exercise 11.2a, that T_1 is contained and downward closed in T_2 .

- (b) Add $c(X \setminus T)$ to T , and make it the largest element.
- (c) Left to the reader.

This shows that the choice axiom implies the well-ordering theorem. The converse is clear.

• **Solution to Exercise 13.6 p. 83.** We must show that the topologies \mathcal{T}_ℓ and \mathcal{T}_K are incomparable.

Claim: $[2, 3] \notin \mathcal{T}_K$. Proof. If not we would have $2 \in (a, b) \setminus K \subset [2, 3)$ for some a and b , hence $a < 2$ and $a \leq 2$, contradiction.

Claim: $(-1, 1) \setminus K \notin \mathcal{T}_\ell$. Proof. If not we would have $0 \in [a, b) \subset (-1, 1) \setminus K \subset [2, 3)$ for some a and b , hence $a \leq 0 < b$, hence $a < \frac{1}{n} < b$ for some n , contradiction.

• **Solution to Exercise 13.7 p. 83.** Let us use the following notation:

\mathcal{T}_s := standard topology,

\mathcal{T}_K := topology of \mathbb{R}_K ,

\mathcal{T}_{fc} := finite complement topology,

\mathcal{T}_u := upper limit topology (having the sets $(a, b]$ as basis),

\mathcal{T}_∞ := topology having the sets $(-\infty, a)$ as basis.

We denote the corresponding topological spaces by $\mathbb{R}_s, \mathbb{R}_K, \mathbb{R}_{fc}, \mathbb{R}_u$ and \mathbb{R}_∞ . Finally we write $\mathcal{B}_s, \mathcal{B}_K, \mathcal{B}_u$ and \mathcal{B}_∞ for the obvious bases.

The inclusions between these five topologies on \mathbb{R} can be summarized by the diagram

$$\begin{array}{ccc} & u & \\ & K & \\ & s & \\ fc & & \infty, \end{array}$$

where “ i below j ” means “ $\mathcal{T}_i \subsetneq \mathcal{T}_j$ ”², and “ i and j on the same level” means “ \mathcal{T}_i and \mathcal{T}_j are incomparable”.

Preliminary comments: It is easy to see that the elements of \mathcal{T}_∞ are \emptyset , the intervals $(-\infty, a)$, and \mathbb{R} , and to observe that $\mathcal{T}_\infty \cap \mathcal{T}_{fc} = \{\emptyset, \mathbb{R}\}$. It is also easy to compare the standard topology \mathcal{T}_s to the others: the elements of \mathcal{T}_{fc} and \mathcal{T}_∞ are clearly open in \mathbb{R}_s , and it is plain that the intervals (a, b) (which are the elements on \mathcal{B}_s) are open in \mathbb{R}_K and in \mathbb{R}_∞ (note that $(a, b) = \bigcup_{d < b} (a, d]$). Clearly, $(-1, 1) \setminus K \in \mathcal{T}_K$ and $(a, b] \in \mathcal{T}_u$ are not open in \mathbb{R}_s . Moreover $(2, 3]$ is in \mathcal{T}_u but not in \mathcal{T}_K . So, it only remains to prove $\mathcal{T}_K \subset \mathcal{T}_u$.

Let x be in $(a, b) \setminus K$. It suffices to show that there is a c such that $x \in (c, x] \subset (a, b) \setminus K$. If $x \leq 0$ we set $c := a$. If $\frac{1}{n+1} < x < \frac{1}{n}$ we set $c := \frac{1}{n+1}$. If $x > 1$ we set $c := \max(1, a)$.

²I denote inclusion by \subset and proper inclusion by \subsetneq . I know that, in some sense, it would be more coherent to use \subseteq for inclusion, but I prefer to do it that way, and hope the reader will not be confused.