A few comments about "Topology" by Munkres

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As the title indicates, we make a comments about the book **Topology** by James R. Munkres. This is a work in progress. The last version of this text is available here: `https://www.overleaf.com/read/kdwwjvqjrzwb#9fe3a6`. Another version is available here: `https://github.com/Pierre-Yves-Gaillard/About-Topology-by-Munkres`.

• **Definition of $\mathbb{R}$ p. 31.** The object $\mathbb{R}$ is defined by assuming that there exists a set $\mathbb{R}$ having certain properties. We take this assumption for granted. Then it is easy to see that there are several sets having these properties. So, let $\mathbb{R}'$ be a set having the same properties as $\mathbb{R}$. Let $\mathbb{Z}'_+, \mathbb{Z}'$ and $\mathbb{Q}'$ be to $\mathbb{R}'$ what $\mathbb{Z}_+, \mathbb{Z}$ and $\mathbb{Q}$ are to $\mathbb{R}$.

Theorem 1. *There is a unique morphism of fields from $f : \mathbb{R} \to \mathbb{R}'$. This morphism is an isomorphism of ordered fields, and it induces isomorphisms $\mathbb{Z}_+ \to \mathbb{Z}'_+, \mathbb{Z} \to \mathbb{Z}'$ and $\mathbb{Q} \to \mathbb{Q}'$.*

Lemma 2. *There is a unique map $g : \mathbb{Z}_+ \to \mathbb{Z}'_+$ such that $g(0) = 0$ and $g(n+1) = g(n)+1$ for all n in $\mathbb{Z}_+$. Similarly, there is a unique map $h : \mathbb{Z}'_+ \to \mathbb{Z}_+$ such that $h(0) = 0$ and $h(n+1) = h(n)+1$ for all n in $\mathbb{Z}'_+$.*

Proof. For $i \in \mathbb{Z}_+$ and $\varphi : \{1, \dots, i\} \to \mathbb{Z}'_+$ define $\rho(\varphi) \in \mathbb{Z}'_+$ by $\rho(\varphi) := \varphi(i) + 1$. Then the first statement follows from the Principle of Recursive Definition (Theorem 4 p. 3). The proof of the second statement is similar. □

Proof of Theorem 1. In the notation of Lemma 2, set $u := h \circ g$. Then $u : \mathbb{Z}_+ \to \mathbb{Z}_+$ satisfies $u(0) = 0$ and $u(n+1) = u(n) + 1$ for all n in $\mathbb{Z}_+$. One can easily prove that $u(n) = n$ by induction. The same argument works for $g \circ h$. This shows that $g : \mathbb{Z}_+ \to \mathbb{Z}'_+$ and $h : \mathbb{Z}'_+ \to \mathbb{Z}_+$ are inverse isomorphisms. Then we extend g to morphisms $\mathbb{Z} \to \mathbb{Z}'$ and $\mathbb{Q} \to \mathbb{Q}'$, and similarly for h, and, arguing as before, we show that these morphisms are isomorphisms. More precisely, we see that there is a unique morphism $\mathbb{Z} \to \mathbb{Z}'$ extending g, and that this morphism is an isomorphism, and similarly for the morphism $\mathbb{Q} \to \mathbb{Q}'$. So we can make the identifications $\mathbb{Z}_+ = \mathbb{Z}'_+, \mathbb{Z} = \mathbb{Z}', \mathbb{Q} = \mathbb{Q}'$. To show that there is a unique morphism of fields $\mathbb{R} \to \mathbb{R}'$, and that this morphism is an isomorphism (inducing the identity of $\mathbb{Q}$), we argue as in Section *Appendix to Chapter 1* in *A few comments about "Principles of Mathematical Analysis" by Rudin*, available at `https://zenodo.org/records/13955297`. □

Theorem 7.8. p. 50 of the book. Recall the statement:

Theorem 3 (Theorem 7.8. p. 50 of the book)**.** *Let A be a set. There is no injective map $f : \mathcal{P}(A) \to A$, and there is no surjective map $g : A \to \mathcal{P}(A)$.*

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Here is my favourite way of phrasing the argument showing that there is no surjective map $g : A \to \mathcal{P}(A)$. Let $g : A \to \mathcal{P}(A)$ be a map, and set $B := \{a \mid a \notin g(a)\}$, so that we have, for all a in A,

$$a \in B \iff a \notin g(a).$$

Let a_0 be in A. If we had $g(a_0) = B$, we would get, for all a in A,

$$a \in g(a_0) \iff a \notin g(a),$$

and we immediately that setting $a := a_0$ yields a contradiction. This shows that B is not in the range of g.

• **Exercise 7.6. p. 51 of the book.** We say that two sets A and B **have the same cardinality** if there is a bijection of A with B.

(a) Show that if $B \subset A$ and if there is an injection

$$f : A \to B,$$

then A and B have the same cardinality. [Hint: Define $A_1 = A, B_1 = B$, and for $n > 1$, $A_n = f(A_{n-1})$ and $B_n = f(B_{n-1})$. (Recursive definition again!) Note that $A_1 \supset B_1 \supset A_2 \supset B_2 \supset A_3 \supset \cdots$ Define a bijection $h : A \to B$ by the rule

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_n \setminus B_n \text{ for some } n, \\ x & \text{otherwise.]} \end{cases}$$

(b) *Theorem (Schroeder-Bernstein theorem).* If there are injections $f : A \to C$ and $g : C \to A$, then A and C have the same cardinality.

Solution. (a) We will freely use the following two obvious facts:

(F1) For $x \in A$ and $n \in \mathbb{Z}_+$ we have

$$x \in A_n \iff f(x) \in A_{n+1} \text{ and } x \in B_n \iff f(x) \in B_{n+1}.$$

(F2) We have $\bigcap_{n \geq 1} A_n = \bigcap_{\geq 1} B_n =: I$.

Setting $A'_n := A_n \setminus B_n, B'_n := B_n \setminus A_{n+1}$, we get

$$A = \left(\bigcup_{n\geq 1} A'_n\right) \cup \left(\bigcup_{n\geq 1} B'_n\right) \cup I,$$

and this union is disjoint. We also have

$$B = \left(\bigcup_{n\geq 2} A'_n\right) \cup \left(\bigcup_{n\geq 1} B'_n\right) \cup I.$$

The injection f induces bijections $f_n : A'_n \to A'_{n+1}$ (here we are using (F1)). To define a bijection $h : A \to B$, it suffices to define three bijections

$$u : \bigcup_{n\geq 1} A'_n \to \bigcup_{n\geq 2} A'_n, \quad v : \bigcup_{n\geq 1} B'_n \to \bigcup_{n\geq 1} B'_n, \quad w : I \to I.$$

We define u by $u(x) = f_n(x)$ if $x \in A'_n$, and take v and w to be the identity maps.

(b) We set $B := g(C) \subset A$ and define $f' : A \to B$ by $f'(a) := g(f(a))$. Then $f' : A \to B$ satisfies the assumptions for $f : A \to B$ in (a).

• **Exercise 8.7. p. 56 of the book.** Prove Theorem 8.4 p. 54.

Solution. Recall the statement of Theorem 8.4.

Theorem 4 (Principle of Recursive Definition, Theorem 8.4 of the book)**.** *Let A be a set; let a_0 be an element of A. Suppose ρ is a function that assigns, to each function f mapping a nonempty section of the positive integers into A, an element of A. Then there exists a unique function*

$$h : \mathbb{Z}^+ \to A$$

such that

$$\begin{aligned} h(1) &= a_0, \\ h(i) &= \rho(h|\{1, \dots, i-1\}) \text{ for } i > 1. \end{aligned} \tag{$*$}$$

The formula $(*)$ is called a recursion formula for h. It specifies $h(1)$, and it expresses the value of h at $i > 1$ in terms of the values of h for positive integers less than i.

The book gives a detailed proof of the particular case when $\rho(h|\{1, \dots, i-1\})$ is equal to $\min(C \setminus h(\{1, \dots, i-1\}))$, where "min" means "*minimum*", and C is an infinite set. A close inspection of this proof reveals that the sole property of the element c of C defined by the equality $c := \min(C \setminus h(\{1, \dots, i-1\}))$ is that it depends only on the restriction $h|\{1, \dots, i-1\}$. This implies that, if, in the proof given by the book, we replace "$\min(C \setminus h(\{1, \dots, i-1\}))$" with "$\rho(h|\{1, \dots, i-1\})$", then we obtain a proof of Theorem 4.

• **Exercise 10.7 p. 67.** Let J be a well-ordered set. A subset J_0 of J is said to be **inductive** if for every $\alpha \in J$,

$$(S_\alpha \subset J_0) \Rightarrow \alpha \in J_0.$$

Theorem 5 (The Principle of Transfinite Induction)**.** *If J is a well-ordered set and J_0 is an inductive subset of J, then $J_0 = J$.*

Solution. If $J_0 \neq J$, let α be the least element of $J \setminus J_0$. We get $S_\alpha \subset J_0$, and thus $\alpha \in J_0$, contradiction.

• **Exercise 10.10 p. 67 of the book.** Prove the following Theorem:

Theorem. Let J and C be well-ordered sets; assume that there is no surjective function mapping a section of J onto C. Then there exists a unique function $h : J \to C$ satisfying the equation

$$h(x) = \min(C \setminus h(S_x)) \tag{$*$}$$

for each $x \in J$, where S_x is the section of J by x.

Solution.

(a) If h and k map sections of J, or all of J, into C and satisfy $(*)$ for all x in their respective domains, show that $h(x) = k(x)$ for all x in both domains.

(b) If there exists a function $h : S_\alpha \to C$ satisfying $(*)$, show that there exists a function $k : S_\alpha \cup \{\alpha\} \to C$ satisfying $(*)$.

(c) If $K \subset J$ and for all $\alpha \in K$ there exists a function $h_\alpha : S_\alpha \to C$ satisfying $(*)$, show that there exists a function

$$k : \bigcup_{\alpha \in K} S_\alpha \to C$$

satisfying $(*)$.

(d) Show by transfinite induction that for every $\beta \in J$, there exists a function $h_\beta : S_\beta \to C$ satisfying $(*)$. [Hint: If β has an immediate predecessor α, then $S_\beta = S_\alpha \cup \{\alpha\}$. If not, S_β is the union of all S_α with $\alpha < \beta$.]

(e) Prove the theorem.

Solution.

(a) Otherwise there would be a least x such that $h(x) \neq k(x)$, we would get $h(S_x) = k(S_x)$, and $(*)$ would yield a contradiction.

(b) We define k by $k(x) = h(x)$ if $x < \alpha$ and $k(x) = \min(C \setminus h(S_x))$ if $x = \alpha$, and verify that k satisfies $(*)$.

(c) Set $k(x) = h_\alpha(x)$ if $x \in S_\alpha$. To show that $k(x)$ is well defined, we must check that $\beta > \alpha$ implies $h_\beta(x) = h_\alpha(x)$. But this follows from (a).

(d) Let I be the set of all $\beta \in J$ such that there is a map $h_\beta : S_\beta \to C$ satisfying $(*)$. It suffices to show that I is inductive. So, assume that β is in J and that $S_\beta \subset I$. We must show $\beta \in I$. To do that, we use (b) if β has an immediate predecessor, and we use (c) if not.

(e) We define h by

$$h(x) = \begin{cases} \min(C \setminus h_x(S_x)) & \text{if } x = \max(J) \\ h_{x+1}(x) & \text{if } x \neq \max(J), \end{cases}$$

where "$x \neq \max(J)$" means "$x \neq \max(J)$ if J has a maximum", and $x + 1$ is the least element greater than x. Let us show that h satisfies $(*)$, that is, $h(x) = \min(C \setminus h(S_x))$. We can assume $x \neq \max(J)$ (in the above sense). We must show $h_{x+1}(x) = \min(C \setminus h(S_x))$. Since we have $h_{x+1}(x) = \min(C \setminus h_{x+1}(S_x))$ by (d) it suffices to prove $h(S_x) = h_{x+1}(S_x)$. Let y be in S_x, that is, $y \in J$ and $y < x$. It is enough to verify $h(y) = h_{x+1}(y)$, that is, $h_{y+1}(y) = h_{x+1}(y)$. We have $y + 1 < x + 1$, and thus $S_{y+1} \subset S_{x+1}$, and (a) implies $h_{x+1}|S_{y+1} = h_{y+1}$. This proves $h_{y+1}(y) = h_{x+1}(y)$, which is what we wanted.

• **Supplementary Exercise 11.1 p. 72 of the book.**

Theorem 6 (General principle of recursive definition)**.** *Let J be a well-ordered set; let C be a set. Let $\mathcal{F}$ be the set of all functions mapping sections of J into C. Given a function $\rho : \mathcal{F} \to C$, there exists a unique function $h : J \to C$ such that $h(\alpha) = \rho(h|S_\alpha)$ for each $\alpha \in J$.*

[Hint: Follow the pattern outlined in Exercise 10 of §10.]

Solution. A close inspection of the solution to Exercise 10 of §10 reveals that the sole property of the element c of C defined by the equality $c := \min(C \setminus h(S_x))$ is that it depends only on the restriction $h|S_x$. This implies that, if, in the proof given by the book, we replace "$\min(C \setminus h(S_x))$" with "$\rho(h|S_x)$", then we obtain a proof of Theorem 6.

Here is a slightly different way of proving the General Principle of Recursive Definition. We state and prove Theorem 8 below, which we call General Principle of Transfinite Induction, and which generalizes both the usual Principle of Transfinite Induction (Exercise 10.7 p. 67 of the book and Theorem 5 p. 3. above) and the General Principle of Recursive Definition (Supplementary Exercise 1 p. 72 of the book and Theorem 6 above).

For each ordered set X and each x in X we denote the subset $\{y \in X \mid y < x\}$ by $X(x)$. (This is the so-called *section by x*.) Let X be a well-ordered set, let A be a set, let

$$\rho : \bigcup_{x \in X} A^{X(x)} \to A,$$

where $A^{X(x)}$ stands for the set of all maps from $X(x)$ to A. (Note that the sets $A^{X(x)}$ with x in X are disjoint.)

Theorem 7 (General Principle of Recursive Definition)**.** *There is a unique map $f : X \to A$ such that $f(x) = \rho(f|X(x))$ for all x in X.*

The main ingredient to prove Theorem 7 is

Theorem 8 (General Principle of Transfinite Induction)**.** *If $P(X)$ is a property that a well-ordered set X may or may not have, and if $P(X)$ holds whenever $P(X(x))$ holds for all x in X, then $P(X)$ holds for all well-ordered set X.*

Before proving Theorem 8 recall the usual Principle of Transfinite Induction:

Theorem 9 (Principle of Transfinite Induction)**.** *Let U be a well-ordered set. If $Q(u)$ is a property that an element u of U may or may not have, and if $Q(u)$ holds whenever $Q(v)$ holds for all $v < u$, then $Q(u)$ holds for all u in U.*

This is Exercise 10.7 p. 67 of the book.

Proof of Theorem 8. Let X be a well-ordered set. We assume that $P(X)$ holds whenever $P(X(x))$ holds for all x in X, and we want to prove $P(X)$. Let $\mathcal{D}$ be the set of all downward closed subsets of X. Then we have $\mathcal{D} = \{X(x) \mid x \in X\} \cup \{X\}$, and $X(x) \subsetneq X(y)$ if and only if $x < y$, and $X(x) \subsetneq X$ for all x, and $\mathcal{D}$ is well-ordered by proper inclusion. We want to apply the Principle of Transfinite Induction (Theorem 9). To this end we set $U := \mathcal{D}$ and, for $D \in \mathcal{D}$ we define $Q(D)$ as being $P(D)$. Then Theorem 9 tells us that $P(D)$ holds for all D in $\mathcal{D}$, and thus in particular for $D = X \in \mathcal{D}$. □

Proof of Theorem 7. We denote the statement of Theorem 7 by $P(X)$, and we want to apply Theorem 8. So we assume that $P(X(x))$ holds for all x in X. It suffices to prove $P(X)$.

Case 1: X has a largest element $\infty \in X$. By assumption, for all $x < \infty$ there is a unique map $f_x : X(x) \to A$ such that $f(y) = \rho(f|X(y))$ for all $y < x$. It is easy to check that each f_x is the restriction of f_∞ to $X(x)$, and that, if we define $f : X \to A$ by $f(\infty) := \rho(f_\infty)$ and $f(x) := f_\infty(x)$ if $x < \infty$, then f is the unique solution to our problem.

Case 2: $X = \bigcup_{x \in X} X(x)$. We have maps $f_x : X(x) \to A$ as above, and it is easy to check that that map $f : X \to A$ defined by $f(x) := f_{x+1}(x)$, where $x + 1$ is the successor of x, is the unique solution to our problem. □

To see why the General Principle of Transfinite Induction (Theorem 8) generalizes the Principle of Transfinite Induction (Theorem 9), note that we can define $P(X)$ in terms of the $Q(u)$ by decreeing the $P(X)$ holds if and only if

$$\Big((\exists\ u \in U)\ \big(X = U(u)\big)\Big) \implies Q(u).$$

• **Supplementary Exercise 11.2 p. 72 of the book.**

(a) Let J and E be well-ordered sets; let $h : J \to E$. Show the following two statements are equivalent:

(i) h is order preserving and its image is E or a section of E.

(ii) $h(\alpha) =$ smallest $[E - h(S_\alpha)]$ for all α.

[Hint: Show that each of these conditions implies that $h(S_\alpha)$ is a section of E; conclude that it must be the section by $h(\alpha)$.]

(b) If E is a well-ordered set, show that no section of E has the order type of E, nor do two different sections of E have the same order type. [Hint: Given J, there is at most one order-preserving map of J into E whose image is E or a section of E.]

Solution.

(a) For all $X \subset E$ set $X^c := E \setminus X$. For the sake of prudence, we change (ii) to:

(ii') $h(S_x) \neq E$ and $h(x) = \min(h(S_x)^c)$ for all x.

We want to show that (i) and (ii') are equivalent.

(i) implies (ii'). We prove $h(S_x) \neq E$ by noting that $h(S_x) = E$ we would entail $h(x) = h(y)$ for some $y < x$, contradiction. To prove $h(x) = \min(h(S_x)^c)$, assume by contradiction that we have $h(x) \neq \min(h(S_x)^c) =: e$. If $h(x) < e$, then $h(x) \notin h(S_x)^c$, that is, $h(x) \in h(S_x)$, and we reach a contradiction as above. If $e < h(x)$, then $e = h(y)$ for some $y < x$, that is, $\min(h(S_x)^c) = e = h(y) \in h(S_x)$, contradiction.

(ii') implies (i). We assume (ii'), and, in particular, that h is weakly increasing. To show that h is increasing, suppose $x < y$ and $h(x) = h(y)$ (we cannot have $h(x) > h(y)$ because h is weakly increasing). Since $h(x) = h(y) = \min(h(S_y)^c)$, we have $h(x) \in h(S_y)^c$, but $h(x) \in h(S_y)$, contradiction. Finally, $h(J)$ is downward closed because $e < h(x) = \min(h(S_x)^c)$ implies $e \in h(S_x) \subset h(J)$.

In the statement of the Exercise, the condition that J is well-ordered can be changed from an assumption to a conclusion.

(b) Let a be in E, and assume there is an isomorphism of well-ordered sets $h : S_a \to E$. It suffices to derive a contradiction. Let $i : S_a \to E$ be the inclusion. By (a) h and i satisfy the same recursion relation. By the Theorem about the General Principle of Definition by Recursion, we have $h = i$, and thus $a \in h(S_a) = i(S_a) = S_a$, contradiction.

• **Supplementary Exercise 11.3 p. 73 of the book.** Let J and E be well-ordered sets; suppose there is an order-preserving map $k : J \to E$. Using Exercises 1 and 2, show that J has the order type of E or a section of E. [Hint: Choose $e_0 \in E$. Define $h : J \to E$ by the recursion formula

$$h(\alpha) = \text{smallest } [E - h(S_\alpha)] \text{ if } h(S_\alpha) \neq E,$$

and $h(\alpha) = e_0$ otherwise. Show that $h(\alpha) \leq k(\alpha)$ for all α; conclude that $h(S_\alpha) \neq E$ for all α.]

Solution. We can assume $E \neq \varnothing$. Let e_0 be in E. Let x be in J. We define $h : J \to E$ as in the hint.

Claim 1: $h(x) \leq k(x)$ for all x.

Claim 2: $h(y) \leq k(y)$ for all y in S_x implies $h(S_x) \neq E$.

Proof of Claim 2. For all y in S_x we have $h(y) \leq k(y) < k(x)$, and in particular $k(x) \neq h(y)$. This implies $k(x) \notin h(S_x)$.

Proof of Claim 1. Assume by contradiction $h(x) > k(x)$ for some x. We can assume that x is minimum for this condition. For $y < x$ we have $h(y) \leq k(y)$, hence $h(S_x) \neq E$ by Claim 2.

Claims 1 and 2 imply $h(x) = \min(h(S_x)^c)$ for all x, hence h is increasing and $h(J)$ is downward closed by Supplementary Exercise 11.2 above, hence J has the order type of E or a section of E.

• **Supplementary Exercise 11.4 p. 73 of the book.** Use Exercises 1–3 to prove the following:

(a) If A and B are well-ordered sets, then exactly one of the following three conditions holds: A and B have the same order type, or A has the order type of a section of B, or B has the order type of a section of A. [Hint: Form a well-ordered set containing both A and B, as in Exercise 8 of §10; then apply the preceding exercise.]

(b) Suppose that A and B are well-ordered sets that are uncountable, such that every section of A and of B is countable. Show A and B have the same order type.

Solution. (a) For any element x of any ordered set X, let $X_{<x}$ denote the corresponding section, and let us set $X_{<\infty} := X$. Let C be the well-ordered set containing both A and B, described in Exercise 8 of §10, and let $k : A \to C$ and $\ell : B \to C$ be the natural increasing maps. By the previous Exercise, we have isomorphisms $A \simeq C_{<x}$ and $B \simeq C_{<y}$ for some x and y in $C \cup \{\infty\}$. We can assume $x \leq y$. Then $x = y$ implies $A \simeq B$. If $x < y$, we get

$$A \simeq C_{<x} = (C_{<y})_{<x} \subsetneq C_{<y} \simeq B.$$

This implies $A \simeq B_{<b}$ for some b in B. The fact that the various cases are exclusive follows from Supplementary Exercise 11.2b.

(b) Follows from (a).

Here is an important consequence of (a):

Theorem 10 (Comparability Theorem)**.** *If A and B are sets, then exactly one of the following three conditions holds:*

(i) *there is a bijection $A \to B$,*

(ii) *there is an injection $A \to B$ and a surjection $B \to A$,*

(iii) *there is an injection $B \to A$ and a surjection $A \to B$.*

• **Supplementary Exercise 11.5 p. 73 of the book.** Let X be a set; let $\mathcal{A}$ be the collection of all pairs $(A, <)$, where A is a subset of X and $<$ is a well-ordering of A. Define

$$(A, <) \prec (A', <')$$

if $(A, <)$ equals a section of $(A', <')$.

(a) Show that $\prec$ is a strict partial order on A.

(b) Let $\mathcal{B}$ be a subcollection of A that is simply ordered by $\prec$. Define B' to be the union of the sets B, for all $(B, <) \in \mathcal{B}$; and define $<'$ to be the union of the relations $<$, for all $(B, <) \in \mathcal{B}$. Show that $(B', <')$ is a well-ordered set.

Solution. Left to the reader.

• **Supplementary Exercise 11.6 p. 73 of the book.** Use Exercises 1 and 5 to prove the following:

Theorem. The maximum principle is equivalent to the well-ordering theorem.

Solution. The fact that the well-ordering theorem implies the maximum principle is proved on p. 70 of the book. Let us prove the converse. In the setting of Supplementary Exercise 11.5b, take $\mathcal{B}$ maximal. Then it suffices to show that $B' = X$. If it was not so, we could add to B' a new element x and make it the largest element of $B' \cup \{x\}$, which would then be a well-ordered set larger than B', contradiction.

• **Supplementary Exercise 11.7 p. 73 of the book.** Use Exercises 1–5 to prove the following:

Theorem. The choice axiom is equivalent to the well-ordering theorem.

Proof. Let X be a set; let c be a fixed choice function for the nonempty subsets of X. If T is a subset of X and $<$ is a relation on T, we say that $(T, <)$ is a tower in X if $<$ is a well-ordering of T and if for each $x \in T$,

$$x = c(X - S_x(T)),$$

where $S_x(T)$ is the section of T by x.

(a) Let $(T_1, <_1)$ and $(T_2, <_2)$ be two towers in X. Show that either these two ordered sets are the same, or one equals a section of the other. [Hint: Switching indices if necessary, we can assume that $h : T_1 \to T_2$ is order preserving and $h(T_1)$ equals either T_2 or a section of T_2. Use Exercise 2 to show that $h(x) = x$ for all x.]

(b) If $(T,<)$ is a tower in X and $T \neq X$, show there is a tower in X of which $(T,<)$ is a section.

(c) Let $\{(T_k,<_k)|k \in K\}$ be the collection of all towers in X. Let

$$T = \bigcup_{k \in K} T_k \text{ and } < = \bigcup_{k \in K} (<_k).$$

Show that $(T,<)$ is a tower in X. Conclude that $T = X$.

Solution. (a) The map h in the hint exists by Supplementary Exercise 4a. Let us show $h(x) = x$ for all x thanks to Supplementary Exercise 2. Assume by contradiction that we have $h(x) \neq x$ for some x in T_1, which we can suppose to be minimum for this condition. The map h induces an isomorphism $T_1 \simeq h(T_1)$, implying $h(T_{1,<y}) = T_{2,h(y)}$ for all y in T_1. By the choice of x we get $T_{1,<x} = T_{2,<h(x)}$. Since T_1 and T_2 are towers, this entails

$$h(x) = c(X \setminus T_{2,<h(x)}) = c(X \setminus T_{1,<x}) = x,$$

contradiction. The fact that $h(x) = x$ for all x in T_1 implies, by Supplementary Exercise 11.2a, that T_1 is contained and downward closed in T_2.

(b) Add $c(X \setminus T)$ to T, and make it the largest element.

(c) Left to the reader.

This shows that the choice axiom implies the well-ordering theorem. The converse is clear.

Lemma 13.1 p. 80 of the book. Recall the statement:

Lemma 11 (Lemma 13.1 p. 80 of the book)**.** *Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Here is the proof given in the book:

Proof. Given a collection of elements of $\mathcal{B}$, they are also elements of $\mathcal{T}$. Because $\mathcal{T}$ is a topology, their union is in $\mathcal{T}$. Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$ an element B_x of $\mathcal{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathcal{B}$. □

To avoid the Axiom of Choice, just replace $U = \bigcup_{x \in U} B_x$ with $U = \bigcup_{B \subset U} B$, where the union runs over the $B \in \mathcal{B}$ such that $B \subset U$.

• **Solution to Exercise 13.6 p. 83 of the book.** We must show that the topologies $\mathcal{T}_\ell$ and $\mathcal{T}_K$ are incomparable.

Claim: $[2,3) \notin \mathcal{T}_K$. Proof. If not we would have $2 \in (a,b) \setminus K \subset [2,3)$ for some a and b, hence $a < 2$ and $a \leq 2$, contradiction.

Claim: $(-1,1) \setminus K \notin \mathcal{T}_\ell$. Proof. If not we would have $0 \in [a,b) \subset (-1,1) \setminus K \subset [2,3)$ for some a and b, hence $a \leq 0 < b$, hence $a < \frac{1}{n} < b$ for some n, contradiction.

• **Solution to Exercise 13.7 p. 83 of the book.** Let us use the following notation:

$\mathcal{T}_s :=$ standard topology,

$\mathcal{T}_K :=$ topology of $\mathbb{R}_K$,

$\mathcal{T}_{fc} :=$ finite complement topology,

$\mathcal{T}_u :=$ upper limit topology (having the sets $(a, b]$ as basis),

$\mathcal{T}_\infty :=$ topology having the sets $(-\infty, a)$ as basis.

We denote the corresponding topological spaces by $\mathbb{R}_s, \mathbb{R}_K, \mathbb{R}_{fc}, \mathbb{R}_u$ and $\mathbb{R}_\infty$. Finally we write $\mathcal{B}_s, \mathcal{B}_K, \mathcal{B}_u$ and $\mathcal{B}_\infty$ for the obvious bases.

The inclusions between these five topologies on $\mathbb{R}$ can be summarized by the diagram

$$\begin{array}{ccc} & u & \\ & K & \\ & s & \\ fc & & \infty, \end{array}$$

where "i below j" means "$\mathcal{T}_i \subsetneq \mathcal{T}_j$"[2], and "$i$ and j on the same level" means "$\mathcal{T}_i$ and $\mathcal{T}_j$ are incomparable".

Preliminary comments: It is easy to see that the elements of $\mathcal{T}_\infty$ are $\varnothing$, the intervals $(-\infty, a)$, and $\mathbb{R}$, and to observe that $\mathcal{T}_\infty \cap \mathcal{T}_{fc} = \{\varnothing, \mathbb{R}\}$. It is also easy to compare the standard topology $\mathcal{T}_s$ to the others: the elements of $\mathcal{T}_{fc}$ and $\mathcal{T}_\infty$ are clearly open in $\mathbb{R}_s$, and it is plain that the intervals (a, b) (which are the elements on $\mathcal{B}_s$) are open in $\mathbb{R}_K$ and in $\mathbb{R}_\infty$ (note that $(a, b) = \bigcup_{d<b}(a, d]$). Clearly, $(-1, 1) \setminus K \in \mathcal{T}_K$ and $(a, b] \in \mathcal{T}_u$ are not open in $\mathbb{R}_s$. Moreover $(2, 3]$ is in $\mathcal{T}_u$ but not in $\mathcal{T}_K$. So, it only remains to prove $\mathcal{T}_K \subset \mathcal{T}_u$.

Let x be in $(a, b) \setminus K$. It suffices to show that there is a c such that $x \in (c, x] \subset (a, b) \setminus K$. If $x \leq 0$ we set $c := a$. If $\frac{1}{n+1} < x < \frac{1}{n}$ we set $c := \frac{1}{n+1}$. If $x > 1$ we set $c := \max(1, a)$.

Subspace topology p. 88 of the book. Munkres writes:

Definition. Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the **subspace topology**. With this topology, Y is called a **subspace** of X; its open sets consist of all intersections of open sets of X with Y.

It is easy to see that $\mathcal{T}_Y$ is a topology. It contains $\varnothing$ and Y because

$$\varnothing = Y \cap \varnothing \quad \text{and} \quad Y = Y \cap X,$$

where $\varnothing$ and X are elements of $\mathcal{T}$. The fact that it is closed under finite intersections and arbitrary unions follows from the equations

$$(U_1 \cap Y) \cap \cdots \cap (U_n \cap Y) = (U_1 \cap \cdots \cap U_n) \cap Y,$$

$$\bigcup_{\alpha \in J}(U_\alpha \cap Y) = \left(\bigcup_{\alpha \in J} U_\alpha\right) \cap Y.$$

[2] I denote inclusion by $\subset$ and proper inclusion by $\subsetneq$. I know that, in some sense, it would be more coherent to use $\subseteq$ for inclusion, but I prefer to do it that way, and hope the reader will not be confused.

Proof. Given a collection of elements of $\mathcal{B}$, they are also elements of $\mathcal{T}$. Because $\mathcal{T}$ is a topology, their union is in $\mathcal{T}$. Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$ an element B_x of $\mathcal{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathcal{B}$. $\square$

End of the excerpt.

The Axiom of Choice is used to handle arbitrary unions. One can avoid it by proceeding as follows.

Given $A \subset Y \subset X$, set $\mathcal{U} := \{U \in \mathcal{T} \mid U \cap Y \subset A\}$, and define the open subset U_A of X by $U_A := \bigcup_{U \in \mathcal{U}} U$. Then we have

$$A \subset U_A \iff U_A \cap Y = A.$$

Proof: We have $Y \cap U_A = Y \cap \bigcup_{U \in \mathcal{U}} U = \bigcup_{U \in \mathcal{U}} Y \cap U \subset A$, so

$$A = U_A \cap Y \iff A \subset U_A \cap Y \iff A \subset U_A. \qquad \square$$

And we decree that A is open in Y if and only if the above equivalent conditions are satisfied. This is equivalent to the usual definition. Proof: If $A = U_A \cap Y$, then A is open in Y in the usual sense. If $A = U \cap Y$ for some U open in X, then $U \in \mathcal{U}$, hence $U \subset U_A$, and we get $A = U \cap Y \subset U_A \cap Y \subset A$, hence $A = U_A \cap Y$. Observe that, in general, $U_A \cap Y$ is the **interior** of A in Y.

Note that things are even nicer if we use closed subsets instead of open ones. Indeed, given $A \subset Y \subset X$ as above, there is a least closed subset C_A of X such that $A \subset C_A \cap Y$, and C_A is the closure of A in X. What I find remarkable is that C_A depends only on A and X, but not on Y. (I don't know if there is a conceptual reason for that.) (To see that U_A depends on Y in general, let X be nonempty and let A be empty. If $Y = \varnothing$, then $U_A = X$, but if $Y = X$, then $U_A = \varnothing$.)

Theorem 17.2 p. 94 of the book. We give a slightly different proof of the indicated Theorem. First recall the statement.

Theorem 12 (Theorem 17.2 p. 94 of the book)**.** *Let X be a topological space, Y a subspace, and A a subset of Y. Then A is closed in Y if and only $A = Y \cap C$ for some closed subset of X.*

Proof. It suffices to show that $Y \setminus A = Y \cap U$, where U is an open subset of X, if and only if $A = Y \cap (X \setminus U)$. Hence it is enough to prove that we have $Y \setminus (Y \cap U) = Y \cap (X \setminus U)$, or equivalently $Y \setminus U = Y \cap (X \setminus U)$, for all subset U of X. But this is clear. $\square$

Theorem 17.4 p. 95 of the book. Recall the statement:

Theorem 13 (Theorem 17.4 p. 95 of the book)**.** *Let X be a topological space, Y a subspace, and A a subset of Y. Then A is closed in Y if and only $A = Y \cap \overline{A}$, where $\overline{A}$ is the closure of A in X.*

Here is a slightly different proof.

Proof. Let C be as in Theorem 12. We have $A \subset C$, hence $A \subset \overline{A} \subset C$, hence

$$A = Y \cap A \subset Y \cap \overline{A} \subset Y \cap C = A,$$

hence $A = Y \cap \overline{A}$. $\square$