A few comments about "Topology" by Munkres

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As the title indicates, we make a comments about the book **Topology** by James R. Munkres. This is a work in progress. The last version of this text is available here: `https://www.overleaf.com/read/kdwwjvqjrzwb#9fe3a6`. Another version is available here: `https://github.com/Pierre-Yves-Gaillard/About-Topology-by-Munkres`.

1 Chapter 1. Set Theory and Logic

• **Definition of $\mathbb{R}$ p. 31.** The object $\mathbb{R}$ is defined by assuming that there exists a set $\mathbb{R}$ having certain properties. We take this assumption for granted. Then it is easy to see that there are several sets having these properties. So, let $\mathbb{R}'$ be a set having the same properties as $\mathbb{R}$. Let $\mathbb{Z}'_+, \mathbb{Z}'$ and $\mathbb{Q}'$ be to $\mathbb{R}'$ what $\mathbb{Z}_+, \mathbb{Z}$ and $\mathbb{Q}$ are to $\mathbb{R}$.

Theorem 1. *There is a unique morphism of fields from $f : \mathbb{R} \to \mathbb{R}'$. This morphism is an isomorphism of ordered fields, and it induces isomorphisms $\mathbb{Z}_+ \to \mathbb{Z}'_+, \mathbb{Z} \to \mathbb{Z}'$ and $\mathbb{Q} \to \mathbb{Q}'$.*

Lemma 2. *There is a unique map $g : \mathbb{Z}_+ \to \mathbb{Z}'_+$ such that $g(0) = 0$ and $g(n+1) = g(n)+1$ for all n in $\mathbb{Z}_+$. Similarly, there is a unique map $h : \mathbb{Z}'_+ \to \mathbb{Z}_+$ such that $h(0) = 0$ and $h(n+1) = h(n)+1$ for all n in $\mathbb{Z}'_+$.*

Proof. For $i \in \mathbb{Z}_+$ and $\varphi : \{1, \dots, i\} \to \mathbb{Z}'_+$ define $\rho(\varphi) \in \mathbb{Z}'_+$ by $\rho(\varphi) := \varphi(i) + 1$. Then the first statement follows from the Principle of Recursive Definition (Theorem 4 p. 3). The proof of the second statement is similar. □

Proof of Theorem 1. In the notation of Lemma 2, set $u := h \circ g$. Then $u : \mathbb{Z}_+ \to \mathbb{Z}_+$ satisfies $u(0) = 0$ and $u(n + 1) = u(n) + 1$ for all n in $\mathbb{Z}_+$. One can easily prove that $u(n) = n$ by induction. The same argument works for $g \circ h$. This shows that $g : \mathbb{Z}_+ \to \mathbb{Z}'_+$ and $h : \mathbb{Z}'_+ \to \mathbb{Z}_+$ are inverse isomorphisms. Then we extend g to morphisms $\mathbb{Z} \to \mathbb{Z}'$ and $\mathbb{Q} \to \mathbb{Q}'$, and similarly for h, and, arguing as before, we show that these morphisms are isomorphisms. More precisely, we see that there is a unique morphism $\mathbb{Z} \to \mathbb{Z}'$ extending g, and that this morphism is an

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isomorphism, and similarly for the morphism $\mathbb{Q} \to \mathbb{Q}'$. So we can make the identifications $\mathbb{Z}_+ = \mathbb{Z}'_+, \mathbb{Z} = \mathbb{Z}', \mathbb{Q} = \mathbb{Q}'$. To show that there is a unique morphism of fields $\mathbb{R} \to \mathbb{R}'$, and that this morphism is an isomorphism (inducing the identity of $\mathbb{Q}$), we argue as in Section *Appendix to Chapter 1* in *A few comments about "Principles of Mathematical Analysis" by Rudin*, available at `https://zenodo.org/records/13955297`. □

Theorem 7.8. p. 50 of the book. Recall the statement:

Theorem 3 (Theorem 7.8. p. 50 of the book)**.** *Let A be a set. There is no injective map $f : \mathcal{P}(A) \to A$, and there is no surjective map $g : A \to \mathcal{P}(A)$.*

Here is my favourite way of phrasing the argument showing that there is no surjective map $g : A \to \mathcal{P}(A)$. Let $g : A \to \mathcal{P}(A)$ be a map, and set $B := \{a \mid a \notin g(a)\}$, so that we have, for all a in A,

$$a \in B \iff a \notin g(a).$$

Let a_0 be in A. If we had $g(a_0) = B$, we would get, for all a in A,

$$a \in g(a_0) \iff a \notin g(a),$$

and we immediately that setting $a := a_0$ yields a contradiction. This shows that B is not in the range of g.

• **Exercise 7.6. p. 51 of the book.** We say that two sets A and B **have the same cardinality** if there is a bijection of A with B.

(a) Show that if $B \subset A$ and if there is an injection

$$f : A \to B,$$

then A and B have the same cardinality. [Hint: Define $A_1 = A, B_1 = B$, and for $n > 1$, $A_n = f(A_{n-1})$ and $B_n = f(B_{n-1})$. (Recursive definition again!) Note that $A_1 \supset B_1 \supset A_2 \supset B_2 \supset A_3 \supset \cdots$ Define a bijection $h : A \to B$ by the rule

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_n \setminus B_n \text{ for some } n, \\ x & \text{otherwise.]} \end{cases}$$

(b) *Theorem (Schroeder-Bernstein theorem).* If there are injections $f : A \to C$ and $g : C \to A$, then A and C have the same cardinality.

Solution. (a) We will freely use the following two obvious facts:

(F1) For $x \in A$ and $n \in \mathbb{Z}_+$ we have

$$x \in A_n \iff f(x) \in A_{n+1} \text{ and } x \in B_n \iff f(x) \in B_{n+1}.$$

(F2) We have $\bigcap_{n\geq 1} A_n = \bigcap_{\geq 1} B_n =: I$.

Setting $A'_n := A_n \setminus B_n, B'_n := B_n \setminus A_{n+1}$, we get

$$A = \left(\bigcup_{n\geq 1} A'_n\right) \cup \left(\bigcup_{n\geq 1} B'_n\right) \cup I,$$

and this union is disjoint. We also have

$$B = \left(\bigcup_{n\geq 2} A'_n\right) \cup \left(\bigcup_{n\geq 1} B'_n\right) \cup I.$$

The injection f induces bijections $f_n : A'_n \to A'_{n+1}$ (here we are using (F1)). To define a bijection $h : A \to B$, it suffices to define three bijections

$$u : \bigcup_{n\geq 1} A'_n \to \bigcup_{n\geq 2} A'_n, \quad v : \bigcup_{n\geq 1} B'_n \to \bigcup_{n\geq 1} B'_n, \quad w : I \to I.$$

We define u by $u(x) = f_n(x)$ if $x \in A'_n$, and take v and w to be the identity maps.

(b) We set $B := g(C) \subset A$ and define $f' : A \to B$ by $f'(a) := g(f(a))$. Then $f' : A \to B$ satisfies the assumptions for $f : A \to B$ in (a).

• **Exercise 8.7. p. 56 of the book.** Prove Theorem 8.4 p. 54.

Solution. Recall the statement of Theorem 8.4.

Theorem 4 (Principle of Recursive Definition, Theorem 8.4 of the book)**.** *Let A be a set; let a_0 be an element of A. Suppose ρ is a function that assigns, to each function f mapping a nonempty section of the positive integers into A, an element of A. Then there exists a unique function*

$$h : \mathbb{Z}^+ \to A$$

such that

$$\begin{aligned} h(1) &= a_0, \\ h(i) &= \rho(h|\{1, \ldots, i-1\}) \text{ for } i > 1. \end{aligned} \tag{$*$}$$

The formula $(*)$ is called a recursion formula for h. It specifies $h(1)$, and it expresses the value of h at $i > 1$ in terms of the values of h for positive integers less than i.

The book gives a detailed proof of the particular case when $\rho(h|\{1, \ldots, i-1\})$ is equal to $\min(C \setminus h(\{1, \ldots, i-1\}))$, where "min" means "*minimum*", and C is an infinite set. A close inspection of this proof reveals that the sole property of the element c of C defined by the equality $c := \min(C \setminus h(\{1, \ldots, i-1\}))$ is that it depends only on the restriction $h|\{1, \ldots, i-1\}$. This implies that, if, in the proof given by the book, we replace "$\min(C \setminus h(\{1, \ldots, i-1\}))$" with "$\rho(h|\{1, \ldots, i-1\})$", then we obtain a proof of Theorem 4.

• **Exercise 10.7 p. 67.** Let J be a well-ordered set. A subset J_0 of J is said to be **inductive** if for every $\alpha \in J$,

$$(S_\alpha \subset J_0) \Rightarrow \alpha \in J_0.$$

Theorem 5 (The Principle of Transfinite Induction). *If J is a well-ordered set and J_0 is an inductive subset of J, then $J_0 = J$.*

Solution. If $J_0 \neq J$, let α be the least element of $J \setminus J_0$. We get $S_\alpha \subset J_0$, and thus $\alpha \in J_0$, contradiction.

• **Exercise 10.10 p. 67 of the book.** Prove the following Theorem:

Theorem. Let J and C be well-ordered sets; assume that there is no surjective function mapping a section of J onto C. Then there exists a unique function $h : J \to C$ satisfying the equation

$$h(x) = \min(C \setminus h(S_x)) \tag{$*$}$$

for each $x \in J$, where S_x is the section of J by x.

Solution.

(a) If h and k map sections of J, or all of J, into C and satisfy $(*)$ for all x in their respective domains, show that $h(x) = k(x)$ for all x in both domains.

(b) If there exists a function $h : S_\alpha \to C$ satisfying $(*)$, show that there exists a function $k : S_\alpha \cup \{\alpha\} \to C$ satisfying $(*)$.

(c) If $K \subset J$ and for all $\alpha \in K$ there exists a function $h_\alpha : S_\alpha \to C$ satisfying $(*)$, show that there exists a function

$$k : \bigcup_{\alpha \in K} S_\alpha \to C$$

satisfying $(*)$.

(d) Show by transfinite induction that for every $\beta \in J$, there exists a function $h_\beta : S_\beta \to C$ satisfying $(*)$. [Hint: If β has an immediate predecessor α, then $S_\beta = S_\alpha \cup \{\alpha\}$. If not, S_β is the union of all S_α with $\alpha < \beta$.]

(e) Prove the theorem.

Solution.

(a) Otherwise there would be a least x such that $h(x) \neq k(x)$, we would get $h(S_x) = k(S_x)$, and $(*)$ would yield a contradiction.

(b) We define k by $k(x) = h(x)$ if $x < \alpha$ and $k(x) = \min(C \setminus h(S_x))$ if $x = \alpha$, and verify that k satisfies $(*)$.

(c) Set $k(x) = h_\alpha(x)$ if $x \in S_\alpha$. To show that $k(x)$ is well defined, we must check that $\beta > \alpha$ implies $h_\beta(x) = h_\alpha(x)$. But this follows from (a).

(d) Let I be the set of all $\beta \in J$ such that there is a map $h_\beta : S_\beta \to C$ satisfying $(*)$. It suffices to show that I is inductive. So, assume that β is in J and that $S_\beta \subset I$. We must show $\beta \in I$. To do that, we use (b) if β has an immediate predecessor, and we use (c) if not.

(e) We define h by

$$h(x)=\begin{cases}\min(C\setminus h_x(S_x)) & \text{if } x=\max(J)\\ h_{x+1}(x) & \text{if } x\neq\max(J),\end{cases}$$

where "$x \neq \max(J)$" means "$x \neq \max(J)$ if J has a maximum", and $x+1$ is the least element greater than x. Let us show that h satisfies $(*)$, that is, $h(x)=\min(C\setminus h(S_x))$. We can assume $x \neq \max(J)$ (in the above sense). We must show $h_{x+1}(x)=\min(C\setminus h(S_x))$. Since we have $h_{x+1}(x)=\min(C\setminus h_{x+1}(S_x))$ by (d) it suffices to prove $h(S_x)=h_{x+1}(S_x)$. Let y be in S_x, that is, $y\in J$ and $y<x$. It is enough to verify $h(y)=h_{x+1}(y)$, that is, $h_{y+1}(y)=h_{x+1}(y)$. We have $y+1<x+1$, and thus $S_{y+1}\subset S_{x+1}$, and (a) implies $h_{x+1}|S_{y+1}=h_{y+1}$. This proves $h_{y+1}(y)=h_{x+1}(y)$, which is what we wanted.

• Supplementary Exercise 11.1 p. 72 of the book.

Theorem 6 (General principle of recursive definition)**.** *Let J be a well-ordered set; let C be a set. Let $\mathcal{F}$ be the set of all functions mapping sections of J into C. Given a function $\rho:\mathcal{F}\to C$, there exists a unique function $h:J\to C$ such that $h(\alpha)=\rho(h|S_\alpha)$ for each $\alpha\in J$.*

[Hint: Follow the pattern outlined in Exercise 10 of §10.]

Solution. A close inspection of the solution to Exercise 10 of §10 reveals that the sole property of the element c of C defined by the equality $c:=\min(C\setminus h(S_x))$ is that it depends only on the restriction $h|S_x$. This implies that, if, in the proof given by the book, we replace "$\min(C\setminus h(S_x))$" with "$\rho(h|S_x)$", then we obtain a proof of Theorem 6.

Here is a slightly different way of proving the General Principle of Recursive Definition. We state and prove Theorem 8 below, which we call General Principle of Transfinite Induction, and which generalizes both the usual Principle of Transfinite Induction (Exercise 10.7 p. 67 of the book and Theorem 5 p. 4. above) and the General Principle of Recursive Definition (Supplementary Exercise 1 p. 72 of the book and Theorem 6 above).

For each ordered set X and each x in X we denote the subset $\{y\in X \mid y<x\}$ by $X(x)$. (This is the so-called *section by x*.) Let X be a well-ordered set, let A be a set, let

$$\rho:\bigcup_{x\in X}A^{X(x)}\to A,$$

where $A^{X(x)}$ stands for the set of all maps from $X(x)$ to A. (Note that the sets $A^{X(x)}$ with x in X are disjoint.)

Theorem 7 (General Principle of Recursive Definition)**.** *There is a unique map $f:X\to A$ such that $f(x)=\rho(f|X(x))$ for all x in X.*

The main ingredient to prove Theorem 7 is

Theorem 8 (General Principle of Transfinite Induction)**.** *If $P(X)$ is a property that a well-ordered set X may or may not have, and if $P(X)$ holds whenever $P(X(x))$ holds for all x in X, then $P(X)$ holds for all well-ordered set X.*

Before proving Theorem 8 recall the usual Principle of Transfinite Induction:

Theorem 9 (Principle of Transfinite Induction)**.** *Let U be a well-ordered set. If $Q(u)$ is a property that an element u of U may or may not have, and if $Q(u)$ holds whenever $Q(v)$ holds for all $v < u$, then $Q(u)$ holds for all u in U.*

This is Exercise 10.7 p. 67 of the book.

Proof of Theorem 8. Let X be a well-ordered set. We assume that $P(X)$ holds whenever $P(X(x))$ holds for all x in X, and we want to prove $P(X)$. Let $\mathcal{D}$ be the set of all downward closed subsets of X. Then we have $\mathcal{D} = \{X(x) \mid x \in X\} \cup \{X\}$, and $X(x) \subsetneqq X(y)$ if and only if $x < y$, and $X(x) \subsetneqq X$ for all x, and $\mathcal{D}$ is well-ordered by proper inclusion. We want to apply the Principle of Transfinite Induction (Theorem 9). To this end we set $U := \mathcal{D}$ and, for $D \in \mathcal{D}$ we define $Q(D)$ as being $P(D)$. Then Theorem 9 tells us that $P(D)$ holds for all D in $\mathcal{D}$, and thus in particular for $D = X \in \mathcal{D}$. $\square$

Proof of Theorem 7. We denote the statement of Theorem 7 by $P(X)$, and we want to apply Theorem 8. So we assume that $P(X(x))$ holds for all x in X. It suffices to prove $P(X)$.

Case 1: X has a largest element $\infty \in X$. By assumption, for all $x < \infty$ there is a unique map $f_x : X(x) \to A$ such that $f(y) = \rho(f|X(y))$ for all $y < x$. It is easy to check that each f_x is the restriction of f_∞ to $X(x)$, and that, if we define $f : X \to A$ by $f(\infty) := \rho(f_\infty)$ and $f(x) := f_\infty(x)$ if $x < \infty$, then f is the unique solution to our problem.

Case 2: $X = \bigcup_{x \in X} X(x)$. We have maps $f_x : X(x) \to A$ as above, and it is easy to check that that map $f : X \to A$ defined by $f(x) := f_{x+1}(x)$, where $x + 1$ is the successor of x, is the unique solution to our problem. $\square$

To see why the General Principle of Transfinite Induction (Theorem 8) generalizes the Principle of Transfinite Induction (Theorem 9), note that we can define $P(X)$ in terms of the $Q(u)$ by decreeing the $P(X)$ holds if and only if

$$\Big((\exists\ u \in U)\ \big(X = U(u)\big)\Big) \implies Q(u).$$

• Supplementary Exercise 11.2 p. 72 of the book.

(a) Let J and E be well-ordered sets; let $h : J \to E$. Show the following two statements are equivalent:

(i) h is order preserving and its image is E or a section of E.

(ii) $h(\alpha) =$ smallest $[E - h(S_\alpha)]$ for all α.

[Hint: Show that each of these conditions implies that $h(S_\alpha)$ is a section of E; conclude that it must be the section by $h(\alpha)$.]

(b) If E is a well-ordered set, show that no section of E has the order type of E, nor do two different sections of E have the same order type. [Hint: Given J, there is at most one order-preserving map of J into E whose image is E or a section of E.]

Solution.

(a) For all $X \subset E$ set $X^c := E \setminus X$. For the sake of prudence, we change (ii) to:

(ii') $h(S_x) \neq E$ and $h(x) = \min(h(S_x)^c)$ for all x.

We want to show that (i) and (ii') are equivalent.

(i) implies (ii'). We prove $h(S_x) \neq E$ by noting that $h(S_x) = E$ we would entail $h(x) = h(y)$ for some $y < x$, contradiction. To prove $h(x) = \min(h(S_x)^c)$, assume by contradiction that we have $h(x) \neq \min(h(S_x)^c) =: e$. If $h(x) < e$, then $h(x) \notin h(S_x)^c$, that is, $h(x) \in h(S_x)$, and we reach a contradiction as above. If $e < h(x)$, then $e = h(y)$ for some $y < x$, that is, $\min(h(S_x)^c) = e = h(y) \in h(S_x)$, contradiction.

(ii') implies (i). We assume (ii'), and, in particular, that h is weakly increasing. To show that h is increasing, suppose $x < y$ and $h(x) = h(y)$ (we cannot have $h(x) > h(y)$ because h is weakly increasing). Since $h(x) = h(y) = \min(h(S_y)^c)$, we have $h(x) \in h(S_y)^c$, but $h(x) \in h(S_y)$, contradiction. Finally, $h(J)$ is downward closed because $e < h(x) = \min(h(S_x)^c)$ implies $e \in h(S_x) \subset h(J)$.

In the statement of the Exercise, the condition that J is well-ordered can be changed from an assumption to a conclusion.

(b) Let a be in E, and assume there is an isomorphism of well-ordered sets $h : S_a \to E$. It suffices to derive a contradiction. Let $i : S_a \to E$ be the inclusion. By (a) h and i satisfy the same recursion relation. By the Theorem about the General Principle of Definition by Recursion, we have $h = i$, and thus $a \in h(S_a) = i(S_a) = S_a$, contradiction.

• **Supplementary Exercise 11.3 p. 73 of the book.** Let J and E be well-ordered sets; suppose there is an order-preserving map $k : J \to E$. Using Exercises 1 and 2, show that J has the order type of E or a section of E. [Hint: Choose $e_0 \in E$. Define $h : J \to E$ by the recursion formula

$$h(\alpha) = \text{smallest } [E - h(S_\alpha)] \text{ if } h(S_\alpha) \neq E,$$

and $h(\alpha) = e_0$ otherwise. Show that $h(\alpha) \leq k(\alpha)$ for all α; conclude that $h(S_\alpha) \neq E$ for all α.]

Solution. We can assume $E \neq \varnothing$. Let e_0 be in E. Let x be in J. We define $h : J \to E$ as in the hint.

Claim 1: $h(x) \leq k(x)$ for all x.

Claim 2: $h(y) \leq k(y)$ for all y in S_x implies $h(S_x) \neq E$.

Proof of Claim 2. For all y in S_x we have $h(y) \leq k(y) < k(x)$, and in particular $k(x) \neq h(y)$. This implies $k(x) \notin h(S_x)$.

Proof of Claim 1. Assume by contradiction $h(x) > k(x)$ for some x. We can assume that x is minimum for this condition. For $y < x$ we have $h(y) \leq k(y)$, hence $h(S_x) \neq E$ by Claim 2.

Claims 1 and 2 imply $h(x) = \min(h(S_x)^c)$ for all x, hence h is increasing and $h(J)$ is downward closed by Supplementary Exercise 11.2 above, hence J has the order type of E or a section of E.

• **Supplementary Exercise 11.4 p. 73 of the book.** Use Exercises 1–3 to prove the following:

(a) If A and B are well-ordered sets, then exactly one of the following three conditions holds: A and B have the same order type, or A has the order type of a section of B, or B has the order type of a section of A. [Hint: Form a well-ordered set containing both A and B, as in Exercise 8 of §10; then apply the preceding exercise.]

(b) Suppose that A and B are well-ordered sets that are uncountable, such that every section of A and of B is countable. Show A and B have the same order type.

Solution. (a) For any element x of any ordered set X, let $X_{<x}$ denote the corresponding section, and let us set $X_{<\infty} := X$. Let C be the well-ordered set containing both A and B, described in Exercise 8 of §10, and let $k : A \to C$ and $\ell : B \to C$ be the natural increasing maps. By the previous Exercise, we have isomorphisms $A \simeq C_{<x}$ and $B \simeq C_{<y}$ for some x and y in $C \cup \{\infty\}$. We can assume $x \leq y$. Then $x = y$ implies $A \simeq B$. If $x < y$, we get

$$A \simeq C_{<x} = (C_{<y})_{<x} \subsetneqq C_{<y} \simeq B.$$

This implies $A \simeq B_{<b}$ for some b in B. The fact that the various cases are exclusive follows from Supplementary Exercise 11.2b.

(b) Follows from (a).

Here is an important consequence of (a):

Theorem 10 (Comparability Theorem)**.** *If A and B are sets, then exactly one of the following three conditions holds:*

(i) *there is a bijection $A \to B$,*

(ii) *there is an injection $A \to B$ and a surjection $B \to A$,*

(iii) *there is an injection $B \to A$ and a surjection $A \to B$.*

• **Supplementary Exercise 11.5 p. 73 of the book.** Let X be a set; let $\mathcal{A}$ be the collection of all pairs $(A, <)$, where A is a subset of X and $<$ is a well-ordering of A. Define

$$(A, <) \prec (A', <')$$

if $(A, <)$ equals a section of $(A', <')$.

(a) Show that $\prec$ is a strict partial order on A.

(b) Let $\mathcal{B}$ be a subcollection of A that is simply ordered by $\prec$. Define B' to be the union of the sets B, for all $(B, <) \in \mathcal{B}$; and define $<'$ to be the union of the relations $<$, for all $(B, <) \in \mathcal{B}$. Show that $(B', <')$ is a well-ordered set.

Solution. Left to the reader.

• **Supplementary Exercise 11.6 p. 73 of the book.** Use Exercises 1 and 5 to prove the following:

Theorem. The maximum principle is equivalent to the well-ordering theorem.

Solution. The fact that the well-ordering theorem implies the maximum principle is proved on p. 70 of the book. Let us prove the converse. In the setting of Supplementary Exercise 11.5b, take $\mathcal{B}$ maximal. Then it suffices to show that $B' = X$. If it was not so, we could add to B' a new element x and make it the largest element of $B' \cup \{x\}$, which would then be a well-ordered set larger than B', contradiction.

• **Supplementary Exercise 11.7 p. 73 of the book.** Use Exercises 1–5 to prove the following:

Theorem. The choice axiom is equivalent to the well-ordering theorem.

Proof. Let X be a set; let c be a fixed choice function for the nonempty subsets of X. If T is a subset of X and $<$ is a relation on T, we say that $(T, <)$ is a tower in X if $<$ is a well-ordering of T and if for each $x \in T$,

$$x = c(X - S_x(T)),$$

where $S_x(T)$ is the section of T by x.

(a) Let $(T_1, <_1)$ and $(T_2, <_2)$ be two towers in X. Show that either these two ordered sets are the same, or one equals a section of the other. [Hint: Switching indices if necessary, we can assume that $h : T_1 \to T_2$ is order preserving and $h(T_1)$ equals either T_2 or a section of T_2. Use Exercise 2 to show that $h(x) = x$ for all x.]

(b) If $(T, <)$ is a tower in X and $T \neq X$, show there is a tower in X of which $(T, <)$ is a section.

(c) Let $\{(T_k, <_k) | k \in K\}$ be the collection of all towers in X. Let

$$T = \bigcup_{k \in K} T_k \text{ and } < = \bigcup_{k \in K} (<_k).$$

Show that $(T, <)$ is a tower in X. Conclude that $T = X$.

Solution. (a) The map h in the hint exists by Supplementary Exercise 4a. Let us show $h(x) = x$ for all x thanks to Supplementary Exercise 2. Assume by contradiction that we have $h(x) \neq x$ for some x in T_1, which we can suppose to be minimum for this condition. The map h induces an isomorphism $T_1 \simeq h(T_1)$, implying $h(T_{1,<y}) = T_{2,h(y)}$ for all y in T_1. By the choice of x we get $T_{1,<x} = T_{2,<h(x)}$. Since T_1 and T_2 are towers, this entails

$$h(x) = c(X \setminus T_{2,<h(x)}) = c(X \setminus T_{1,<x}) = x,$$

contradiction. The fact that $h(x) = x$ for all x in T_1 implies, by Supplementary Exercise 11.2a, that T_1 is contained and downward closed in T_2.

(b) Add $c(X \setminus T)$ to T, and make it the largest element.

(c) Left to the reader.

This shows that the choice axiom implies the well-ordering theorem. The converse is clear.

2 Chapter 2. Topological Spaces and Continuous Functions

Lemma 13.1 p. 80 of the book. Recall the statement:

Lemma 11 (Lemma 13.1 p. 80 of the book)**.** *Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Here is the proof given in the book:

Proof. Given a collection of elements of $\mathcal{B}$, they are also elements of $\mathcal{T}$. Because $\mathcal{T}$ is a topology, their union is in $\mathcal{T}$. Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$ an element B_x of $\mathcal{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathcal{B}$. □

To avoid the Axiom of Choice when showing that $U \in \mathcal{T}$ is a union of elements of $\mathcal{B}$, just note that U is the union of all those B in $\mathcal{B}$ which are contained in U.

About Lemma 13.2 p. 80 of the book. Recall the statement:

Lemma 12 (Lemma 13.2 of the book)**.** *Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.*

The proof shows the following:

Lemma 13. *Let X be a topological space and $\mathcal{B}$ a basis. Suppose that $\mathcal{C}$ is a set of open sets of X such that for each basic set B in $\mathcal{B}$ and each x in B, there is an element C of $\mathcal{C}$ such that $x \in C \subset B$. Then $\mathcal{C}$ is also a basis for the topology of X.*

• **Solution to Exercise 13.6 p. 83 of the book.** We must show that the topologies $\mathcal{T}_\ell$ and $\mathcal{T}_K$ are incomparable.

Claim: $[2,3) \notin \mathcal{T}_K$. Proof. If not we would have $2 \in (a,b) \setminus K \subset [2,3)$ for some a and b, hence $a < 2$ and $a \le 2$, contradiction.

Claim: $(-1,1) \setminus K \notin \mathcal{T}_\ell$. Proof. If not we would have $0 \in [a,b) \subset (-1,1) \setminus K \subset [2,3)$ for some a and b, hence $a \le 0 < b$, hence $a < \frac{1}{n} < b$ for some n, contradiction.

• **Solution to Exercise 13.7 p. 83 of the book.** Let us use the following notation:

$\mathcal{T}_s :=$ standard topology,

$\mathcal{T}_K :=$ topology of $\mathbb{R}_K$,

$\mathcal{T}_{fc} :=$ finite complement topology,

$\mathcal{T}_u :=$ upper limit topology (having the sets $(a,b]$ as basis),

$\mathcal{T}_\infty :=$ topology having the sets $(-\infty, a)$ as basis.

We denote the corresponding topological spaces by $\mathbb{R}_s, \mathbb{R}_K, \mathbb{R}_{fc}, \mathbb{R}_u$ and $\mathbb{R}_\infty$. Finally we write $\mathcal{B}_s, \mathcal{B}_K, \mathcal{B}_u$ and $\mathcal{B}_\infty$ for the obvious bases.

The inclusions between these five topologies on $\mathbb{R}$ can be summarized by the diagram

$$\begin{array}{ccc} & u & \\ & K & \\ & s & \\ fc & & \infty, \end{array}$$

where "i below j" means "$\mathcal{T}_i \subsetneqq \mathcal{T}_j$", and "$i$ and j on the same level" means "$\mathcal{T}_i$ and $\mathcal{T}_j$ are incomparable".

Preliminary comments: It is easy to see that the elements of $\mathcal{T}_\infty$ are $\varnothing$, the intervals $(-\infty, a)$, and $\mathbb{R}$, and to observe that $\mathcal{T}_\infty \cap \mathcal{T}_{fc} = \{\varnothing, \mathbb{R}\}$. It is also easy to compare the standard topology $\mathcal{T}_s$ to the others: the elements of $\mathcal{T}_{fc}$ and $\mathcal{T}_\infty$ are clearly open in $\mathbb{R}_s$, and it is plain that the intervals (a,b) (which are the elements on $\mathcal{B}_s$) are open in $\mathbb{R}_K$ and in $\mathbb{R}_\infty$ (note that $(a,b) = \bigcup_{d<b}(a,d]$). Clearly, $(-1,1) \setminus K \in \mathcal{T}_K$ and $(a,b] \in \mathcal{T}_u$ are not open in $\mathbb{R}_s$. Moreover $(2,3]$ is in $\mathcal{T}_u$ but not in $\mathcal{T}_K$. So, it only remains to prove $\mathcal{T}_K \subset \mathcal{T}_u$.

Let x be in $(a,b) \setminus K$. It suffices to show that there is a c such that $x \in (c,x] \subset (a,b) \setminus K$. If $x \le 0$ we set $c := a$. If $\frac{1}{n+1} < x < \frac{1}{n}$ we set $c := \frac{1}{n+1}$. If $x > 1$ we set $c := \max(1,a)$.

Exercise 13.8c p. 83 of the book. Show that the collection

$$\mathcal{C} = \{[a,b) \mid a < b,\ a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Solution. It is easy to check that $\mathcal{C}$ is indeed a basis. Let us show that it generates a topology different from the lower limit topology. Otherwise we would have $[\sqrt{2}, 2) = \bigcup_{i \in I}[a_i, b_i)$ with $a_i < b_i$ and a_i and b_i rational for all i. This implies $a_i \ge \sqrt{2}$, that is $a_i > \sqrt{2}$, for all i, hence $\sqrt{2} \notin \bigcup_{i \in I}[a_i, b_i)$, contradiction.

Subspace topology p. 88 of the book. Munkres writes:

Definition. Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the **subspace topology**. With this topology, Y is called a **subspace** of X; its open sets consist of all intersections of open sets of X with Y.

It is easy to see that $\mathcal{T}_Y$ is a topology. It contains $\varnothing$ and Y because

$$\varnothing = Y \cap \varnothing \quad \text{and} \quad Y = Y \cap X,$$

where $\varnothing$ and X are elements of $\mathcal{T}$. The fact that it is closed under finite intersections and arbitrary unions follows from the equations

$$(U_1 \cap Y) \cap \cdots \cap (U_n \cap Y) = (U_1 \cap \cdots \cap U_n) \cap Y,$$

$$\bigcup_{\alpha \in J} (U_\alpha \cap Y) = \left(\bigcup_{\alpha \in J} U_\alpha \right) \cap Y.$$

Proof. Given a collection of elements of $\mathcal{B}$, they are also elements of $\mathcal{T}$. Because $\mathcal{T}$ is a topology, their union is in $\mathcal{T}$. Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$ an element B_x of $\mathcal{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathcal{B}$. $\square$

End of the excerpt.

The Axiom of Choice is used to handle arbitrary unions. One can avoid it by proceeding as follows.

Given $A \subset Y \subset X$, set $\mathcal{U} := \{U \in \mathcal{T} \mid U \cap Y \subset A\}$, and define the open subset U_A of X by $U_A := \bigcup_{U \in \mathcal{U}} U$. Then we have

$$A \subset U_A \iff U_A \cap Y = A.$$

Proof: We have $Y \cap U_A = Y \cap \bigcup_{U \in \mathcal{U}} U = \bigcup_{U \in \mathcal{U}} Y \cap U \subset A$, so

$$A = U_A \cap Y \iff A \subset U_A \cap Y \iff A \subset U_A. \qquad \square$$

And we decree that A is open in Y if and only if the above equivalent conditions are satisfied. This is equivalent to the usual definition. Proof: If $A = U_A \cap Y$, then A is open in Y in the usual sense. If $A = U \cap Y$ for some U open in X, then $U \in \mathcal{U}$, hence $U \subset U_A$, and we get $A = U \cap Y \subset U_A \cap Y \subset A$, hence $A = U_A \cap Y$. Observe that, in general, $U_A \cap Y$ is the **interior** of A in Y.

Note that things are even nicer if we use closed subsets instead of open ones. Indeed, given $A \subset Y \subset X$ as above, there is a least closed subset C_A of X such that $A \subset C_A \cap Y$, and C_A is the closure of A in X. What I find remarkable is that C_A depends only on A and X, but not on Y. (I don't know if there is a conceptual reason for that.) (To see that U_A depends on Y in general, let X be nonempty and let A be empty. If $Y = \varnothing$, then $U_A = X$, but if $Y = X$, then $U_A = \varnothing$.)

Exercise 16.5 p. 92 of the book. Let X and X' denote a single set in the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively; let Y and Y' denote a single set in the topologies $\mathcal{U}$ and $\mathcal{U}'$, respectively. Assume these sets are nonempty.

(a) Show that if $\mathcal{T}' \supset \mathcal{T}$ and $\mathcal{U}' \supset \mathcal{U}$, then the product topology on $X' \times Y'$ is finer than the product topology on $X \times Y$.

(b) Does the converse of (a) hold? Justify your answer.

Solution. Part (a) is straightforward. The answer to the question in (b) is Yes. Here is the justification. We denote the respective product topologies by $\mathcal{V}$ and $\mathcal{V}'$. Assume $\mathcal{V}' \supset \mathcal{V}$. It suffices to show $\mathcal{T}' \supset \mathcal{T}$. Let U be in $\mathcal{T}$. It is enough to prove $U \in \mathcal{T}'$. The set $U \times Y$ is in $\mathcal{V} \subset \mathcal{V}'$, that is,

$$U \times Y = \bigcup_{i \in I} (U_i' \times V_i')$$

with $U_i' \in \mathcal{T}'$ and $V_i' \in \mathcal{U}'$ for all i. It suffices to show $U = \bigcup_{i \in I} U_i'$. Let u be in U. Pick some y in Y. Then $u \times y = u_i' \times v_i'$ for some i and some $u_i' \in U_i'$ and $v_i' \in V_i'$, so $u = u_i'$ is in U_i'. Conversely, let u_i' be in U_i' for some i. Then $u_i' \times v_i' = u \times y$ for some u in U and y in Y. In particular $u_i' = u \in U$.

Exercise 16.7 p. 92 of the book. Let X be an ordered set. If Y is a proper subset of X that is convex in X, does it follow that Y is an interval or a ray in X?

Solution. No. Example: $X := \{1, 2\} \times \mathbb{Z}$ with the dictionary order, $Y := \mathbb{Z} \times \{1\}$.

Exercise 16.8 p. 92 of the book. If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. In each case it is a familiar topology.

Solution. Let $\mathcal{T}_1$ be the topology of $\mathbb{R}_\ell \times \mathbb{R}$, let $\mathcal{T}_2$ be the topology of $\mathbb{R}_\ell \times \mathbb{R}_\ell$, and let L_i, for $i = 1, 2$, be the line L equipped with the topology induces by $\mathcal{T}_i$. I think what Munkres wants us to realize is that L_i is **homeomorphic** to $\mathbb{R}$ with the standard topology, or to $\mathbb{R}$ with the lower limit topology, or to $\mathbb{R}$ with the discrete topology, depending on i and the direction of L. But of course the word "homeomorphic" is introduced much later in the book, so, strictly speaking, the question, as stated, does not make sense. We will solve the following interpretation of the exercise:

Exercise 16.8'. For $i = 1, 2$ let $p_i : \mathbb{R}^2 \to \mathbb{R}$ be the ith canonical projection. We set $i(L) := 2$ if L is vertical, and $i(L) = 1$ otherwise. Then the restriction $r_L : L \to \mathbb{R}$ of $p_{i(L)}$ to L is bijective. Let $\mathcal{T}_o$ be the standard topology of $\mathbb{R}$ (the subscript "o" stands for "order topology"), let $\mathbb{R}_o$ be $\mathbb{R}$ equipped with $\mathcal{T}_o$, let $\mathcal{T}_\ell$ be the lower limit topology of $\mathbb{R}$, let $\mathbb{R}_\ell$ be $\mathbb{R}$ equipped with $\mathcal{T}_\ell$, let $\mathcal{T}_{\ell o}$ be the topology of $\mathbb{R}_\ell \times \mathbb{R}_o$, let $\mathcal{T}_{\ell\ell}$ be the topology of $\mathbb{R}_\ell \times \mathbb{R}_\ell$, let $\mathcal{U}_{\ell o}$ be the topology induced on L by $\mathcal{T}_{\ell o}$, let $\mathcal{U}_{\ell\ell}$ be the topology induced on L by $\mathcal{T}_{\ell\ell}$, let $\mathcal{T}_{\ell o L}$ be the topology on $\mathbb{R}$ obtained by transporting $\mathcal{T}_{\ell o}$ along r_L, and let $\mathcal{T}_{\ell\ell L}$ be the topology on $\mathbb{R}$ obtained by transporting $\mathcal{T}_{\ell\ell}$ along r_L. Describe the topologies $\mathcal{T}_{\ell o L}$ and $\mathcal{T}_{\ell\ell L}$.

Solution to Exercise 16.8'. We decree that the slope of a vertical line is $+\infty$. Let $s \in \mathbb{R} \cup \{+\infty\}$ be the slope of L, and let $\mathcal{T}_d$ be the discrete topology of $\mathbb{R}$. We claim

$$\mathcal{T}_{\ell o L} = \begin{cases} \mathcal{T}_o & \text{if } s = +\infty \\ \mathcal{T}_\ell & \text{if } s \in \mathbb{R}, \end{cases}$$

$$\mathcal{T}_{\ell\ell L} = \begin{cases} \mathcal{T}_d & \text{if } s < 0 \\ \mathcal{T}_\ell & \text{otherwise.} \end{cases}$$

To prove this, we first analyze the intersections

$$L \cap \big(([a, b) \times (c, d)\big) \text{ and } L \cap \big(([a, b) \times [c, d)\big)$$

for $a, b, c, d \in \mathbb{R}, a < b, c < d$, where L is fixed and a, b, c and d vary. We obtain a set of subsets of L, which we transport to $\mathbb{R}$ by r_L, and we take the topology on $\mathbb{R}$ generated by these subsets. The details are tedious, but very easy, and left to the reader.

Exercise 16.9 p. 92 of the book. Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.

Solution. First recall the definition of the order topology:

Let X be a set with a simple order relation; assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

1. All open intervals (a, b) in X.
2. All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.
3. All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.

The collection $\mathcal{B}$ is a basis for a topology on X, which is called the **order topology**.

The dictionary order topology on $\mathbb{R} \times \mathbb{R}$ is given by the basis elements $(a \times b, c \times d)$ with $a \times b < c \times d$. One checks easily that the intervals of the form $(a \times b, a \times c)$ with $b < c$ also form a basis for the order topology (see Example 2 p. 85 of the book). This shows that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$. It is strictly finer than the standard topology on $\mathbb{R}^2$.

Exercise 16.10 p. 92 of the book. Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.

Solution. Several topological spaces X are (explicitly or implicitly) involved in the above statement. We denote the topology of such an X by $\mathcal{T}(X)$, and if $\mathcal{T}(X)$ comes with a preferred basis, we denote it by $\mathcal{B}(X)$. If X is an ordered set, we denote by X_o the set X equipped with the order topology $\mathcal{T}(X_o)$, and by $\mathcal{B}(X_o)$ the basis defining $\mathcal{T}(X_o)$ (see the Solution to Exercise 16.9 p. 92 above). If S is a set, we denote by S_d the set S equipped with the discrete topology, and we regard the set $\mathcal{B}(S_d)$ of all singletons contained in S as the preferred basis of $\mathcal{T}(S_d)$. If X and Y are topological spaces with preferred basis $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, we define $\mathcal{B}(X \times Y)$ as the basis consisting of the $B \times C$ with $B \in \mathcal{B}(X)$ and $C \in \mathcal{B}(Y)$ (see Theorem 15.1 p. 86). If X is a topological space with preferred basis $\mathcal{B}(X)$ and Y is a subspace of X, we define $\mathcal{B}(Y)$ as the basis consisting of the $B \cap Y$ with $B \in \mathcal{B}(X)$ (see Theorem 16.1 p. 89). If X and Y are ordered set, we denote by $(X \times Y)_o$ the set $X \times Y$ equipped with the dictionary order topology.

Recall the following facts from the book. By Example 1 p. 85, $\mathbb{R}$ equipped with its standard topology coincides with $\mathbb{R}_o$. By Example 1 p. 90, $I \subset \mathbb{R}$ equipped with the subspace topology is equal to I_o. In Example 3 p. 90, $(I^2)_o$ is denoted I_o^2, and is called the **ordered square**. Remember also that in the Solution to Exercise 16.9 p. 92 we saw that $(\mathbb{R}^2)_o = \mathbb{R}_d \times \mathbb{R}_o$ (equipped with the product topology).

Using these facts and Theorem 16.3 p. 89, it is easy to see that the three topological spaces of the exercise are respectively $(I_o)^2, (I^2)_o$ and $I_d \times I_o$. In particular, the six sets

$$\mathcal{T}((I_o)^2), \quad \mathcal{T}((I^2)_o), \quad \mathcal{T}(I_d \times I_o), \quad \mathcal{B}((I_o)^2), \quad \mathcal{B}((I^2)_o), \quad \mathcal{B}(I_d \times I_o)$$

are well-defined.

We claim

$$\mathcal{T}((I_o)^2) \cup \mathcal{T}((I^2)_o) \subsetneqq \mathcal{T}(I_d \times I_o) \quad \text{and} \quad \mathcal{T}((I_o)^2) \not\subset \mathcal{T}((I^2)_o) \not\subset \mathcal{T}((I_o)^2).$$

Let $U \subset I \times I$. Then $U \in \mathcal{T}(I_d \times I_o)$ if and only if

$$U = \bigcup_{x \in I} (\{x\} \times U_x)$$

for some family $(U_x)_{x \in I}$ of members of $\mathcal{T}(I_o)$.

The inclusion $\mathcal{T}((I_o)^2) \cup \mathcal{T}((I^2)_o) \subset \mathcal{T}(I_d \times I_o)$ follows immediately from the above observations.

The following criterium to prove that a subset of I^2 is not in $\mathcal{T}((I^2)_o$ will be handy.

Criterium: If U is an open subset of $(I^2)_o$ containing 0×1, then U contains $\varepsilon \times 0$ for some $\varepsilon \in I$.

Thanks to this criterium, we see that

$$\{0\} \times I = [0 \times 0, 0 \times 1] \in \mathcal{T}(I_d \times I_o) \setminus (\mathcal{T}((I_o)^2) \cup \mathcal{T}((I^2)_o)),$$

implying $\mathcal{T}((I_o)^2) \cup \mathcal{T}((I^2)_o) \neq \mathcal{T}(I_d \times I_o)$.

A similar argument shows that $I \times (0, 1] \in \mathcal{T}((I_o)^2) \setminus \mathcal{T}((I^2)_o)$, and thus that $\mathcal{T}((I_o)^2) \not\subset \mathcal{T}((I^2)_o)$.

Finally, to prove $\mathcal{T}((I^2)_o) \not\subset \mathcal{T}((I_o)^2)$, note that $\{0\} \times (0, 1) = (0 \times 0, 0 \times 1) \in \mathcal{T}((I^2)_o) \setminus \mathcal{T}((I_o)^2)$.

Theorem 17.2 p. 94 of the book. We give a slightly different proof of the indicated Theorem. First recall the statement.

Theorem 14 (Theorem 17.2 p. 94 of the book)**.** *Let X be a topological space, Y a subspace, and A a subset of Y. Then A is closed in Y if and only $A = Y \cap C$ for some closed subset of X.*

Proof. It suffices to show that $Y \setminus A = Y \cap U$, where U is an open subset of X, if and only if $A = Y \cap (X \setminus U)$. Hence it is enough to prove that we have $Y \setminus (Y \cap U) = Y \cap (X \setminus U)$, or equivalently $Y \setminus U = Y \cap (X \setminus U)$, for all subset U of X. But this is clear. $\square$

Theorem 17.4 p. 95 of the book. Recall the statement:

Theorem 15 (Theorem 17.4 p. 95 of the book)**.** *Let X be a topological space, Y a subspace, and A a subset of Y. Then the closure $\overline{A}^Y$ of A in Y equals $\overline{A} \cap Y$, where $\overline{A}$ is the closure of A in X.*

Here is a slightly different proof.

Proof. Set
$$\mathcal{B} = \{B \mid B \text{ closed in } Y,\ B \supset A\}.$$
For all B in $\mathcal{B}$ put
$$\mathcal{C}(B) = \{C \mid C \text{ closed in } X,\ C \supset B\}.$$
Finally write
$$\mathcal{C} = \{C \mid C \text{ closed in } X,\ C \supset A\}.$$
We get
$$\mathcal{C} = \bigcup_{B \in \mathcal{B}} \mathcal{C}(B)$$
and
$$\overline{A}^Y = \bigcap_{B \in \mathcal{B}} B = \bigcap_{B \in \mathcal{B}} \bigcap_{C \in \mathcal{C}(B)} (Y \cap C) = \bigcap_{C \in \mathcal{C}} (Y \cap C) = Y \cap \bigcap_{C \in \mathcal{C}} C = Y \cap \overline{A}.$$
□

Theorem 17.6 p. 97 of the book. Recall the statement:

Theorem 16 (Theorem 17.6 p. 97 of the book)**.** *Let A be a subset of the topological space X; let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.*

Here is a slightly different proof.

Proof. In this proof we denote the closure $\overline{B}$ of a subset B of X by B^-. To show $A' \subset A^-$, note that if x is in A', then x is in $(A \setminus \{x\})^- \subset A^-$. It only remains to prove $A^- \subset A \cup A'$. Assume by contradiction that there is an x in A^- which is not in $A \cup A'$. We have in particular $x \notin (A \setminus \{x\})^-$, that is, $x \in U := X \setminus (A \setminus \{x\})^-$ with U open. The fact that x is in $\overline{A}$ implies that U intersects A. Let a be such an intersection point. Since x is not in A but a is, we have $a \neq x$, hence a is in $A \setminus \{x\}$, which is disjoint from U, contradiction. □

Theorem 17.8 p. 99 of the book. Recall the statement:

Theorem 17 (Theorem 17.8 p. 99 of the book)**.** *Every finite point set in a Hausdorff space X is closed.*

Here is a slightly different proof.

Proof. Let x_0 be a point of X, let $\mathcal{U}$ be the set of all those open subsets of X which do not contain x_0, let U be the union of the members of $\mathcal{U}$, and let x be in $X \setminus \{x_0\}$. It suffices to show that $x \in U$, that is, $x \in V$ for some V in $\mathcal{U}$, which is clear. □

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Theorem 17.9 p. 99 of the book. Recall the statement:

Theorem 18 (Theorem 17.8 p. 99 of the book)**.** *Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Here is a slightly different proof.

Proof. It suffices to show that the following two conditions are equivalent:

(1) there is a neighborhood U of x such that $U \cap A = \{x\}$,

(2) there is a neighborhood V of x such that $V \cap A$ is finite.

Clearly (1) implies (2). To prove the converse it suffices to set $U := (V \setminus A) \cup \{x\}$. (Since X is T_1, the subset V, being obtained from U by removing finitely many points, is open.) $\square$

Exercise 17.3 p. 100 of the book. Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Solution. Setting $U := X \setminus A, V := Y \setminus B$, we get

$$(X \times Y) \setminus (A \times B) = (X \times V) \cup (U \times Y).$$

Exercise 17.5 p. 100 of the book. Let X be an ordered set in the order topology. Show that $\overline{(a,b)} \subset [a,b]$. Under what conditions does equality hold?

Solution. To prove the indicated inclusion, it suffices to show that

$$[a,b] \text{ is closed.} \tag{1}$$

To prove (1), let c be in $X \setminus [a,b]$. Assume first $c < a$. If there is a d less than c, then (d,a) contains c, and is open and disjoint from $[a,b]$. If c is the least element of X, then $[c,a)$ contains c, and is open and disjoint from $[a,b]$. The case $c > b$ is similar.

For the second question, note that, by (1), the set $\overline{(a,b)}$ is equal to (a,b), to $[a,b)$, to $(a,b]$, or to $[a,b]$. Thus it suffices to determine when a or b is in $\overline{(a,b)}$. We claim:

(a) the point a is not in $\overline{(a,b)}$ if and only if it has an immediate successor,

(b) the point b is not in $\overline{(a,b)}$ if and only if it has an immediate predecessor.

To prove this, we can, by Theorems 16.4 p. 91 and 17.4 p. 95, assume that $X = [a,b]$. To prove (a), note that

$$a \notin \overline{(a,b)}$$

$$\iff \text{there is a } c \text{ in } [a,b] \text{ with } a \in [a,c) \text{ and } [a,c) \cap (a,b) = \varnothing$$

$$\iff \text{there is a } c \text{ in } [a,b] \text{ which is the immediate successor of } a.$$

The proof of (b) is similar.

Exercise 17.6 p. 101 of the book. Let A, B, and A_α denote subsets of a space X. Prove the following:

(a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(c) $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$; give an example where equality fails.

Solution. (a) We have $A \subset \overline{B}$, and thus $\overline{A} \subset \overline{B}$.

(b) For x in X the following conditions are equivalent:

(A) $x \notin \overline{A \cup B}$,

(B) some neighborhood U of x does not intersect $A \cup B$,

(C) some neighborhood U of x intersects neither A nor B,

(D) (some neighborhood V of x does not intersect A) and (some neighborhood W of x does not intersect B),

(E) $x \notin \overline{A}$ and $x \notin \overline{B}$,

(F) $x \notin \overline{A} \cup \overline{B}$,

the implication (D) $\Longrightarrow$ (C) being obtained by setting $U := V \cap W$.

(c) For x in X we have:

$$\begin{aligned} & x \notin \overline{\bigcup A_\alpha} \\ \Longrightarrow\ & \text{some neighborhood } U \text{ of } x \text{ does not intersect } \bigcup A_\alpha \\ \Longrightarrow\ & \text{some neighborhood } U \text{ of } x \text{ intersects no } A_\alpha \\ \Longrightarrow\ & x \notin \overline{A_\alpha} \text{ for all } \alpha \\ \Longrightarrow\ & x \notin \bigcup \overline{A_\alpha}. \end{aligned}$$

Exercise 17.8 p. 108 of the book. Let A, B, and A_α denote subsets of a space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\subset$ or $\supset$ holds.

(a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

(b) $\overline{\bigcap A_\alpha} = \bigcap \overline{A_\alpha}$

(c) $\overline{A \setminus B} = \overline{A} \setminus \overline{B}$.

Solution. We claim that (b) holds, and will clearly imply (a). Proof: Let x be in X. Then

$$x \in \overline{\bigcap A_\alpha}$$

$$\iff \text{ every neighborhood of } x \text{ intersects } \bigcap A_\alpha$$

$$\iff \text{ every neighborhood of } x \text{ intersects } A_\alpha \text{ for all } \alpha$$

$$\iff x \text{ is in } \overline{A_\alpha} \text{ for all } \alpha,$$

$$\iff x \text{ is in } \bigcap \overline{A_\alpha}.$$

(c) If $X = A = \mathbb{R}$ and $B = \mathbb{Q}$, then we have $\overline{A \setminus B} = \mathbb{R}$ and $\overline{A} \setminus \overline{B} = \varnothing$. Thus our only hope is to have $\overline{A} \setminus \overline{B} \subset \overline{A \setminus B}$ for all A, B. We can rewrite this inclusion as $\overline{A} \cap \overline{B}^c \subset \overline{A \cap B^c}$ for all A, B, where C^c means $X \setminus C$. In view of (a), this is equivalent to $\overline{B}^c \subset \overline{B^c}$ for all B. To prove this, it suffices to show that $\overline{B}^c$ is the interior of B^c. Let U be an open subset of X contained in B^c. It is enough to prove $U \subset \overline{B}^c$. We have $U^c \supset B$. Since U^c is closed, this implies $U^c \supset \overline{B}$, and thus $U \subset \overline{B}^c$.

Exercise 17.9 p. 101 of the book. Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \overline{A} \times \overline{B}.$$

Solution. We have $\overline{A \times B} \subset \overline{A} \times \overline{B}$ by Exercise 17.3. To prove the converse inclusion, let $z = x \times y$ be in $\overline{A} \times \overline{B}$, that is, $x \in \overline{A}, y \in \overline{B}$, and let W be an open subset of $X \times Y$ containing z. It suffices to show $W \cap (A \times B) \neq \varnothing$. There is an open subset U of X containing x and an open subset V of Y containing y such that $U \times V \subset W$. It is enough to prove $(U \times V) \cap (A \times B) \neq \varnothing$, that is $(U \cap A) \times (V \cap B) \neq \varnothing$. But this follows from the assumption that $x \in \overline{A}$ and $y \in \overline{B}$.

Exercise 17.10 p. 101 of the book. Show that every order topology is Hausdorff.

Solution. Let x, y be in X with $x < y$. If there is a z in (x, y) we separates x and y with the open rays $(-\infty, z)$ and $(z, +\infty)$. If (x, y) is empty, we separates x and y with the open rays $(-\infty, y)$ and $(z, +\infty)$.

Exercise 17.13 p. 101 of the book. Show that X is Hausdorff if and only if the *diagonal* $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Solution. Let $\overline{\Delta}$ be the closure of the diagonal, and let $z = x \times y$ be in X^2. We claim: $z \notin \overline{\Delta} \iff x$ and y can be separated by disjoint open sets.

$\implies$: There is an open subset W of X^2 such that $z \in W$ and $W \cap \Delta = \varnothing$. There are open subsets U and V of X such that $z \in U \times V \subset W$. We have $x \in U, y \in V$. If u was in $U \cap V$, then $u \times u$ would be in $\Delta \cap W$, contradiction.

$\impliedby$: If there are disjoint open subsets U and V of X such that $x \in U, y \in V$, then z is in $U \times V$, and it suffices to show $(U \times V) \cap \Delta = \varnothing$. But $(U \times V) \cap \Delta$ is the diagonal of $U \cap V$, which is empty.

Exercise 17.14 p. 101 of the book. In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n = \frac{1}{n}$ converge?

Solution. Set $S := \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$, and not that any nonempty open subset U of $\mathbb{R}$ (with the finite complement topology) intersects S. This means that $\mathbb{R}$ is the set of limits of the sequence $\frac{1}{n}$.

Exercise 17.15 p. 101 of the book. Show the T_1 axiom is equivalent to the condition that for each pair of points of X, each has a neighborhood not containing the other.

Solution. (In the statement of the Exercise, "each has a neighborhood not containing the other", means "each point has a neighborhood not containing the other point".) Recall that X is T_1 if and only if the finite subsets of X are closed. If X is T_1 and x and y are distinct points of X, the open subsets $X \setminus \{x\}$ and $X \setminus \{y\}$ do the job. If X is *not* T_1, there distinct points x and y of X such that y is in the closure of $\{x\}$, and any neighborhood of y contains x.

Exercise 17.16 p. 101 of the book. Consider the five topologies on $\mathbb{R}$ given in Exercise 7 of §13.

(a) Determine the closure of the set $K = \left\{\frac{1}{n} \mid n \in \mathbb{Z}_+\right\}$ under each of these topologies.

(b) Which of these topologies satisfy the Hausdorff axiom? the T_1 axiom?

Solution. Recall that the five topologies are:

$\mathcal{T}_s :=$ standard topology,

$\mathcal{T}_K :=$ topology of $\mathbb{R}_K$,

$\mathcal{T}_{fc} :=$ finite complement topology,

$\mathcal{T}_u :=$ upper limit topology (having the sets $(a, b]$ as basis),

$\mathcal{T}_\infty :=$ topology having the sets $(-\infty, a)$ as basis.

Recall also that $\mathcal{T}_K$ was defined as follows: Let K be defined as above. The topology $\mathcal{T}_K$ generated by all open intervals (a, b), along with all sets of the form $(a, b) \setminus K$ will be called the K-topology on $\mathbb{R}$. When $\mathbb{R}$ is given this topology, we denote it by $\mathbb{R}_K$. We define $\mathbb{R}_s, \mathbb{R}_{fc}, \mathbb{R}_u$ and $\mathbb{R}_\infty$ similarly.

(a) Let C_* be the closure of K in $\mathbb{R}_*$. Clearly, $C_s = C_\infty = K \cup \{0\}$ and $C_{fc} = \mathbb{R}$ (see Exercise 17.14). We claim $C_K = K$. Indeed, the inclusion $C_K \subset K \cup \{0\}$ is easy, and we have $0 \notin C_u$ because 0 is in the open set $(-1, 1) \setminus K$, which is disjoint from K. We claim $C_u = K$. The argument is the same, with $(-1, 0]$ instead of $(-1, 1) \setminus K$.

(b) Clearly $\mathbb{R}_s, \mathbb{R}_K$ and $\mathbb{R}_u$ are Hausdorff, and $\mathbb{R}_{fc}$ is T_1 but not Hausdorff. Finally, $\mathbb{R}_\infty$ is also T_1 but not Hausdorff. The fact that it is T_1 is clear. It is not Hausdorff because all the nonempty open sets are basic, and the intersection of any two basic sets is basic.

Exercise 17.17 p. 101 of the book. Consider the lower limit topology on $\mathbb{R}$ and the topology given by the basis $\mathcal{C}$ of Exercise 8 of §13. Determine the closures of the intervals $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in these two topologies.

Solution. Recall the definition of $\mathcal{C}$:

$$\mathcal{C} = \{[a, b) \mid a < b,\ a \text{ and } b \text{ rational}\}.$$

For $\subset \mathbb{R}$ let $\overline{S}^{\,\ell}$ and $\overline{S}^{\,\mathcal{C}}$ denote respectively the closure in each of the two topologies in the statement. Then we have

$$\overline{(0, \sqrt{2})}^{\,\ell} = [0, \sqrt{2}), \quad \overline{(0, \sqrt{2})}^{\,\mathcal{C}} = [0, \sqrt{2}], \quad \overline{(\sqrt{2}, 3)}^{\,\ell} = [\sqrt{2}, 3) = \overline{(\sqrt{2}, 3)}^{\,\mathcal{C}}$$

The justifications are left to the reader.

Exercise 17.18 p. 101 of the book. Determine the closures of the following subsets of the ordered square:

$$\begin{aligned}
A &= \left\{\tfrac{1}{n} \times 0 \mid n \in \mathbb{Z}_+\right\} \\
B &= \left\{(1 - \tfrac{1}{n}) \times \tfrac{1}{2} \mid n \in \mathbb{Z}_+\right\} \\
C &= \{x \times 0 \mid 0 < x < 1\} \\
D &= \left\{x \times \tfrac{1}{2} \mid 0 < x < 1\right\} \\
E &= \left\{\tfrac{1}{2} \times y \mid 0 < y < 1\right\}.
\end{aligned}$$

Solution. Let X be the ordered square. For any $S \subset X$ we set $S' := \overline{S} \setminus S$. We have

$$A' = \{0 \times 1\}, \quad B' = \{1 \times 0\}, \quad C' = [0, 1) \times \{1\}, \quad D' = (0, 1] \times \{0\}, \quad E' = \{\tfrac{1}{2} \times 0, \tfrac{1}{2} \times 1\}.$$

Exercise 17.19 p. 102 of the book. If $A \subset X$, we define the **boundary** of A by the equation

$$\operatorname{Bd} A = \overline{A} \cap (\overline{X - A})$$

(a) Show that $\operatorname{Int} A$ and $\operatorname{Bd} A$ are disjoint, and $\overline{A} = \operatorname{Int} A \cup \operatorname{Bd} A$.

(b) Show that $\operatorname{Bd} A = \varnothing \Leftrightarrow A$ is both open and closed.

(c) Show that U is open $\Leftrightarrow \operatorname{Bd} U = \overline{U} - U$.

(d) If U is open, is it true that $U = \operatorname{Int}(\,\overline{U}\,)$? Justify your answer.

Solution. In this Solution we use the following notation: $A^c := X \setminus A$, $A^- :=$ closure of A, $A^o :=$ interior of A, $A^b :=$ boundary of A. Note that A^b is closed, that $A^{cb} = A^b$, and that $A^b = A^- \cap A^{c-} = A^- \cap A^{o\,c} = A^- \setminus A^o$. In particular, the equality

$$A^b = A^- \setminus A^o$$

implies (a) and (b).

(c) We must show $U^o = U \Leftrightarrow U^- \setminus U^o = U^- \setminus U$, which is clear.

(d) The answer is no, a counterexample being give by $\mathbb{R} \setminus \{0\}$ in $\mathbb{R}$.

Exercise 17.20 p. 102 of the book. Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$:

(a) $A = \{x \times y \mid y = 0\}$

(b) $B = \{x \times y \mid x > 0 \text{ and } y \neq 0\}$

(c) $C = A \cup B$

(d) $D = \{x \times y \mid x \text{ is rational}\}$

(e) $E = \{x \times y \mid 0 < x^2 - y^2 \leq 1\}$

(f) $F = \{x \times y \mid x \neq 0 \text{ and } y \leq 1/x\}$.

Solution. We use the same notations as in Exercise 17.19.

(a) $A^- = A$ (A is closed), $A^o = \varnothing,\ A^b = A$.

(b) $B^- = \{x \times y \mid x \geq 0,\ B^o = B$ (B is open), $B^b = \{x \times y \mid x = 0\} \cup \{x \times y \mid x \geq 0 \text{ and } y = 0\}$.

(c) $C^- = \{x \times y \mid x \geq 0\},\ C^o = \{x \times y \mid x > 0\},\ C^b = \{x \times y \mid xy = 0 \text{ and } x \leq 0\}$.

(d) $D^- = \mathbb{R}^2,\ D^o = \varnothing,\ D^b = \mathbb{R}^2$.

(e) $E^- = \{x \times y \mid 0 \leq x^2 - y^2 \leq 1\},\ E^o = \{x \times y \mid 0 < x^2 - y^2 < 1\},\ E^b = \{x \times y \mid x^2 - y^2 \in \{0, 1\}\}$.

(f) $F^- = \{x \times y \mid (x \neq 0 \text{ and } y \leq 1/x) \text{ or } x = 0\},\ F^o = \{x \times y \mid x \neq 0 \text{ and } y < 1/x\}$,
$F^b = \{x \times y \mid (x \neq 0 \text{ and } y = 1/x) \text{ or } x = 0\}$.

Theorem 18.1 p. 104 of the book. Recall the statement:

Theorem 19 (Theorem 18.1 of the book)**.** *Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:*

(1) *f is continuous.*

(2) *For every subset A of X, one has $f(\overline{A}) \subset \overline{f(A)}$.*

(3) *For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.*

(4) *For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.*

If the condition in (4) holds for the point x of X, we say that f is ***continuous at the point*** *x.*

The proof that (4) implies (1) is written as follows: "Let V be an open set of Y; let x be a point of $f^{-1}(V)$. Then $f(x) \in V$, so that by hypothesis there is a neighborhood U_x of x such that $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$. It follows that $f^{-1}(V)$ can be written as the union of the open sets U_x, so that it is open." One can avoid using the Axiom of Choice by the following wording: $f^{-1}(V)$ can be written as the union of the open sets U such that $x \in U \subset f^{-1}(V)$, so that it is open. (A similar comment was made after Lemma 11.)

A corollary to Theorem 18.1 p. 104 of the book.

Corollary 20. *Let X and X' be topological spaces, let $\mathcal{B}$ and $\mathcal{B}'$ be respective basis for the topology of X and X', let $f : X \to X'$ be a map, and x a point of X. Then f is continuous at x if and only if, for all basic neighborhood $B' \in \mathcal{B}'$ of $f(x)$ there is a basic neighborhood $B \in \mathcal{B}$ of x such that $f(B) \subset B'$.*

Proof. Let f be continuous at x, and let B' be a basic neighborhood of $f(x)$. Then there is a neighborhood U of x such that $f(U) \subset B'$, and any basic neighborhood B of x contained in U will satisfy $f(B) \subset B'$. Conversely, given a neighborhood U' of $f(x)$, pick a basic neighborhood B' of $f(x)$ contained in U', and note that there is a basic neighborhood B of x such that $f(B) \subset B' \subset U'$. □

Exercise 18.7 p. 111 of the book. (a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is "continuous from the right," that is,

$$\lim_{x \to a^+} f(x) = f(a)$$

for each $a \in \mathbb{R}$. Show that f is continuous when considered as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$.

(b) Can you conjecture what functions $f : \mathbb{R} \to \mathbb{R}$ are continuous when considered as maps from $\mathbb{R}$ to $\mathbb{R}_\ell$? As maps from $\mathbb{R}_\ell$ to $\mathbb{R}_\ell$? We shall return to this question in Chapter 3.

Solution. (a) Assume $f(x) \in (a, b)$ for some $x, a, b \in \mathbb{R}$. It suffices to show that there is a d in $\mathbb{R}$ such that $x \in [x, d)$ and $[x, d) \subset f^{-1}((a, b))$, that is $x \in [x, d)$ and $f([x, d)) \subset (a, b)$. We have

$$\lim_{y \to x^+} f(y) = f(x).$$

Set $\varepsilon := \min(b - f(x), f(x) - a)$. There is a $\delta > 0$ such that $y \in [x, x + \delta)$ implies $|f(y) - f(x)| < \varepsilon$, and thus $f(y) \in (a, b)$. This shows that $d := x + \delta$ does the job.

Exercise 18.8 p. 111 of the book. Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.

(a) Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X.

(b) Let $h : X \to Y$ be the function

$$h(x) = \min\{f(x), g(x)\}$$

Show that h is continuous. [Hint: Use the pasting lemma.]

Solution. I think that X is implicitly assumed to be a topological space.

(a) Define $u : X \to Y^2$ by $u(x) := f(x) \times g(x)$. Note that u is continuous. Set

$$B := \{y \times z \in Y^2 \mid y \leq z\}.$$

Then the set in the statement of (a) is $u^{-1}(B)$. Thus, it suffices to show that B is closed. Set $U := \{y \times z \in Y^2 \mid y > z\}$. We must check that U is open. Let $y \times z$ be in U. There are basic open sets $V, W \subset Y$ such $y \in V, z \in W$ and $y' > z'$ for all $y' \in V$ and all $z' \in W$.

Exercise 18.13 p. 112 of the book. Let $A \subset X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \overline{A} \to Y$, then g is uniquely determined by f.

Solution. Let $h : \overline{A} \to Y$ be another continuous extension of f, assume by contradcition that $g(x_0) \neq h(x_0)$ for some x_0 in $\overline{A}$, and define $u : \overline{A} \to Y^2$ by $u(x) := g(x) \times h(x)$. In particular u is continuous. By Exercise 17.13 p. 101 of the book, $U := \{y \times y' \in Y^2 \mid y \neq y'\}$ is open. Hence $u^{-1}(U)$ is open, and it is nonempty because it contains x_0, hence we have $a \in u^{-1}(U)$, that is, $g(a) \neq h(a)$ for some a in A, contradicting the equalities $g(a) = f(a) = h(a)$.

Exercise 19.1 p. 118 of the book. Prove Theorem 19.2.

Solution. Recall the statement:

Theorem 21 (Theorem 19.2 of the book)**.** *Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form*

$$\prod_{\alpha \in J} B_\alpha$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$. The collection of all sets of the same form, where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology $\prod_{\alpha \in J} X_\alpha$.

Proof. For each $\alpha \in J$ let U_α be open in X_α, set $U := \prod_{\alpha \in J} U_\alpha$, and let $x = (x_\alpha)_{\alpha \in J}$ be in U. By Lemma 13 p. 10 it suffices to show that there is a set B of the form $\prod_{\alpha \in J} B_\alpha$, where $B_\alpha \in \mathcal{B}_\alpha$ for each α such that $x \in B \subset U$, that is, $x_\alpha \in B_\alpha \subset U_\alpha$ for all α. But this is clear. The case of the product topology is similar. $\square$

Exercise 19.2 p. 118 of the book. Prove Theorem 19.3.

Solution. Recall the statement:

Theorem 22 (Theorem 19.3 of the book)**.** *Let A_α be a subspace of X_α, for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the product topology.*

Proof. Consider first the box topology. Let $\mathcal{T}_b$ and $\mathcal{T}_s$ be respectively the box topology and the subspace topology on $\prod A_\alpha$ (when X is given the box topology), let $\mathcal{B}_b$ be the basis of $\mathcal{T}_b$ consisting of all the products $\prod U_\alpha$ with U_α open in A_α, that is, of all the products $\prod(A_\alpha \cap V_\alpha)$ with V_α open X_α, and let $\mathcal{B}_s$ be the basis of $\mathcal{T}_s$ consisting of all the intersections $(\prod A_\alpha) \cap (\prod W_\alpha)$ with W_α open in X_α. It suffices to show $\mathcal{B}_b = \mathcal{B}_s$, or, more concretely, $\prod(A_\alpha \cap V_\alpha) = (\prod A_\alpha) \cap (\prod V_\alpha)$. Let $x = (x_\alpha)$ be in $\prod X_\alpha$. Then

$$x \in \prod(A_\alpha \cap V_\alpha) \iff x_\alpha \in A_\alpha \cap V_\alpha \text{ for all } \alpha \iff x \in (\prod A_\alpha) \cap (\prod V_\alpha).$$

The case of the product topology is similar. □

Exercise 19.3 p. 118 of the book. Prove Theorem 19.4.

Solution. Recall the statement:

Theorem 23 (Theorem 19.4 of the book)**.** *If each space X_α is Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies.*

Proof. Let x and y be distinct points of $\prod X_\alpha$, let α satisfy $x_\alpha \neq y_\alpha$, and let U_α and V_α be two disjoint open subsets of X_α such that $x_\alpha \in U_\alpha$ and $y_\alpha \in v_\alpha$. For $\beta \neq \alpha$ set $U_\beta = V_\beta = X_\beta$. Then $\prod U_\gamma$ and $\prod V_\gamma$ are two disjoint open subsets of $\prod X_\gamma$ (in both topologies) such that $x \in \prod U_\gamma$ and $y \in \prod V_\gamma$. □

Exercise 19.6 p. 118 of the book. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \ldots$ converges to $\pi_\alpha(\mathbf{x})$ for each α. Is this fact true if one uses the box topology instead of the product topology?

Solution. If $x_1, x_2, \ldots$ converges in either topology, then $x_{1\alpha}, x_{2\alpha}, \ldots$ converges for all α because π_α is continuous. If $x_{1\alpha}, x_{2\alpha}, \ldots$ converges for all α, then $x_1, x_2, \ldots$ converges in the product topology. Indeed, for all α let x_α be a limit of $x_{1\alpha}, x_{2\alpha}, \ldots$, and let $U = \prod U_\alpha$ be a neighborhood of $(x_\alpha)_{\alpha \in J}$. Since there is a finite subset F of J such that $U_\alpha = X_\alpha$ for α not in J, we see that x_n will be in U for all n large enough.

Exercise 19.7 p. 118 of the book. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.

Solution. The subset $\mathbb{R}^\infty$ is closed in $\mathbb{R}^\omega$ equipped with the box topology. Indeed, let x be in $\mathbb{R}^\omega \setminus \mathbb{R}^\infty$. Then there are infinitely many i such that $x_i \neq 0$. For each such i set $U_i := (-|x_i|/2, |x_i|/2)$. If $x_i = 0$ set $U_i := \mathbb{R}$. Finally set $U := \prod U_i$. Then U is a neighborhood of x in the box topology, and U is disjoint from $\mathbb{R}^\infty$. The closure of $\mathbb{R}^\infty$ in the product topology is $\mathbb{R}^\omega$. Indeed, let x be in $\mathbb{R}^\omega$, and let $U = \prod U_i$ be a neighborhood of x. There is a finite subset F of ω such that $U_i = \mathbb{R}$ for i not in F. Define $y \in \mathbb{R}^\omega$ by $y_i = x_i$ if $i \in F$ and $y_i = 0$ otherwise. Then y is in $U \cap \mathbb{R}^\infty$.

Exercise 19.8 p. 118 of the book. Given sequences $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ of real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation

$$h((x_1, x_2, \ldots)) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \ldots)$$

Show that if $\mathbb{R}^\omega$ is given the product topology, h is a homeomorphism of $\mathbb{R}^\omega$ with itself. What happens if $\mathbb{R}^\omega$ is given the box topology?

Solution. In fact h is a homeomorphism on both cases. The details are left to the reader.

Exercise 19.9 p. 118 of the book. Show that the choice axiom is equivalent to the statement that for any indexed family $\{A_\alpha\}_{\alpha \in J}$ of nonempty sets, with $J \neq \varnothing$, the cartesian product

$$\prod_{\alpha \in J} A_\alpha$$

is not empty.

Solution. Let $(A_\alpha)_{\alpha \in J}$ be as above. The choice axiom says that $\prod A_\alpha \neq \varnothing$ if the A_α are *disjoint*. The statement in the exercise clearly implies the choice axiom. To prove the converse, set $B_\alpha := A_\alpha \times \{\alpha\}$ for all α. Then the B_α are disjoint, and the choice axiom implies $\prod B_\alpha \neq \varnothing$. Let b be in $\prod B_\alpha$. Then each b_α is of the form (a_α, α) for some a_α in A_α, and $(a_\alpha)_{\alpha \in J}$ is in $\prod A_\alpha$.

Exercise 19.10 p. 118 of the book. Let A be a set; let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of spaces; and let $\{f_\alpha\}_{\alpha \in J}$ be an indexed family of functions $f_\alpha : A \to X_\alpha$.

(a) Show there is a unique coarsest topology $\mathcal{T}$ on A relative to which each of the functions f_α is continuous.

(b) Let
$$\mathcal{S}_\beta = \left\{ f_\beta^{-1}(U_\beta) \mid U_\beta \text{ is open in } X_\beta \right\}$$
and let $\mathcal{S} = \bigcup \mathcal{S}_\beta$. Show that s is a subbasis for $\mathcal{T}$.

(c) Show that a map $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and only if each map $f_\alpha \circ g$ is continuous.

(d) Let $f : A \to \prod X_\alpha$ be defined by the equation
$$f(a) = (f_\alpha(a))_{\alpha \in J}$$
let Z denote the subspace $f(A)$ of the product space $\prod X_\alpha$. Show that the image under f of each element of $\mathcal{T}$ is an open set of Z.

Solution. We define $\mathcal{T}$ as the topology generated by $\mathcal{S}$. Then (a) and (b) are clear. To prove (c), it suffices to show that $g^{-1}(f_\alpha^{-1}(U_\alpha))$ is open whenever U_α is open in X_α. But we have $g^{-1}(f_\alpha^{-1}(U_\alpha)) = (f_\alpha \circ g)^{-1}(U_\alpha)$, which is open because $f_\alpha \circ g$ is continuous. Here is a counterexample to (d). Set $A = J = \{1\}, X_1 = \{1\}$ and equip X_1 with the indiscrete topology.

About the definitions p. 119 of the book.

Corollary 20 p. 23 above was a first corollary to Theorem 18.1 p. 104 of the book. Here is a second one:

Corollary 24. *Let X and X' be metric spaces, let $f : X \to X'$ be a map, and x a point of X. Then f is continuous at x if and only if, for all positive ε there is a positive δ such that $f(B(x, \delta)) \subset B(f(x), \varepsilon)$.*

Proof. This follows immediately from Corollary 20. □

If the convention of denoting the couple (x, y) by $x \times y$ was strictly applied, we should write $d(x \times y)$ instead of $d(x, y)$ for the distance between x and y.

Exercise 20.1 p. 126 of the book.

(a) In $\mathbb{R}^n$, define

$$d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$

Show that d' is a metric that induces the usual topology of $\mathbb{R}^n$. Sketch the basis elements under d' when $n = 2$.

(b) More generally, given $p \geq 1$, define

$$d'(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Assume that d' is a metric. Show that it induces the usual topology on $\mathbb{R}^n$.

Solution. (b) For $p \geq 1$ and $x \in \mathbb{R}^n$ we set $|x|_p := (|x_1|^p + \cdots + |x_n|^p)^{1/p}$ and $|x|_\infty := \max_i |x_i|$. For $x, y \in \mathbb{R}^n$ and $1 \leq p \leq \infty$ we set $d_p(x, y) := |y = x|_p$. It was shown in the proof of Theorem 20.3 p. 123 of the book that d_∞ is a metric, and that it induces the product topology on $\mathbb{R}^n$. We assume that d_p is a metric for $1 \leq p < \infty$, and we must prove that it also induces the product topology on $\mathbb{R}^n$. We claim

$$1 \leq \frac{d_p(x, y)}{d_\infty(x, y)} \leq n^{1/p} =: a \tag{2}$$

for all $x \neq y$ in $\mathbb{R}^n$. We first show that this implies that d_p induces the product topology on $\mathbb{R}^n$. We abbreviate B_{d_p} by B_p. The above display implies

$$d_\infty(x, y) \leq d_p(x, y) \leq a d_\infty(x, y),$$

hence $B_p(x,r) \subset B_\infty(x,r) \subset B_p(x,ar)$, hence d_p induces the product topology on $\mathbb{R}^n$. We prove (2). It suffices to show

$$1 \leq \frac{|x|_p^p}{|x|_\infty^p} \leq n$$

for all $x \neq 0$. We can assume $0 \leq x_1 \leq \cdots \leq x_n = 1$. We must show

$$1 \leq \sum_i x_i^p \leq n,$$

which is clear.

Even if this is not requested, we show that d_p is a metric for $p > 1$. We will be content to prove the triangle inequality, the other properties being easy. The argument below is almost a copy-and-paste of the post `https://math.stackexchange.com/a/2283804/660` by Felix Benning.

Let $p, q > 1$. We have

$$\frac{1}{p} + \frac{1}{q} = 1 \iff p + q = pq \iff pq - (p+q) + 1 = 1 \iff (p-1)(q-1) = 1$$

$$\iff p - 1 = \frac{1}{q-1} \iff p\,(q-1) = q \iff p - \frac{p}{q} = 1.$$

Young's Inequality: $xy \leq \dfrac{x^p}{p} + \dfrac{y^q}{q}$ for $x, y \geq 0$.

Proof. We can assume $x, y > 0$. Set $f(x) = \dfrac{x^p}{p} + \dfrac{y^q}{q} - xy$. It suffices to show $f(x) \geq 0$ for $x > 0$. We have

$$f'(x) = x^{p-1} - y, \quad f''(x) = (p-1)\,x^{p-2} > 0, \quad f'(x) = 0 \iff x = y^{\frac{1}{p-1}},$$

$$f'(x) = 0 \iff x = y^{q-1}, \quad f(y^{q-1}) = \frac{y^{p(q-1)}}{p} + \frac{y^q}{q} - y^q = \left(\frac{1}{p} + \frac{1}{q} - 1\right) y^q = 0$$

We see that f' is increasing and vanishes at y^{q-1}, hence f' is negative on $(0, y^{q-1})$ and positive on (y^{q-1}, ∞), hence f is decreasing on $(0, y^{q-1})$, zero at y^{q-1}, and increasing on (y^{q-1}, ∞). This implies $f(x) \geq 0$ for $x > 0$, as required. $\square$

Rewrite Young's Inequality as $uv \leq \dfrac{u^p}{p} + \dfrac{v^q}{q}$ for $u, v \geq 0$.

Hölder's Inequality: $\sum |x_i\, y_i| \leq |x|_p\, |y|_q$, for $x_i, y_i \in \mathbb{R},\ i = 1, \ldots, n$.

Proof. Setting $u_i = \dfrac{|x_i|}{|x|_p}$ and $v_i = \dfrac{|y_i|}{|y|_q}$ and using Young's inequality, we get

$$\frac{|x_i\, y_i|}{|x|_p\, |y|_q} \leq \frac{|x_i|^p}{p\,|x|_p^p} + \frac{|y_i|^q}{q\,|y|_q^q}$$

$$\implies \quad \sum \frac{|x_i\, y_i|}{|x|_p\, |y|_q} \leq \sum \frac{|x_i|^p}{p\,|x|_p^p} + \sum \frac{|y_i|^q}{q\,|y|_q^q}$$

$$\implies \quad \frac{\sum x_i\, y_i}{|x|_p\, |y|_q} \leq \frac{1}{p} + \frac{1}{q} = 1. \ \square$$

We prove the triangle inequality $|x+y|_p \leq |x|_p + |y|_p$. Setting $q := \dfrac{p}{p-1}$ we get

$$\begin{aligned}
|x+y|_p^p &= \sum |x_i \ + y_i|^p \leq \sum |x_i + y_i|^{p-1} |x_i| + \sum |x_i + y_i|^{p-1} |y_i| \\
&\leq \left(\sum |x_i + y_i|^{(p-1)q}\right)^{\frac{1}{q}} \left(\sum |x_i|^p\right)^{\frac{1}{p}} + \left(\sum |x_i + y_i|^{(p-1)q}\right)^{\frac{1}{q}} \left(\sum |y_i|^p\right)^{\frac{1}{p}} \\
&= \left(\sum |x_i + y_i|^p\right)^{\frac{1}{p}\frac{p}{q}} (|x|_p + |y|_p) \\
&= (|x+y|_p)^{\frac{p}{q}} \ (|x|_p + |y|_p),
\end{aligned}$$

hence

$$|x+y|_p = |x+y|_p^{p-\frac{p}{q}} \leq |x|_p + |y|_p.$$

Exercise 20.2 p. 126 of the book. Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

Solution. Define $d(x_1 \times x_2, y_1 \times y_2)$ by decreeing $d(t \times x_2, t \times y_2) = \min(|y_2 - x_2|, 1)$ and $d(x_1 \times x_2, y_1 \times y_2) = 2$ if $x_1 \neq y_1$.

Exercise 20.3 p. 126 of the book. Let X be a metric space with metric d.

(a) Show that $d : X \times X \to \mathbb{R}$ is continuous.

(b) Let X' denote a space having the same underlying set as X. Show that if $d : X' \times X' \to \mathbb{R}$ is continuous, then the topology of X' is finer than the topology of X.

One can summarize the result of this exercise as follows: If X has a metric d, then the topology induced by d is the coarsest topology relative to which the function d is continuous.

Solution. (a) Let $a, b \in \mathbb{R}$ with $a < b$. It suffices to show that $d^{-1}((a,b))$ is open. Let $x_0 \times y_0 \in X^2$ set $c := d(x_0, y_0)$ and assume

$$a < c < b. \tag{3}$$

It suffices to show that there is a positive ε such that $d(x_0, x), d(y_0, y) < \varepsilon$ implies

$$a < d(x,y) < b. \tag{4}$$

Let δ be positive. We have

$$d(x,y) \leq d(x,x_0) + d(x_0,y_0) + d(y_0,y) \leq c + 2\delta.$$

A similar computation shows $c \leq d(x,y) + 2\delta$, that is, $c - 2\delta \leq d(x,y)$, and thus

$$c - 2\delta \leq d(x,y) \leq c + 2\delta. \tag{5}$$

In view of (3), (4) and (5), it suffices to check that, for δ small enough, we have $a < c - 2\delta$ and $c + 2\delta < b$. Clearly $\delta = \min(\frac{c-a}{2}, \frac{b-c}{2})$ does the job, and we can set $\varepsilon := \delta$.

(b) Let x be in X and r be positive. It suffices to show that $B(x,r)$ is open in X'. Note that $U := d^{-1}((-1,r))$ is open in X'. Let $p_2 : X^2 \to X$ be the second projection. We have

$$B(x,r) = p_2\big(U \cap (\{x\} \times X)\big)$$

Let y be in $B(x,r)$. It suffices to find an open subset W of X' such that $y \in W \subset B(x,r)$. By definition of the product topology, there are open subsets V, W of X' such that $x \in V, y \in W$ and $V \times W \subset U$, and we get $y \in W \subset B(x,r)$, as required.

Exercise 20.4 p. 127 of the book. Consider the product, uniform, and box topologies on $\mathbb{R}^\omega$.

(a) In which topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous?

$$f(t) = (t, 2t, 3t, \ldots)$$
$$g(t) = (t, t, t, \ldots)$$
$$h(t) = \left(t, \frac{1}{2}t, \frac{1}{3}t, \ldots\right)$$

(b) In which topologies do the following sequences converge?

$$\begin{aligned}
&\mathbf{w}_1 = (1,1,1,1,\ldots), && \mathbf{x}_1 = (1,1,1,1,\ldots)\\
&\mathbf{w}_2 = (0,2,2,2,\ldots), && \mathbf{x}_2 = \left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots\right)\\
&\mathbf{w}_3 = (0,0,3,3,\ldots), && \mathbf{x}_3 = \left(0, 0, \frac{1}{3}, \frac{1}{3} \ldots\right)\\
&\ldots\ldots && \\
&\mathbf{y}_1 = (1,0,0,0,\ldots), && \mathbf{z}_1 = (1,1,0,0,\ldots)\\
&\mathbf{y}_2 = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, \ldots\right), && \mathbf{z}_2 = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, \ldots\right)\\
&\mathbf{y}_3 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \ldots\right), && \mathbf{z}_3 = \left(\frac{1}{3}, \frac{1}{3}, 0, 0, \ldots\right)\\
&\ldots\ldots &&
\end{aligned}$$

Solution. (a) We equip $\mathbb{R}^\omega$ with its natural $\mathbb{R}$-vector space structure, and observe that the three topologies considered are invariant by translation. Since f, g and h are linear, they are continuous if and only if they are continuous at 0. We can use Corollary 20 p. 23, and, for the uniform topology, Corollary 24 p. 27. It is easy to see that f, g and h are continuous for the product topology, and discontinuous for the box topology. So, let us endow $\mathbb{R}^\omega$ with the uniform topology, that is, the topology given by the metric $\overline{\rho}$. Let ε be positive and less than 1. We must decide of there is a positive δ such that $k((-\delta,\delta)) \subset B_{\overline{\rho}}(0,\varepsilon)$ for k equal to f, g or h. Since we have $\overline{\rho}(0, f(\frac{\delta}{2})) = 1$, we see that f is discontinuous. If k equal to g or h, and $|t|$ is less than $\frac{\varepsilon}{2}$, we have $\overline{\rho}(0, g(t)) < \varepsilon$, and we can set $\delta := \frac{\varepsilon}{2}$. This shows that g and h are continuous.

(b) Assume that one of the sequences u_n in the statement tends to $a \in \mathbb{R}^\omega$ in the topology $\mathcal{T}$. Since each projection $\mathbb{R}^\omega \to \mathbb{R}$ is continuous, we have $\lim_{n\to\infty} u_{ni} = a_i$ for all i. Since $\lim_{n\to\infty} u_{ni} = 0$, this implies $a = 0$. Conversely, it is easy to see that, if $\lim_{n\to\infty} u_{ni} = a_i$ for all i, then u_n tends to 0 in the product topology. Clearly z_n tends to 0 in each of the three topologies.

We show that y_n does not tend to 0 in the box topology. Set

$$U := \prod_{i=1}^{\infty} (-i^{-2}, i^{-2}).$$

Then U is open in the box topology and contains 0. Then $y_n \in U$ for some n would imply $|y_{ni}| < i^{-2}$ for all i, and thus $|y_{nn}| < n^{-2}$, that is $n^{-1} < n^{-2}$, contradiction. As a result, y_n does not tend to 0 in the box topology. A similar argument applies to w_n and x_n.

It only remans to handle the uniform topology. In this case, u_n tends to 0 if and only if $\overline{\rho}(0, u_n)$ does, that is, u_n tends to 0 if and only if there is a positive integer n_0 and a sequence $a_1, a_2, \ldots$ of positive numbers such that $n \geq n_0$ and $i \in \omega$ imply $|u_{ni}| \leq a_n$. In particular w_n does not tend to 0, but the other sequences do.

Exercise 20.5 p. 127 of the book. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the uniform topology? Justify your answer.

Solution. The closure in question is the set C of all sequences tending to 0. Let us show that C is closed. Let $x_1, x_2, \ldots$ be a sequence not tending to 0. There is a positive ε such that the set $\{n \in \omega \mid |x_n| > \varepsilon\}$ is infinite, and $B_{\overline{\rho}}(x, \varepsilon/2) \cap \mathbb{R}^\infty = \varnothing$, so C is closed. Let x be in C, let ε be positive, and lets us prove $B_{\overline{\rho}}(x,\varepsilon) \cap \mathbb{R}^\infty \neq \varnothing$. There is a n_0 in ω such that $n > n_0$ implies $|x_n| < \varepsilon/2$. Define $y \in \mathbb{R}^\infty$ by $y_n = x_n$ if $n \leq n_0$ and $y_n = 0$ if $n > n_0$. Then y is in $B_{\overline{\rho}}(x,\varepsilon)$.

Exercise 20.6 p. 127 of the book. Let $\overline{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. Given $\mathbf{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$ and given $0 < \varepsilon < 1$, let

$$U(\mathbf{x}, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon) \times \cdots$$

(a) Show that $U(\mathbf{x}, \varepsilon)$ is not equal to the ε-ball $B_{\overline{\rho}}(\mathbf{x}, \varepsilon)$.

(b) Show that $U(\mathbf{x}, \varepsilon)$ is not even open in the uniform topology.

(c) Show that

$$B_{\overline{\rho}}(\mathbf{x}, \varepsilon) = \bigcup_{\delta < \varepsilon} U(\mathbf{x}, \delta).$$

Solution. (a) Define $y \in \mathbb{R}^\omega$ by $y_n := x + \frac{n-1}{n}\,\varepsilon$. Then y is in $U(x,\varepsilon)$ but not in $B_{\overline{\rho}}(x,\varepsilon)$.

(b) We can assume $x = 0$. Again, define $y \in \mathbb{R}^\omega$ by $y_n := \frac{n-1}{n}\,\varepsilon$. Then y is in $U(0,\varepsilon)$. Let δ be positive, and set $z_n := y_n + \frac{\delta}{2} = \frac{n-1}{n}\varepsilon + \frac{\delta}{2}$. Then z is in $B_{\overline{\rho}}(y,\delta)$ but not in $U(0,\varepsilon)$.

(c) We can again assume $x = 0$. The inclusion $\bigcup_{\delta<\varepsilon} U(0,\delta) \subset B_{\overline{\rho}}(0,\varepsilon)$ is clear. To prove the converse inclusion, let y be in $B_{\overline{\rho}}(0,\varepsilon)$, and let α satisfy $\overline{\rho}(0,y) < \alpha < \varepsilon$. There is an n_0 such that $|y_n| < \alpha$ for $n > n_0$, and a δ such that $\alpha < \delta < \varepsilon$ and $|y_n| < \delta$ for all n, that is, $y \in U(0,\delta)$.

Exercise 20.7 p. 127 of the book. Consider the map $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ defined in Exercise 8 of §19; give $\mathbb{R}^\omega$ the uniform topology. Under what conditions on the numbers a_i and b_i is h continuous? a homeomorphism?

Solution. Recall that h is defined by $h(x_1, x_2, \ldots) = (a_1x_1 + b_1, a_2x_2 + b_2, \ldots)$ with $a_i > 0$ for all i.

We claim that h is continuous if and only if the a_i are bounded, and that h is a homeomorphism if and only if the a_i and the a_i^{-1} are bounded.

Set $h'(x_1, x_2, \ldots) = (a_1x_1, a_2x_2, \ldots)$. It is easy to see that h is continuous if and only if h' is continuous if and only if h' is continuous at 0, and that h is a homeomorphism if and only if h' and h'^{-1} are continuous at 0. In other words, we can assume $b_i = 0$ for all i. We shall use Corollary 24 p. 27 above.

If $a_i < c$ for all i and ε is positive and less than 1, then $h(B(0,\varepsilon/c)) \subset B(0,\varepsilon)$.

Assume that the a_i are not bounded, let ε be as above, and let δ be positive and less than 1. Then $(\frac{\delta}{2}, \frac{\delta}{2}, \ldots) \in B(0,\delta)$ but $h(\frac{\delta}{2}, \frac{\delta}{2}, \ldots) \notin B(0,\varepsilon)$; in particular $h(B(0,\delta)) \not\subset B(0,\varepsilon)$.

Finally, not that $h^{-1}(x_1, x_1, \ldots) = (a_1^{-1}x_1, a_2^{-1}x_2, \ldots)$. This proves the claim.

Exercise 20.8 p. 127 of the book. Let X be the subset of $\mathbb{R}^\omega$ consisting of all sequences $\mathbf{x}$ such that $\sum x_i^2$ converges. Then the formula

$$d(\mathbf{x},\mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{1/2}$$

defines a metric on X. (See Exercise 10.) On X we have the three topologies it inherits from the box, uniform, and product topologies on $\mathbb{R}^n$. We have also the topology given by the metric d, which we call the ℓ^2-topology. (Read "little ell two.")

(a) Show that on X, we have the inclusions

$$\text{box topology} \supset \ell^2\text{-topology} \supset \text{uniform topology}.$$

(b) The set $\mathbb{R}^\infty$ of all sequences that are eventually zero is contained in X. Show that the four topologies that $\mathbb{R}^\infty$ inherits as a subspace of X are all distinct.

(c) The set

$$H = \prod_{n \in \mathbb{Z}_+} [0, 1/n]$$

is contained in X; it is called the Hilbert cube. Compare the four topologies that H inherits as a subspace of X.

Solution. (a) Let I be the interval $(0,1)$. If $\varepsilon_* = (\varepsilon_1, \varepsilon_2, \ldots) \in I^\omega$, set

$$B(\varepsilon_*) := X \cap ((-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_2, \varepsilon_2) \times \cdots) = \{x \in X \mid |x_i| < \varepsilon_i \ \forall\ i\}.$$

If $\varepsilon \in I$, set $B_2(\varepsilon) := B_d(0, \varepsilon)$ and $B_u(\varepsilon) := X \cap B_{\overline{\rho}}(0, \varepsilon)$. An easy argument implies that it suffices to show:

(A) for all $\varepsilon \in I$ there is a $\delta_* \in I^\omega$ such that $B(\delta_*) \subset B_2(\varepsilon)$,

(B) for all $\varepsilon \in I$ there is a $\delta \in I$ such that $B_2(\delta) \subset B_u(\varepsilon)$.

To prove (A) (resp. (B)) set $\delta_n := 2^{-n/2}\varepsilon$ (resp. $\delta = \varepsilon$).

(b) Let us denote the box, ℓ^2, uniform and product topologies on X by $\mathcal{T}_b, \mathcal{T}_2, \mathcal{T}_u, \mathcal{T}_p$ respectively, and denote the respective topologies induced on $\mathbb{R}^\infty$ by $\mathcal{T}_b', \mathcal{T}_2', \mathcal{T}_u', \mathcal{T}_p'$. By (a) we have $\mathcal{T}_b' \supset \mathcal{T}_2' \supset \mathcal{T}_u' \supset \mathcal{T}_p'$, and we must show $\mathcal{T}_b' \supsetneqq \mathcal{T}_2' \supsetneqq \mathcal{T}_u' \supsetneqq \mathcal{T}_p'$. It suffices to prove $\mathcal{T}_b' \neq \mathcal{T}_2' \neq \mathcal{T}_u' \neq \mathcal{T}_p'$. In the notation of the solution to (a) set

$$C(\varepsilon_*) := \mathbb{R}^\infty \cap B(\varepsilon_*), \quad C_2(\varepsilon) := \mathbb{R}^\infty \cap B_2(\varepsilon), \quad C_u(\varepsilon) := \mathbb{R}^\infty \cap B_u(\varepsilon),$$

and, for $\varepsilon \in I$ and $n \in \omega$ set $C(\varepsilon, n) := \{x \in \mathbb{R}^\infty \mid |x_i| < \varepsilon \ \forall\ i \leq n\}$. Then the $C(\varepsilon_*)$ and their translates form a basis of $\mathcal{T}_b'$, the $C_2(\varepsilon)$ and their translates form a basis of $\mathcal{T}_2'$, the $C_u(\varepsilon)$ and their translates form a basis of $\mathcal{T}_u'$, and the $C(\varepsilon, n)$ and their translates form a basis of $\mathcal{T}_p'$. We show:

- $\mathcal{T}_b' \neq \mathcal{T}_2'$: It suffices to check that $C_2(\delta) \not\subset C(1, \frac{1}{2}, \frac{1}{3}, \ldots)$ for all positive δ. In fact, let $i \in \omega$ be larger than $\frac{2}{\delta}$, let e_i be the i-th vector of the canonical basis of $\mathbb{R}^\infty$, and observe that $\frac{\delta}{2}e_i$ is in $C_2(\delta)$ but not in $C(1, \frac{1}{2}, \frac{1}{3}, \ldots)$.

- $\mathcal{T}_2' \neq \mathcal{T}_u'$: It suffices to check that $C_u(\delta) \not\subset C_2(1)$ for all positive δ. Indeed, let $n \in \omega$ be larger than $\frac{4}{\delta^2}$ and, in the above notation, set $x = \frac{\delta}{2}\sum_{i=1}^n e_i$, and observe that x is in $C_u(\delta)$ but not in $C_2(1)$.

- $\mathcal{T}_u' \neq \mathcal{T}_p'$: It suffices to check that $C(\delta, n) \not\subset C_u(1)$ for all positive δ and all $n \in \omega$. This follows from the fact that e_{n+1} is in $C(\delta, n)$ but not in $C_u(1)$.

(c) Recall that we denote the box, ℓ^2, uniform and product topologies on X by $\mathcal{T}_b, \mathcal{T}_2, \mathcal{T}_u, \mathcal{T}_p$ respectively, and denote the respective topologies induced on H by $\mathcal{T}_b'', \mathcal{T}_2'', \mathcal{T}_u'', \mathcal{T}_p''$. We claim $\mathcal{T}_b'' \supsetneqq \mathcal{T}_2'' = \mathcal{T}_u'' = \mathcal{T}_p''$. In view of (a) and (b) it suffices to show $\mathcal{T}_b'' \neq \mathcal{T}_2'' = \mathcal{T}_p''$. For $x \in X$ define $t_x : X \to X$ by $t_x(y)_i := x_i + y_i$, and set

$$D(x, \varepsilon_*) := \mathbb{R}^\infty \cap t_x(C(\varepsilon_*)), \quad D_2(x, \varepsilon) := \mathbb{R}^\infty \cap t_x(C_2(\varepsilon)),$$

$$D_u(x, \varepsilon) := \mathbb{R}^\infty \cap t_x(C_u(\varepsilon)), \quad D(x, \varepsilon, n) := \mathbb{R}^\infty \cap t_x(C(\varepsilon, n))$$

for $x \in H, \varepsilon_* \in I^\omega, \varepsilon \in I$ and $n \in \omega$. Then the $D(x, \varepsilon_*)$ form a basis of $\mathcal{T}_b''$, the $D_2(x, \varepsilon)$ form a basis of $\mathcal{T}_2''$, the $D_u(x, \varepsilon)$ form a basis of $\mathcal{T}_u''$, and the $D(x, \varepsilon, n)$ form a basis of $\mathcal{T}_p''$.

• $\mathcal{T}_b'' \neq \mathcal{T}_2''$: It suffices to show that there is an $\varepsilon_* \in I^\omega$ such that $D_2(0, \delta) \not\subset D(0, \varepsilon_*)$ for all $\delta \in I$. We claim $D_2(0, \delta) \not\subset D(0, (2^{-1}, 2^{-2}, 2^{-3}, \ldots))$ for all $\delta \in I$. To prove this, let δ be in I. Set $x_i := 2^{-i/2}$. We get $d(0, x) = 1$ and $d(0, \frac{\delta}{2}\, x) = \frac{\delta}{2}$, so $\frac{\delta}{2}\, x \in D_2(0, \delta)$. We claim $\frac{\delta}{2}\, x \notin D(0, \varepsilon_*)$. To prove this it suffices to show $\frac{\delta}{2}\, x_i \geq \varepsilon_i$, that is $\frac{\delta}{2}\, 2^{-i/2} \geq 2^{-i}$, that is $2^{i/2} \geq \frac{2}{\delta}$, for i large enough, which is clear.

• $\mathcal{T}_2'' = \mathcal{T}_p''$: Let x be in H and ε be in I. It suffices to show that there an $n \in \omega$ such that $D(x, \frac{1}{n}, n) \subset D_2(\varepsilon)$. Let $n \in \omega$ and $y \in D(x, \frac{1}{n}, n)$ be arbitrary. Set $z_i := y_i - x_i$ for all i. We want a condition on n which implies $y \in D_2(\varepsilon)$, that is, $\sum_{i=1}^\infty z_i^2 < \varepsilon^2$. We have

$$\sum_{i=1}^\infty z_i^2 = \sum_{i=1}^n z_i^2 + \sum_{i=n+1}^\infty z_i^2 < n\,\frac{1}{n^2} + \sum_{i=n+1}^\infty \frac{1}{i^2} = \frac{1}{n} + \sum_{i=n+1}^\infty \frac{1}{i^2} =: f(n),$$

and we can pick an n such that $f(n) < \varepsilon^2$.

Exercise 20.9 p. 128 of the book. Show that the euclidean metric d on $\mathbb{R}^n$ is a metric, as follows: If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, define

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, \ldots, x_n + y_n) \\ c\mathbf{x} &= (cx_1, \ldots, cx_n) \\ \mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + \cdots + x_n y_n \end{aligned}$$

(a) Show that $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z})$.

(b) Show that $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$. [Hint: If $\mathbf{x}, \mathbf{y} \neq 0$, let $a = 1/\|\mathbf{x}\|$ and $b = 1/\|\mathbf{y}\|$, and use the fact that $\|a\mathbf{x} \pm b\mathbf{y}\| \geq 0$.]

(c) Show that $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. [Hint: Compute $(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y})$ and apply (b).]

(d) Verify that d is a metric.

Solution. Left to the reader.

Exercise 20.10 p. 128 of the book. Let X denote the subset of $\mathbb{R}^n$ consisting of all sequences $(x_1, x_2, \ldots)$ such that $\sum x_i^2$ converges. (You may assume the standard facts about infinite series. In case they are not familiar to you, we shall give them in Exercise 11 of the next section.)

(a) Show that if $\mathbf{x}, \mathbf{y} \in X$, then $\sum |x_i\, y_i|$ converges. [Hint: Use (b) of Exercise 9 to show that the partial sums are bounded.]

(b) Let $c \in \mathbb{R}$. Show that if $\mathbf{x}, \mathbf{y} \in X$, then so are $\mathbf{x} + \mathbf{y}$ and $c\mathbf{x}$.

(c) Show that

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{1/2}$$

is a well-defined metric on X.

Solution. Left to the reader.

Exercise 20.11 p. 128 of the book. Show that if d is a metric for X, then

$$d'(x, y) = d(x, y)/(1 + d(x, y))$$

is a bounded metric that gives the topology of X. [Hint: If $f(x) = x/(1 + x)$ for $x > 0$, use the mean-value theorem to show that $f(a + b) - f(b) \leq f(a)$.]

Solution. We prove the hint and leave the rest to the reader. We can assume $0 < a \leq b$. We have $f'(x) = \frac{1}{(1+x)^2}$. The mean-value theorem implies $f(a + b) - f(b) = af'(c) = \frac{a}{(1+c)^2}$ for some $c \in (b, a + b)$. It suffices to show $f'(c) \leq \frac{f(a)}{a}$, or, equivalently, $\frac{a}{f(a)} \leq \frac{1}{f'(c)}$. We have

$$\frac{a}{f(a)} = 1 + a \leq 1 + b \leq (1 + c)^2 = \frac{1}{f'(c)} \ .$$