

Existence of a least increasing surjection from an artinian poset onto a set of ordinals¹

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Let X be a set. If $g : X \rightarrow S$ and $h : X \rightarrow T$ are surjections and S and T are sets of ordinals, say that $g \leq h$ if $g(x) \leq h(x)$ for all x . If X is a poset, say that g is **increasing** if $x < y$ implies $g(x) < g(y)$, and that it is **weakly increasing** if $x \leq y$ implies $g(x) \leq g(y)$.

Theorem 1. *If X is an artinian poset, then there is a least increasing surjection f from X onto a set of ordinals.*

The theorem will follow from Propositions 3 and 4 below.

If β is a ordinal, we write $[0, \beta)$ (resp. $[0, \beta]$) for the set of all ordinals $< \beta$ (resp. $\leq \beta$). If S is a set of ordinals, we denote by $\alpha(S)$ the least ordinal not in S , and by $\omega(S)$ the least ordinal larger than all the elements of S . We obviously have $\alpha(S) \leq \omega(S)$.

Lemma 2. *The following conditions are equivalent:*

- (a) $\alpha(S) = \omega(S)$,
- (b) $S = [0, \beta)$ for some ordinal β ,
- (c) $S = [0, \alpha(S))$,
- (d) $S = [0, \omega(S))$,
- (e) no element of S is larger than $\alpha(S)$.

The proof is left to the reader.

Recall that X is an artinian poset. For x in X , set $X_x := \{y \in X \mid y < x\}$. Let κ be a cardinal larger than the cardinality of X , and set $A := [0, \kappa]$. For x in X let A^{X_x} be the set of all maps from X_x to A . Define $r_x : A^{X_x} \rightarrow A$ by letting $r_x(h)$ be the minimum of κ and $\alpha(h(X_x))$, and $s_x : A^{X_x} \rightarrow A$ by letting $s_x(h)$ be the minimum of κ and $\omega(h(X_x))$. If $h : X \rightarrow A$ is a map, we denote by $h|_{X_x}$ the restriction of h to X_x . By the last statement in

<https://github.com/Pierre-Yves-Gaillard/The-Transfinite-Recursion-Theorem>

there is a unique map $f : X \rightarrow A$ such that $f(x) = r_x(f|_{X_x})$ for all x in X , and a unique map $g : X \rightarrow A$ such that $g(x) = s_x(g|_{X_x})$ for all x in X . For all x in X set $\alpha_x := \alpha(f(X_x))$ and $\omega_x := \omega(g(X_x))$.

Proposition 3. *We have:*

- (f) $f = g$,
- (g) $f(X_x) = [0, \alpha_x) = [0, \omega_x)$ for all x .
- (h) $f(x) = \alpha_x = \omega_x$ for all x .

Proof. Note that f and g are weakly increasing. Let (f') be the condition (obviously equivalent to (f)) that $f(x) = g(x)$ for all x . Assume by contradiction that at least one of the three conditions

¹available at <https://github.com/Pierre-Yves-Gaillard/Artin-poset-to-ordinals>.

(f'), (g), (h) is false. Let x be minimal among the elements for which (f'), (g) or (h) fails. Note that $f(X_y) = g(X_y)$ for $y \leq x$.

We have $\alpha_x \leq \omega_x$. If $\alpha_x = \omega_x$, then the lemma implies $f(X_x) = [0, \alpha_x)$, and (f'), (g) and (h) hold for x , contradiction. We conclude that $\alpha_x < \omega_x$. By (e) there is a y less than x such that $f(y) > \alpha_x \notin f(X_x)$.

Claim: We have $f(z) < \alpha_x$ for $z < y$.

Proof: Otherwise we would have $\alpha_x < f(z)$, and thus

$$\alpha_x \in [0, f(z)) = [0, \alpha_z) = [0, \omega_z) = f(X_z) \subset f(X_x),$$

contradiction. This proves the claim.

By the claim α_x is larger than each element of $f(X_y)$, hence $\alpha_x \geq \omega_y$, hence $f(y) = \omega_y \leq \alpha_x < f(y)$, contradiction. \square

Proposition 4. *We have:*

- (i) $f(X) = [0, \alpha(f(X))) = [0, \omega(f(X)))$,
- (j) $f : X \rightarrow [0, \alpha(f(X)))$ is the least increasing surjection from X onto a set of ordinals.

Proof. Define the poset Y by the following conditions: $Y = X \cup \{\infty\}$, $\infty \notin X$, X is a sub-poset of Y . Then Y is artinian and satisfies $Y_\infty = X$, and we can prove (i) by applying Proposition 3 to Y (instead of X). More precisely, let $f : X \rightarrow S$ (resp. $g : X \rightarrow S$) be the least increasing surjection from X (resp. Y) onto a set of ordinals. We have $f(X) = g(X) = [0, \alpha(f(X))) = [0, \omega(f(X)))$, the first equality following from the fact that $Y_\infty = X$, the second and third equalities following from (g) above. Then (j) is a consequence of (i) and Proposition 3. \square

The following corollary to Propositions 3 and 4 is well known.

Corollary 5. *Let β be an ordinal, X a subset of $[0, \beta)$ and $f : X \rightarrow [0, \gamma)$ the least increasing surjection of X onto a set of ordinals. Then we have $f(x) \leq x$ for all x , and $\gamma \leq \beta$.*

Proof. It suffices to show $f(x) \leq x$ for all x . Suppose not, and let x be the least element of X satisfying $f(x) > x$. We get $x < f(x) = \alpha(f(X_x)) = \omega(f(X_x))$ and $f(X_x) = [0, f(x)) \ni x$, hence $x = f(y)$ for some y less than x , hence $f(y) \leq y < x = f(y)$, contradiction. \square

We end with a loosely related question. If X and Y are posets and $f : X \rightarrow Y$ and $g : Y \rightarrow X$ are increasing surjections, are X and Y necessarily isomorphic? The following example shows that the answer is no. Let X and Y be the posets respectively given by the Hasse diagrams

$$\begin{array}{cccc} b_1 & b_2 & b_3 & \dots \\ | & | & | & \\ a_1 & a_2 & a_3 & \end{array}$$

and

$$\begin{array}{cccc} d_1 & d_2 & d_3 & d_4 & \dots \\ \backslash & / & | & | & \\ c_1 & & c_3 & c_4 & \end{array}$$

We see that if we identify a_1 and a_2 in X we get a poset isomorphic to Y , and if we identify d_1 and d_2 in Y we get a poset isomorphic to X .