Existence of a least increasing surjection from an artinian poset onto a set of ordinals[1]

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Let X be a set. If $g : X \to S$ and $h : X \to T$ are surjections and S and T are sets of ordinals, say that $g \le h$ if $g(x) \le h(x)$ for all x. If X is a poset, say that g is **increasing** if $x < y$ implies $g(x) < g(y)$, and that it is **weakly increasing** if $x \le y$ implies $g(x) \le g(y)$.

Theorem 1. *If X is an artinian poset, then there is a least increasing surjection f from X onto a set of ordinals.*

The theorem will follow from Propositions 3 and 4 below.

If β is a ordinal, we write $[0, \beta)$ (resp. $[0, \beta]$) for the set of all ordinals $< \beta$ (resp. $\le \beta$). If S is a set of ordinals, we denote by $\alpha(S)$ the least ordinal not in S, and by $\omega(S)$ the least ordinal larger than all the elements of S. We obviously have $\alpha(S) \le \omega(S)$.

Lemma 2. *The following conditions are equivalent:*

(a) $\alpha(S) = \omega(S)$,

(b) $S = [0, \beta)$ *for some ordinal* β,

(c) $S = [0, \alpha(S))$,

(d) $S = [0, \omega(S))$,

(e) *no element of* S *is larger than* $\alpha(S)$.

The proof is left to the reader.

Recall that X is an artinian poset. For x in X, set $X_x := \{y \in X \mid y < x\}$. Let κ be a cardinal larger that the cardinality of X, and set $A := [0, \kappa]$. For x in X let A^{X_x} be the set of all maps from X_x to A. Define $r_x : A^{X_x} \to A$ by letting $r_x(h)$ be the minimum of κ and $\alpha(h(X_x))$, and $s_x : A^{X_x} \to A$ by letting $s_x(h)$ be the minimum of κ and $\omega(h(X_x))$. If $h : X \to A$ is a map, we denote by $h|X_x$ the restriction of h to X_x. By the last statement in

`https://github.com/Pierre-Yves-Gaillard/The-Transfinite-Recursion-Theorem`

there is a unique map $f : X \to A$ such that $f(x) = r_x(f|X_x)$ for all x in X, and a unique map $g : X \to A$ such that $g(x) = s_x(g|X_x)$ for all x in X.

Proposition 3. *We have:*

(f) $f = g$,

(g) $f(X_x) = [0, \alpha(f(X_x))) = [0, \omega(f(X_x)))$ *for all* x.

(h) $f(x) = \alpha(f(X_x)) = \omega(f(X_x))$ *for all* x.

Proof. Note that f and g are weakly increasing. Let (f') be the condition (obviously equivalent to (f)) that $f(x) = g(x)$ for all x. Assume by contradiction that at least one of the three conditions

[1] available at `https://github.com/Pierre-Yves-Gaillard/Artin-poset-to-ordinals`.

(f'), (g), (h) is false. Let x be minimal among the elements for which (f'), (g) or (h) fails. For $y \leq x$ note that $f(X_y) = g(X_y)$, and set $\alpha_y := \alpha(f(X_y))$ and $\omega_y := \omega(f(X_y))$.

We have $\alpha_x \leq \omega_x$. If $\alpha_x = \omega_x$, then the lemma implies $f(X_x) = [0, \alpha_x)$, and (f'), (g) and (h) hold for x, contradiction. We conclude that $\alpha_x < \omega_x$. By (e) there is a y less than x such that $f(y) > \alpha_x \notin f(X_x)$.

Claim: We have $f(z) < \alpha_x$ for $z < y$.

Proof: Otherwise we would have $\alpha_x < f(z)$, and thus

$$\alpha_x \in [0, f(z)) = [0, \alpha_z) = [0, \omega_z) = f(X_z) \subset f(X_x),$$

contradiction. This proves the claim.

By the claim α_x is larger than each element of $f(X_y)$, hence $\alpha_x \geq \omega_y$, hence $f(y) = \omega_y \leq \alpha_x < f(y)$, contradiction. □

Proposition 4. *We have:*

(i) $f(X) = [0, \alpha(f(X))) = [0, \omega(f(X)))$,

(j) $f : X \to [0, \alpha(f(X)))$ *is the least increasing surjection from X onto a set of ordinals.*

Proof. Define the poset Y be the following conditions: $Y = X \cup \{\infty\}, \infty \notin X, X$ is a sub-poset of Y. Then Y is artinian and satisfies $Y_\infty = X$, and we can prove (i) by applying Proposition 3 to Y (instead of X). Then (j) is a consequence of (i) and Proposition 3. □

The following corollary to Propositions 3 and 4 is well known.

Corollary 5. *Let β be an ordinal, X a subset of $[0, \beta)$ and $f : X \to [0, \gamma)$ the least increasing surjection of X onto a set of ordinals. Then we have $f(x) \leq x$ for all x, and $\gamma \leq \beta$.*

Proof. It suffices to show $f(x) \leq x$ for all x. Suppose not, and let x be the least element of X satisfying $f(x) > x$. We get $x < f(x) = \alpha(f(X_x)) = \omega(f(X_x))$ and $f(X_x) = [0, f(x)) \ni x$, hence $x = f(y)$ for some y less than x, hence $f(y) \leq y < x = f(y)$, contradiction. □