Number of elements of a finite set[1]

Pierre-Yves Gaillard

The statements proved here are almost obvious and not too difficult to find in the literature. I wrote this short text to make them as easily available as possible.

Let $\mathbb{N}$ be the set of nonnegative integers, and for each $n \in \mathbb{N}$ set $[n] := \{1, 2, \dots, n\}$ (in particular $[0] = \varnothing$). Let a bijection $[n] \to X$, where X is a set, be called a **count** of X, and let n be called the **result** of the count. We say that X is **finite** if it admits a count $[n] \to X$ for some $n \in \mathbb{N}$.

Theorem 1. *If a set X is finite, then any two counts of X give the same result.*

If this is the case and the result is n, we say that X **has n elements**. If X and Y are two sets, write $X \simeq Y$ to indicate that there is a bijection $X \to Y$.

Lemma 2. *Let X and Y be two sets; let k be in $\mathbb{N}$; let $a_1, \dots, a_k$ be distinct elements of X; let $b_1, \dots, b_k$ be distinct elements of Y; and consider the following conditions:*

(a) $X \simeq Y$,

(b) $X \setminus \{a_1, \dots, a_k\} \simeq Y \setminus \{b_1, \dots, b_k\}$,

(c) $X = \{a_1, \dots, a_k\} \Leftrightarrow Y = \{b_1, \dots, b_k\}$.

Then we have: (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c).

Proof. We clearly have (a) $\Leftarrow$ (b) $\Rightarrow$ (c). Let us show that (a) $\Rightarrow$ (b). The case $k = 0$ is obvious. Assume $k = 1$. If Z is any set and $a, b \in Z$, then we define the map $[Z, a, b] : Z \to Z$ by

$$[Z, a, b](z) = \begin{cases} b & \text{if } z = a \\ a & \text{if } z = b \\ z & \text{otherwise.} \end{cases} \tag{1}$$

Then $[Z, a, b]$ is its own inverse; in particular it is bijective. By assumption there is a bijection $f : X \to Y$. Then $[Y, f(a_1), b_1] \circ f$, being a bijection from X to Y mapping a_1 to b_1, induces a bijection from $X \setminus \{a_1\}$ to $Y \setminus \{b_1\}$. If $k > 1$ we have, assuming (as we may) that the implication holds for $k - 1$,

$$X \setminus \{a_1, \dots, a_k\} = (X \setminus \{a_1, \dots, a_{k-1}\}) \setminus \{a_k\} \simeq (Y \setminus \{b_1, \dots, b_{k-1}\}) \setminus \{b_k\} = Y \setminus \{b_1, \dots, b_k\}.$$

$\square$

Lemma 3. *If $[k] \to Z$ is a count of a set Z, and if $c_1, \dots, c_k$ are distinct elements of Z, then we have $Z = \{c_1, \dots, c_k\}$.*

[1] This text is available at
`https://github.com/Pierre-Yves-Gaillard/Number-of-elements-of-a-finite-set`.

Proof. Use Lemma 2 with $X = [k], Y = Z$ and $a_i = i, b_i = c_i$ for $i = 1, \ldots, k$. $\square$

Proposition 4. *If $m, n \in \mathbb{N}$ satisfy $[m] \simeq [n]$, then we have $m = n$.*

Proof. We assume $m \leq n$ (as we may, by symmetry $\simeq$) and apply Lemma 3, (a) $\Rightarrow$ (c), with $k = m, Z = [n]$ and $c_i = i$ for $i = 1, \ldots, m$. $\square$

Theorem 1 now follows immediately from Proposition 4.

We end with two additional results. Let us denote the number of elements of a finite set X by $|X|$.

Theorem 5. *If Y is a proper subset a finite set X (i.e. $Y \subsetneqq X$), then Y is finite and satisfies $|Y| < |X|$.*

Theorem 6 (Pigeonhole Principle)**.** *If X and Y are finite set, if $|Y| < |X|$, and if $f : X \to Y$ is a map, then f is* not *injective.*

Proof of Theorem 5. We can assume $X = [n]$ with $n > 0$. Let i be in $[n] \setminus Y$ and, using Notation (1) set $Z := [[n], i, n](Y)$. Then $|Z| = |Y|$ and Z is a subset of $[n]$ contained in $[n-1]$. An obvious induction completes the proof. $\square$

Proof of Theorem 6. If f was injective, Theorem 5 would imply $|X| = |f(X)| \leq |Y| < |X|$, contradiction. $\square$

For completeness' sake we add the "Co-Pigeonhole Principle":

Proposition 7 ("Co-Pigeonhole Principle")**.** *If X and Y are finite set, if $|X| < |Y|$, and if $f : X \to Y$ is a map, then f is* not *surjective.*

Proof. Assume by contradiction that f *is* surjective. We can suppose $X = [n]$. Define $g : Y \to X$ by $g(y) = \min f^{-1}(y)$. Then g is injective and Theorem 5 implies $|X| < |Y| = |g(Y)| \leq |X|$, contradiction. $\square$

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