

# Number of elements of a finite set<sup>1</sup>

Pierre-Yves Gaillard

The statements proved here are almost obvious and not too difficult to find in the literature. I wrote this short text to make them as easily available as possible.

Let  $\mathbb{N}$  be the set of nonnegative integers, and, for each  $n \in \mathbb{N}$  set  $[n] := \{1, 2, \dots, n\}$  (in particular  $[0] = \emptyset$ ). If  $X$  is a set, say that a bijection  $[n] \rightarrow X$  is a **count** of  $X$ , and that  $n$  is the **result** of the count. Say that  $X$  is **finite** if it admits a count  $[n] \rightarrow X$  for some  $n \in \mathbb{N}$ . If  $X$  and  $Y$  are two sets, write  $X \simeq Y$  to indicate that there is a bijection  $X \rightarrow Y$ .

**Theorem 1.** *If a set  $X$  is finite, then any two counts of  $X$  give the same result.*

If such is the case and the result is  $n$ , we say that  $X$  **has  $n$  elements**. The theorem will follow from the proposition below:

**Proposition 2.** *If  $m, n \in \mathbb{N}$  satisfy  $[m] \simeq [n]$ , then we have  $m = n$ .*

If  $X$  is a set and  $a, b$  are in  $X$ , then we define the map  $[X, a, b] : X \rightarrow X$  by

$$[X, a, b](x) = \begin{cases} b & \text{if } x = a \\ a & \text{if } x = b \\ x & \text{otherwise.} \end{cases}$$

The  $[X, a, b]$  is its own inverse; in particular it is bijective.

**Lemma 3.** *If  $X$  and  $Y$  are sets such that  $X \simeq Y$ , and if  $a$  is in  $X$  and  $b$  in  $Y$ , then there is a bijection  $X \rightarrow Y$  mapping  $a$  to  $b$ .*

*Proof.* By assumption there is a bijection  $f : X \rightarrow Y$ . Then  $[Y, f(a), b] \circ f$  does the job.  $\square$

**Lemma 4.** *If  $X$  and  $Y$  are sets such that  $X \simeq Y$ , and if  $a$  is in  $X$  and  $b$  in  $Y$ , then we have  $X \setminus \{a\} \simeq Y \setminus \{b\}$ .*

*Proof.* By Lemma 3 there is a bijection  $f : X \rightarrow Y$  mapping  $a$  to  $b$ . Then  $f$  induces a bijection  $X \setminus \{a\} \rightarrow Y \setminus \{b\}$ .  $\square$

**Lemma 5.** *If  $m$  and  $n$  are positive integers such that  $[m] \simeq [n]$ , then we have  $[m - 1] \simeq [n - 1]$ .*

*Proof.* Apply Lemma 4 with  $X = [m], Y = [n], a = m, b = n$ .  $\square$

*Proof of Proposition 2.* Let  $m, n \in \mathbb{N}$  satisfy  $[m] \simeq [n]$ . We must show  $m = n$ . We can assume  $m \leq n$ . Lemma 5 implies

$$[m - 1] \simeq [n - 1], [m - 2] \simeq [n - 2], \dots, [0] \simeq [n - m],$$

and thus  $m = n$ , as required.  $\square$

---

<sup>1</sup>This text is available at  
<https://github.com/Pierre-Yves-Gaillard/Number-of-elements-of-a-finite-set>.