

# Number of elements of a finite set<sup>1</sup>

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The statements proved here are almost obvious and not too difficult to find in the literature. I wrote this short text to make them as easily available as possible.

Let  $\mathbb{N}$  be the set of nonnegative integers, and for each  $n \in \mathbb{N}$  set  $[n] := \{1, 2, \dots, n\}$  (in particular  $[0] = \emptyset$ ). If  $X$  is a set, say that a bijection  $[n] \rightarrow X$  is a **count** of  $X$ , and that  $n$  is the **result** of the count. Say that  $X$  is **finite** if it admits a count  $[n] \rightarrow X$  for some  $n \in \mathbb{N}$ .

**Theorem 1.** *If a set  $X$  is finite, then any two counts of  $X$  give the same result.*

If such is the case and the result is  $n$ , we say that  $X$  **has  $n$  elements**. If  $X$  and  $Y$  are two sets, write  $X \simeq Y$  to indicate that there is a bijection  $X \rightarrow Y$ .

**Lemma 2.** *Let  $X$  and  $Y$  be two sets; let  $k$  be a positive integer; let  $a_1, \dots, a_k$  be distinct elements of  $X$ ; let  $b_1, \dots, b_k$  be distinct elements of  $Y$ ; and consider the following conditions:*

- (a)  $X \simeq Y$ ,
- (b)  $X \setminus \{a_1, \dots, a_k\} \simeq Y \setminus \{b_1, \dots, b_k\}$ ,
- (c)  $X = \{a_1, \dots, a_k\} \Leftrightarrow Y = \{b_1, \dots, b_k\}$ .

Then we have: (a)  $\Leftrightarrow$  (b)  $\Rightarrow$  (c).

*Proof.* We clearly have (a)  $\Leftarrow$  (b)  $\Rightarrow$  (c). We show that (a)  $\Rightarrow$  (b) when  $k = 1$ . If  $Z$  is any set and  $a, b \in Z$ , then we define the map  $[Z, a, b] : Z \rightarrow Z$  by

$$[Z, a, b](z) = \begin{cases} b & \text{if } z = a \\ a & \text{if } z = b \\ z & \text{otherwise.} \end{cases}$$

Then  $[Z, a, b]$  is its own inverse; in particular it is bijective. By assumption there is a bijection  $f : X \rightarrow Y$ . Then  $[Y, f(a_1), b_1] \circ f$ , being a bijection from  $X$  to  $Y$  mapping  $a_1$  to  $b_1$ , induces a bijection from  $X \setminus \{a_1\}$  to  $Y \setminus \{b_1\}$ . For  $k > 1$  we have, assuming the implication holds for  $k - 1$ ,

$$X \setminus \{a_1, \dots, a_k\} = (X \setminus \{a_1, \dots, a_{k-1}\}) \setminus \{a_k\} \simeq (Y \setminus \{b_1, \dots, b_{k-1}\}) \setminus \{b_k\} = Y \setminus \{b_1, \dots, b_k\}.$$

□

**Proposition 3.** *If  $m, n \in \mathbb{N}$  satisfy  $[m] \simeq [n]$ , then we have  $m = n$ .*

*Proof.* We assume  $m \leq n$  (as we may) and apply Lemma 2, (a)  $\Rightarrow$  (c), with  $X = [m]$ ,  $Y = [n]$ ,  $k = m$  and  $a_i = b_i = i$  for  $i = 1, \dots, m$ . □

Theorem 1 follows from Proposition 3.

<sup>1</sup>This text is available at

<https://github.com/Pierre-Yves-Gaillard/Number-of-elements-of-a-finite-set>.