

Number of elements of a finite set¹

Pierre-Yves Gaillard

The statements proved here are almost obvious and not too difficult to find in the literature. I wrote this short text to make them as easily available as possible.

Let \mathbb{N} be the set of nonnegative integers, and for each $n \in \mathbb{N}$ set $[n] := \{1, 2, \dots, n\}$ (in particular $[0] = \emptyset$). Let a bijection $[n] \rightarrow X$, where X is a set, be called a **count** of X , and let n be called the **result** of the count. We say that X is **finite** if it admits a count $[n] \rightarrow X$ for some $n \in \mathbb{N}$.

Theorem 1. *If a set X is finite, then any two counts of X give the same result.*

If this is the case and the result is n , we say that X **has n elements**. If X and Y are two sets, write $X \simeq Y$ to indicate that there is a bijection $X \rightarrow Y$.

Lemma 2. *Let X and Y be two sets; let k be in \mathbb{N} ; let a_1, \dots, a_k be distinct elements of X ; let b_1, \dots, b_k be distinct elements of Y ; and consider the following conditions:*

- (a) $X \simeq Y$,
- (b) $X \setminus \{a_1, \dots, a_k\} \simeq Y \setminus \{b_1, \dots, b_k\}$,
- (c) $X = \{a_1, \dots, a_k\} \Leftrightarrow Y = \{b_1, \dots, b_k\}$.

Then we have: (a) \Leftrightarrow (b) \Rightarrow (c).

Proof. We clearly have (a) \Leftarrow (b) \Rightarrow (c). Let us show that (a) \Rightarrow (b). The case $k = 0$ is obvious. Assume $k = 1$. If Z is any set and $a, b \in Z$, then we define the map $[Z, a, b] : Z \rightarrow Z$ by

$$[Z, a, b](z) = \begin{cases} b & \text{if } z = a \\ a & \text{if } z = b \\ z & \text{otherwise.} \end{cases}$$

Then $[Z, a, b]$ is its own inverse; in particular it is bijective. By assumption there is a bijection $f : X \rightarrow Y$. Then $[Y, f(a_1), b_1] \circ f$, being a bijection from X to Y mapping a_1 to b_1 , induces a bijection from $X \setminus \{a_1\}$ to $Y \setminus \{b_1\}$. If $k > 1$ we have, assuming (as we may) that the implication holds for $k - 1$,

$$X \setminus \{a_1, \dots, a_k\} = (X \setminus \{a_1, \dots, a_{k-1}\}) \setminus \{a_k\} \simeq (Y \setminus \{b_1, \dots, b_{k-1}\}) \setminus \{b_k\} = Y \setminus \{b_1, \dots, b_k\}.$$

□

Proposition 3. *If $m, n \in \mathbb{N}$ satisfy $[m] \simeq [n]$, then we have $m = n$.*

Proof. We assume $m \leq n$ (as we may, by symmetry \simeq) and apply Lemma 2, (a) \Rightarrow (c), with $X = [m], Y = [n], k = m$ and $a_i = b_i = i$ for $i = 1, \dots, m$. □

Theorem 1 now follows immediately from Proposition 3.

¹This text is available at
<https://github.com/Pierre-Yves-Gaillard/Number-of-elements-of-a-finite-set>.