

# Number of elements of a finite set<sup>1</sup>

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The statements proved here are almost obvious and not too difficult to find in the literature. I wrote this short text to make them as easily available as possible.

Let  $\mathbb{N}$  be the set of nonnegative integers, and for each  $n \in \mathbb{N}$  set  $[n] := \{1, 2, \dots, n\}$  (in particular  $[0] = \emptyset$ ). Let a bijection  $[n] \rightarrow X$ , where  $X$  is a set, be called a **count** of  $X$ , and let  $n$  be called the **result** of the count. We say that  $X$  is **finite** if it admits a count  $[n] \rightarrow X$  for some  $n \in \mathbb{N}$ .

**Theorem 1.** *If a set  $X$  is finite, then any two counts of  $X$  give the same result.*

If this is the case and the result is  $n$ , we say that  $X$  **has  $n$  elements**. If  $X$  and  $Y$  are two sets, write  $X \simeq Y$  to indicate that there is a bijection  $X \rightarrow Y$ .

**Lemma 2.** *Let  $X$  and  $Y$  be two sets; let  $k$  be in  $\mathbb{N}$ ; let  $a_1, \dots, a_k$  be distinct elements of  $X$ ; let  $b_1, \dots, b_k$  be distinct elements of  $Y$ ; and consider the following conditions:*

- (a)  $X \simeq Y$ ,
- (b)  $X \setminus \{a_1, \dots, a_k\} \simeq Y \setminus \{b_1, \dots, b_k\}$ ,
- (c)  $X = \{a_1, \dots, a_k\} \Leftrightarrow Y = \{b_1, \dots, b_k\}$ .

*Then we have:* (a)  $\Leftrightarrow$  (b)  $\Rightarrow$  (c).

*Proof.* We clearly have (a)  $\Leftrightarrow$  (b)  $\Rightarrow$  (c). Let us show that (a)  $\Rightarrow$  (b). The case  $k = 0$  is obvious. Assume  $k = 1$ . If  $Z$  is any set and  $a, b \in Z$ , then we define the map  $[Z, a, b] : Z \rightarrow Z$  by

$$[Z, a, b](z) = \begin{cases} b & \text{if } z = a \\ a & \text{if } z = b \\ z & \text{otherwise.} \end{cases} \quad (1)$$

Then  $[Z, a, b]$  is its own inverse; in particular it is bijective. By assumption there is a bijection  $f : X \rightarrow Y$ . Then  $[Y, f(a_1), b_1] \circ f$ , being a bijection from  $X$  to  $Y$  mapping  $a_1$  to  $b_1$ , induces a bijection from  $X \setminus \{a_1\}$  to  $Y \setminus \{b_1\}$ . If  $k > 1$  we have, assuming (as we may) that the implication holds for  $k - 1$ ,

$$X \setminus \{a_1, \dots, a_k\} = (X \setminus \{a_1, \dots, a_{k-1}\}) \setminus \{a_k\} \simeq (Y \setminus \{b_1, \dots, b_{k-1}\}) \setminus \{b_k\} = Y \setminus \{b_1, \dots, b_k\}.$$

□

**Lemma 3.** *If  $[k] \rightarrow Z$  is a count of a set  $Z$ , and if  $c_1, \dots, c_k$  are distinct elements of  $Z$ , then we have  $Z = \{c_1, \dots, c_k\}$ .*

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<sup>1</sup>This text is available at

<https://github.com/Pierre-Yves-Gaillard/Number-of-elements-of-a-finite-set>.

*Proof.* Use Lemma 2 with  $X = [k]$ ,  $Y = Z$  and  $a_i = i, b_i = c_i$  for  $i = 1, \dots, k$ .  $\square$

**Proposition 4.** *If  $m, n \in \mathbb{N}$  satisfy  $[m] \simeq [n]$ , then we have  $m = n$ .*

*Proof.* We assume  $m \leq n$  (as we may, by symmetry  $\simeq$ ) and apply Lemma 3, (a)  $\Rightarrow$  (c), with  $k = m$ ,  $Z = [n]$  and  $c_i = i$  for  $i = 1, \dots, m$ .  $\square$

Theorem 1 now follows immediately from Proposition 4.

We end with two additional results. Let us denote the number of elements of a finite set  $X$  by  $|X|$ .

**Theorem 5.** *If  $Y$  is a proper subset a finite set  $X$  (i.e.  $Y \subsetneq X$ ), then  $Y$  is finite and satisfies  $|Y| < |X|$ .*

**Theorem 6** (Pigeonhole Principle). *If  $X$  and  $Y$  are finite set, if  $|Y| < |X|$ , and if  $f : X \rightarrow Y$  is a map, then  $f$  is not injective.*

*Proof of Theorem 5.* We can assume  $X = [n]$  with  $n > 0$ . Let  $i$  be in  $[n] \setminus Y$  and, using Notation (1) set  $Z := [[n], i, n](Y)$ . Then  $|Z| = |Y|$  and  $Z$  is a subset of  $[n]$  contained in  $[n - 1]$ . An obvious induction completes the proof.  $\square$

*Proof of Theorem 6.* If  $f$  was injective, Theorem 5 would imply  $|X| = |f(X)| \leq |Y| < |X|$ , contradiction.  $\square$

For completeness' sake we add the “Co-Pigeonhole Principle”:

**Proposition 7** (“Co-Pigeonhole Principle”). *If  $X$  and  $Y$  are finite set, if  $|X| < |Y|$ , and if  $f : X \rightarrow Y$  is a map, then  $f$  is not surjective.*

*Proof.* Assume by contradiction that  $f$  is surjective. We can suppose  $X = [n]$ . Set

$$Z := \{\min f^{-1}(f(i)) \mid i \in X\} \subset X.$$

Then  $f$  induces a bijection  $Z \xrightarrow{\sim} Y$  and Theorem 5 implies  $|X| < |Y| = |Z| \leq |X|$ , contradiction.  $\square$