Number of elements of a finite set[1]

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The statements proved here are almost obvious and not too difficult to find in the literature. I wrote this short text to make them as easily available as possible.

Let $\mathbb{N}$ be the set of nonnegative integers, and, for each $n \in \mathbb{N}$ set $[n] := \{1, 2, \ldots, n\}$ (in particular $[0] = \varnothing$). If X is a set, say that a bijection $[n] \to X$ is a **count** of X, and that n is the **result** of the count. Say that X is **finite** if it admits a count $[n] \to X$ for some $n \in \mathbb{N}$.

Theorem 1. *If a set X is finite, then any two counts of X give the same result.*

If such is the case and the result is n, we say that X **has n elements**. If X and Y are two sets, write $X \simeq Y$ to indicate that there is a bijection $X \to Y$. The theorem will follow from the proposition below:

Proposition 2. *If $m, n \in \mathbb{N}$ satisfy $[m] \simeq [n]$, then we have $m = n$.*

If X is a set and a, b are in X, then we define the map $[X, a, b] : X \to X$ by

$$[X, a, b](x) = \begin{cases} b & \text{if } x = a \\ a & \text{if } x = b \\ x & \text{otherwise.} \end{cases}$$

The $[X, a, b]$ is its own inverse; in particular it is bijective.

Lemma 3. *If X and Y are sets such that $X \simeq Y$, and if a is in X and b in Y, then there is a bijection $X \to Y$ mapping a to b.*

Proof. By assumption there is a bijection $f : X \to Y$. Then $[Y, f(a), b] \circ f$ does the job. □

Lemma 4. *If X and Y are sets such that $X \simeq Y$, and if a is in X and b in Y, then we have $X \setminus \{a\} \simeq Y \setminus \{b\}$.*

Proof. By Lemma 3 there is a bijection $f : X \to Y$ mapping a to b. Then f induces a bijection $X \setminus \{a\} \to Y \setminus \{b\}$. □

Lemma 5. *If m and n are* positive *integers such that $[m] \simeq [n]$, then we have $[m - 1] \simeq [n - 1]$.*

Proof. Apply Lemma 4 with $X = [m], Y = [n], a = m, b = n$. □

Proof of Proposition 2. Let $m, n \in \mathbb{N}$ satisfy $[m] \simeq [n]$. We must show $m = n$. We can assume $m \leq n$. Lemma 5 implies

$$[m - 1] \simeq [n - 1],\ [m - 2] \simeq [n - 2],\ \ldots,\ [0] \simeq [n - m],$$

and thus $m = n$, as required. □

[1] This text is available at
`https://github.com/Pierre-Yves-Gaillard/Number-of-elements-of-a-finite-set`.