Transfinite Recursion

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The only purpose of this short text[1] is to point out that the Transfinite Recursion Theorem is a particular case of the Transfinite Induction Theorem.

For any member x of any ordered set X, we denote by X_x the set of all those y which are less than x, that is $X_x = \{y \in X \mid y < x\}$.

Say that a subset S of an ordered set X is **inductive** if, for all x in X, the condition $X_x \subset S$ implies $x \in S$.

Transfinite Induction Theorem. Let S be a subset of a well-ordered set X. Then $S = X$ if and only if S is inductive.

The proof is well-known and easy.

Transfinite Recursion Theorem. Let X be well-ordered, let A be a set, and, for all x in X, let A^{X_x} be the set of all maps from X_x to A. Finally, for all x in X, let $r_x : A^{X_x} \to A$ be a map. Then there is a unique map $f : X \to A$ such that $f(x) = r_x(f|X_x)$ for all x in X. Here $f|X_x$ denotes the restriction of f to X_x.

Lemma. The map f in the above theorem is unique.

Proof. Let f and g be two maps from X to A satisfying the required conditions. It is easy to see that $\{x \in X \mid f(x) = g(x)\}$ is inductive. $\square$

Proof of the Transfinite Recursion Theorem. Let ω satisfy $\omega \notin X$, set $Y := X \cup \{\omega\}$, and decree $\omega > x$ for all x in X. Then Y is again well-ordered, and $X = Y_\omega$. We define the subset T of Y as follows: let y be in Y. Then y is in T if and only if there is a map $f_y : Y_y \to A$ such that $f_y(x) = r_x(f_y|X_x)$ for all x in Y_y. By the Lemma, there is at most one such map f_y.

Claim: T is inductive. Proof. Let y be in Y. We can assume that the f_x for $x < y$ have already been defined, and we must define f_y, that is, we must define $f_y(x)$ for $x < y$. If $x < z < y$ for some z we set $f_y(x) := f_z(x)$ and the Lemma shows that this does not depend on the choice of z. If there is no such z, we set $f_y(x) := r_x(f_x)$. It is easy to check that f_y meets our requirements.

By the Transfinite Recursion Theorem, this implies that f_ω exists, and it is clear that $f := f_\omega$ does the job. $\square$

[1] available at `https://github.com/Pierre-Yves-Gaillard/The-Transfinite-Recursion-Theorem`

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