Existence of a least increasing surjection from an artinian poset onto a set of ordinals[1]

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Let X be a set. If $g : X \to S$ and $h : X \to T$ are surjections and S and T are sets of ordinals, say that $g \le h$ if $g(x) \le h(x)$ for all x. If X is a poset, say that g is **increasing** if $x < y$ implies $g(x) < g(y)$.

Theorem 1. *If X is an artinian poset, then there is a least increasing surjection f from X onto a set of ordinals. For x in X, set $X_x := \{y \in X \mid y < x\}$. Then f is the unique surjection from X onto a set S of ordinals such that, setting $\alpha_x := \sup f(X_x)$, we have*

$$f(x) = \begin{cases} \alpha_x + 1 & \text{if } \alpha_x \in f(X_x) \\ \\ \alpha_x & \text{if } \alpha_x \notin f(X_x) \end{cases} \tag{1}$$

for all x in X.

Lemma 2. *There is at most one such surjection f.*

Proof. Let f and g satisfy (1) and let x be a minimal element of X such that $f(x) \ne g(x)$. Then (1) yields a contradiction. □

Lemma 3. *Let $A \subset X$ be the set of all x in X such that there is a surjection $f_x : X_x \to S(x)$ s.t. $S(x)$ is a set of ordinals, and, setting $\alpha_{xy} := \sup f_x(X_y)$ for all y in X_x, we have*

$$f_x(y) = \begin{cases} \alpha_{xy} + 1 & \text{if } \alpha_{xy} \in f_x(X_y) \\ \\ \alpha_{xy} & \text{if } \alpha_{xy} \notin f_x(X_y) \end{cases} \tag{2}$$

for all y in X_x. Then $A = X$.

Proof. We assume by contradiction $A \ne X$. Let x be minimal in $X \setminus A$. So f_x doesn't exist, but f_y exists for all $y < x$. We will define f_x. Let y be less than x. We must define $f_x(y)$. If $y < z < x$ for some z, we set $f_x(y) := f_z(y)$. By Lemma 2, this is well-defined. Assume now that no such z exists. Then we set $\beta_y = \sup f_y(X_y)$ and

$$f_x(y) = \begin{cases} \beta_y + 1 & \text{if } \beta_y \in f_y(X_y) \\ \\ \beta_y & \text{if } \beta_y \notin f_y(X_y). \end{cases} \tag{3}$$

Now f_x is defined, and we must show that it satisfies (2). Let y be in X_x.

If $y < z < x$ for some z, then $f_x(y)$ is defined by $f_x(y) := f_z(y)$, and we have $\alpha_{xy} = \alpha_{zy}$. Since $f_z(y)$ satisfies (2) with z instead of x by assumption, we are done.

If no such z exists, then $f_x(y)$ is defined by (3), and it suffices to show $\beta_y = \alpha_{xy}$. Let t be in X_y. It suffices to show $f_x(t) = f_y(t)$, but this follows immediately from the definition of $f_x(t)$.

This shows that f_x satisfies (2), contradiction. □

Proof of Theorem 1. It suffices to show that there is a surjection f from X onto a set S of ordinals satisfying (1). Set $Y := X \cup \{\infty\}$ with $\infty \notin X$ and $\infty > x$ for all x in X (and $x \le x'$ in Y iff $x \le x'$ in X for all x, x' in X). In particular, Y is artinian, and we have $Y_\infty = X$. Applying Lemma 3 to Y, we get a surjection $f_\infty : X \to S$ satisfying (1). □

[1]This text is available at `https://github.com/Pierre-Yves-Gaillard/Artin-poset-to-ordinals`. Here is a related text: `https://github.com/Pierre-Yves-Gaillard/The-Transfinite-Recursion-Theorem`.

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