Transfinite Recursion

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The only purpose of this short text[1] is to point out that the Transfinite Recursion Theorem is a particular case of the Transfinite Induction Theorem (and that these theorems generalize immediately from the well-order setting to the artinian setting).

For any element x of any ordered set X, we denote by X_x the set of all those y which are less than x, that is $X_x = \{y \in X \mid y < x\}$.

Say that a subset S of an ordered set X is **inductive** if, for all x in X, the condition $X_x \subset S$ implies $x \in S$.

Transfinite Induction Theorem. Let S be a subset of a well-ordered set X. Then $S = X$ if and only if S is inductive.

Proof. If $S = X$, then S is trivially inductive. If $S \subsetneqq X$, let x be the least element of $X \setminus S$, and note that S is not inductive because we have $X_x \subset S$ but $x \notin S$. $\square$

Transfinite Recursion Theorem. Let X be well-ordered, let A be a set, and, for all x in X, let A^{X_x} be the set of all maps from X_x to A. Finally, for all x in X, let $r_x : A^{X_x} \to A$ be a map. Then there is a unique map $f : X \to A$ such that $f(x) = r_x(f|X_x)$ for all x in X. Here $f|X_x$ denotes the restriction of f to X_x.

Lemma. The map f in the above theorem is unique.

Proof. Let f and g be two maps from X to A satisfying the required conditions. It is easy to see that $\{x \in X \mid f(x) = g(x)\}$ is inductive. $\square$

Proof of the Transfinite Recursion Theorem. Let ω satisfy $\omega \notin X$, set $Y := X \cup \{\omega\}$, and equip Y with the unique order such that $\omega > x$ for all x in X and the order of X is induced by that of Y. Then Y is again well-ordered, and $X = Y_\omega$. We define the subset T of Y as follows. Let y be in Y. Then y is in T if and only if there is a map $f_y : Y_y \to A$ such that $f_y(x) = r_x(f_y|X_x)$ for all x in Y_y. By the Lemma, there is at most one such map f_y.

Claim: T is inductive. Proof. Let y be in Y. We can assume that the f_x for $x < y$ have already been defined, and we must define f_y, that is, we must define $f_y(x)$ for $x < y$. If $x < z < y$ for some z we set $f_y(x) := f_z(x)$ and the Lemma shows that this does not depend on the choice of z. If there is no such z, we set $f_y(x) := r_x(f_x)$. It is easy to check that f_y meets our requirements.

By the Transfinite Recursion Theorem, this implies that f_ω exists, and it is clear that $f := f_\omega$ does the job. $\square$

Here is a mild generalization.

Recall that a poset X is **artinian** if each nonempty finite subset of X has a minimal element. If x is an element of a poset X, we define X_x as above, and, if S is a subset of X, we define the condition that S is **inductive** as above.

Artinian Induction Theorem. Let S be a subset of an artinian poset X. Then $S = X$ if and only if S is inductive.

[1] available at `https://github.com/Pierre-Yves-Gaillard/The-Transfinite-Recursion-Theorem`

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Artinian Recursion Theorem. Let X be an artinian poset, let A be a set, and, for all x in X, let A^{X_x} be the set of all maps from X_x to A. Finally, for all x in X, let $r_x : A^{X_x} \to A$ be a map. Then there is a unique map $f : X \to A$ such that $f(x) = r_x(f|X_x)$ for all x in X. Here $f|X_x$ denotes the restriction of f to X_x.

The proofs are almost the same as in the well-ordered case. Let me just spell out the part where I wrote "If $x < z < y$ for some z we set $f_y(x) := f_z(x)$ and the Lemma shows that this does not depend on the choice of z". Assume $x < z < y$ and $x < z' < y$. Then, by the Lemma, f_z and $f_{z'}$ coincide on the artinian poset $X_z \cap X_{z'}$ (even if z and z' are not comparable), and we get $f_z(y) = f_{z'}(y)$, as required.

Note that there is a natural way of attaching an ordinal $f(x)$ to each element x of an artinian poset X. More precisely, $f(x)$ is defined by artinian recursion as follows. If x is minimal, set $f(x) = 0$. Suppose x is not minimal, assume, as we can by the Artinian Recursion Theorem, that $f(y)$ is defined for all $y < x$, and set

$$f(x) = \begin{cases} \sup f(X_x) + 1 & \text{if } \sup f(X_x) \in f(X_x) \\ \\ \sup f(X_x) & \text{if } \sup f(X_x) \notin f(X_x). \end{cases}$$

(Observe that $f(X)$ is the set of all those ordinals which are less than a given ordinal.)

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