

# Transfinite Recursion

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The only purpose of this short text<sup>1</sup> is to point out that the Transfinite Recursion Theorem is a particular case of the Transfinite Induction Theorem.

For any member  $x$  of any ordered set  $X$ , we denote by  $X_x$  the set of all those  $y$  which are less than  $x$ , that is  $X_x = \{y \in X \mid y < x\}$ .

Say that a subset  $S$  of an ordered set  $X$  is **inductive** if, for all  $x$  in  $X$ , the condition  $X_x \subset S$  implies  $x \in S$ .

**Transfinite Induction Theorem.** Let  $S$  be a subset of a well-ordered set  $X$ . Then  $S = X$  if and only if  $S$  is inductive.

*Proof.* If  $S = X$ , then  $S$  is trivially inductive. If  $S \subsetneq X$ , let  $x$  be the least element of  $X \setminus S$ , and note that  $S$  is not inductive because we have  $X_x \subset S$  but  $x \notin S$ .  $\square$

**Transfinite Recursion Theorem.** Let  $X$  be well-ordered, let  $A$  be a set, and, for all  $x$  in  $X$ , let  $A^{X_x}$  be the set of all maps from  $X_x$  to  $A$ . Finally, for all  $x$  in  $X$ , let  $r_x : A^{X_x} \rightarrow A$  be a map. Then there is a unique map  $f : X \rightarrow A$  such that  $f(x) = r_x(f|X_x)$  for all  $x$  in  $X$ . Here  $f|X_x$  denotes the restriction of  $f$  to  $X_x$ .

**Lemma.** The map  $f$  in the above theorem is unique.

*Proof.* Let  $f$  and  $g$  be two maps from  $X$  to  $A$  satisfying the required conditions. It is easy to see that  $\{x \in X \mid f(x) = g(x)\}$  is inductive.  $\square$

*Proof of the Transfinite Recursion Theorem.* Let  $\omega$  satisfy  $\omega \notin X$ , set  $Y := X \cup \{\omega\}$ , and equip  $Y$  with the unique order such that  $\omega > x$  for all  $x$  in  $X$  and the order of  $X$  is induced by that of  $Y$ . Then  $Y$  is again well-ordered, and  $X = Y_\omega$ . We define the subset  $T$  of  $Y$  as follows. Let  $y$  be in  $Y$ . Then  $y$  is in  $T$  if and only if there is a map  $f_y : Y_y \rightarrow A$  such that  $f_y(x) = r_x(f_y|X_x)$  for all  $x$  in  $Y_y$ . By the Lemma, there is at most one such map  $f_y$ .

Claim:  $T$  is inductive. Proof. Let  $y$  be in  $Y$ . We can assume that the  $f_x$  for  $x < y$  have already been defined, and we must define  $f_y$ , that is, we must define  $f_y(x)$  for  $x < y$ . If  $x < z < y$  for some  $z$  we set  $f_y(x) := f_z(x)$  and the Lemma shows that this does not depend on the choice of  $z$ . If there is no such  $z$ , we set  $f_y(x) := r_x(f_x)$ . It is easy to check that  $f_y$  meets our requirements.

By the Transfinite Recursion Theorem, this implies that  $f_\omega$  exists, and it is clear that  $f := f_\omega$  does the job.  $\square$

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<sup>1</sup>available at <https://github.com/Pierre-Yves-Gaillard/The-Transfinite-Recursion-Theorem>