TEOREMA DEL LIMITE CENTRALE

DEFINIZIONE

Data una successione di variabili casuali $X_1, \cdots X_n$ identicamente distribuite e indipendenti con:

$$E[X_k] = \mu \quad e \quad Var(X_k) = \sigma^2$$

$$\Downarrow$$

$$Y = \sum_{k=1}^{n} X_k \quad converge\ in\ distribuzione\ ad\ una\ gaussiana\ per\ n \longrightarrow \infty$$

DIMOSTRAZIONE (SEMPLIFICATA)

Per facilitare la dimostrazione introduciamo 2 ipotesi aggiuntive:

- $\sigma^2 = 1$
- $\mu = 0$

Tesi:

$$Y \sim N\left(0, \sqrt{n}\right) \Leftrightarrow \frac{Y - n\mu}{\sigma\sqrt{n}} = \frac{Y}{\sqrt{n}} \sim N(0,1)$$

Definiamo:

$$L(s) = \ln \Phi_{X_k}(s)$$

Verifichiamo che:

$$\Phi_{X_k}(s) = 1 \Longrightarrow L(0) = 0$$

$$L'(0) = \frac{1}{\Phi_{X_k}(s)} \left(\frac{d}{ds} \Phi_{X_k}(s) \right) \Bigg|_{s=0} = \frac{\Phi'_{X_k}(s)}{\Phi_{X_k}(s)} \Bigg|_{s=0} = E[X_k] = \mu = 0$$

$$L''(0) = \frac{d}{ds} \frac{\Phi'_{X_k}(s)}{\Phi_{X_k}(s)} \Bigg|_{s=0} = \frac{\Phi''_{X_k}(s)\Phi_{X_k}(s) - \left(\Phi'_{X_k}(s)\right)^2}{\left(\Phi_{X_k}(s)\right)^2} \Bigg|_{s=0} = \frac{\Phi''_{X_k}(s)}{\Phi_{X_k}(s)} - \left(\frac{\Phi'_{X_k}(s)}{\Phi_{X_k}(s)} \right)^2 \Bigg|_{s=0} = E\left[{X_k}^2\right] - E[X_k]^2 = \sigma^2 = 1$$

Dimostriamo che $\frac{Y}{\sqrt{n}}$ si comporta come una gaussiana:

$$\frac{Y}{\sqrt{n}} = \sum_{k=1}^{n} \frac{X_k}{\sqrt{n}} \Longrightarrow \Phi_{\sum_{k=1}^{n} \frac{X_k}{\sqrt{n}}}(t) = \left[\Phi_{\frac{X_k}{\sqrt{n}}}(t) \right]^n = \left[\Phi_{X_k}\left(\frac{t}{\sqrt{n}} \right) \right]^n$$

$$se \lim_{n\to\infty} \Phi_{\frac{Y}{\sqrt{n}}}(t) = \Phi_Z(t) = e^{\frac{t^2}{2}}\ allora \lim_{n\to\infty} \ln \Phi_{\frac{Y}{\sqrt{n}}}(t) = \lim_{n\to\infty} \ln \left[\Phi_{X_k}\left(\frac{t}{\sqrt{n}} \right) \right]^n = \frac{t^2}{2} \Longrightarrow \lim_{n\to\infty} nL\left(\frac{t}{\sqrt{n}} \right) = \frac{t^2}{2}$$

Verifichiamo tale risultato:

$$\lim_{n\to\infty} nL\left(\frac{t}{\sqrt{n}} \right) = \lim_{n\to\infty} \frac{L\left(\frac{t}{\sqrt{n}} \right)}{n^{-1}} = \frac{0}{0} \to De\ L'Hôpital \to \lim_{n\to\infty} - \frac{L'\left(\frac{t}{\sqrt{n}} \right)\left(-\frac{t}{2} \right) n^{-\frac{3}{2}}}{n^{-2}} = \lim_{n\to\infty} - \frac{L'\left(\frac{t}{\sqrt{n}} \right)\left(-\frac{t}{2} \right)}{n^{-\frac{1}{2}}} =$$

$$\frac{0}{\infty} \to De\ L'Hôpital \to \lim_{n\to\infty} + \frac{L''\left(\frac{t}{\sqrt{n}} \right)\left(-\frac{t}{2} \right)^2 n^{-\frac{3}{2}}}{\frac{1}{2} n^{-\frac{3}{2}}} = \frac{\frac{t^2}{4}}{\frac{1}{2}} = \frac{t^2}{2}$$