

# A Theoretical Analysis of Catastrophic Forgetting

Pierre Alquier



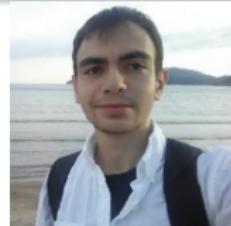
Center for  
Advanced Intelligence Project

New Trends in Statistical Learning II  
Porquerolles, June 2022



Thang Doan

Bosch research



Mehdi Abbana Bennani

Aqemia



Bogdan Mazoure

MILA / McGill University



Guillaume Rabusseau

MILA / Université de Montréal



Doan, T., Bennani, M. A., Mazoure, B., Rabusseau, G. & Alquier, P. (2021). A theoretical analysis of catastrophic forgetting through the NTK overlap matrix. *AISTATS'2021*.

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- Continual learning problem
- Catastrophic forgetting

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# Notations

Regression/classification problem :

- objects  $x \in \mathcal{X}$ ,
- labels  $y \in \mathcal{Y} \subset \mathbb{R}$ ,
- predictors  $f_w : \mathcal{X} \rightarrow \mathcal{Y}$ ,  $w \in \mathbb{R}^d$ , objective : neural networks.

## Difficulties of “continual learning”

- $d$  is huge, → we need a lot of data.
- the dataset is huge, → impossible to store all the data.
- we will learn  $w$  sequentially based on a data stream  $(x_t, y_t)$ , → the  $x_t$  come from a real life data collection process that makes them non-identically distributed..

# Online learning theory

Online learning theory provides algorithms to learn from data streams, with theoretical guarantees.

## Online Gradient Algorithm

- $w_1 := 0$ ,
- $w_{t+1} = w_t - \eta_t \nabla_{w=w_t} \ell(y_t, f_w(x_t))$ .

## Regret bound for OGA

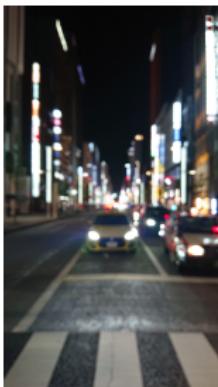
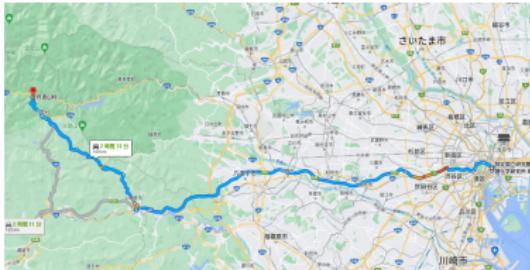
If  $\ell$  is  $L$ -Lipschitz + convex, one can calibrate  $\eta_t$  such that

$$\frac{1}{T} \sum_{t=1}^T \ell(y_t, f_{w_t}(x_t)) - \inf_{\|w\| \leq B} \frac{1}{T} \sum_{t=1}^T \ell(y_t, f_w(x_t)) \leq BL \sqrt{\frac{2}{T}}.$$

# Example : training a self-driving car

Decide an itinerary

- from RIKEN AIP (Tokyo)
- to Tabayama.



# Observation

$$y_t = f_{w^*}(x_t)$$

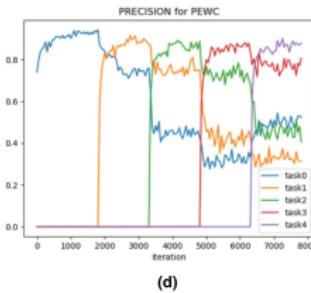
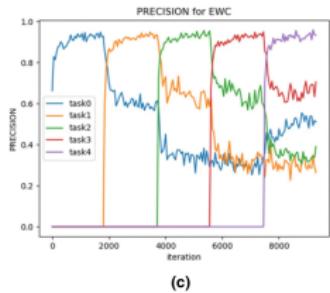
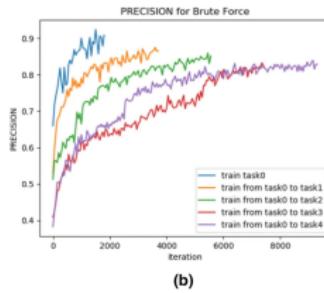
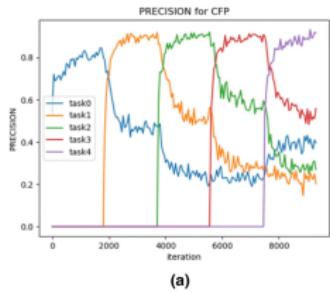
- $y = 1, \dots, \tau_1$  :
  - $x_t$  i.i.d from  $P_1 \rightarrow$  we learn  $w_1$ .
- $y = \tau_1 + 1, \dots, \tau_2$  :
  - $x_t$  i.i.d from  $P_2 \rightarrow$  we update  $w_1$  to  $w_2$ .
- ...
- $x \sim P_1$  :
  - $f_{w_K}(x)$  is a much worse prediction than  $f_{w_1}(x)$ .
  - we **forgot** how to deal with objects  $x \sim P_1$ .

# What is the problem with online learning theory?

$$\frac{1}{T} \sum_{t=1}^T \ell(y_t, f_{w_t}(x_t)) - \inf_{\|w\| \leq B} \frac{1}{T} \sum_{t=1}^T \ell(y_t, f_w(x_t)) \leq BL\sqrt{\frac{2}{T}}.$$

- tells you  $f_{w_t}(x_t)$  predicts well  $y_t$  (on average over  $t$ ), *not* that  $f_{w_T}(x_t)$  predicts well  $y_t$ .
- *online-to-batch* bounds : averaging  $\bar{w}_t = \frac{1}{t} \sum_{s=1}^t w_s$  is proven to work well for out-of-sample prediction... in the i.i.d case !

# An example



Hong, D. Y., Li, Y. & Shin, B. S. (2019). Predictive EWC : mitigating catastrophic forgetting of neural network through pre-prediction of learning data. *Journal of Ambient Intelligence and Humanized Computing*.

# Some references



French, R. M. (1999). Catastrophic forgetting in connectionist networks. *Trends in cognitive sciences*.



Kirkpatrick, J., Pascanu, R., Rabinowitz, N., Veness, J., Desjardins, G., Rusu, A. A., Milan, K., Quan, J., Ramalho, T., Grabska-Barwinska, A. & Hassabis, D. (2017). Overcoming catastrophic forgetting in neural networks. *Proceedings of the National Academy of Sciences*.



Kemker, R., McClure, M., Abitino, A., Hayes, T. & Kanan, C. (2018). Measuring catastrophic forgetting in neural networks. *AAAI'2018*.

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# Linear model – notations

- initialization :  $w_{\tau_0} = 0$ .
- task  $\tau_k$  given as a block :

$$Y_{\tau_k} := \begin{pmatrix} y_{\tau_k+1} \\ \vdots \\ y_{\tau_{k+1}} \end{pmatrix} \text{ and } X_{\tau_k} := \begin{pmatrix} x_{\tau_k+1}^T \\ \vdots \\ x_{\tau_{k+1}}^T \end{pmatrix}$$

- update :

$$\begin{aligned} w_{\tau_k} &= \arg \min_{w \in \mathbb{R}^d} \left\{ \|Y_{\tau_k} - X_{\tau_k} w\|^2 + \lambda \cdot \|w - w_{\tau_{k-1}}\|^2 \right\} \\ &= w_{\tau_{k-1}} + (X_{\tau_k}^T X_{\tau_k} + \lambda \cdot I)^{-1} X_{\tau_k}^T \underbrace{(Y_{\tau_k} - X_{\tau_k} w_{\tau_{k-1}})}_{=\tilde{Y}_{\tau_k}}. \end{aligned}$$

# Definition of forgetting

Definition - forgetting of task  $i$  at the end of task  $j$

For  $s \leq t$  we put

$$\Delta^{\tau_s \rightarrow \tau_t} := \|X_{\tau_s} w_{\tau_t} - X_{\tau_s} w_{\tau_s}\|^2.$$

- $X_{\tau_t} = U_{\tau_t} \Sigma_{\tau_t} V_{\tau_t}^T$  be the SVD of  $X_{\tau_t}$ ,
- $O^{\tau_s \rightarrow \tau_t} = V_{\tau_s}^T V_{\tau_t}$  the overlap matrix,
- $M_{\tau_t} := \Sigma_{\tau_t} (\Sigma_{\tau_t} + \lambda \cdot I)^{-1} U_{\tau_t}^T$ .

## Theorem

For any  $t > s$ ,

$$\Delta^{\tau_s \rightarrow \tau_t} = \left\| \sum_{k=s+1}^t U_{\tau_k} \Sigma_{\tau_k} O^{\tau_s \rightarrow \tau_k} M_{\tau_k} \tilde{Y}_{\tau_k} \right\|^2.$$

# Upper bound on forgetting

## Corollary

$$\sqrt{\Delta_{\tau_s \rightarrow \tau_t}} \leq \|\Sigma_{\tau_s}\|_{\text{op}} \sum_{k=s+1}^t \|\mathbf{O}_{\tau_s \rightarrow \tau_t}^{\tau_s \rightarrow \tau_t}\|_{\text{op}} \left\| \mathbf{M}_{\tau_k} \tilde{\mathbf{Y}}_{\tau_k} \right\|$$

With  $\mathbf{V}_{\tau_t} = (\mathbf{V}_{\tau_t}[1] | \mathbf{V}_{\tau_t}[2] | \dots)$  we have

$$\mathbf{O}_{i,j}^{\tau_s \rightarrow \tau_t} = \cos(\mathbf{V}_{\tau_s}[i], \mathbf{V}_{\tau_t}[j])$$

and  $\|\mathbf{O}_{\tau_s \rightarrow \tau_t}^{\tau_s \rightarrow \tau_t}\|_{\text{op}} = \cos(\alpha)$  where  $\alpha$  is the Dixmier angle between the span of  $\mathbf{V}_{\tau_t}$  and the span of  $\mathbf{V}_{\tau_s}$ .



Dixmier, J. (1949). Étude sur les variétés et les opérateurs de Julia, avec quelques applications.  
*Bulletin de la SMF*.

# A recent improvement



Evron, I., Moroshko, E., Ward, R., Srebro, N. & Soudry, D. (2022). How catastrophic can catastrophic forgetting be in linear regression? *COLT'22*.

- simplified setting, allows an refinement of the analysis,
- note : I find their results very elegant, so I presented the previous result using *some* of their notations.

In their paper :

- $\lambda = 0$ , there is  $w^*$  such that  $Y_{\tau_s} = X_{\tau_s} w^*$  (no noise).
- the  $X_{\tau_s}$  are normalized  $\Rightarrow \|\Sigma_{\tau_s}\|_{\text{op}} \leq 1$ .

# Consequences of the simplifications

Define the orthogonal projection  $P_{\tau_k} = I - X_{\tau_k}(X_{\tau_k}^T X_{\tau_k})^{-1} X_{\tau_k}^T$ ,

$$\begin{aligned} \text{then } w_{\tau_k} - w^* &= P_{\tau_k}(w_{\tau_{k-1}} - w^*) \\ &= P_{\tau_k} \dots P_{\tau_1} (\underbrace{w_{\tau_0} - w^*}_{=0}), \end{aligned}$$

$$\begin{aligned} \text{and } \Delta^{\tau_s \rightarrow \tau_t} &= \|X_{\tau_s} w_{\tau_t} - X_{\tau_s} w_{\tau_s}\|^2 \\ &= \|X_{\tau_s} w_{\tau_t} - Y_{\tau_s}\|^2 \\ &= \|X_{\tau_s} w_{\tau_t} - X_{\tau_s} w^*\|^2 \\ &= \|X_{\tau_s} P_{\tau_t} \dots P_{\tau_1} w^*\|^2 \\ &\leq \|(I - P_{\tau_s}) P_{\tau_t} \dots P_{\tau_1} w^*\|^2. \end{aligned}$$

# Average forgetting : worst case

Definition - average forgetting at task  $t$

$$F(t) := \frac{1}{t} \sum_{s=1}^t \|X_{\tau_s} w_{\tau_t} - X_{\tau_s} w_{\tau_s}\|^2 = \frac{1}{t} \sum_{s=1}^t \Delta^{\tau_s \rightarrow \tau_t}$$

$$F(t) = \frac{1}{t} \sum_{s=1}^t \|X_{\tau_s} P_{\tau_t} \dots P_{\tau_1} w^*\|^2$$

They design a situation where :

$$F(t) \geq 1 - \mathcal{O}\left(\frac{1}{\sqrt{t}}\right).$$

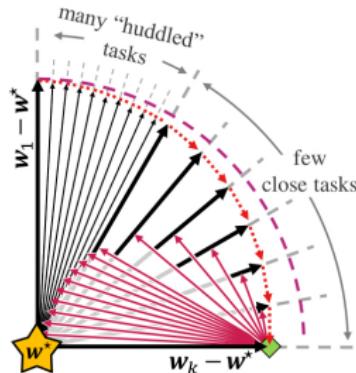


Figure from Evron et al. (2022).

# Situations where forgetting do not occur

Evron *et al.* (2022) then argue that in general, forgetting is not that bad :

- cyclic tasks :  $\tau_1, \dots, \tau_T, \tau_1, \dots, \tau_T, \dots$  After seeing  $t$  tasks,

$$F(t) \leq \min \left( \frac{T^2}{\sqrt{t}}, \frac{T^2(d - \max\{\text{rank}(X_{\tau_s})\})}{t} \right),$$

- randomized tasks :  $\tau_{I_1}, \tau_{I_2}, \dots$  where the  $I_i$  are i.i.d uniform in  $\{1, \dots, T\}$ , then after seeing  $t$  tasks,

$$\mathbb{E}[F(t)] \leq \frac{9 \left( d - \frac{1}{T} \sum_{s=1}^T \text{rank}(X_{\tau_s}) \right)}{t}.$$

→ however, this requires to store the tasks, or, at least, to be able to learn them many times...

# Conclusion of the theoretical analysis

What we learnt so far

- catastrophic forgetting can happen even in linear models,
- depends on the geometry and order of the tasks.

Open questions :

- noisy case,
- nonlinear case,
- tasks not by block // not aware that a new task begins,
- other algorithms... (we propose a few in the next section),
- theoretical limitations :



Knoblauch, J., Hisham, H. & Diethe, T. (2020). Optimal continual learning has perfect memory and is NP-hard. *ICML'2020*.

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# Orthogonal updates



Doan, T., Bennani, M. A. & Sugiyama, M. (2020). Generalisation guarantees for continual learning with orthogonal gradient descent. *ICML'2020 Workshop on Lifelong Learning*.

$$\begin{aligned} w_{\tau_k} = \arg \min_{w \in \mathbb{R}^d} & \left\{ \|Y_{\tau_k} - X_{\tau_k} w\|^2 + \lambda \cdot \|w - w_{\tau_{k-1}}\|^2 \right\} \\ & V_{\tau_1}^T (w - w_{\tau_{k-1}}) = 0 \\ & \vdots \\ & V_{\tau_{k-1}}^T (w - w_{\tau_{k-1}}) = 0 \\ = w_{\tau_{k-1}} & + \Pi_k (X_{\tau_k}^T X_{\tau_k} + \lambda \cdot I)^{-1} X_{\tau_k}^T (Y_{\tau_k} - X_{\tau_k} w_{\tau_{k-1}}) \end{aligned}$$

where  $\Pi_k$  is the orthogonal projection on  $\ker(V_{\tau_1}^T | \dots | V_{\tau_{k-1}}^T)$ .

$$\Delta^{\tau_s \rightarrow \tau_t} = 0.$$

But the procedure requires to store  $V_{\tau_1}, V_{\tau_2}, \dots$

# Data compression (1/2)

$$\text{In general, } \underbrace{\mathbf{X}_{\tau_t}}_{N_t \times d} = \underbrace{\mathbf{U}_{\tau_t}}_{N_t \times N_t} \underbrace{\boldsymbol{\Sigma}_{\tau_t}}_{N_t \times N_t} \underbrace{\mathbf{V}_{\tau_t}^T}_{N_t \times d}.$$

Data compression : replace  $\mathbf{V}_{\tau_t}$  by  $\hat{\mathbf{V}}_{\tau_t}$  ( $d \times n$ ,  $n \ll N_t$ ) :

- “OGD” :  $\hat{\mathbf{X}}_{\tau_t}$  :  $n$  rows sampled from  $\mathbf{X}_{\tau_t}$ ,  $\hat{\mathbf{X}}_{\tau_t} = \hat{\mathbf{U}}_{\tau_t} \hat{\boldsymbol{\Sigma}}_{\tau_t} \hat{\mathbf{V}}_{\tau_t}^T$ .



Farajtabar, M., Azizan, N., Mott, A. & Li, A. (2020). Orthogonal gradient descent for continual learning. *AISTATS'2020*.

- instead of random rows, “memorable observations” :



Pan, P. , Swaroop, S. , Immer, A., Eschenhagen, R., Turner, R. & Khan, M. E. (2020). Continual Deep Learning by Functional Regularisation of Memorable Past. *NeurIPS'2020*.

Different framework, but the philosophy would here lead to select high-leverage observations.

## Data compression (2/2)

Data compression : replace  $V_{\tau_t}$  by  $\hat{V}_{\tau_t}$  ( $n \times d$ ,  $n \ll N_t$ ) :

- our proposal, “PCA-OGD” : PCA on  $X_{\tau_t}$ , that is

$$V_{\tau_t}^T = \left( \frac{\hat{V}_{\tau_t}^T}{*} \right).$$

- $\hat{\Pi}_t :=$  orthogonal projection on  $\ker(\hat{V}_{\tau_1}^T | \dots | \hat{V}_{\tau_{t-1}}^T)$ .
- $\hat{O}^{\tau_s \rightarrow \tau_t} = V_{\tau_s}^T \hat{\Pi}_t V_{\tau_t}$

$$\sqrt{\Delta^{\tau_s \rightarrow \tau_t}} \leq \|\Sigma_{\tau_s}\|_{\text{op}} \sum_{k=s+1}^t \left\| \hat{O}^{\tau_s \rightarrow \tau_t} \right\|_{\text{op}} \left\| M_{\tau_k} \tilde{Y}_{\tau_k} \right\|$$

# Experiments on the MNIST dataset

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0  
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1  
2 2 2 2 2 2 2 2 2 2 2 2 2 2 2  
3 3 3 3 3 3 3 3 3 3 3 3 3 3 3  
4 4 4 4 4 4 4 4 4 4 4 4 4 4 4  
5 5 5 5 5 5 5 5 5 5 5 5 5 5 5  
6 6 6 6 6 6 6 6 6 6 6 6 6 6 6  
7 7 7 7 7 7 7 7 7 7 7 7 7 7 7  
8 8 8 8 8 8 8 8 8 8 8 8 8 8 8  
9 9 9 9 9 9 9 9 9 9 9 9 9 9 9

Neural network with the NTK approximation :

$$f_w(x) \simeq f_{w_0}(x) + \langle \nabla_{w=w_0} f_w(x), w - w_0 \rangle$$

# Experiments : impact of $\|O^{\tau_s \rightarrow \tau_t}\|_{\text{op}}$

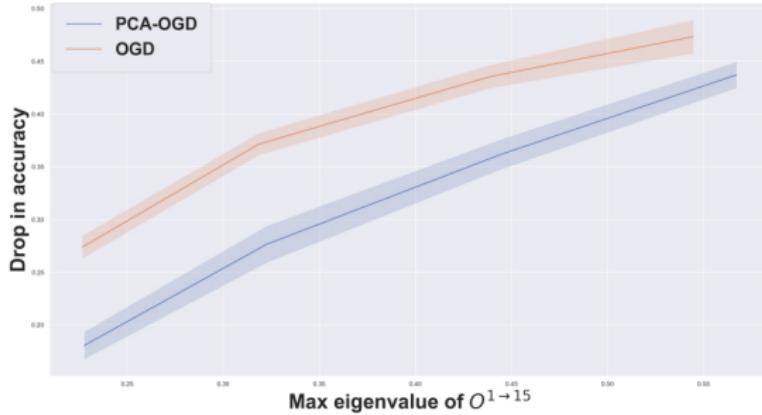
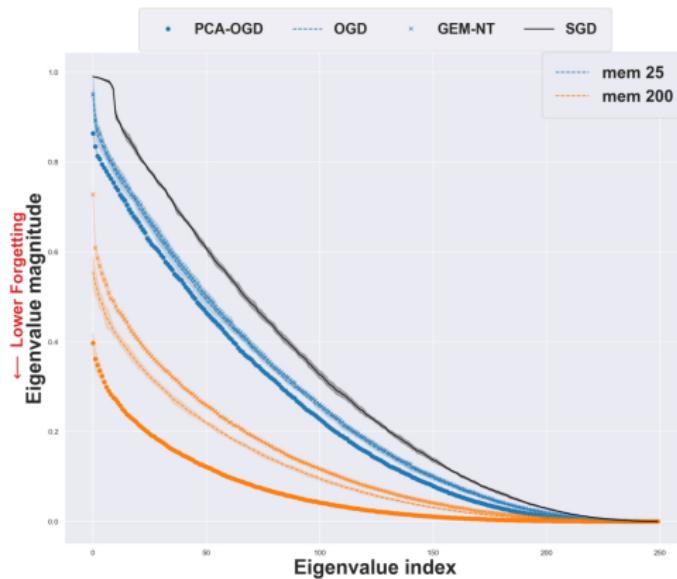


Figure 2: Drop in performance with respect to the maximum eigenvalue for Rotated MNIST (averaged over 5 seeds  $\pm 1$  std).

# Experiments : evaluation of $\|\hat{O}^{\tau_s \rightarrow \tau_t}\|_{\text{op}}$



# Experiments : performances

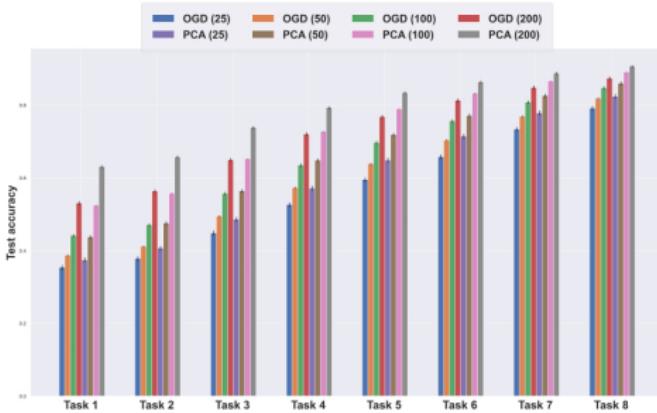


Figure 4: Final accuracy on **Rotated MNIST** for different memory size (averaged over 5 seeds  $\pm 1$  std). OGD needs twice as much memory as PCA-OGD in order to achieve the same performance (i.e compare OGD (200) and PCA (100)).

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終わり

ありがとうございます。