Concentration and robustness of discrepancy-based ABC

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The Maximum Likelihood Estimator (MLE)

Let $X_1, \ldots, X_n$ be i.i.d in $\mathcal{X}$ from a probability distribution P_0.

Statistical inference :

- propose a model $(P_\theta, \theta \in \Theta)$, assume $P_0 = P_{\theta_0}$.
- compute $\hat{\theta}_n = \hat{\theta}_n(X_1, \ldots, X_n)$.

Letting p_θ denote the density of P_θ, then

$$\hat{\theta}_n^{MLE} = \arg\max_{\theta \in \Theta} L_n(\theta), \text{ where } L_n(\theta) = \prod_{i=1}^{n} p_\theta(X_i).$$

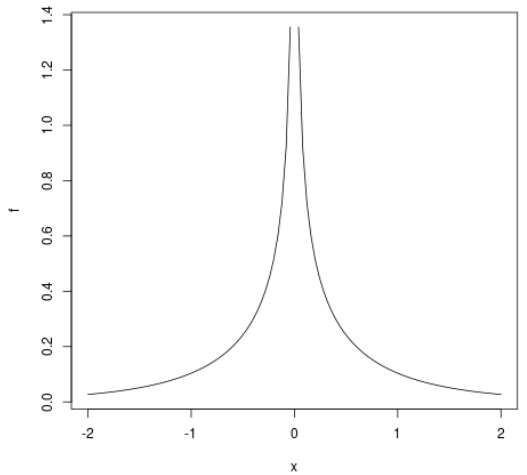
Example : $P_{(m,\sigma)} = \mathcal{N}(m, \sigma^2)$ then

$$\hat{m} = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ and } \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{m})^2.$$

MLE not unique / not consistent

Example :

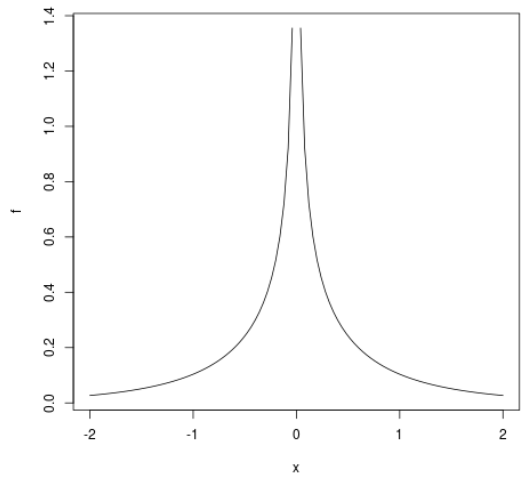
$$p_\theta(x) = \frac{\exp(-|x-\theta|)}{2\sqrt{\pi|x-\theta|}},$$



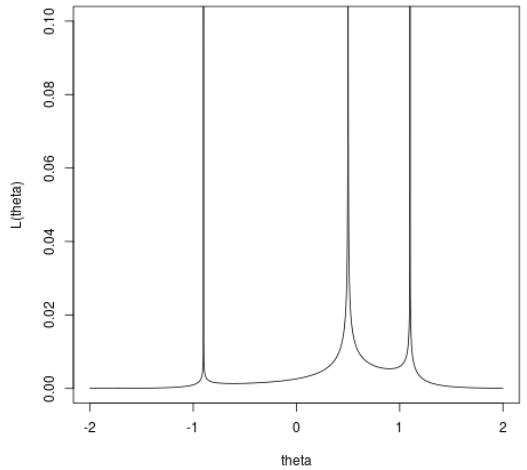
MLE not unique / not consistent

Example :

$$p_\theta(x) = \frac{\exp(-|x-\theta|)}{2\sqrt{\pi|x-\theta|}},$$



$$L_n(\theta) = \frac{\exp\left(-\sum_{i=1}^n |X_i-\theta|\right)}{(2\sqrt{\pi})^n \prod_{i=1}^n \sqrt{|X_i-\theta|}}.$$



MLE fails in the presence of outliers

What is an outlier ?

Huber proposed the contamination model : with probability ε, X_i is not drawn from P_{θ_0} but from Q that can be anything :

$$P_0 = (1-\varepsilon)P_{\theta_0} + \varepsilon Q.$$

Example : $P_\theta = \mathcal{U}nif[0,\theta]$, then

$$L_n(\theta) = \frac{1}{\theta^n}\prod_{i=1}^{n} \mathbb{1}_{\{0\leq X_i\leq\theta\}} \Rightarrow \hat{\theta} = \max_{1\leq i\leq n} X_i.$$

In the case of the following contamination, the MLE is extremely far from the truth :

$$P_0 = (1-\varepsilon).\mathcal{U}nif[0,1] + \varepsilon.\mathcal{N}(10^{10},1)...$$

Minimum Distance Estimation

$$\text{Empirical distribution : } \hat{P}_n := \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i}.$$

Minimum Distance Estimation (MDE)

Let $d(\cdot, \cdot)$ be a metric on probability distributions.

$$\hat{\theta}_d := \arg\min_{\theta \in \Theta} d\left(P_\theta, \hat{P}_n\right).$$

Wolfowitz, J. (1957). The minimum distance method. *The Annals of Mathematical Statistics*.

Idea : MDE with an adequate d leads to robust estimation.

Bickel, P. J. (1976). Another look at robustness : a review of reviews and some new developments. *Scandinavian Journal of Statistics*. Discussion by Sture Holm.

Parr, W. C. & Schucany, W. R. (1980). Minimum distance and robust estimation. *JASA*.

Yatracos, Y. G. (1985). Rates of convergence of minimum distance estimators and Kolmogorov's entropy. *Annals of Statistics*.

Integral Probability Semimetrics

Integral Probability Semimetrics (IPS)

Let $\mathcal{F}$ be a set of real-valued, measurable functions and put

$$d_{\mathcal{F}}(P,Q) = \sup_{f\in\mathcal{F}} \left| \mathbb{E}_{X\sim P}[f(X)] - \mathbb{E}_{X\sim Q}[f(X)] \right|.$$

Müller, A. (1997). Integral probability metrics and their generating classes of functions. *Applied Probability*.

- assumptions required in order to ensure that $d_{\mathcal{F}}(P,Q) = 0 \Rightarrow P = Q$ (that is, $d_{\mathcal{F}}$ is a metric).
- assumptions required in order to ensure that $d_{\mathcal{F}} < +\infty$.

Non-asymptotic bound for MDE

Theorem 1

- $X_1, \ldots, X_n$ i.i.d from P_0,
- for any $f \in \mathcal{F}$, $\sup_{x \in \mathcal{X}} |f(x)| \leq 1$.

Then

$$\mathbb{E}\left[d_{\mathcal{F}}(P_{\hat{\theta}_{d_{\mathcal{F}}}}, P_0)\right] \leq \inf_{\theta \in \Theta} d_{\mathcal{F}}(P_\theta, P_0) + 4.\mathrm{Rad}_n(\mathcal{F}).$$

Rademacher complexity

$$\mathrm{Rad}_n(\mathcal{F}) := \sup_P \mathbb{E}_{Y_1,\ldots,Y_n \sim P} \mathbb{E}_{\epsilon_1,\ldots,\epsilon_n} \left[\sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \epsilon_i f(Y_i)\right].$$

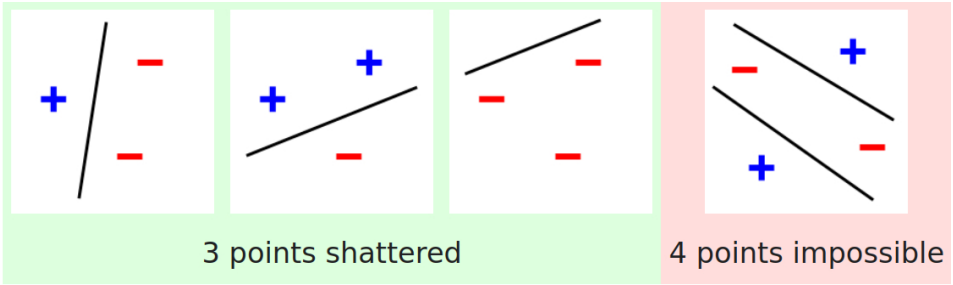
where $\epsilon_1, \ldots, \epsilon_n$ are i.i.d Rademacher variables :

$$\mathbb{P}(\epsilon_1 = 1) = \mathbb{P}(\epsilon_1 = -1) = 1/2.$$

Example 1 : set of indicators

$$\mathbb{1}_A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A. \end{cases}$$

Image from Wikipedia.



Reminder - Vapnik-Chervonenkis dimension

Assume that $\mathcal{F} = \{\mathbb{1}_A, A \in \mathcal{A}\}$ for some $\mathcal{A} \subseteq \mathcal{P}(\mathcal{X})$,

- $S_{\mathcal{F}}(x_1, \ldots, x_n) := \{(f(x_1), \ldots, f(x_n)), f \in \mathcal{F}\}$,
- $\mathrm{VC}(\mathcal{F}) := \max\{n : \exists x_1, \ldots, x_n, |S_{\mathcal{F}}(x_1, \ldots, x_n)| = 2^n\}$.

Theorem (Bartlett and Mendelson)

$$\mathrm{Rad}_n(\mathcal{F}) \leq \sqrt{\frac{2.\mathrm{VC}(\mathcal{F}) \log(n+1)}{n}}.$$

Bartlett, P. L. & Mendelson, S. (2002). Rademacher and Gaussian complexities : Risk bounds and structural results. *JMLR*.

Example 1 : KS and TV distances

Two classical examples :

- $\mathcal{A} = \{\text{all measurable sets in } \mathcal{X}\}$, then $d_{\mathcal{F}}(\cdot, \cdot)$ is the total variation distance $\mathrm{TV}(\cdot, \cdot)$.
  - $\mathrm{VC}(\mathcal{F}) = +\infty$ when $|\mathcal{X}| = +\infty$,
  - in general, $\mathrm{Rad}_n(\mathcal{F}) \not\to 0$.
- $\mathcal{X} = \mathbb{R}$, $\mathcal{A} = \{(-\infty, x], x \in \mathbb{R}\}$, then $d_{\mathcal{F}}(\cdot, \cdot)$ is the Kolmogorov-Smirnov distance $\mathrm{KS}(\cdot, \cdot)$.
  - KS distance was actually proposed by S. Holm for robust estimation,
  - $\mathrm{VC}(\mathcal{F}) = 1$.

$$\mathbb{E}\left[\mathrm{KS}(P_{\hat{\theta}_{\mathrm{KS}}}, P_0)\right] \leq \inf_{\theta \in \Theta} \mathrm{KS}(P_\theta, P_0) + 4.\sqrt{\frac{2\log(n+1)}{n}}.$$

Example 2 : Maximum Mean Discrepancy (MMD)

- Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be a RKHS with kernel
$$k(x, y) = \langle \phi(x), \phi(y) \rangle_{\mathcal{H}} .$$
- If $\|\phi(x)\|_{\mathcal{H}} = k(x, x) \leq 1$ then $\mathbb{E}_{X \sim P}[\phi(X)]$ is well-defined .
- The map $P \mapsto \mathbb{E}_{X \sim P}[\phi(X)]$ is one-to-one if k is *characteristic*.
- For example, $k(x, y) = \exp(-\|x - y\|^2/\gamma^2)$ works.

Definition - MMD

$$\mathrm{MMD}_k(P, Q) = \sup_{\substack{f \in \mathcal{H} \\ \|f\|_{\mathcal{H}} \leq 1}} \left| \mathbb{E}_{X \sim P}[f(X)] - \mathbb{E}_{X \sim Q}[f(X)] \right|$$
$$= \left\| \mathbb{E}_{X \sim P}[\phi(X)] - \mathbb{E}_{X \sim Q}[\phi(X)] \right\|_{\mathcal{H}} .$$

Example 2 : MMD

$$\mathcal{F} = \{f \in \mathcal{H} : \|f\|_{\mathcal{H}} \leq 1\} \Rightarrow \mathrm{Rad}_n(\mathcal{F}) \leq \sqrt{\frac{\sup_x k(x,x)}{n}}.$$

Theorem 2

For k bounded by 1 and characteristic,

$$\mathbb{E}\left[\mathrm{MMD}_k(P_{\hat{\theta}_{\mathrm{MMD}_k}}, P_0)\right] \leq \inf_{\theta \in \Theta} \mathrm{MMD}_k(P_\theta, P_0) + \frac{2}{\sqrt{n}}.$$

Joint work with Badr-Eddine Chérief-Abdellatif (Oxford).

Chérief-Abdellatif, B.-E. and Alquier, P. Finite Sample Properties of Parametric MMD Estimation : Robustness to Misspecification and Dependence. **Bernoulli**, 2022.

Example 2 : MMD

We actually have

$$\mathrm{MMD}_k^2(P_\theta, \hat{P}_n) = \mathbb{E}_{X,X' \sim P_\theta}[k(X, X')] - \frac{2}{n}\sum_{i=1}^{n} \mathbb{E}_{X \sim P_\theta}[k(X_i, X)] + \frac{1}{n^2}\sum_{1 \leq i,j \leq n} k(X_i, X_j)$$

and so

$$\nabla_\theta \mathrm{MMD}_k^2(P_\theta, \hat{P}_n) = 2\mathbb{E}_{X,X' \sim P_\theta}\left\{ \left[k(X, X') - \frac{1}{n}\sum_{i=1}^{n} k(X_i, X) \right] \nabla_\theta [\log p_\theta(X)] \right\}$$

that can be approximated by sampling from P_θ.

Example 2 : MMD

Dziugaite, G. K., Roy, D. M., & Ghahramani, Z. (2015). Training generative neural networks via maximum mean discrepancy optimization. *UAI 2015*.

define the estimator and used it to train GANs.



Briol, F. X., Barp, A., Duncan, A. B., & Girolami, M. (2019). Statistical Inference for Generative Models with Maximum Mean Discrepancy. *Preprint arXiv :1906.05944*.

assumptions $\Rightarrow$ $\sqrt{n}(\hat{\theta}_{\mathrm{MMD}_k} - \theta_0) \rightsquigarrow \mathcal{N}(0, V_0(k))$.

Example 3 : Wasserstein

Another classical metric belongs to the IPS family :

$$\mathrm{W}_\delta(P, Q) = \sup_{\substack{f\,:\,\mathcal{X}\to\mathbb{R} \\ \mathrm{Lip}(f)\leq 1}} \left| \mathbb{E}_{X\sim P}[f(X)] - \mathbb{E}_{X\sim Q}[f(X)] \right|$$

where $\mathrm{Lip}(f) := \sup_{x\neq y} |f(x) - f(y)|/\delta(x, y)$.

- In general, $\mathrm{Rad}_n(\mathcal{F}) \nrightarrow 0$, so will not converge in full generality as with MMD and KS.
- However, nice results can be proven under additional assumptions :

Bernton, E., Jacob, P. E., Gerber, M. & Robert, C. P. (2019). On parameter estimation with the Wasserstein distance. *Information and Inference : A Journal of the IMA*.

MDE and robustness

Reminder

$$\mathbb{E}\left[d_{\mathcal{F}}(P_{\hat{\theta}_{d_{\mathcal{F}}}}, P_0)\right] \leq \inf_{\theta\in\Theta} d_{\mathcal{F}}(P_\theta, P_0) + 4.\mathrm{Rad}_n(\mathcal{F}).$$

Huber's contamination model : $P_0 = (1-\varepsilon)P_{\theta_0} + \varepsilon Q$.

$$\begin{aligned} d_{\mathcal{F}}(P_{\theta_0}, P_0) &= \sup_{f\in\mathcal{F}} \left|\mathbb{E}_{X\sim P_{\theta_0}} f(X) - (1-\varepsilon)\mathbb{E}_{X\sim P_{\theta_0}} f(X) - \varepsilon\mathbb{E}_{X\sim Q} f(X)\right| \\ &= \sup_{f\in\mathcal{F}} \left|\varepsilon\mathbb{E}_{X\sim P_{\theta_0}} f(X) - \varepsilon\mathbb{E}_{X\sim Q} f(X)\right| \\ &= \varepsilon.d_{\mathcal{F}}(P_{\theta_0}, Q) \leq 2\varepsilon \qquad \text{if for any } f\in\mathcal{F},\ \sup_x |f(x)| \leq 1 \end{aligned}$$

Corollary - in Huber's contamination model

$$\mathbb{E}\left[d_{\mathcal{F}}(P_{\hat{\theta}_{d_{\mathcal{F}}}}, P_{\theta_0})\right] \leq 4\varepsilon + 4.\mathrm{Rad}_n(\mathcal{F}).$$

MDE and robustness : toy experiment

Model : $\mathcal{N}(\theta, 1)$, $X_1, \ldots, X_n$ i.i.d $\mathcal{N}(\theta_0, 1)$, $n = 100$ and we repeat the exp. 200 times. Kernel $k(x, y) = \exp(-|x - y|)$.

	$\hat{\theta}_{MLE}$	$\hat{\theta}_{\mathrm{MMD}_k}$	$\hat{\theta}_{\mathrm{KS}}$
mean abs. error	0.081	0.094	0.088

Now, $\varepsilon = 2\%$ of the observations drawn from a Cauchy.

mean abs. error	0.276	0.095	0.088

Now, $\varepsilon = 1\%$ are replaced by $1,000$.

mean abs. error	10.008	0.088	0.082

Contents

Generalized posteriors

Posterior

$$\pi(\theta|X_1,\dots,X_n) \propto L_n(\theta)\pi(\theta).$$

Generalized posterior

$$\hat{\pi}_{\beta,R_n}(\theta) \propto \exp(-\beta.R_n(\theta))\pi(\theta).$$

- old idea in ML (PAC-Bayes, forecasting with expert advice...) and in statistics (Gibbs posteriors...)
- popularized / extended and studied by :


Bissiri, P. G., Holmes, C. C. & Walker, S. G. (2016). A general framework for updating belief distributions. *JRSS-B*.


Knoblauch, J., Jewson, J. & Damoulas, T. (2022). An Optimization-centric View on Bayes' Rule : Reviewing and Generalizing Variational Inference. *JMLR* (to appear).

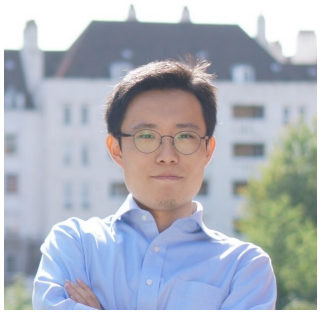
Generalizing the posterior with IPS

Generalized posterior with IPS

$$\hat{\pi}_{\beta,R_n}(\theta) \propto \exp(-\beta.d_{\mathcal{F}}(P_\theta, \hat{P}_n))\pi(\theta).$$

- in the MMD case : non-asymptotic result in

Chérief-Abdellatif, B.-E. and Alquier, P. (2020). MMD-Bayes : Robust Bayesian Estimation via Maximum Mean Discrepancy. *Proceedings of AABI*.



- asymptotic results to come very soon in a joint paper with Takuo Matsubara (Newcastle) and Jeremias Knoblauch (UCL).

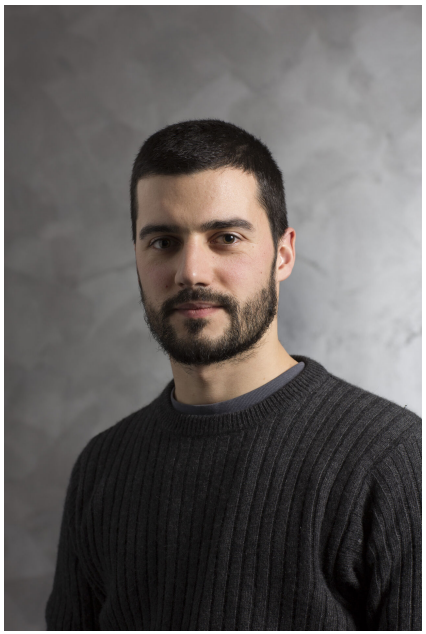
- both papers discuss computation via variational approximations or MCMC.

Contents

ABC with IPS

What follows is based on a joint work with :

Sirio Legramanti
(University of Bergamo)



Daniele Durante
(Bocconi University, Milan)

Reminder on ABC

Approximate Bayesian Computation (ABC)

input : sample $X_1^n = (X_1, \dots, X_n)$, model $(P_\theta, \theta \in \Theta)$, prior π, statistic S, distance δ and threshold ϵ.

(i) sample $\theta \sim \pi$,

(ii) sample $Y_1^n = (Y_1, \dots, Y_n)$ i.i.d. from P_θ :

- if $\delta(S(X_1^n), S(Y_1^n)) \leq \epsilon$ return θ,
- else goto (i).

- how close is the distribution of the output to the posterior $\pi(\theta | X_1, \dots, X_n)$?
- reverse point of view : what are the properties of the "generalized posterior" we sample from ?

ABC with IPS

Here, we study the situation :

- $S(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$ the empirical distribution,
- $\delta(P, Q) = d_{\mathcal{F}}(P, Q)$.

IPS–ABC

input : sample $X_1^n = (X_1, \ldots, X_n)$, model $(P_\theta, \theta \in \Theta)$, prior π, set of functions $\mathcal{F}$ and threshold ϵ. Put $\hat{P}_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$.

(i) sample $\theta \sim \pi$,

(ii) sample $Y_1^n = (Y_1, \ldots, Y_n)$ i.i.d. from P_θ and put $\hat{P}_n^Y = \frac{1}{n} \sum_{i=1}^n \delta_{Y_i}$,

- if $d_{\mathcal{F}}(\hat{P}_n, \hat{P}_n^Y) \leq \epsilon$ return θ,
- else goto (i).

Notation : the output $\vartheta \sim \hat{\pi}_{n,\epsilon}^{\mathcal{F}}(\cdot)$.

Properties of $\hat{\pi}^{\mathcal{F}}_{n,\epsilon}(\cdot)$

3 questions :

1. $\hat{\pi}^{\mathcal{F}}_{n,\epsilon}(\theta) \xrightarrow[\epsilon \searrow ?]{?} \pi(\theta|X_1^n)$.
2. $\hat{\pi}^{\mathcal{F}}_{n,\epsilon}(\theta) \xrightarrow[n\to\infty]{?} ?$
3. $\hat{\pi}^{\mathcal{F}}_{n,\epsilon_n}(\cdot) \xrightarrow[n\to\infty]{?} \delta_{\theta_0}$ if $P_0 = P_{\theta_0}$.

Contraction of the ABC posterior

$$\epsilon_* := \inf_{\theta \in \Theta} d_{\mathcal{F}}(P_\theta, P_0).$$

Theorem 3

Assume :

- for all $\epsilon > 0$, $\pi(\{\theta : d_{\mathcal{F}}(P_\theta, P_0) \leq \epsilon_* + \epsilon\}) \geq c\epsilon^d$.
- $\forall f \in \mathcal{F}$, $\sup_{x \in \mathcal{X}} |f(x)| \leq 1$.
- $\mathrm{Rad}_n(\mathcal{F}) \xrightarrow[n \to \infty]{} 0$.

Let ϵ_n be any sequence such that $\epsilon_n / \mathrm{Rad}_n(\mathcal{F}) \to 0$ and $n\epsilon_n \to \infty$. Then, with probability $\to 1$ on the sample, for any $M_n \to \infty$,

$$\hat{\pi}^{\mathcal{F}}_{n, \epsilon_* + \epsilon_n} \left(d_{\mathcal{F}}(P_\theta, P_0) \leq \epsilon_* + \frac{4\epsilon_n}{3} + \mathrm{Rad}_n(\mathcal{F}) + \sqrt{\frac{\log \frac{M_n}{\epsilon_n^d}}{n}} \right) \geq 1 - \frac{2.3^d}{cM_n}.$$

Example : MMD-ABC with bounded kernel

As an example, consider MMD_k when $k(x,x) \leq 1$, as

Park, M., Jitkrittum, W. & Sejdinovic, D. (2016). K2-ABC : Approximate Bayesian Computation with kernel embeddings. *AISTATS*.

Corollary

Assume :

- for all $\epsilon > 0$, $\pi(\{\theta : d_{\mathcal{F}}(P_\theta, P_0) \leq \epsilon_* + \epsilon\}) \geq c\epsilon^d$.

Let $1/n \ll \epsilon_n \ll 1/\sqrt{n}$. Then, with probability $\to 1$ on the sample, for any $M_n \to \infty$,

$$\hat{\pi}^{\mathcal{F}}_{n,\epsilon_*+\epsilon_n}\left(\mathrm{MMD}_k(P_\theta, P_0) \leq \epsilon_* + \frac{4\epsilon_n}{3} + \frac{1+\sqrt{\log \frac{M_n}{\epsilon_n^d}}}{\sqrt{n}}\right) \geq 1 - \frac{2.3^d}{cM_n}.$$

A result without Rademacher complexity

Theorem 4

Assume :

- for all $\epsilon > 0$, $\pi(\{\theta : d_{\mathcal{F}}(P_\theta, P_0) \leq \epsilon_* + \epsilon\}) \geq c\epsilon^d$,
- $d_{\mathcal{F}}(\hat{P}_n, P_0) \xrightarrow{P_0 \text{ a.s.}} 0$,
- $\mathbb{P}_{Y_1^n \sim P_\theta}(d_{\mathcal{F}}(\hat{P}_n^Y, P_\theta) > \epsilon) \leq c(\theta) f_n(\epsilon)$ where $f_n(\epsilon) \xrightarrow[n\to\infty]{} 0$.

Let $\epsilon_n \to 0$ and $f_n(\epsilon_n) \to 0$. Then, with probability $\to 1$ on the sample, for some $C > 0$ and any $M_n \to \infty$,

$$\hat{\pi}^{\mathcal{F}}_{n,\epsilon_*+\epsilon_n}\left(d_{\mathcal{F}}(P_\theta, P_0) \leq \epsilon_* + \frac{4\epsilon_n}{3} + f_n^{-1}\left(\frac{\epsilon_n^d}{M_n}\right)\right) \geq 1 - \frac{C}{M_n}.$$

Example : MMD-ABC with unbounded kernel

Corollary

Assume :

- for all $\epsilon > 0$, $\pi(\{\theta : d_{\mathcal{F}}(P_\theta, P_0) \leq \epsilon_* + \epsilon\}) \geq c\epsilon^d$,
- $\mathbb{E}_{Z\sim Q}[k(Z,Z)] < +\infty$ for $Q = P_0$ and any $Q \in \{P_\theta, \theta \in \Theta\}$.

Let $1/n^{2d} \ll \epsilon_n \ll 1$. Then, with probability $\to 1$ on the sample, there is a $C > 0$ such that for any $M_n \to \infty$,

$$\hat{\pi}^{\mathcal{F}}_{n,\epsilon_*+\epsilon_n}\left(\mathrm{MMD}_k(P_\theta, P_0) \leq \epsilon_* + \frac{4\epsilon_n}{3} + \frac{1}{n}\sqrt{\frac{M_n}{\epsilon_n^d}}\right) \geq 1 - \frac{C}{M_n}.$$

Example : Wasserstein-ABC

Bernton, E., Jacob, P. E., Gerber, M. & Robert, C. P. (2019). Approximate Bayesian Computation with the Wasserstein distance. *JRSS-B*.

considered ABC with the Wasserstein distance, and proved Theorem 4 in this case (note that our Theorem 4 is a restatement of their result for a general IPS).

However, it not easy to prove non-trivial bounds :

$$\mathbb{P}_{Y_1^n \sim P_\theta}(d_{\mathcal{F}}(\hat{P}_n^Y, P_\theta) > \epsilon) \leq c(\theta) f_n(\epsilon),$$

and the examples they cite require $\mathcal{X}$ to be a **bounded space**.

Weed, J. & Bach, F. (2019). Sharp asymptotic and finite-sample rates of convergence of empirical measures in Wasserstein distance. *Bernoulli*.

$n \to \infty$, fixed ϵ

Jiang, B., Wu, T.-Y. & Wong, W. H. (2018). Approximate Bayesian computation with Kullback-Leibler divergence as data discrepancy. *AISTATS*.

provides a general result that can be directly used here :

Theorem (simplified version)

Assume :

- $\mathrm{Rad}_n(\mathcal{F}) \to 0$,

then for any measurable B,

$$\hat{\pi}^{\mathcal{F}}_{n,\epsilon}(\theta \in B) \xrightarrow[n\to\infty]{\text{a.s.}} \pi(\theta \in B | d_{\mathcal{F}}(P_\theta, P_0) \leq \epsilon_* + \epsilon).$$

$\epsilon \searrow 0$, fixed n

Bernton, E., Jacob, P. E., Gerber, M. & Robert, C. P. (2019). Approximate Bayesian Computation with the Wasserstein distance. *JRSS-B*.

provides a general result that can be directly used here :

Theorem (simplified version)

Assume :

- P_θ has continuous, bounded densities p_θ, and $P_0 = P_{\theta_0}$,
- $d_{\mathcal{F}}$ is continous, and is a metric,

then a.s. with respect to the sample, for any measurable B,

$$\hat{\pi}^{\mathcal{F}}_{n,\epsilon}(\theta \in B) \xrightarrow[\epsilon \searrow 0]{} \pi(\theta \in B | X_1^n).$$

Experimental results

	$\mathbb{E}_{\pi_n}(\theta)$			$\mathrm{var}_{\pi_n}(\theta)$			$\pi_n(\theta \in [0.95, 1.05])$			time
	5%	10%	20%	5%	10%	20%	5%	10%	20%	
MMD	0.99	0.95	0.87	0.03	0.03	0.03	0.22	0.22	0.19	3"
Wasserstein	0.98	0.78	0.66	0.03	0.05	0.05	0.23	0.12	0.09	6"
KL	0.91	0.91	0.84	0.11	0.11	0.09	0.11	0.11	0.12	5"
"Exact" Gaussian	1.01	0.83	0.71	0.01	0.01	0.01	0.38	0.10	0.01	

Table 1: For $\mathcal{D}$-ABC based, respectively, on MMD, Wasserstein and KL discrepancies, runtimes and summaries of the ABC posterior for θ under the Gaussian model with sample size $n = 100$ and varying percentages of Cauchy contamination. Results are also compared with the "Exact" Gaussian posterior when updated with all data, including contaminated ones.

Other current and future directions

- go beyond IPS, see (among others) f-divergences or energy statistics in

Frazier, D. T. (2020). *Robust and efficient Approximate Bayesian Computation : A minimum distance approach*. Preprint arXiv.

Nguyen, H. D., Arbel, J., Lü, H. and Forbes, F. (2020). Approximate Bayesian computation via the energy statistic. *IEEE Access*.

- solve practical issues : choice of ϵ_n, choice of k in MMD_k.
- semi-parametric models (with J.-D. Fermanian (ENSAE Paris), A. Derumigny (TU Delft) and M. Gerber (Bristol).



Alquier, P., Chérief-Abdellatif, B.-E., Derumigny, A. and Fermanian, J.-D. Estimation of copulas via Maximum Mean Discrepancy. **JASA**, to appear.

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La fin

終わり

ありがとう ございます。