

Regret bounds for generalized Bayes updates

Pierre Alquier



Center for
Advanced Intelligence Project

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The sequential prediction problem

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Objective : make sure that we learn to predict well **as soon as possible**.

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Objective : make sure that we learn to predict well **as soon as possible**. Keep

$$\sum_{t=1}^T \ell(\hat{y}_t, y_t)$$

as small as possible.

1st approach : “follow the regularized leader”

- set of predictors : $\{f_\theta, \theta \in \Theta\}$.
- $\ell_t(\theta) := \ell(f_\theta(x_t), y_t)$.

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Follow The Regularized Leader – FTRL

$$\theta^t := \arg \min_{\theta} \left\{ \sum_{s=1}^{t-1} \ell_s(\theta) + \frac{\text{pen}(\theta)}{\eta} \right\}.$$

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Quadratic penalty + linearization :

$$\theta^t := \arg \min_{\theta} \left\{ \sum_{s=1}^{t-1} \langle \theta, \nabla \ell_s(\theta^s) \rangle + \frac{\|\theta\|^2}{2\eta} \right\}.$$

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Differentiate :

$$0 = \sum_{s=1}^{t-1} \nabla \ell_s(\theta^s) + \frac{\theta^t}{\eta}.$$

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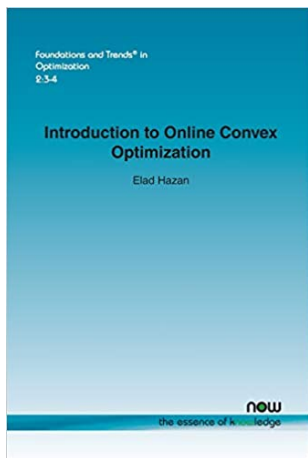
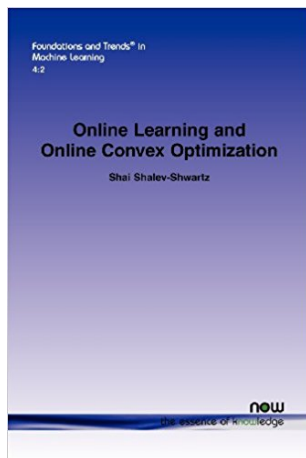
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$$\theta^t := \arg \min_{\theta} \left\{ \sum_{s=1}^{t-1} \ell_s(\theta) + \frac{\text{pen}(\theta)}{\eta} \right\}.$$

Online Gradient Algorithm – OGA

$$\theta^t := \theta^{t-1} - \eta \nabla \ell_{t-1}(\theta^{t-1}).$$

Theoretical properties of FTRL & OGA



2nd approach : (generalized) Bayes

Generalized Bayes, multiplicative weights, Exponential Weight Aggregation (EWA)...

EWA

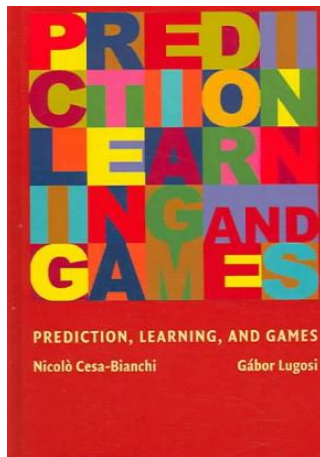
$$\rho^t(\theta) \propto \exp \left[-\eta \sum_{s=1}^{t-1} \ell_s(\theta) \right] \pi(\theta)$$

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EWA

$$\rho^t(\theta) \propto \exp \left[-\eta \sum_{s=1}^{t-1} \ell_s(\theta) \right] \pi(\theta)$$



EWA as FTRL

It is known that

$$\rho^t = \arg \min_{\rho \in \mathcal{P}(\Theta)} \left\{ \sum_{s=1}^{t-1} \underbrace{\mathbb{E}_{\theta \sim \rho} [\ell_s(\theta)]}_{=: L_s(\rho)} + \underbrace{\frac{\text{KL}(\rho || \pi)}{\eta}}_{=: \frac{\text{pen}(\rho)}{\eta}} \right\}.$$

That is, EWA is a special case of FTRL.

$$\text{KL}(\rho || \pi) = \begin{cases} \mathbb{E}_{\theta \sim \rho} \left[\log \left(\frac{d\rho}{d\pi}(\theta) \right) \right] & \text{if } \rho \ll \pi \\ +\infty & \text{otherwise.} \end{cases}$$

1st objective

We will study a more general version of FTRL on ρ :

$$\rho^t = \arg \min_{\rho \in \mathcal{P}(\Theta)} \left\{ \sum_{s=1}^{t-1} \mathbb{E}_{\theta \sim \rho} [\ell_s(\theta)] + \frac{D(\rho || \pi)}{\eta} \right\},$$

for more general divergences D .



P. Alquier. *Non-exponentially Weighted Aggregation : Regret Bounds for Unbounded Loss Functions*. Accepted for ICML 2021.

2nd objective

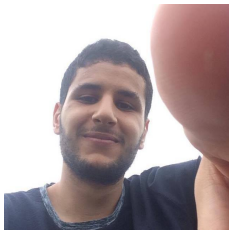
EWA is often non feasible in practice. We will thus modify it : we will constrain ρ^t to belong to a feasible set of probability distributions (e.g. : Gaussian).



B.-E. Chérif-Abdellatif, P. Alquier, M. E. Khan (2019). *A regret bound for online variational inference*. 11th Asian Conference on Machine Learning (ACML).

Co-authors

Badr-Eddine
Chérif-Abdellatif



Emtiyaz Khan



Approximate Bayesian Inference team

[https : //team — approx — bayes.github.io/](https://team-approx-bayes.github.io/)

1 Generalized Bayes update

- Formula for the posterior : non-exponential weights
- Regret bound

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- The algorithms : SVA and SVB
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Reminder

$$\rho^t = \arg \min_{\rho \in \mathcal{P}(\Theta)} \left\{ \sum_{s=1}^{t-1} \mathbb{E}_{\theta \sim \rho} [\ell_s(\theta)] + \frac{D_\phi(\rho || \pi)}{\eta} \right\},$$

Reminder

$$\rho^t = \arg \min_{\rho \in \mathcal{P}(\Theta)} \left\{ \sum_{s=1}^{t-1} \mathbb{E}_{\theta \sim \rho} [\ell_s(\theta)] + \frac{D_\phi(\rho || \pi)}{\eta} \right\},$$

where

$$D_\phi(\rho || \pi) = \begin{cases} \mathbb{E}_{\theta \sim \pi} \left[\phi \left(\frac{d\rho}{d\pi}(\theta) \right) \right] & \text{if } \rho \ll \pi \\ +\infty & \text{otherwise,} \end{cases}$$

and $\phi : \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{+\infty\}$ with :

- ϕ convex,
- $\phi(1) = 0$,
- $\inf_{x \geq 0} \phi(x) > -\infty$.

Differential of the convex conjugate

Assume that ϕ is differentiable, strictly convex. Put

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & \text{if } x \geq 0, \\ +\infty & \text{if } x < 0. \end{cases}$$

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Then

$$\tilde{\phi}^* = \sup_{x \in \mathbb{R}} [xy - \tilde{\phi}(x)] = \sup_{x \geq 0} [xy - \phi(x)]$$

is differentiable and for any $y \in \mathbb{R}$,

$$\nabla \tilde{\phi}^*(y) = \arg \max_{x \geq 0} \{xy - \phi(x)\}.$$

Formula for ρ^t

Assume moreover that $\tilde{\phi}^*(\lambda - a) - \lambda \rightarrow \infty$ when $\lambda \rightarrow \infty$, for any $a \geq 0$. Then :

$$\lambda_t = \arg \min_{\lambda \in \mathbb{R}} \left\{ \int \tilde{\phi}^* \left(\lambda - \eta \sum_{s=1}^{t-1} \ell_s(\theta) \right) \pi(d\theta) - \lambda \right\}$$

exists, and

$$\rho^t(d\theta) = \nabla \tilde{\phi}^* \left(\lambda_t - \eta \sum_{s=1}^{t-1} \ell_s(\theta) \right) \pi(d\theta).$$

The classical example : KL and exponential weights

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$$\rho^t(d\theta) = \exp \left[\lambda_t - \eta \sum_{s=1}^{t-1} \ell_s(\theta) - 1 \right] \pi(d\theta).$$

$$\rho^t(d\theta) = \frac{\exp \left[-\eta \sum_{s=1}^{t-1} \ell_s(\theta) \right] \pi(d\theta)}{\int \exp \left[-\eta \sum_{s=1}^{t-1} \ell_s(\vartheta) \right] \pi(d\vartheta)}.$$

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$$\rho^t(d\theta) = \left[\frac{\lambda_t - \eta \sum_{s=1}^{t-1} \ell_s(\theta)}{2} \right]_+ \pi(d\theta).$$

Some references

- the formula was known for a finite Θ :



M. D. Reid, R. M. Frongillo, R. C. Williamson, N. Mehta (2015). *Generalized mixability via entropic duality*. COLT.

- the proof for the general case relies on :



R. Agrawal, T. Horel (2020). *Optimal bounds between f -divergences and integral probability metrics*. ICML.

- comparable PAC-Bayes bounds (no online update) :



P. Alquier and B. Guedj (2018). *Simpler PAC-Bayesian bounds for hostile data*. Machine Learning.

- defense of the generalized Bayes update :



J. Knoblauch, J. Jewson, T. Damoulas (2019). *Generalized variational inference : Three arguments for deriving new posteriors..* Preprint arXiv.

- more : see the paper.

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Theorem

$$\sum_{t=1}^T \mathbb{E}_{\theta \sim \rho^t}[\ell_t(\theta)] \leq \inf_{\rho \in \mathcal{P}(\Theta)} \left\{ \sum_{t=1}^T \mathbb{E}_{\theta \sim \rho}[\ell_t(\theta)] + \frac{\eta L^2 T}{\alpha} + \frac{D_\phi(\rho||\pi)}{\eta} \right\}.$$

Bound for EWA

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and the two conditions reduce to :

$$|\ell_t(\theta)| \leq L \text{ on the support of } \pi.$$

Bound with χ^2

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but the two conditions reduce to :

$$\int \ell_t(\theta)^2 \pi(d\theta) \leq L^2.$$

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Motivation



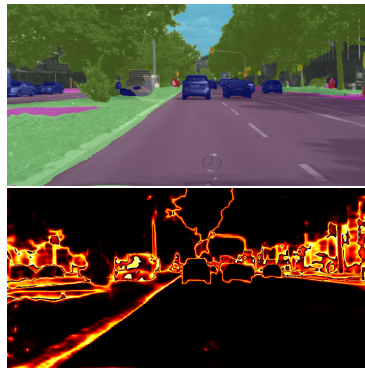
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- 1 proposes a fast algorithm to approximate the posterior,
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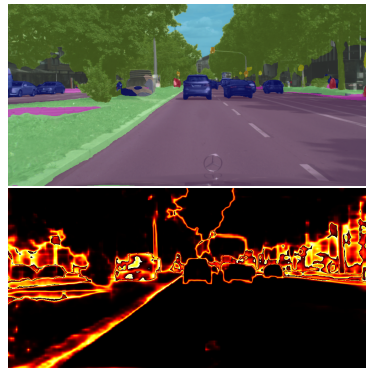
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Objective : provide a theoretical analysis of this algorithm.

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Linearization gives :

SVA

$$\mu^t = \arg \min_{\mu \in M} \left\{ \sum_{s=1}^{t-1} \langle \mu, \nabla \mathbb{E}_{\theta \sim q_{\mu^s}} [\ell_s(\theta)] \rangle + \frac{D_\phi(q_\mu, \pi)}{\eta} \right\}.$$

Streaming Variational Bayes (SVB)

(OGA) can actually be obtained via :

$$\theta^t := \arg \min_{\theta} \left\{ \sum_{s=1}^{t-1} \langle \theta, \nabla \ell_s(\theta^s) \rangle + \frac{\|\theta\|^2}{2\eta} \right\}$$

OR

$$\theta^t := \arg \min_{\theta} \left\{ \langle \theta, \nabla \ell_{t-1}(\theta^{t-1}) \rangle + \frac{\|\theta - \theta^{t-1}\|^2}{2\eta} \right\}$$

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SVB

$$\mu^t = \arg \min_{\mu \in M} \left\{ \left\langle \mu, \nabla \mathbb{E}_{\theta \sim q_{\mu^{t-1}}} [\ell_{t-1}(\theta)] \right\rangle + \frac{D_{\phi}(q_{\mu}, q_{\mu^{t-1}})}{\eta} \right\}.$$

SVA & SVB are tractable, and not equivalent

Example : Gaussian prior $\theta \sim \pi = \mathcal{N}(0, s^2 I)$, $D_\phi = \text{KL}$ and mean-field Gaussian approximation, $\mu = (m, \sigma)$.

$$\begin{aligned}\text{SVA : } m_{t+1} &\leftarrow m_t - \eta s^2 \bar{g}_{m_t}, & g_{t+1} &\leftarrow g_t + \bar{g}_{\sigma_t}, \\ \sigma_{t+1} &\leftarrow h(\eta s g_{t+1}) s, \\ \text{SVB : } m_{t+1} &\leftarrow m_t - \eta \sigma_t^2 \bar{g}_{m_t}, \\ \sigma_{t+1} &\leftarrow \sigma_t h(\eta \sigma_t \bar{g}_{\sigma_t})\end{aligned}$$

where $h(x) := \sqrt{1 + x^2} - x$ is applied componentwise, as well as the multiplication of two vectors, and

$$\begin{aligned}\bar{g}_{m_t} &= \frac{\partial}{\partial m} \mathbb{E}_{\theta \sim \pi_{m_t, \sigma_t}} [\ell_t(\theta)], \\ \bar{g}_{\sigma_t} &= \frac{\partial}{\partial \sigma} \mathbb{E}_{\theta \sim \pi_{m_t, \sigma_t}} [\ell_t(\theta)].\end{aligned}$$

Theoretical analysis of SVA

Two assumptions :

- 1 $\mu \mapsto \mathbb{E}_{\theta \sim q_\mu}[\ell_t(\theta)]$ is L -Lipschitz and convex.

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For example true when q_μ is Gaussian with $\mu = (m, \Sigma)$ and $D_\phi = \text{KL}$.

Theoretical analysis of SVA

Theorem

Under the previous assumptions SVA leads to

$$\sum_{t=1}^T \mathbb{E}_{\theta \sim q_{\mu_t}} [\ell_t(\theta)] \\ \leq \inf_{\mu \in M} \left\{ \sum_{t=1}^T \mathbb{E}_{\theta \sim q_{\mu}} [\ell_t(\theta)] + \frac{\eta L^2 T}{\alpha} + \frac{D_{\phi}(q_{\mu}, \pi)}{\eta} \right\}.$$

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Application to Gaussian approximation with KL :

$$\sum_{t=1}^T \mathbb{E}_{\theta \sim q_{\mu_t}} [\ell_t(\theta)] \leq \inf_{\theta} \sum_{t=1}^T \ell_t(\theta) + (1 + o(1)) L \sqrt{dT \log(T)}.$$

Theoretical analysis of SVB

Theorem 2

Using **Gaussian approximations** and $D_\phi = \text{KL}$, assuming the loss is convex, L -Lipschitz and the parameter space bounded (diameter = D), SVB with adequate η leads to

$$\sum_{t=1}^T \ell_t \left(\mathbb{E}_{\theta \sim q_{\mu_t}}(\theta) \right) \leq \inf_{\theta} \sum_{t=1}^T \ell_t(\theta) + DL\sqrt{2T}.$$

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If, moreover, the loss is H -strongly convex,

$$\sum_{t=1}^T \ell_t \left(\mathbb{E}_{\theta \sim q_{\mu_t}}(\theta) \right) \leq \inf_{\theta} \sum_{t=1}^T \ell_t(\theta) + \frac{L^2(1 + \log(T))}{H}.$$

Test on a simulated dataset

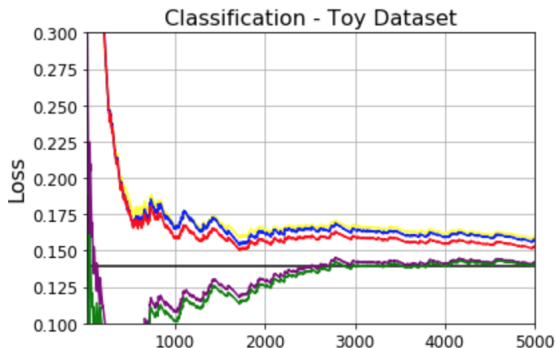


Figure – Average cumulative losses on different datasets for classification and regression tasks with OGA (yellow), OGA-EL (red), SVA (blue), SVB (purple) and NGVI (green).

Test on the Breast dataset

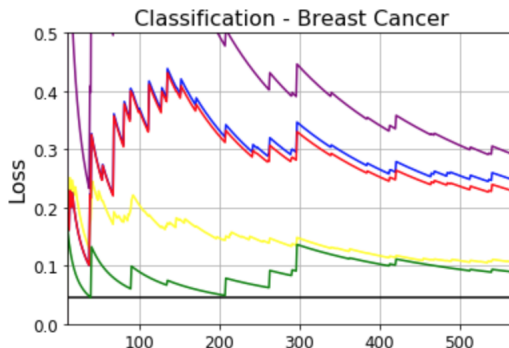


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Uses exponential family approximations $\{q_\mu, \mu \in M\}$ where m is the mean parameter. Denoting λ the natural parameter (with $\lambda = F(\mu)$),

$$\lambda^t = (1 - \rho)\lambda^{t-1} + \rho \nabla_\mu \mathbb{E}_{\theta \sim q_{\mu^{t-1}}} [\ell_t(\theta)],$$



M. E. Khan, D. Nielsen (2018). *Fast yet Simple Natural-Gradient Descent for Variational Inference in Complex Models*. ISITA.

Thank you !