Unanimity of two selves in decision making

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Abstract

We propose a new model of incomplete preferences under uncertainty, which we call *unanimous dual-self preferences*. Act f is considered more desirable than act g when, and only when, both the evaluation of an optimistic self, computed as the welfare level attained in a best-case scenario, and that of a pessimistic self, computed as the welfare level attained in a worst-case scenario, rank f above g. Our comparison criterion involves multiple priors, as best and worst cases are determined among sets of probability distributions, and is, generically, less conservative than *Bewley preferences* and *twofold multi-prior preferences*, the two ambiguity models that are closest to ours.[1]

Keywords: Decision theory; Incomplete preference; Multiple-selves; Non-obvious manipulability.

JEL classification: D01; D81; D90

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[1]These preferences are introduced in Bewley (2002) and Echenique et al. (2022), respectively.

1 Introduction

The complexity of everyday decisions is likely to result in agents only having a vague understanding of the (possibly risky) options they are supposed to compare. We propose and characterize an incomplete decision criterion under uncertainty that captures cognitive limitations in a new way.

We study preferences over acts $f : S \to X$, which are mappings from states of the world to outcomes, and we introduce and axiomatize preferences admitting the following representation:

$$f \succ g \iff \begin{cases} \min_{p\in C} \int u(f)dp > \min_{p\in C} \int u(g)dp \\ \max_{p\in D} \int u(f)dp > \max_{p\in D} \int u(g)dp \end{cases},$$

where u is a numerical representation of preferences over the outcome space (unique up to affine transformation) and C and D are two non-disjoint (convex compact) sets of probability distributions over the state space, interpreted as sets of different *scenarios*. Thus, a decision maker (DM) following such a decision criterion ranks act f above act g if and only if f provides higher expected utility than g in the worst-case scenario in C as well as in the best-case scenario in D. We shall give special attention in our analysis to the more transparent case in which $C = D$.

Our criterion is based on the conjunction of an optimistic (or ambiguity-seeking) assessment and of a pessimistic (or ambiguity-averse) assessment.[2] In this perspective, we interpret a decision maker with such a preference as requiring that her *optimistic self* and her *pessimistic self* be *unanimous* for her to rank some act above another one. That is why we refer to a preference relation admitting such a representation as a *unanimous dual-self preference*, using a similar formulation to Chandrasekher et al. (2022).[3]

Recently, both within the behavioral mechanism design literature and within the decision theory literature, comparisons of options in terms of their best-and-worst-case scenarios have been analysed as consequences of the DM's inability to engage in contingent reasoning, that is, their inability to comprehensively understand how the underlying state space determines the outcomes yielded by each option. In the context of strategic interaction, for example, agents might be able to recognize the set of possible payoffs that

[2]As C and D are not disjoint, the expected utility in the best-case scenario is higher.
[3]In which *dual-self expected utility* is introduced.

they might get by choosing some strategy, but fail to understand how others' strategies will influence which of these payoffs they will obtain. This form of cognitive limitation turns out to be of first order importance in explaining the departure of agents' choices from rational behaviour.[4] From a theoretical perspective, we observe several attempts —discussed below— to model failure of contingent reasoning through options being evaluated exclusively according to the *range* of (expected) utility levels they induce, special attention being given to the endpoints of this range. This is precisely how the criterion of interest in this paper works: agents assess the desirability of a given act according to the endpoints of the corresponding range of expected utility, this range being computed over two non-disjoint compact and convex sets of probability measures. More specifically, the conjunction of a best-case evaluation and of a worst-case evaluation at work in our criterion is akin to the one at work in the notion of *obvious manipulations* (Troyan and Morrill (2020)), defined for revelation games in which the uncertainty faced by an agent concerns others' messages. Thus, this work may be seen as a generalisation of the intuition explored in Troyan and Morrill (2020) to a decision theoretic framework involving multiple priors, just as Echenique et al. (2022) may be seen as such a generalisation of the intuition behind Li's notion of *obvious strategy-proofness* (Li (2017)). Consequently, our criterion is, generically, less conservative than the one of Echenique et al. (2022) (see Proposition 1), and, in particular, it is monotonic: an act which state-by-state dominates an other one must be ranked above.

Our axiomatic approach aims at enriching the analysis of incomplete preferences in non-deterministic environments. As Aumann (1962) acknowledged[5], the incompleteness of the criteria agents might use is a prevalent aspect of real-world decision-making, all the more so when uncertainty is involved. Yet, the preferences we characterize induce a partial order, reflecting, through the unanimity requirement imposed on the optimistic and the pessimistic assessments, the necessity to have *sufficient conviction* when comparing acts. Our axiomatization maintains the assumption that preferences are complete over constant acts, deemed as the simplest ones. It also imposes, as in Bewley (2002), the standard monotonicity property we already mentioned, and the classical *C-independence* axiom

[4] Various classic anomalies observed in different environments, such as overbidding in auctions or the Ellsberg paradox for choice over lotteries, can be explained by failures in contingent reasoning (see Esponda and Vespa (2023)).

[5] "*Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from a normative viewpoint.*"

introduced by Gilboa and Schmeidler (1989).[6] The incompleteness of unanimous dual-self preferences is rooted in two axioms. The first one (see Axiom 6), underscores the role of constant acts as references for decision making: if the DM is unable to compare act g with constant act x whenever she is unable to compare x to f, then she is not able to compare f and g. *Importantly, this axiom is satisfied by the vast majority of incomplete criteria defined on single acts proposed in the literature.* According to Axiom 7, if *i)* the DM cannot compare f to the constant x, while she declares x more desirable than g and, on the other hand, *ii)* she cannot compare g to the constant act y, while she declares f more desirable than y, then she declares f more desirable than g.

In order to account for typical situations in which agents *have to* choose between two options, even if they lack evidence to rank them, we study the completion of unanimous dual-self preferences. We identify conditions on this extension under which it admits an α-maxmin expected utility representation. To the best of our knowledge, the α-maxmin expected utility model is, among the theoretical attempts to incorporate empirical findings about "mixed attitudes"[7] toward ambiguity (see Trautmann and van de Kuilen (2015) for a survey), the one that has received the most attention. Recently, Frick, Iijima and Le Yaouanq (2022) provided a characterization of this model based on the *objective/subjective rationality* framework of Gilboa et al. (2010): we follow the same path and obtain $\alpha-$maxmin expected utility criteria as the subjective *invariant biseparable* complete extensions of objective unanimous dual-self preferences.

Finally, we apply this new criterion in the context of decision making on the basis of experts' opinions, which covers, for example, situations in which the DM is in charge of deciding complex fiscal or environmental policies. We then propose and axiomatize a rule for aggregating diverse and potentially conflicting opinions from a panel of experts, under the assumption that both the DM and the experts have unanimous dual-self preferences.

Our paper is organized as follows: we define the formal framework and introduce our criterion in Section 2. In Section 3, we give the main representation result and explore the case of main interest, in which the sets of scenarios used in the optimistic and in the pessimistic assessment are the same. We compare, in Section 4 , the degree of incompleteness of our criterion to that of Bewley preferences and of twofold preferences, and we provide a way to compare the ambiguity attitudes of two unanimous dual-self

[6] We also require the convexity of both lower and upper contour sets at constant acts, a property which was introduced in Echenique et al. (2022).

[7] By this expression, we refer to the general idea, supported by experimental evidence, that agents do not display complete aversion to ambiguity, nor complete preference for ambiguity.

preferences. In Section 5, we investigate the completion of our decision criterion. Section 6 is dedicated to the problem of aggregating experts' opinions, under the assumption that both the DM and experts have unanimous dual-self preferences. The conclusions are presented in Section 7. All proofs can be found in the appendix.

1.1 Related literature

This paper contributes to the existing decision-theoretic literature by introducing preferences that can accommodate a broader range of attitudes towards ambiguity. The decision criterion we axiomatize builds upon the idea of a dual-self preference in which the evaluations of a pessimistic (or uncertainty-averse) self and of an optimistic (or uncertainty-seeking) self are compared. In recent works, Chandrasekher et al. (2022) and Xia (2020) have provided axiomatizations for preferences involving two selves, called by the former *dual-self expected utility* (DSEU). Their representation differs from ours in that the agent's final decision is to be interpreted as the result of a specific *leader-follower game* between an optimistic self and a pessimistic self, whereas, in our representation, it is induced by a requirement of unanimity imposed by the agent herself on the assessments of her two selves.

Our representation is also motivated by the concept of obvious manipulation proposed in the context of mechanism design by Troyan and Morrill (2020). A revelation mechanism is said to be *non-obviously manipulable* if, for any agent and any potential untruthful report from her, revealing her own type leads to a more desirable outcome in both of the following cases: when the others' reports are the most favourable to her, and when they are the least favourable. In our model, in the same spirit, a default option —such as an untruthful report in the previous example— is only abandoned for an alternative if this alternative leads to preferred outcomes in both the best and the worst scenarios among given sets of probability measures. The relation of our contribution to the concept of obvious manipulation mirrors that of Echenique et al. (2022) to the concept of obvious dominance, due to Li (2017): informally, when the set of scenarios according to which all acts are evaluated is the simplex, act f is preferred to act g by a *twofold multi-prior preference* if and only if f obviously dominates g, and, on the other hand, f is preferred to g by a *unanimous dual-self preference* if and only if f dominates g in the sense of Troyan and Morrill (2020).[8] We explore in more details throughout the paper the connection

[8]With this same set of scenarios, one can also recover the concept of strategy-proofness from *Bewley preferences* (Bewley (2002)).

between our paper and Echenique et al. (2022).

Unanimous dual-self preferences define a partial order on acts. Pioneering work by Aumann (1962), Bewley (2002) and Dubra, Maccheroni and Ok (2004) studied the representation of incomplete preferences under risk and uncertainty. Incomplete preferences in non-deterministic environments have been the object of a growing literature: see, for example, Nascimento and Riella (2011) Galaabaatar and Karni (2012), Efe, Ortoleva and Riella (2012), Faro (2015), Minardi and Savochkin (2015), Hill (2016), Karni (2020), Cusumano and Miyashita (2021) and Echenique et al. (2022). We contribute to this body of research by introducing a novel axiomatic model, differing from previous approaches in that it characterizes incomplete preferences through the comparison of the best and worst potential outcomes associated with each alternative.

In line with several recent papers (Kopylov (2009), Cerreia-Vioglio (2016), Faro and Lefort (2019), Grant, Rich and Stecher (2021)), we complete our approach by introducing our decision criterion in the *objective and subjective rationality* framework, proposed by Gilboa et al. (2010) —referred to as GMMS in the following. The GMMS approach demonstrates how the Knightian decision model presented by Bewley (2002) and the maxmin expected utility model developed by Ghirardato, Maccheroni and Marinacci (2004) complement each other, and can be analyzed in a unifying model in which the agent's decision involves both an *objective* preference relation and a *subjective* one, the latter being an order extension of the former. Our paper establishes a representation in which the subjective relation follows the α-maxmin model, as in Frick, Iijima and Le Yaouanq (2022), while the objective relation has a unanimous dual-self representation, based on a simple extension property.

2 Setup and representation

2.1 Model

Our analysis is conducted in the classical framework proposed by Anscombe and Aumann (1963). Uncertainty is modelled through a set S of *states of the world*, endowed with an algebra Σ of subsets of S called *events*, and a non-empty set of consequences X, which is a non-singleton convex subset of a real vector space. *A simple act* is defined as a function $f : S \to X$ which takes finitely many values and is measurable with respect to Σ; we denote by $\mathcal{F}$ the set of all simple acts. The *mixture* of two simple acts f and g, for any $\alpha \in [0, 1]$, denoted by $\alpha f + (1 - \alpha)g$, is then defined by setting, for each $s \in S$,

$[\alpha f + (1-\alpha)g](s) = \alpha f(s) + (1-\alpha)g(s)$. With the usual slight abuse of notation, for every $x \in X$, we use x to denote the constant act defined by $f_x(s) = x$ for all $s \in S$. We use Δ to denote the set of all finitely additive probability distributions on (S, Σ), endowed with the weak* topology.[9] We refer to a measure $p \in \Delta$ as a *scenario* according to which simple acts are evaluated.

We consider a DM whose preference is represented by a binary relation $\succ \subseteq \mathcal{F} \times \mathcal{F}$. It is a partial ranking over simple acts and we use the standard notation $f \succ g$ to denote $(f, g) \in \succ$. If $f \not\succ g$ and $g \not\succ f$, we write $f \bowtie g$, and say that f and g are *incomparable.*[10] We interpret $f \succ g$ as reflecting the fact that the DM considers f is more desirable than g *with sufficient conviction.* Alternatively, $f \succ g$ can be interpreted as meaning that the DM would choose f *over the default act* g if she had to compare these acts, and thus, $f \bowtie g$ as meaning that she is not willing to abandon the default g for f.

We use the standard notation by denoting the set of vectors whose k elements are non-negative by $\mathbb{R}^k_+$, the set of vectors whose k elements are positive by $\mathbb{R}^k_{++}$, for any natural number k. For a given set A, $|A|$ denotes the cardinality of A.

2.2 Unanimous dual-self preferences

Our representation involves multiple priors:[11] the DM has a set of relevant beliefs according to which she evaluates acts.

Definition 1. A binary relation $\succ$ is a *unanimous dual-self* preference if

$$f \succ g \iff \begin{cases} \min_{p \in C} \int u(f)dp > \min_{p \in C} \int u(g)dp \\ \max_{p \in D} \int u(f)dp > \max_{p \in D} \int u(g)dp \end{cases},$$

where u is an affine function defined on X, and C and D are two compact and convex subsets of Δ with $C \cap D \neq \emptyset$.

The representation is *concordant* if $C = D$.

[9] The set of finitely additive bounded measures on (S, Σ) is the dual of the set of all measurable real-valued bounded functions on (S, Σ). Thus the weak* topology on Δ is defined according to the following convergence notion: we say that a sequence $\{p_n\}$ of elements of Δ converges to $p \in \Delta$ if for all measurable bounded function $\varphi : S \to \mathbb{R}$, $\int \varphi dp_n$ converges to $\int \varphi dp$.

[10] Accordingly, we say that f and g are *comparable* if either $f \succ g$ or $g \succ f$.

[11] Etner, Jeleva and Tallon (2012) and Gilboa and Marinacci (2016) both provide a review of the ways in which ambiguity, and ambiguity attitudes, have been modeled in order to offer alternatives to the traditional Bayesian framework. Multiple prior models stand out as one of the main lines of research.

We sometimes write that $\succ$ admits the representation (u, C, D) to refer to the unanimous dual-self representation given in Definition 1. We obtain in our axiomatization the uniqueness up to affine transformation of u, and the uniqueness of C and D. *We sometimes write, then, as a shortcut, that $\succ$ admits the unique representation (u, C, D).*[12]

Take such preference relation $\succ$, with representation (u, C, D). A simple act f is preferred to a simple act g if and only if f gives a higher expected utility than g when they are evaluated according to their best-case scenario in D, *and* gives a higher expected utility than g when they are evaluated according to their worst-case scenario in C. Accordingly, the DM evaluates any simple act f in terms of the extreme points of the range $\{\int u(f)d\mu : \mu \in C \cup D\}$ of possible expected utility levels. As C and D are non-disjoint, compact and convex sets, we will say that f induces the interval $R(f) = \{\int u(f)d\mu : \mu \in C \cup D\}$. Consider another simple act $g \in \mathcal{F}$ and suppose that $R(f) = [a, b]$ and $R(g) = [c, d]$. Then f is preferred to g if and only if $a > c$ and $b > d$.

In the same way as several other criteria involving comparisons of ranges studied in decision theory and mechanism design, which we discuss below, unanimous dual-self preferences can be interpreted as resulting from a cognitively limited understanding of uncertainty, or, alternatively, from the restriction to partial information about possible scenarios. Precisely, the DM is only aware of the range of evaluations a given act may result in, and focuses on the extreme cases: the best-case scenario, on which on optimistic evaluation of the act would be based, and the worst-case scenario, on which a pessimistic evaluation would be based.

This interpretation is reminiscent of that of *dual-self expected utility*, characterized by Chandrasekher et al. (2022), in which two conflicting forces influencing the DM's evaluation, Optimism and Pessimism, play a sequential game: first, Optimism chooses a set of beliefs with the goal of maximizing the DM's expected utility, then Pessimism chooses a belief from the set chosen by Optimism with the goal of minimizing expected utility. The representation we derive do involve optimistic and pessimistic selves, but our criterion does not result from their strategic interaction. For our DM to consider with sufficiently strong conviction that act f is more desirable than act g, it is necessary, and sufficient, that her optimistic self and pessimistic self, whose preferences are respectively represented by $h \mapsto \max_{p \in C} \int u(h)dp$ and $h \mapsto \min_{p \in C} \int u(h)dp$ on $\mathcal{F}$, be *unanimous* over the ranking of f and g.[13]

[12] As opposed to writing that $\succ$ admits representation (u, C, D), where u is unique, *up to affine transformation*, and C and D are unique.

[13] Thus, there is, *a priori*, no formal connection between the two models.

At this point, it is interesting to describe how the way ranges are compared according to unanimous dual-self preferences can be formally related to the way in which they are compared according to *twofold preferences*, introduced and axiomatised by Echenique et al. (2022).

Definition 2. (Echenique et al. (2022)) A binary relation $\succ$ is a (multi-prior) twofold preference if

$$f \succ g \iff \min_{p \in C} \int u(f)dp > \max_{p \in D} \int u(g)dp$$

where u is an affine function defined on X, C and D are two compact and convex subsets of Δ with $C \cap D \neq \emptyset$. The representation is said *concordant* if $C = D$.[14]

Consider $\succ_1$ a unanimous dual-self preference and $\succ_2$ a twofold preference with the same representing utility function u on X and the same set of scenarios $C, D \subseteq \Delta$. For any two (not necessarily closed) real intervals I and I', we say that I lies above I' whenever $u > v$ for any $u \in I, v \in I'$. One has $f \succ_2 g$ if and only if the closed interval $R(f)$ lies above the closed interval $R(g)$, and $f \succ_1 g$ if and only if $R(f) \setminus R(g)$ lies above the interval $R(g) \setminus R(f)$. In words, $f \succ_2 g$ if and only if any attainable evaluation from f is greater than any attainable evaluation from g, while $f \succ_1 g$ if and only if any evaluation that is attainable from f but not from g is larger than any evaluation that is attainable from g but not from f.

When the preference $\succ$ is not concordant, different collections of scenarios are considered under the optimistic evaluation and under the pessimistic one. This discrepancy reflects the difference between the degree of preference for uncertainty of the optimistic evaluation and the degree of aversion to uncertainty of the pessimistic one. We explore this interpretation on the basis of Proposition 2 in section 4.2.

Unanimous dual-self preferences have the property that if the outcome of an act is considered more desirable than the outcome of another act in each state of the world, then the first act is preferred to the second one —referred to as monotonicity. Echenique et al. (2022), inspired by Li (2017), capture failure of contingent reasoning through the violation of this property. On the other hand, the dominance criterion proposed by Troyan and Morrill (2020), which we incorporate in a decision theoretic framework here, for which the authors also provide a characterization in terms of failure of contingent reasoning, satisfies such a monotonicity property.

[14] They obtain the uniqueness, up to affine transformation, of u, and the uniqueness of C and D in their axiomatization.

3 Representation results

3.1 Characterization of unanimous dual-self preferences

3.1.1 Axioms

We now proceed to an axiomatic characterization of unanimous dual-self preferences based on the following seven axioms. Axioms 1, 2 and 3 are standard, and Axiom 5 has often been considered as a minimal rationality requirement in decision theory.

Let us insist on the fact that the last five axioms, in which constant acts play a central role, are all derived from the basic idea that constant acts, because they are simpler, are relevant reference points for decision making.

Only Axioms 5 and 7 are neither adopted in Echenique et al. (2022) or implied by axioms in Echenique et al. (2022).

Axiom 1. Relation $\succ$ is asymmetric and transitive, and the restriction of $\succ$ to X is non-trivial and negatively transitive.[15]

Axiom 2. For every triple $(f, g, h) \in \mathcal{F}^3$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succ h\}$ and $\{\alpha \in [0,1] : h \succ \alpha f + (1-\alpha)g\}$ are open.

Axiom 3. For every $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in (0,1)$, $f \succ g$ if and only if $\alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)x$.

The interpretation of the assumptions in Axiom 1 is well-known. In particular, on X, $\succ$ is the asymmetric part of a complete and transitive relation. Axiom 2 is the standard Archimedean continuity condition adopted in models of decision under uncertainty. Axiom 3 is the independence axiom proposed by Gilboa and Schmeidler (1989) in their seminal paper as a weakening of the independence axiom at play in the characterization of subjective expected utility.

Axiom 4. For every $x \in X$, the sets $\{f \in \mathcal{F} : f \succ x\}$ and $\{f \in \mathcal{F} : x \succ f\}$ are convex.

Axiom 4 is identical to Axiom 4 in Echenique et al. (2022). At a high level, it states that comparisons to a given constant act should not be sensitive to hedging. More precisely, a relation satisfying Axiom 4 exhibits either aversion for uncertainty or preference for uncertainty based on whether uncertainty concerns the alternative or the default act. Recall that for $f, g \in \mathcal{F}$, we interpret $[f \succ g]$ as the comparison in which g is a default

[15] Negative transitivity of $\succ$ means that for all $x, y, z \in X$, if $x \not\succ y$ and $y \not\succ z$ then $x \not\succ z$.

act, so that $\{f \in \mathcal{F} : f \succ x\}$ is the set of acts for which the DM is willing to abandon the default constant act x. The convexity of $\{f \in \mathcal{F} : f \succ x\}$ is interpreted in terms of uncertainty aversion: when a reference act x is constant, an act obtained through hedging between two acts that are preferred to x is always preferred to x. The convexity of $\{f \in \mathcal{F} : x \succ f\}$ is interpreted in terms of preference for uncertainty: when two uncertain acts, considered as default acts, are considered worse than an alternative constant act x, that is, when the DM is willing to abandon each of these default acts for x, then an act obtained through hedging between the two does not offer a default act that is more desirable than x.

Axiom 5. For every $f, g \in \mathcal{F}$, if $f(s) \succ g(s)$ for all $s \in S$, then $f \succ g$.

According to Axiom 5, if the outcome of an act is considered more desirable than the outcome of another act in each state of the world, then the first act is preferred to the second one

Axiom 6. For every $f, g \in \mathcal{F}$, if *for every* $x \in X$, $f \bowtie x$ implies $g \bowtie x$, then $f \bowtie g$.

Remark 1. Almost all (the asymmetric part of) the incomplete criteria comparing single acts mentioned in Section 1.1, that is, almost all the incomplete criteria defined in a classical Anscombe-Aumann framework mentioned in Section 1.1, satisfy Axiom 6. More precisely, the (asymmetric part of) the criteria proposed in Bewley (2002), Nascimento and Riella (2011), Efe, Ortoleva and Riella (2012), Faro (2015), Cusumano and Miyashita (2021) and Echenique et al. (2022) all satisfy Axiom 6 (see the appendix).[16]

This fact is not surprising: Axiom 6 underscores the role of constant acts as reference acts based on which comparisons of more complex acts are made; and this view is vastly endorsed in the literature. According to this axiom, for the DM to express a preference between acts f and g, it is necessary that there exist a constant act x that the DM prefers to one of these acts, while she cannot compare x, with sufficient conviction, with the other act.[17]

Axiom 7. For every $f, g \in \mathcal{F}$, and for every $x, y \in X$, if $f \bowtie x$, $g \bowtie y$, $x \succ g$, and $f \succ y$ then $f \succ g$.

[16]There is one incomplete criterion mentioned in Section 1.1 that is defined on single acts and that may not satisfy Axiom 6, the one proposed in Hill (2016).

[17]Note that we do not impose that whenever there exists $x \in X$ such that $f \bowtie x$ and $g \bowtie x$, then $f \bowtie g$ (which is Axiom 5 in Echenique et al. (2022)). Actually, our criterion does not satisfy such property.

While the DM cannot compare f to the constant x, she declares x more desirable than g. On the other hand, while she cannot compare g to the constant act y, she declares f more desirable than y. Axiom 7 implies that in the presence of such consonant conclusions as to the comparison of f and g to constant acts x and y, the DM considers f, with sufficient conviction, more desirable than g.

We sometimes refer to the classical Axioms 2, 3 and 5 as continuity, certainty independence and monotonicity.

3.1.2 First characterization theorem

Theorem 1. *A binary relation $\succ$ satisfies Axioms 1-7 if and only if there exist*

- *an affine function $u : X \to \mathbb{R}$, unique up to positive affine transformation,*
- *a unique pair (C, D) of non-disjoint convex compact subsets of Δ,*

such that, for all $f, g \in \mathcal{F}$,

$$f \succ g \Leftrightarrow \begin{cases} \min_{p \in C} \int u(f)dp > \min_{p \in C} \int u(g)dp \\ \max_{p \in D} \int u(f)dp > \max_{p \in D} \int u(g)dp \end{cases},$$

that is, $\succ$ admits the unanimous dual-self representation (u, C, D), where C and D are unique, and u is unique up to positive affine transformation.

We now give a brief sketch of the proof and highlight some interesting properties of $\succ$ that we derive.[18] First of all, Axioms 1-3 guarantee that there exists an affine function $u : X \to \mathbb{R}$, unique up to affine transformation, representing $\succ$ on X.

The proof consists in defining two binary relations on $\mathcal{F}$, denoted $\succ_p$ and $\succ_o$, such that for any $f, g \in \mathcal{F}$, $f \succ g$ if and only if $f \succ_p g$ and $f \succ_o g$ —we provide the precise definitions of these relations below. In that perspective, the following two lemmas are crucial.

Lemma. *For every $f \in \mathcal{F}$, the set $\{x \in X : x \bowtie f\}$ is non-empty.*

Lemma. *For every $f \in \mathcal{F}$, and $x, y, z \in X$, if $x \bowtie f$, $f \succ y$, and $z \succ f$, then $z \succ x \succ y$.*

This second result has an interesting interpretation. While the DM cannot assert with sufficient conviction that f is more desirable than the constant act x, she considers with

[18] The following lemmas are not presented here in the order in which they are proved.

sufficient conviction that f is more desirable than the constant act y and worse than the constant act z. We show that in such a case, the DM considers, with sufficient conviction, that z is more desirable than x, and that x is more desirable than y.

From the original relation $\succ$, we define two preference relations on $\mathcal{F}$ as follows:

$$g \succ_p f \iff g \succ x \text{ and } x \bowtie f \text{ for some } x \in X,$$
$$g \succ_o f \iff x \bowtie g \text{ and } x \succ f \text{ for some } x \in X.$$

The subscripts p and o are used to denote respectively a pessimistic and an optimistic assessment, based on $\succ$, where these two terms are justified given the way the incomparability to a constant act is treated. Let us describe the interpretation of $\succ_p$: this relation is pessimistic in the sense that for the default act f, whenever there is a constant act x such that f cannot be compared with sufficient conviction to x, while g is considered more desirable than x with sufficient conviction, then $\succ_p$ declares f to be worse than g.

We then proceed by showing that $\succ_p$ and $\succ_o$ are asymmetric and negatively transitive. This enables us to define $\sim_p$ by $f \sim_p g$ if and only if $f \not\succ_p g$ and $g \not\succ_p f$, for all $f, g \in \mathcal{F}$, and to define $\succsim_p$ by $f \succsim_p g$ if and only if either $f \succ_p g$ or $f \sim_p g$, for all $f, g \in \mathcal{F}$. We define in the same way $\sim_o$ and $\succsim_o$. Then it is clear that $\succsim_p$ and $\succsim_o$ are weak orders,[19] and we show that they are continuous and monotone, that they satisfy the classical properties of certainty independence, and, respectively, aversion to ambiguity and preference for ambiguity.[20]

As a consequence, $\succsim_p$ can be represented by the function $f \mapsto \min_{p \in C} \int u_p(f) dp$, and $\succsim_o$ can represented by the function $f \mapsto \max_{p \in D} \int u_o(f) dp$, where C and D are non-empty convex compact subsets of Δ, and u_p and u_o are two affine functions on X. We conclude that there is no loss of generality in assuming $u_p = u_o = u$, and that $C \cap D \neq \emptyset$, using the separating hyperplane theorem on these subsets of Δ endowed with the weak* topology.

Note that in this sketch of proof, the relation between $\succ$ and the two weak orders $\succ_p$ and $\succ_o$ is established before the minmax and maxmax representations of $\succ_p$ and $\succ_o$: Axioms 1-3 and Axioms 5-7 are necessary and sufficient for a general representation that we describe in the next section.

[19]They are non-trivial asymmetric and negatively transitive binary relations.

[20]Definitions of these properties for weak orders are provided in the appendix.

3.1.3 Intermediary representation result

When $\succ$ satisfies Axioms 1-3 and Axioms 5-7, we can still define the pessimistic and optimistic relations $\succ_p$ and $\succ_o$ on $\mathcal{F}$,

$$g \succ_p f \iff g \succ x \text{ and } x \bowtie f \text{ for some } x \in X,$$
$$g \succ_o f \iff x \bowtie g \text{ and } x \succ f \text{ for some } x \in X,$$

and obtain that $f \succ g$ if and only if $f \succ_p g$ *and* $f \succ_o g$.

We take the terminology used in Ghirardato, Maccheroni and Marinacci (2004) and Frick, Iijima and Le Yaouanq (2022). Accordingly, $I : \mathbb{R}^S \to \mathbb{R}$ is said to be *constant-linear* if, for every $\varphi \in \mathbb{R}^S$, $a \in \mathbb{R}_+$, and $b \in \mathbb{R}$, $I(a\varphi + b) = aI(\varphi) + b$, where, with a slight abuse of notation, we use b to denote the constant function $\phi : s \in S \mapsto b \in \mathbb{R}$. It is said *monotonic* if it weakly preserves the usual partial order of $\mathbb{R}^S$.

Theorem 2. *A binary relation $\succ$ satisfies Axioms 1-3 and Axioms 5-7 if and only if there exist*

- *an affine function $u : X \to \mathbb{R}$, unique up to positive affine transformation,*
- *a unique pair of monotonic constant linear functionals $I_p, I_o : \mathbb{R}^S \to \mathbb{R}$,*

such that, for all $f, g \in \mathcal{F}$,

$$f \succ g \iff \begin{cases} I_p(u(f)) > I_p(u(g)) \\ I_o(u(f)) > I_o(u(g)) \end{cases} .$$

The proof of this theorem follows from the proof of Theorem 1 and Lemma 6, used in the proof of Theorem 4 below.

3.2 Characterization of concordant unanimous dual-self preferences

3.2.1 Axioms

The necessary and sufficient conditions identified in Echenique et al. (2022) for the identity $C = D$ to hold in their twofold multiprior preference representation are also necessary and sufficient in our representation.[21] Before introducing them, let us specify that, as

[21] The proof of the following result is a direct adaptation of the proof of Proposition 1 in their paper.

suggested in the sketch of the proof of Theorem 1, when $\succ$ satisfies Axioms 1-3, we define on X the relation $\succsim$ by $x \succsim y$ if and only if $y \not\succ x$ for all $x, y \in X$. Clearly, $\succsim$ on X is asymmetric and negatively transitive; and $\bowtie$ is equivalent to $\sim$, the symmetric part of $\succsim$, on X.

We use the notion of complementary acts (Siniscalchi (2009)) to identify comparisons that are, under Axioms 1-7, characteristic of the uncertainty aversion of the agent's pessimistic self, and of the preference for uncertainty of her optimistic self, respectively. Simple acts f and g are *complementary* if they perfectly hedge against each other in the sense that their equal-weight-mixture is equivalent to a constant act:

$$\frac{1}{2}f(s) + \frac{1}{2}g(s) \sim \frac{1}{2}f(s') + \frac{1}{2}g(s') \text{ for all } s, s' \in S.$$

Axiom 8. If f and g in $\mathcal{F}$ are complementary, then $f \succ \frac{1}{2}f + \frac{1}{2}g$ implies $\frac{1}{2}f + \frac{1}{2}g \succ g$.

Consider two complementary $f, g \in \mathcal{F}$, and a preference $\succ$ with representation (u, C, D) on $\mathcal{F}$. Assume $f \succ \frac{1}{2}f + \frac{1}{2}g$, as in Axiom 8 and let $x \in X$ denote a constant act such that $x \sim \frac{1}{2}f + \frac{1}{2}g$. If, in addition, $g \succ \frac{1}{2}f + \frac{1}{2}g$, then $f \succ x$ and $g \succ x$, and thus $\frac{1}{2}f + \frac{1}{2}g \succ x$; a contradiction.

In other words, if $f \succ \frac{1}{2}f + \frac{1}{2}g$, then either $\frac{1}{2}f + \frac{1}{2}g \bowtie g$ or $\frac{1}{2}f + \frac{1}{2}g \succ g$. Axiom 8 requires that the second case hold, and this requirement is interpreted as a consequence of the simplicity of constant acts. Indeed, this second case implies $f \succ g$, so Axiom 8 states that whenever $f \succ \frac{1}{2}f + \frac{1}{2}g$, then $f \succ g$, that is, it states that it should always be easier for the DM to assess whether f is more desirable than the essentially constant act $\frac{1}{2}f + \frac{1}{2}g$ than to assess whether f is more desirable than g.

The interpretation of Axiom 9 is similar: it states that for complementary acts $f, g \in \mathcal{F}$, it should always be easier for the DM to assess whether the essentially constant act $\frac{1}{2}f + \frac{1}{2}g$ is more desirable than g than to assess whether f is more desirable than g.

Axiom 9. If f and g in $\mathcal{F}$ are complementary, then $\frac{1}{2}f + \frac{1}{2}g \succ g$ implies $f \succ \frac{1}{2}f + \frac{1}{2}g$.

3.2.2 Second characterization theorem

Theorem 3. *The following statements hold:*

(i) A unanimous dual-self preference $\succ$, with unique representation (u, C, D), satisfies Axiom 8 if and only if $D \subseteq C$.

(ii) A unanimous dual-self preference $\succ$, with unique representation (u, C, D), satisfies Axiom 9 if and only if $C \subseteq D$.

In particular, a binary relation $\succ$ satisfies Axioms 1-9 if and only if there exist

- *an affine function $u : X \to \mathbb{R}$, unique up to positive affine transformation,*
- *a unique convex compact subset of Δ, denoted C, such that, for all $f, g \in \mathcal{F}$,*

$$f \succ g \iff \begin{cases} \min_{p \in C} \int u(f)dp > \min_{p \in C} \int u(g)dp \\ \max_{p \in C} \int u(f)dp > \max_{p \in C} \int u(g)dp \end{cases}.$$

When $\succ$ admits a concordant representation, acts are evaluated according to the minimum and the maximum expected utility level *attained on a common set of* scenarios. On the other hand, when $\succ$ satisfies both Axiom 8 and 9, for any simple complementary acts f and g, $f \succ \frac{1}{2}f + \frac{1}{2}g$ if and only if $\frac{1}{2}f + \frac{1}{2}g \succ g$. In other words, for complementary acts, it is always as easy to determine whether their equal-weight-mixture is more desirable than one of them as it is to determine whether one of them is more desirable than the mixture.

4 Comparison of incomplete criteria

4.1 Degree of incompleteness

We have stated that *unanimous dual-self preferences are generically less conservative than Bewley preferences and twofold preferences.* The criterion we use to determine whether a binary relation is more conservative than another one pertains to their respective *degree of incompleteness.*

Definition 3. Given two preference relations $\succ_1$ and $\succ_2$ on $\mathcal{F}$, we say that $\succ_1$ is more conservative than $\succ_2$ if $\succ_2$ is an extension of $\succ_1$, that is, for all $f, g \in \mathcal{F}$,

$$f \succ_1 g \text{ implies } f \succ_2 g.$$

Definition 2 above introduces twofold preferences. Bewley preferences have the following form:

Definition 4. (Bewley (2002)) A binary relation $\succ$ is a (multi-prior) Bewley preference if

$$f \succ g \iff \int u(f)dp > \int u(g)dp \text{ for all } p \in C,$$

where u is an affine function defined on X, C is a non-empty compact and convex subset of Δ.

Similarly to the expression we used for our criterion, we will say that the the twofold preference $\succ_T$ admits the unique representation (u, C_T, D_T), and that the Bewley preference $\succ_B$ admits the unique representation (u, C_B) to refer to the fact that u is unique up to affine transformation, and C_T, B_T and C_B are unique.

The next proposition identifies necessary and sufficient conditions under which a unanimous dual-self preference relation is an extension of a Bewley or of a twofold preference relation.

Proposition 1. *Let $\succ_U$ be a unanimous dual-self preference with unique representation (u, C_U, D_U), let $\succ_T$ be a twofold multiprior preference with unique representation (u, C_T, D_T), and $\succ_B$ be a Bewley preference with unique representation (u, C_B). Then,*

(i) the preference relation $\succ_B$ is more conservative than $\succ_U$ if and only if $C_U \cup D_U \subseteq C_B$;

(ii) the preference relation $\succ_T$ is more conservative than $\succ_U$ if and only if $C_U \subseteq C_T$ and $D_U \subseteq D_T$.

Remark 2. A direct consequence of this proposition and Proposition 4 in Echenique et al. (2022) is that if $C_U \cup D_U \subseteq C_B \subseteq C_T \cap D_T$, in particular if $C_U = D_U = C_B = C_T = D_T$, then $\succ_T$ is more conservative than $\succ_B$, which is more conservative than $\succ_U$.

4.2 Ambiguity attitudes

We are able to compare ambiguity attitudes displayed by different unanimous dual-self preferences using the classical comparative statics notions of Ghirardato and Marinacci (2002).

Definition 5. Given two preference relations $\succ_1$ and $\succ_2$ on $\mathcal{F}$,

(i) $\succ_1$ is more ambiguity averse than $\succ_2$ if, for every $f \in \mathcal{F}$ and $x \in X$, $f \succ_1 x$ implies $f \succ_2 x$.

(ii) $\succ_1$ is more ambiguity loving than $\succ_2$ if, for every $f \in \mathcal{F}$ and $x \in X$, $x \succ_1 f$ implies $x \succ_2 f$.

An agent is more ambiguity averse than another one if she is less inclined to choose an uncertain act f over a constant act x. On the other hand, an agent is more uncertainty loving than another one if she is more inclined to stick to an uncertain act f than to switch to a constant act x. The next result characterizes ambiguity attitudes for unanimous dual-self preferences.

Proposition 2. *Let $\succ_1$ and $\succ_2$ be two unanimous dual-self preference relations with unique representation (u, C_1, D_1) and (u, C_2, D_2), respectively. Then,*

(i) $\succ_1$ is more ambiguity averse than $\succ_2$ if and only if $C_2 \subseteq C_1$.

(ii) $\succ_1$ is more ambiguity loving than $\succ_2$ if and only if $D_2 \subseteq D_1$.

For a unanimous dual-self representation (u, C, D), the two sets of priors C and D represent the level of pessimism and optimism related to the DM's ambiguity attitudes. More precisely, the relationship $C_2 \subseteq C_1$ means that, in the worst scenario, the level of welfare attained by the agent if she has preference relation $\succ_1$ is lower than the one attained if she has preference relation $\succ_2$. Similarly, $D_2 \subseteq D_1$ means that, in the best scenario, the level of welfare attained by the agent if she has preference relation $\succ_1$ is higher than the one attained if she has preference relation $\succ_2$. Proposition 2 then simply states that an agent is more ambiguity averse (respectively more ambiguity loving) than an other one if and only if she is more pessimistic (respectively more optimistic) when facing ambiguity.

Importantly, based on Proposition 2 *i)*, by comparing the concordant preference $\succ$ with representation (u, C, C) to the non-concordant preference $\succ^1$ with representation (u, C_1, C), with $C_1 \subset C$, we can say that $\succ^1$ *is more ambiguity averse than it is ambiguity loving*. Similarly, the non-concordant representation $\succ^2$ with representation (u, C, D_2), with $D_2 \subset C$, *can be said to be more ambiguity loving than it is ambiguity averse.*

We end this subsection by briefly discussing the relation between the degree of conservatism of a unanimous dual-self preference relation and the attitude towards ambiguity that it displays.

It is easy to see that if $\succ_1$ and $\succ_2$ are unanimous dual-self preferences, and if $\succ_1$ is more conservative than $\succ_2$, then $\succ_1$ is both more ambiguity averse and more ambiguity loving than $\succ_2$. Does the converse statement hold ? This question is all the more natural that if $\succ_1$ and $\succ_2$ are twofold preferences, then $\succ_1$ is more conservative that $\succ_2$ if, *and only if*, $\succ_1$ is more ambiguity averse and more ambiguity loving than $\succ_2$.[22] The following example shows that the answer is negative for unanimous dual-self preferences.

Let $\succ_1$ and $\succ_2$ be two unanimous dual-self preferences with representations (u, C_1, D_1) and (u, C_2, D_2), respectively. We identify conditions under which $C_2 \subseteq C_1$ and $D_2 \subseteq D_1$, but $\succ_1$ is not more conservative than $\succ_2$. Consider $f, g \in \mathcal{F}$ such that there are $s_1, s_2 \in S$ satisfying

$$\begin{cases} u(f(s_1)) > u(g(s_1)) \\ u(f(s_2)) < u(g(s_2)) \\ u(f(s)) = u(g(s)) \text{ for all } s \neq s_1, s_2. \end{cases}$$

Assume that the utility function u is such that $u(f(s_1)) = u(g(s_2)) = 1$ and $u(f(s_2)) = u(g(s_1)) = 0$. Define p_1, p_2 and p_3 as follows

$$\begin{cases} p_1(s_1) = \frac{1}{3}, p_1(s_2) = \frac{2}{3}, p_1(s) = 0 \ \forall s \neq s_1, s_2 \\ p_2(s_1) = 1, p_2(s_2) = 0, p_2(s) = 0 \ \forall s \neq s_1, s_2 \\ p_3(s_1) = \frac{2}{5}, p_3(s_2) = \frac{3}{5}, p_3(s) = 0 \ \forall s \neq s_1, s_2. \end{cases}$$

Now let $C_1 = C_2 = \{p_2\}$, $D_1 = \overline{co}(\{p_1, p_2\})$ and $D_2 = \overline{co}(\{p_1, p_3\})$, where $\overline{co}$ denotes the operator which associates to any subset of Δ its closed convex hull in Δ. One readily obtains:

$$\begin{aligned} \min_{p \in C_1} \int u(f)dp = 1 > 0 = \min_{p \in C_1} \int u(g)dp \\ \max_{p \in D_1} \int u(f)dp = 1 > \frac{2}{3} = \max_{p \in D_1} \int u(g)dp \\ \max_{p \in D_2} \int u(f)dp = \frac{2}{5} < \frac{2}{3} = \max_{p \in D_2} \int u(g)dp, \end{aligned}$$

that is, $f \succ_1 g$ but $f \not\succ_2 g$.

5 Generalized α-maxmin expected utility

We now explore the extension of unanimous dual-self preferences to complete preferences. Building on the *objective-subjective* relations framework proposed in Gilboa et al. (2010),

[22]See Corollary 1 in Echenique et al. (2022).

we obtain an objective rationality foundation of α-maxmin expected utility based on unanimous dual-self preferences. Accordingly, the DM's decision process involves two preference relations on $\mathcal{F}$; we denote them $\succ$ and $\succ^*$. The relation $\succ$ represents the comparisons the DM makes with sufficient conviction, and corresponds to the objective relation in the model of Gilboa et al. (2010). In line with the motivation of this paper, it may be incomplete due to missing relevant information or cognitive limitations. On the other hand, $\succ^*$ represents the rankings that the DM is compelled to make under the burden of choice, and corresponds to the subjective relation in their model. In this context, $\succ^*$ is a complete extension of $\succ$.

Our approach now differs from other foundations of generalised α-maxmin expected utility in that we will assume that $\succ$ is a unanimous dual-self preference.

A weak order on $\mathcal{F}$, *i.e.*, a non-trivial asymmetric and negatively transitive binary relation on $\mathcal{F}$, satisfying Axioms 2, 3 and 5, is refered to in the literature as *invariant biseparable preference*. Such order satisfies the axioms characterizing expected utility, apart from the independence axiom, which is weakened to the certainty independence property introduced in Gilboa and Schmeidler (1989). The preference relation $\succ^*$ will be assumed invariant biseparable. In addition, as we already mentioned, we will require that $\succ^*$ be a completion of $\succ$, that is, that the pair $(\succ, \succ^*)$ satisfy the following property.

Extension. The pair $(\succ_1, \succ_2)$ satisfies the extension property if for all $f, g \in \mathcal{F}$, $f \succ_2 g$ whenever $f \succ_1 g$.

This property corresponds to a special case of Definition 3 in which one of the preference relation is complete.

Definition 6. A preference relation $\succ$ on $\mathcal{F}$ admits an *α-maxmin expected utility representation* if there exist $\alpha \in [0, 1]$, two non-disjoint compact convex subsets C and D of Δ, and a non-constant affine function $u : X \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$,

$$
\begin{aligned}
f \succ g \iff & \alpha \min_{p \in C} \int u(f)dp + (1-\alpha) \max_{p \in D} \int u(f)dp \\
& > \alpha \min_{p \in C} \int u(g)dp + (1-\alpha) \max_{p \in D} \int u(g)dp.
\end{aligned}
$$

We will refer to such representation as as an (u, C, D, α) representation.

Theorem 4. *The following conditions are equivalent when $\succ$ is a unanimous dual-self preference with unique representation (u, C, D):*

(i) $\succ^*$ *is an invariant biseparable preference and the pair* $(\succ, \succ^*)$ *satisfies the extension property.*

(ii) $\succ^*$ *admits an* α*-maxmin expected utility representation* (u, C, D, α) *in which* α *is unique whenever* $\succ$ *is not complete.*

6 Aggregating the opinion of experts with unanimous dual-self preferences

Numerous economic decisions, such as those related to fiscal policy and those addressing climate change, often hinge on the guidance provided by groups of experts or specialists, who frequently hold conflicting opinions. We explore in this section the issue of the aggregation of conflicting opinions among experts.[23] While this topic has received considerable attention (see, for example, Stone (1961), Crès, Gilboa and Vieille (2011), Nascimento (2012), Qu (2017), Amarante and Ghossoub (2021), Stanca (2021), and Dong-Xuan (2023)), our contribution lies in the assumption that both the DM's and the experts' preferences are unanimous dual-self preferences. We propose a novel Pareto condition, stating that if an arbitrary act falls between two constant acts *according to all experts*, it does so also for the DM. We show that this conditions implies than any probability measure ($i.e$ any scenario) deemed plausible by the DM to assess the worst-case (respectively the best-case) is a weighted average of probability measures ($i.e$ scenarios) deemed plausible by the experts to assess the worst-case (respectively the best-case). We also consider a *caution* axiom, positing that if some experts are not able to compare an arbitrary act to a constant one, the DM should also consider these acts incomparable. We show that under this condition, the DM considers plausible for the worst-case evaluation (respectively for the best-case evaluation) any weighted average of probability measures deemed by experts as plausible for the worst-case evaluation (respectively for the best-case evaluation). An obvious implication is that if these two conditions are met, the sets of measures considered by the DM for both evaluations are the convex hulls of the ones considered by the experts.

We assume that both the DM and experts have unanimous dual-self preferences. Let $N = \{1, 2, \dots, n\}$ be a finite set of experts. Expert $j \in N$ has a preference $\succ_j$ on $\mathcal{F}$. We

[23]This approach is different from that of the theory of *aggregation of preferences*, which focuses on the diversity of preferences *over outcomes*. This diversity is ignored here (this statement is made precise in the following paragraph).

use $\succ_0$ to denote the DM's preference on $\mathcal{F}$. We suppose that, for all $i \in N \cup \{0\}$, agent i's preference is a unanimous dual-self preference with unique representation (u, C_i, D_i). We thus assume in particular that there is no diversity of preferences over outcomes, which is a distinctive element of the theory of the aggregation of opinions, compared to the theory of the aggregation of preferences.

Axiom 10 (Pareto for comparability). For every $f \in F$ and $x, y \in X$, if $y \succ_i f \succ_i x$ for all $i \in N$, then $y \succ_0 f \succ_0 x$.

Axiom 11 (Caution for incomparability). For every $f \in F$ and $x \in X$, if there exists $i \in N$ such that $f \bowtie_i x$, then $f \bowtie_0 x$.

We emphasize that these two axioms focus on the comparison of arbitrary acts to constant acts, these comparisons being simpler than those between two arbitrary acts.

As we already mentioned, Pareto for comparability asserts that the DM should follow experts' *unanimous* comparisons to constant acts. More precisely, if *all* experts prefer act f to a constant act x, the DM should also favor f over x. Symmetrically, if *all* experts prefer a constant act y to act f, the DM should also favor y over f

In contrast, caution for incomparability focuses on situations without clear comparisons. Based on the idea that the DM wants to rely on experts' opinions because the decision problem she is facing is *crucial* to her, this condition describes a conservative attitude when aggregating these opinions. It states that if some experts struggle to compare act f to constant act x, the DM should treat these acts as incomparable.

For every $P \subseteq \Delta$, we use $\mathrm{co}(P)$ to denote the convex hull of P.

Proposition 3. *Suppose that for all $i \in N \cup \{0\}$, $\succ_i$ is a unanimous dual-self preference with unique representation (u, C_i, D_i).*

(i) Pareto for comparability is satisfied if and only if $C_0 \subseteq \mathrm{co}\left(\bigcup_{i=1}^n C_i\right)$ and $D_0 \subseteq \mathrm{co}\left(\bigcup_{i=1}^n D_i\right)$.

(ii) Caution for incomparability is satisfied if and only if $\mathrm{co}\left(\bigcup_{i=1}^n C_i\right) \subseteq C_0$ and $\mathrm{co}\left(\bigcup_{i=1}^n D_i\right) \subseteq D_0$.

In particular, when both conditions are met, $C_0 = \mathrm{co}\left(\bigcup_{i=1}^n C_i\right)$ and $D_0 = \mathrm{co}\left(\bigcup_{i=1}^n D_i\right)$.

7 Conclusion

We provided a new perspective on the analysis of incomplete preferences by introducing a novel decision criterion involving multiple priors, based on a requirement of unanimity between an optimistic and a pessimistic evaluation. Our representation of *unanimous dual-self preferences* is a translation and a generalization of the domination concept at work in the notion of obvious manipulations, introduced by Troyan and Morrill (2020), into a decision theoretic framework involving uncertainty and ambiguity. Furthermore, based on the model proposed in Gilboa et al. (2010), we delved into the complete extension of this partial order, and identified conditions under which the α-maxmin expected utility is the subjective complete extension of an objective unanimous dual-self preference. Finally, we applied our criterion to the problem of the aggregation of experts' opinions.

Appendix

A Discussion of Axiom 6

It's easy to see that Axiom 5 in Echenique et al. (2022) implies our Axiom 6 so twofold preferences satisfies Axiom 6. Let us prove that Bewley preferences proposed in Bewley (2002) also satisfies Axiom 6.

Let $\succ$ a Bewley preference relation with representation (u, C). Let $x, y \in X$ and $f, g \in \mathcal{F}$ satisfying the assumptions of Axiom 6. Consider $x \in X$ such that $f \bowtie x$. By definition of Bewley preference, there are $p, p' \in C$ such that $\int u(f)dp' \geq u(x) \geq \int u(f)dp$. Therefore, the set $\{x \in X | f \bowtie x\}$ is the set $\{x \in X | \max_{p \in C} \int u(f)dp \geq u(x) \geq \min_{p \in C} \int u(f)dp\}$. By continuity, there are $\overline{x} \in X$ and $\underline{x} \in X$ such that $u(\overline{x}) = \max_{p \in C} \int u(f)dp$ and $u(\underline{x}) = \min_{p \in C} \int u(f)dp$. Then one gets $f \bowtie \overline{x}$ and $f \bowtie \underline{x}$. By Axiom 6, $g \bowtie \overline{x}$ and $g \bowtie \underline{x}$ which imply that there are $p_1, p_2 \in C$ such that

$$u(\underline{x}) = \min_{p \in C} \int u(f)dp \geq \int u(g)dp_1, \text{ and}$$
$$u(\overline{x}) = \max_{p \in C} \int u(f)dp \leq \int u(g)dp_2$$

That is $g \not\succ f$ and $f \not\succ g$: $g \bowtie f$, which ends the proof.

Other incomplete preferences: Basic adaptations of this simple proof lead to the conclusion that the (asymmetric part of the) criteria proposed by Nascimento and Riella (2011), Efe, Ortoleva and Riella (2012), Faro (2015), and Cusumano and Miyashita (2021) satisfy Axiom 6. Let us note that some of these criteria allow for indifferences: generically denoting them by $\succsim$, we derive a representation of the associated asymmetric part, denoted $\succ$, by using the representation of $\succsim$ and defining $\succ$ by $[f \succ g]$ if and only if $[[f \succsim g]$ and not$[g \succsim f]]$ for all admissible acts $f, g \in \mathcal{F}$.

B Proof of Theorem 1

Only-if part. Assume that $\succ$ satisfies Axioms 1-7.

Consider the restriction of $\succ$ to X and define $\succsim$ by $x \succsim y$ if and only if $y \not\succ x$ for all $x, y \in X$. Clearly, $\succsim$ on X is asymmetric and negatively transitive; and $\bowtie$ is equivalent to $\sim$ on X. By Axiom 3, for every $x, y, z \in X$, $x \succsim y$ if and only if $\alpha x + (1-\alpha)z \succsim \alpha y + (1-\alpha)z$. Thus, by continuity of $\succ$, there exists an affine function $u : X \to \mathbb{R}$, unique up to affine transformation, such that $x \succsim y$ if and only if $u(x) \geq u(y)$. Also, u is non-constant as $\succ$ is non-trivial.

Let us now introduce intermediary results on which our proof is based.

Lemma 1. *For every $f \in \mathcal{F}$, and $x, y, z \in X$, if $f \bowtie x$, and $f \succ y$, and $z \succ f$, then $z \succ x \succ y$.*

Proof. We prove $x \succ y$, as $z \succ x$ is similarly shown. Assume $x \not\succ y$, by contradiction. Having $y \succ x$ would contradict $f \bowtie x$ by the transitivity of $\succ$. Thus, $y \sim x$. There are three posibilities:

Case 1: There exists $x' \in X$ such that $f \bowtie x'$ and $x \succ x'$. Then $y \succ x'$ since $y \sim x$. So $f \succ y$, and $y \succ x'$ which implies $f \succ x'$, a contradiction.

Case 2: There exists $x' \in X$ such that $f \bowtie x'$ and $x' \succ x$. Then, $f \bowtie x'$, $x \sim y$, $f \succ y$, and $x' \succ x$. Applying Axiom 7, we get $f \succ x$, a contradiction.

Case 3: For every $x' \in X$ such that $f \bowtie x'$, $x' \sim x$. Applying Axiom 6 to f and y, one gets $f \bowtie y$, a contradiction.

□

Lemma 2. *$\bowtie$ satisfies certainty independence.*

Proof. Let $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in (0,1)$, the following equivalence relations hold:

$$\begin{aligned} & f \bowtie g \\ \iff & f \not\succ g \text{ and } g \not\succ f \\ \iff & \alpha f + (1-\alpha)x \not\succ \alpha g + (1-\alpha)x \text{ and } \alpha g + (1-\alpha)x \not\succ \alpha f + (1-\alpha)x \\ \iff & \alpha f + (1-\alpha)x \bowtie \alpha g + (1-\alpha)x. \end{aligned}$$

The first and the third ones follow from the definition of $\bowtie$, and the second from the fact that $\succ$ satisfies certainty independence (Axiom 3). □

Lemma 3. *For every $f \in \mathcal{F}$ and $x, y \in X$, if $f \succ x$ and $x \succsim y$, then $f \succ y$.*

Proof. Let $f \in \mathcal{F}$ and $x, y \in X$ such that $f \succ x$ and $x \succsim y$. By contradiction, assume that $f \not\succ y$, then either $y \succ f$ or $y \bowtie f$. If $y \succ f$, then $y \succ x$ (since $\succ$ is transitive), which contradicts the assumption that $x \succsim y$. If $y \bowtie f$, it follows from Lemma 1 that $y \succ x$; a contradiction. □

Lemma 4. *For every $f, g \in \mathcal{F}$ with $f(s) \succsim g(s)$ for all $s \in S$, and for all $x \in X$, if $x \bowtie f$, then $g \not\succ x$; and if $x \bowtie g$, then $x \not\succ f$.*

Proof. Suppose that $f(s) \succsim g(s)$ for all $s \in S$, and suppose by contradiction that there is $x \in X$ such that $x \bowtie f$ and $g \succ x$.[24]

As $\succ$ is non-trivial, there are y and z in X such that $y \succ z$. From Axiom 3 and Lemma 2, $\alpha f(s) + (1-\alpha)z \succsim \alpha g(s) + (1-\alpha)z$ for all $s \in S$ and all $\alpha \in (0,1)$. Let $f^\alpha = \alpha f + (1-\alpha)z$, $g^\alpha = \alpha g + (1-\alpha)z$, and $x^\alpha = \alpha x + (1-\alpha)z$. Note that $f^\alpha(s) \succsim g^\alpha(s)$ for all $s \in S$. Axiom 3 and Lemma 2 imply, for all $\alpha \in (0,1)$,

$$x \bowtie f \iff x^\alpha \bowtie f^\alpha$$
$$g \succ x \iff g^\alpha \succ x^\alpha.$$

Besides, Axiom 2 guarantees that for α close enough to 0, $y \succ g^\alpha(s)$ for all $s \in S$. Now, fix $\alpha \in (0,1)$ such that $y \succ g^\alpha(s)$ for all $s \in S$ and define, for every $\beta \in (0,1)$, $f_\beta \in \mathcal{F}$ by $f_\beta(s) = \beta f^\alpha(s) + (1-\beta)y$ for all $s \in S$. As $u(f^\alpha(s)) \geq u(g^\alpha(s))$ and $u(y) > u(g^\alpha(s))$, $f_\beta(s) \succ g^\alpha(s)$ for all $s \in S$. In addition, by Lemma 2, $x^\alpha \bowtie f^\alpha$ implies $\beta x^\alpha + (1-\beta)y \bowtie f_\beta$. Then, by Axiom 5, $g^\alpha \not\succ \beta x^\alpha + (1-\beta)y$ for every $\beta \in (0,1)$. However, as $g^\alpha \succ x^\alpha$, if β is close enough to 1, Axiom 2 implies that $g^\alpha \succ \beta x^\alpha + (1-\beta)y$, a contradiction. $\square$

Lemma 5. *For every $f \in \mathcal{F}$, the set $\{x \in X : x \bowtie f\}$ is non-empty.*

Proof. By definition of a simple act, for all $f \in \mathcal{F}$, there are x^* and x_* in X such that $x^* \succsim f(s) \succsim x_*$ for all $s \in S$. Since $f(s) \succsim x_*$ for all $s \in S$ and $x_* \bowtie x_*$, Lemma 4 implies that $x_* \not\succ f$. One obtains similarly that $f \not\succ x^*$. Consider the sets $\{\alpha \in [0,1] : f \not\succ \alpha x^* + (1-\alpha)x_*\}$ and $\{\alpha \in [0,1] : \alpha x^* + (1-\alpha)x_* \not\succ f\}$ which are non-empty and closed relative to $[0,1]$ by the continuity of $\succ$. Clearly, their union is $[0,1]$, and the connectedness of $[0,1]$ in turn implies that their intersection is non-empty: there is $\alpha^* \in [0,1]$ such that $\alpha^* x^* + (1-\alpha^*)x_* \bowtie f$. $\square$

From the original relation $\succ$, we define two preference relations as follows:

$$f \succ_p g \iff f \succ x \text{ and } x \bowtie g \text{ for some } x \in X,$$
$$f \succ_o g \iff f \bowtie x \text{ and } x \succ g \text{ for some } x \in X.$$

Step 1. *$f \succ g$ if and only if $f \succ_p g$ and $f \succ_o g$.*

[24] The conclusion that $x \not\succ f$ when $x \bowtie g$ follows easily from the same argument by contradiction.

Let us first prove that for $f, g \in \mathcal{F}$ such that $f \succ g$, one has $f \succ_p g$ and $f \succ_o g$, giving the explicit argument exclusively for $f \succ_p g$, as $f \succ_o g$ is proved symmetrically. By contradiction, assume that $f \not\succ_p g$, then for every $x \bowtie g$, one has $f \not\succ x$, that is, either $f \bowtie x$ or $x \succ f$. But if $x \succ f$, then $x \succ g$ by transitivity, which contradicts $x \bowtie g$. Thus, for every $x \in X$, if $x \bowtie g$, then $x \bowtie f$. By Axiom 6, $f \bowtie g$, a contradiction. We have thus proved $f \succ_p g$.

Suppose now $f \succ_p g$ and $f \succ_o g$, and let us show $f \succ g$. By definition of $\succ_p$ and $\succ_o$, there exist $x \in X$ such that $f \succ x$ and $x \bowtie g$, and $y \in X$ such that $y \bowtie f$ and $y \succ g$. Axiom 7 then implies $f \succ g$.

Step 2. $\succ_p$ *and* $\succ_o$ *are asymmetric and negatively transitive.*

We only prove that $\succ_p$ has these properties, as the argument for $\succ_o$ is similar.

Assume by contradiction $f \succ_p g$ and $g \succ_p f$ for some $f, g \in \mathcal{F}$. That is, there are some $x, y \in X$ such that $f \succ x$, $x \bowtie g$, $g \succ y$, and $y \bowtie f$. By Lemma 1, one concludes that $y \succ x$ since $f \succ x$ and $y \bowtie f$ and $x \succ y$ as $g \succ y$ and $x \bowtie g$. This is impossible since $\succ$ is asymmetric. As a consequence, $\succ_p$ is asymmetric.

Now, assume by contradiction that for some $f, g, h \in \mathcal{F}$ $f \not\succ_p g$, $g \not\succ_p h$, and $f \succ_p h$. By definition of $\succ_p$, there is $x \in X$ such that $f \succ x$ and $x \bowtie h$. Since $g \not\succ_p h$, the following holds: $f \succ x \succ g$. Let $y \in X$ such that $g \bowtie y$. From Lemma 1, $x \succ y$, implying $f \succ y$. But then, $f \succ_p g$, which is a contradiction. Therefore, $\succ_p$ is negatively asymmetric.

Define $\sim_p$ by $f \sim_p g$ if and only if $f \not\succ_p g$ and $g \not\succ_p f$, for all $f, g \in \mathcal{F}$, and define $\succsim_p$ by $f \succsim_p g$ if and only if either $f \succ_p g$ or $f \sim_p g$, for all $f, g \in \mathcal{F}$. $\sim_o$ and $\succsim_o$ are similarly defined. It is clear that $\succsim_p$ and $\succsim_o$ are weak orders, *i.e.*, they are complete and transitive. We say that $\succsim_p$ (resp. $\succsim_o$) is continuous if $\succ_p$ (resp. $\succ_o$) is continuous, which is equivalent to the closedness of $\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim_p h\}$ and $\{\alpha \in [0,1] : h \succsim_p \alpha f + (1-\alpha) g\}$.

Step 3. $\succsim_p$ *and* $\succsim_o$ *are continuous and satisfy monotonicity and certainty independence.*[25]

We only provide the proof that $\succsim_p$ is continuous and satisfies monotonicity and certainty independence, where monotonicity, when allowing for indifference, means that for all $f, g \in \mathcal{F}$ such that $f(s) \succsim_p g(s)$ for all $s \in S$, $f \succsim_p g$, and certainty independance means that for all $f, g \in \mathcal{F}$, all $x \in X$, and all $\alpha \in (0,1)$, $f \succsim_p g$ if and only if $\alpha f + (1-\alpha)x \succsim_p \alpha g + (1-\alpha)x$.

[25]The definition of these properties for a *weak* order are reminded in the following lines.

We first show that $\succsim_p$ is continuous. Let $f, g, h \in \mathcal{F}$ and $x \in X$; denote A_x the set of $\alpha \in [0,1]$ such that $\alpha f + (1-\alpha)g \succ x$ and $x \bowtie h$. Either A_x is empty or it coincides with $\{\lambda \in [0,1] : \lambda f + (1-\lambda)g \succ x\}$. Then A_x is open by the continuity of $\succ$. Therefore, $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succ_p h\} = \cup_{x\in X} A_x$ is open. Similarly, one can show that $\{\alpha \in [0,1] : h \succ_p \alpha f + (1-\alpha)g\}$ is open; thus, $\succsim_p$ is continuous.

Next, we prove that $\succsim_p$ satisfies monotonicity. Let $f, g \in \mathcal{F}$ such that $f(s) \succsim_p g(s)$ for all $s \in S$, which clearly implies $f(s) \succsim g(s)$ for all $s \in S$. Suppose $g \succ_p f$, which means that there exists $x \in X$ such that $g \succ x$ and $x \bowtie f$. This is a direct contradiction as, by Lemma 4, for any $x' \in X$ such that $x' \bowtie f$, $g \not\succ x'$. Thus, $f \succsim_p g$.

Lastly, we establish that $\succsim_p$ satisfies certainty independence. Let $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in (0,1)$. We first show that $f \succsim_p g$ implies $\alpha f + (1-\alpha)x \succsim_p \alpha g + (1-\alpha)x$. Since $\succsim_p$ is a weak order, $f \succsim_p g$ is equivalent to $g \not\succ_p f$, which holds if, and only if, for every $y \in X$ such that $g \succ y$, f and y are comparable. Under Axiom 3, it is sufficient to prove that, for every $y \in X$ such that $\alpha g + (1-\alpha)x \succ y$, $\alpha f + (1-\alpha)x$ and y are comparable. Let $y \in X$ such that $\alpha g + (1-\alpha)x \succ y$ and suppose by contradiction $\alpha f + (1-\alpha)x \bowtie y$.

Claim: For such $y \in X$, there is $\underline{z} \in \{z : z \bowtie f\}$ with $u(\underline{z}) = \inf\{u(z) : z \bowtie f\}$ such that $y \succsim \alpha \underline{z} + (1-\alpha)x$.

We have shown in Step 1 that there exists $\underline{z} \in \{z \in X : z \bowtie f\}$ such that $u(\underline{z}) = \inf\{u(z) : z \bowtie f\}$. Since $\bowtie$ satisfies certainty independence (Lemma 2), $\alpha f + (1-\alpha)x \bowtie \alpha \underline{z} + (1-\alpha)x$.

We claim $y \succsim \alpha \underline{z} + (1-\alpha)x$. Indeed, if there exists $z \in X$ such that $\underline{z} \succ z$, then it follows from Axiom 3 and Lemma 1 that $f \succ z_\beta := \beta \underline{z} + (1-\beta)z$ for every $\beta \in (0,1)$. Using Axiom 3 again yields $\alpha f + (1-\alpha)x \succ \alpha z_\beta + (1-\alpha)x$. It then follows from Lemma 1 that $y \succ \alpha z_\beta + (1-\alpha)x$ for every $\beta \in (0,1)$. Letting β tend to 1, one concludes, since u is affine, that $y \succsim \alpha \underline{z} + (1-\alpha)x$. If $z \succsim \underline{z}$ for every $z \in X$, then $\alpha f(s) + (1-\alpha)x \succsim \alpha \underline{z} + (1-\alpha)x$ for all $s \in S$ (otherwise, Axiom 3 implies $\underline{z} \succ f(s)$, which is a contradiction). Since $\alpha f + (1-\alpha)x \bowtie y$, Lemma 4 implies $\alpha \underline{z} + (1-\alpha)x \not\succ y$, which is equivalent to $y \succsim \alpha \underline{z} + (1-\alpha)x$.

Since $\alpha g + (1-\alpha)x \succ y$ and $y \succsim \alpha \underline{z} + (1-\alpha)x$, Lemma 3 implies

$$\alpha g + (1-\alpha)x \succ \alpha \underline{z} + (1-\alpha)x,$$

which is equivalent to $g \succ \underline{z}$ by Axiom 3. Hence, by definition of $\underline{z}$, $g \succ_p f$, a contradiction. Therefore, $\alpha f + (1-\alpha)x$ and y are comparable, which yields $\alpha f + (1-\alpha)x \succsim_p \alpha g + (1-\alpha)x$.

We now show the converse implication. For $\alpha \in (0,1)$, and any two $f, g \in \mathcal{F}$, $\alpha f + (1-\alpha)x \succsim_p \alpha g + (1-\alpha)x$ if, and only if, for every $y \in X$ such that $\alpha g + (1-\alpha)x \succ y$, $\alpha f + (1-\alpha)x$ and y are comparable. Let $y \in X$ such that $g \succ y$, then one has $\alpha g + (1-\alpha)x \succ \alpha y + (1-\alpha)x$ by Axiom 3. Thus, $\alpha f + (1-\alpha)x$ and $\alpha y + (1-\alpha)x$ are comparable, implying that f and y are comparable by Axiom 3. Hence, $g \not\succ_p f$, which is equivalent to $f \succsim_p g$. We have proved that $\succsim_p$ satisfies certainty independence.

Step 4. *An agent with preferences $\succsim_p$ on $\mathcal{F}$ is averse to ambiguity, i.e., for all $f, g \in \mathcal{F}$, $f \sim_p g$ implies $\alpha f + (1-\alpha)g \succsim_p f$. An agent with preferences $\succsim_o$ on $\mathcal{F}$ loves ambiguity, i.e., for all $f, g \in \mathcal{F}$, $f \sim_p g$ implies $f \succsim_o \alpha f + (1-\alpha)g$.*

We only prove that $\succsim_p$ displays ambiguity aversion. Let $f, g \in \mathcal{F}$ such that $f \sim_p g$, *i.e.*, $f \not\succ_p g$ and $g \not\succ_p f$. In other words, for every $x \in X$ with $f \succ x$, x is comparable with g, and for every $x \in X$ with $g \succ x$, x is comparable with f. Let $x \in X$ such that $f \succ x$. If $x \succ g$, then $f \succ g$, and then, by Step 1, $f \succ_p g$, which is a contradiction; thus, one must have $g \succ x$. This implies $\{x \in X : f \succ x\} \subseteq \{x \in X : g \succ x\}$. Analogously, $\{x \in X : g \succ x\} \subseteq \{x \in X : f \succ x\}$; therefore, $\{x \in X : f \succ x\} = \{x \in X : g \succ x\}$.

Let $\alpha \in (0,1)$, we claim that $\alpha f + (1-\alpha)g \succsim_p f$. Since $\succsim_p$ is a weak order, it is sufficient to prove $f \not\succ_p \alpha f + (1-\alpha)g$, which holds if, for every $x \in X$ such that $f \succ x$, $\alpha f + (1-\alpha)g \succ x$. Yet, we have just proved that $f \succ x$ if and only if $g \succ x$. Axiom 4 then directly entails $\alpha f + (1-\alpha)g \succ x$; which concludes.

Conclusion. It is well-known since Gilboa and Schmeidler (1989) that a weak order defined on $\mathcal{F}$ satisfying the properties presented in Step 3 can be represented by $f \mapsto \min_{p \in C} \int u_p(f)dp$ if it displays ambiguity aversion, such as $\succsim_p$, and by $f \mapsto \max_{p \in D} \int u_o(f)dp$ if it displays love for ambiguity, such as $\succsim_o$, where C and D are unique non-empty convex compact subsets of Δ, u_p and u_o are two affine functions on X, unique up to positive affine transformation. Clearly, for all $x, y \in X$, $x \succsim_p y$ if and only if $x \succsim y$, and $x \succsim_o y$ if and only if $x \succsim y$. Thus, u_p and u_o are positive affine transformations of u, and one may assume that $u_p = u_o = u$. Finally, it remains to prove that $C \cap D \neq \emptyset$.

Claim: C and D are non-disjoint if, and only if, for all $f \in \mathcal{F}$, $min_{p \in C} \int u(f)dp \leq max_{p \in D} \int u(f)dp$.

We only prove the *if part*, the other direction being trivial. We proceed by contraposition. Suppose that $C \cap D = \emptyset$. By the separating hyperplane theorem, there exists a bounded measurable function $\varphi : S \to \mathbb{R}$ such that $\min_{p \in C} \int \varphi dp > \max_{p \in D} \int \varphi dp$. Yet, there exists a sequence of simple functions $\{\varphi_n\}$ that converges (in supnorm topology) to

φ. Since both $\tilde{\varphi} \mapsto \min_{p \in C} \int \tilde{\varphi} dp$ and $\tilde{\varphi} \mapsto \max_{p \in D} \int \tilde{\varphi} dp$ are continuous, there is $n \in \mathbb{N}$ such that $\min_{p \in C} \int \varphi_n dp > \max_{p \in D} \int \varphi_n dp$. As $a\varphi_n + b$ also satisfies this last inequality for all $a > 0$ and $b \in \mathbb{R}$, one can choose $a > 0$ and $b \in \mathbb{R}$ such that $a\varphi_n(s) + b \in u(X)$ for all $s \in S$, which implies $\varphi_n = u(f)$ for some $f \in \mathcal{F}$:

$$\min_{p \in C} \int u(f) dp > \max_{p \in D} \int u(f) dp.$$

As a consequence, the fact that, for all $f \in \mathcal{F}$, $min_{p \in C} \int u(f) dp \leq max_{p \in D} \int u(f) dp$, implies $C \cap D \neq \emptyset$.

Based on this claim, it remains to prove that $min_{p \in C} \int u(f) dp \leq max_{p \in D} \int u(f) dp$ for all $f \in \mathcal{F}$ in order to conclude.

The inequality, $min_{p \in C} \int u(f) dp \leq max_{p \in D} \int u(f) dp$ holds for all $f \in \mathcal{F}$ if and only if, for all $x \in X$, for all $f \in \mathcal{F}$, $f \succ_p x$ implies $f \succ_o x$.

Suppose that for all $x \in X$, for all $f \in \mathcal{F}$, $f \succ_p x$ implies $f \succ_o x$. Suppose, by contradiction, that there is $f \in \mathcal{F}$ such that $\min_{p \in C} \int u(f) dp > \max_{p \in D} \int u(f) dp$. Clearly, one has $u(x_*) \leq \min_{p \in C} \int \varphi_n dp \leq u(x^*)$, where x_* and x^* are defined as in the proof of Lemma 5. Since u is affine and X is convex, the set $u(X)$ is convex. Thus, $\min_{p \in C} \int u(f) dp$ belongs to $u(X)$. Similarly, one can deduce that $\max_{p \in D} \int u(f) dp$ lies in $u(X)$. Then, the convexity of $u(X)$ implies that there exists $x \in X$ such that

$$\min_{p \in C} \int u(f) dp > u(x) > \max_{p \in D} \int u(f) dp,$$

which is a contradiction as it implies, as $min_{p \in C} \int u(x) dp = max_{p \in D} \int u(x) dp = u(x)$, $f \succ_p x$ and $x \succ_o f$. The other direction of the equivalence is trivial.

Yet, for all $x \in X$, and all $f \in \mathcal{F}$, $f \succ_p x$ implies $f \succ x$. Indeed, $f \succ_p x$ if and only if there exists $y \in X$ such that $f \succ y$ and $y \bowtie x$; then Lemma 3 implies $f \succ x$. By step 1, we conclude that $f \succ_o x$. We have thus proved that $min_{p \in C} \int u(f) dp \leq max_{p \in D} \int u(f) dp$ for all $f \in \mathcal{F}$.

If part. Assume that $\succ$ admits a unanimous dual-self representation. One can readily check that Axioms 1 to 5 are satisfied.

For all $f \in \mathcal{F}$, denote $\overline{p}_f \in \arg\max_{p \in D} \int u(f) dp$ and $\underline{p}_f \in \arg\min_{p \in C} \int u(f) dp$. Define also the constant acts $\overline{f} = \int f d\overline{p}_f$ and $\underline{f} = \int f d\underline{p}_f$. Clearly, $f \bowtie \overline{f}$ and $f \bowtie \underline{f}$; moreover,

$$\begin{cases} f \succ x & \iff u(\underline{f}) > u(x), \\ x \succ f & \iff u(x) > u(\overline{f}), \\ f \bowtie x & \iff u(\overline{f}) \geq u(x) \geq u(\underline{f}) \end{cases} . \tag{1}$$

We prove that Axiom 6 is verified by contradiction. Consider $f, g \in \mathcal{F}$ such that for every $x \in X$, $f \bowtie x$ implies $g \bowtie x$. If $f \succ g$, then $u(\overline{f}) > u(\overline{g})$. However, by assumption, $g \bowtie \overline{f}$, which implies $u(\overline{g}) \geq u(\overline{f}) \geq u(\underline{g})$, a contradiction. The same argument applies to prove that $g \succ f$ cannot hold. Therefore, $f \bowtie g$.

Axiom 7 easily obtains from by the comparisons in (1). Indeed, let $f, g \in \mathcal{F}$ and $x, y \in X$ such that $f \bowtie x$, $g \bowtie y$, $x \succ g$, and $f \succ y$. Using (1), one gets

$$\begin{cases} u(\overline{f}) \geq u(x) \geq u(\underline{f}), \\ u(\overline{g}) \geq u(y) \geq u(\underline{g}), \\ u(x) > u(\overline{g}), u(\underline{f}) > u(y) \end{cases} \tag{2}$$

Then $u(\overline{f}) > u(x) > u(\overline{f})$ and $u(\underline{f}) > u(y) > u(\underline{g})$, that is $f \succ g$, by definition of a dual-self preference.

C Proof of Theorem 3

By assumption,

$$f \succ g \iff \begin{cases} \min_{p \in C} \int u(f) dp > \min_{p \in C} \int u(g) dp \\ \max_{p \in D} \int u(f) dp > \max_{p \in D} \int u(g) dp \end{cases} ,$$

where u is an affine function defined on X, unique up to affine transformation, C and D are two unique compact and convex subsets of Δ with $C \cap D \neq \emptyset$. It remains to prove that $\succ$ admitting such a representation satisfies Axiom 8 and and 9 if and only if $C = D$. We show in a very similar way to Echenique et al. (2022) that it satisfies Axiom 8 if and only if $D \subseteq C$ —the other inclusion being equivalent to Axiom 9 is shown in a symmetric way.

Only-if part. Suppose by contraposition that $D \nsubseteq C$: there is some $p^* \in D$ such that $p^* \notin C$. Then, by the separating hyperplane theorem and the argument given in the Conclusion step of the proof of Theorem 1, there is a simple act ψ and $k \in \mathbb{R}$ such that

$$\min_{p \in C} \int u(\psi) dp > k > \int u(\psi) dp^*. \tag{3}$$

By scaling ψ and k appropriately, as u is affine, one can find $f, h \in \mathcal{F}$ and $x \in X$ such that $u(f) = \frac{1}{2}u(\psi), u(h) = -u(\psi)$ and $u(x) = 2k$.[26] Let $g = \frac{1}{2}h + \frac{1}{2}x$:

$$u\Big(\frac{1}{2}f + \frac{1}{2}g\Big) = \frac{1}{4}u(\psi) + \frac{1}{2}\left(-\frac{1}{2}u(\psi) + k\right) = \frac{k}{2},$$

that is, f and g are complementary, and $\frac{1}{2}f(s) + \frac{1}{2}g(s) \sim y$ for some $y \in X$ such that $u(y) = \frac{k}{2}$, for all $s \in S$. Since $u(f) = \frac{1}{2}u(\psi)$, Equation (3) implies

$$\min_{p \in C} \int u(f) \mathrm{d}p = \frac{1}{2} \min_{p \in C} \int u(\psi) \mathrm{d}p > \frac{k}{2} = u(y).$$

In addition, as $D \cap C \neq \emptyset$, $\max_{p \in D} \int u(f)\mathrm{d}p \geq \min_{p \in C} \int u(f)\mathrm{d}p > u(y)$. As a consequence, $f \succ y$.

Futhermore, $u(g) = u\left(\frac{1}{2}h + \frac{1}{2}x\right) = -\frac{1}{2}u(\psi) + k$. Since $p^* \in D$, Equation (3) implies

$$\max_{p \in D} \int u(g) \mathrm{d}p \geq \int u(g) \mathrm{d}p^* = -\frac{1}{2} \int u(\psi) dp^* + k > -\frac{k}{2} + k = \frac{k}{2} = u(y),$$

from which $y \not\succ g$. One thus has $\frac{1}{2}f(s) + \frac{1}{2}g(s) \sim y$ for all $s \in S$, $f \succ y$, and $y \not\succ g$, which is a violation of Axiom 8.

If part. Suppose that $D \subseteq C$. Consider any complementary acts $f, g \in \mathcal{F}$ such that $\frac{1}{2}f(s) + \frac{1}{2}g(s) \sim x$ for some $x \in X$, for all $s \in S$, and assume $f \succ x$, or, $\frac{1}{2}u(f) + \frac{1}{2}u(g) = k$

[26] We abuse notation in a standard way when writing $u(f) = t$, for $t \in \mathbb{R}$, to actually denote $u(f(s)) = t$ for all $s \in S$.

with $u(x) = k$. $f \succ x$ is equivalent to

$$\begin{cases} \min_{p\in C} \int u(f)dp > k \\ \max_{p\in D} \int u(f)dp > k \end{cases} \iff \begin{cases} \frac{1}{2}\min_{p\in C} \int u(f) - u(g)dp > 0 \\ \frac{1}{2}\max_{p\in D} \int u(f) - u(g)dp > 0 \end{cases}$$

$$\iff \begin{cases} \frac{1}{2}\max_{p\in C} \int u(f) - u(g)dp < 0 \\ \frac{1}{2}\min_{p\in D} \int u(f) - u(g)dp < 0 \end{cases}.$$

Since $D \subseteq C$, the last inequalities yield

$$\begin{cases} \frac{1}{2}\max_{p\in D} \int u(f) - u(g)dp < 0 \\ \frac{1}{2}\min_{p\in C} \int u(f) - u(g)dp < 0 \end{cases}.$$

Plugging $u(f) = 2k - u(g)$, one obtains

$$\begin{cases} 2\max_{p\in D} \int u(g) - kdp < 0 \\ 2\min_{p\in C} \int u(g) - kdp < 0 \end{cases} \iff \begin{cases} \max_{p\in D} \int u(g)dp < k \\ \min_{p\in C} \int u(g)dp < k \end{cases}.$$

As $k = u(x)$, this means $x \succ g$. Therefore, $\succ$ satisfies Axiom 8.

D Proof of Proposition 1

i) Let $\succ_U$ be a unanimous dual-self preference with unique representation (u, C_U, D_U), and let $\succ_B$ be Bewley preference with unique representation (u, C_B).

First, suppose that $C_U \cup D_U \subseteq C_B$. If $f \succ_B g$, then for every $p \in C_U \cup D_U$,

$$\int u(f)dp > \int u(g)dp,$$

which implies, as C_U and D_U are not disjoint,

$$\min_{p\in C_U} \int u(f)dp > \min_{p\in C_U} \int u(g)dp,$$
$$\max_{p\in D_U} \int u(f)dp > \max_{p\in D_U} \int u(g)dp.$$

Therefore, $f \succ_U g$. Thus, $\succ_B$ is more conservative than $\succ_U$.

Conversely, suppose $\succ_B$ is more conservative than $\succ_U$ and suppose, by contradiction, that there exists $p^* \in C_U \setminus C_B$. By the separation argument we already used in the Conclusion step of the proof of Theorem 1, there are $f \in \mathcal{F}$ and $x \in X$ such that

$$\int u(f)dp^* > u(x) > \max_{p \in C_B} \int u(f)dp.$$

It follows that $x \succ_B f$ but $x \not\succ_U f$, a contradiction. Similarly, suppose there exists $p^* \in D_U \setminus C_B$. Then there are $f \in \mathcal{F}$ and $x \in X$ such that

$$\min_{p \in C_B} \int u(f)dp > u(x) > \int u(f)dp^*.$$

In this case, we have $f \succ_B x$ but $f \not\succ_U x$, another contradiction. Therefore, $C_U \cup D_U \subseteq C_B$.

ii) Let $\succ_U$ be a unanimous dual-self preference with unique representation (u, C_U, D_U), and $\succ_T$ be a twofold multiprior preference with unique representation (u, C_T, D_T).

First, suppose that $C_U \subseteq C_T$ and $D_U \subseteq D_T$. Since $D_T \cap C_T \neq \emptyset$ and $D_U \cap C_U \neq \emptyset$, $C_U \cap D_T \neq \emptyset$ and $D_U \cap C_T \neq \emptyset$. If $f \succ_T g$, then

$$\min_{p \in C_T} \int u(f)dp > \max_{p \in D_T} \int u(g)dp,$$

which implies

$$\min_{p \in C_U} \int u(f)dp \geq \min_{p \in C_T} \int u(f)dp > \max_{p \in D_T} \int u(g)dp \geq \max_{p \in D_U} \int u(g)dp.$$
$$\max_{p \in D_U} \int u(f)dp \geq \min_{p \in C_T} \int u(f)dp > \max_{p \in D_T} \int u(g)dp \geq \max_{p \in D_U} \int u(g)dp.$$

Since $D_U \cap C_U \neq \emptyset$, one gets

$$\max_{p \in D_U} \int u(f)dp \geq \min_{p \in C_U} \int u(f)dp > \max_{p \in D_U} \int u(g)dp \geq \min_{p \in C_U} \int u(g)dp.$$

Therefore, $f \succ_U g$. Thus, $\succ_T$ is more conservative than $\succ_U$.

Conversely, suppose $\succ_T$ is more conservative than $\succ_U$ and suppose, by contradiction, that there exists $p^* \in C_U \setminus C_T$. There are $f \in \mathcal{F}$ and $x \in X$ such that

$$\min_{p \in C_T} \int u(f)dp > u(x) > \int u(f)dp^*,$$

from which it follows that $f \succ_T x$ but $f \not\succ_U x$, a contradiction. To prove that $D_U \subseteq D_T$, suppose there exists $p^* \in D_U \setminus D_T$. There are $f \in \mathcal{F}$ and $x \in X$ such that

$$\int u(f)dp^* > u(x) > \max_{p \in D_T} \int u(f)dp.$$

In this case, $x \succ_T f$ but $x \not\succ_U f$, another contradiction.

E Proof of Proposition 2

Clearly, for each $i \in \{1,2\}$, and all $x \in X$, $f \succ_i x$ if and only if $f \succ_{ip} x$, where $\succ_{ip}$ is the pessimistic relation defined, as in the proof of Theorem 1, by $f \succ_{ip} g$ if and only if $f \succ_i y$ and $y \bowtie_i g$ for some $y \in X$. Thus, $\succ_1$ is more ambiguity averse than $\succ_2$ if and only if $\succ_{1p}$ is more ambiguity averse than $\succ_{2p}$. When proving Theorem 1, we have shown that $\succ_{ip}$ is represented by a maxmin expected utility functional; therefore, $\succ_{1p}$ is more ambiguity averse than $\succ_{2p}$ if and only if $C_2 \subseteq C_1$.

Similarly, for each $i \in \{1,2\}$, and all $x \in X$, $x \succ_i f$ if and only if $x \succ_{io} f$, where $\succ_{io}$ is the optimistic relation defined, as in the proof of Theorem 1, by $f \succ_{io} g$ if and only if $f \bowtie_i y$ and $y \succ_i g$ for some $y \in X$. As we have proved that $\succ_{io}$ admits a maxmax expected utility representation, one obtains that $\succ_1$ is more ambiguity loving than $\succ_2$ if and only if $D_2 \subseteq D_1$.

F Proof of Theorem 4

We will only prove that (i) implies (ii), the inverse implication being routine.

Lemma 6. *A weak order relation $\succ$ on $\mathcal{F}$ satisfies Axioms 2, 3 and 5 if and only if there exists a monotonic, constant-linear functional $I : \mathbb{R}^S \to \mathbb{R}$ and a non-constant affine function $u : X \to \mathbb{R}$ such that, for all $f, g \in \mathcal{F}$,*

$$f \succ g \iff I(u(f)) > I(u(g)).$$

Moreover, I is unique and u is unique up to positive affine transformation.

Proof. As before, define $\succsim$ by $f \succsim g$ if and only if $g \not\succ f$ for all $f, g \in \mathcal{F}$. Clearly, $\succsim$ is complete and transitive, and $\bowtie$ is an equivalence (see Theorem 2.1 in Fishburn (1970)). The weak order $\succsim$ is continuous if, for all $f, g, h \in \mathcal{F}$, $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim h\}$ and $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha)g\}$ are closed. Clearly, $\succsim$ is continuous and non-trivial. It is monotone if and only if, for every $f, g \in \mathcal{F}$, if $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$. Since Lemma 4 holds, in particular, for a weak order whose asymmetric part satisfies Axioms 2, 3 and 5, and since $\bowtie$ is an equivalence relation, $\succsim$ is monotone.

Now, we check that that $\succsim$ satisfies certainty independence: for every $f, g \in \mathcal{F}$ and $x \in X$,

$$\begin{aligned} f \succsim g &\iff g \not\succ f \\ &\iff \alpha g + (1-\alpha)x \not\succ \alpha f + (1-\alpha)x \\ &\iff \alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x. \end{aligned}$$

As a consequence, by Lemma 1 in Ghirardato, Maccheroni and Marinacci (2004),[27] there exists a monotonic, constant-linear functional $I : \mathbb{R}^S \to \mathbb{R}$ and a non-constant affine function $u : X \to \mathbb{R}$ such that, for all $f, g \in \mathcal{F}$,

$$f \succsim g \iff I(u(f)) \geq I(u(g)).$$

Moreover, I is unique and u is unique up to positive affine transformation. $\square$

Lemma 7. *Suppose that $I, I', I'' : \mathbb{R}^S \to \mathbb{R}$ are monotonic and constant-linear with $I' \leq I''$. Then the following statements are equivalent:*

(i) For every $\phi, \varphi \in \mathbb{R}^S$, if $I'(\phi) > I'(\varphi)$ and $I''(\phi) > I''(\varphi)$, then $I(\phi) > I(\varphi)$.

(ii) There exists $\alpha \in [0,1]$ such that, for every $\varphi \in \mathbb{R}^S$, $I(\varphi) = \alpha I'(\varphi) + (1-\alpha) I''(\varphi)$.

Proof. We only prove that (i) implies (ii); the other implication is easily checked. Let $\phi, \varphi \in \mathbb{R}^S$ such that $I'(\phi) \geq I'(\varphi)$ and $I''(\phi) \geq I''(\varphi)$, we will show that $I(\phi) \geq I(\varphi)$. Since $I, I', I'' : \mathbb{R}^S \to \mathbb{R}$ are constant-linear, one has, for every $n \in \mathbb{N}$,

$$I'(\phi + \frac{1}{n}) = I'(\phi) + \frac{1}{n} > I'(\phi) \geq I'(\varphi) \text{ and } I''(\phi + \frac{1}{n}) = I''(\phi) + \frac{1}{n} > I''(\phi) \geq I''(\varphi).$$

It follows from (i) that $I(\phi) + \frac{1}{n} = I(\phi + \frac{1}{n}) > I(\varphi)$. Letting n go to infinity, one obtains $I(\phi) \geq I(\varphi)$. By Lemma A.3 in Frick, Iijima and Le Yaouanq (2022), there exists $\alpha \in [0,1]$ such that, for every $\varphi \in \mathbb{R}^S$, $I(\varphi) = \alpha I'(\varphi) + (1-\alpha) I''(\varphi)$. $\square$

Assume that $\succ$ is a unanimity dual-self preference, $\succ^*$ is an invariant biseparable preference, and that the pair $(\succ, \succ^*)$ satisfies the extension property. Let $u : X \to \mathbb{R}$ be

[27] Axiom 2 implies the "Archimedean axiom" in Ghirardato, Maccheroni and Marinacci (2004).

an affine function, and let C and D be two compact convex subsets of Δ with $C \cap D \neq \emptyset$ such that

$$f \succ g \iff \begin{cases} \min_{p\in C} \int u(f)dp > \min_{p\in C} \int u(g)dp \\ \max_{p\in D} \int u(f)dp > \max_{p\in D} \int u(g)dp \end{cases}.$$

From the uniqueness result of Theorem 1, u is unique up to positive affine transformation, and C and D are unique.

It follows from Lemma 6 that there exist a monotonic, constant-linear functional $I : \mathbb{R}^S \to \mathbb{R}$ and a non-constant affine function $u' : X \to \mathbb{R}$ such that, for all $f, g \in \mathcal{F}$,

$$f \succ^* g \iff I(u'(f)) > I(u'(g)).$$

Moreover, I is unique and u' is unique up to positive affine transformation.

It trivially follows from the extension property that for every $x, y \in X$, $u(x) = u(y)$ if and only if $u'(x) = u'(y)$, which implies that u is a positive affine transformation of u'. Thus, one can assume without loss of generality $u = u'$.

Define $I' : u(X)^S \to \mathbb{R}$ and $I'' : u(X)^S \to \mathbb{R}$ by $I'(u(f)) = \min_{p\in C} \int u(f)dp$ and $I''(u(f)) = \max_{p\in D} \int u(f)dp$ for all $f \in \mathcal{F}$. Clearly, I' and I'' are monotonic, constant-linear functionals. Thus, they can be uniquely extended to $\mathbb{R}^S$. We still denote these extensions by I' and I''. Since $C \cap D \neq \emptyset$, $I'' \geq I'$.

Now, the extension property implies that, for every $\phi, \varphi \in u(X)^S$, if $I'(\phi) > I'(\varphi)$ and $I''(\phi) > I''(\varphi)$, then $I(\phi) > I(\varphi)$. Because I, I' and I'' are constant-linear, the same implication holds actually for all $\phi, \varphi \in \mathbb{R}^S$. It then follows from Lemma 7 that there exists $\alpha \in [0, 1]$ such that, for every $\varphi \in \mathbb{R}^S$, $I(\varphi) = \alpha I'(\varphi) + (1 - \alpha)I''(\varphi)$. Thus, $I(u(f)) = \alpha \min_{p\in C} \int u(f)dp + (1 - \alpha) \max_{p\in D} \int u(f)dp$ for all $f \in \mathcal{F}$.

Finally, if $\succ$ is incomplete, then there exists $f \in \mathcal{F}^S$ such that $\min_{p\in C} \int u(f)dp < \max_{p\in D} \int u(f)dp$. For every $\alpha' \neq \alpha$, $\alpha \min_{p\in C} \int u(f)dp + (1 - \alpha) \max_{p\in D} \int u(f)dp \neq \alpha' \min_{p\in C} \int u(f)dp + (1 - \alpha') \max_{p\in D} \int u(f)dp$. That is, α is uniquely defined.

G Proof of Proposition 3

Let $C = \mathrm{co}\,(\bigcup_{i=1}^n C_i)$ and $D = \mathrm{co}\,(\bigcup_{i=1}^n D_i)$. Clearly, C and D are two compact convex sets, with $C \cap D \neq \emptyset$. By definition, for all $i \in N$, $C_i \subseteq C$ and $D_i \subseteq D$. Let $\succ$ a unanimous dual-self preference relation on $\mathcal{F}$, with representation (u, C, D).

i) Suppose that Pareto for comparability holds. Let $f \in \mathcal{F}$ and $x, y \in X$ such that $y \succ f \succ x$. Proposition 2 implies that $y \succ_i f \succ_i x$ for all $i \in N$. By Pareto for comparability, $y \succ_0 f \succ_0 x$. This means that $\succ$ is both more ambiguity averse and more ambiguity loving than $\succ_0$. Therefore, applying Proposition 2 again, we obtain $D_0 \subseteq D$ and $C_0 \subseteq C$.

We now prove the converse implication. Assume that $D_0 \subseteq D$ and $C_0 \subseteq C$. Let $f \in \mathcal{F}$ and $x, y \in X$ such that $y \succ_i f \succ_i x$ for all $i \in N$. This is equivalent to

$$\begin{cases} \min_{p \in C_i} \int u(f)dp > u(x) \\ \max_{p \in D_i} \int u(f)dp < u(y) \end{cases} \quad \text{for all } i \in N.$$

Let $p^* \in \arg\max_{p \in D} \int u(f)dp$. Since $D = \operatorname{co}\left(\bigcup_{i=1}^n D_i\right)$, there exists $(p_i)_{i \in N} \in \times_{i \in N} D_i$ and $(\lambda_i)_{i \in N} \in \mathbb{R}^n_+$ such that $\displaystyle\sum_{i=1}^n \lambda_i = 1$ and $p^* = \displaystyle\sum_{i=1}^n \lambda_i p_i$. Thus,

$$\int u(f)dp^* = \sum_{i=1}^n \lambda_i \int u(f)dp_i \leq \sum_{i=1}^n \lambda_i \max_{p \in D_i} \int u(f)dp < u(y).$$

Since $D_0 \subseteq D$,

$$\max_{p \in D_0} \int u(f)dp \leq \max_{p \in D} \int u(f)dp < u(y).$$

that is $y \succ_0 f$. By a similar argument, one obtains $f \succ_0 x$. Therefore $y \succ_0 f \succ_0 x$.

ii) Assume that caution for incomparability holds. Suppose, by contradiction, that there is $p^* \in D \setminus D_0$. Then, there are $f \in \mathcal{F}$ and $x \in X$ such that

$$\int u(f)dp^* > u(x) > \max_{p \in D_0} \int u(f)dp.\text{[28]}$$

Since $D = \operatorname{co}\left(\bigcup_{i=1}^n D_i\right)$, there exists $(p_i)_{i \in N} \in \times_{i \in N} D_i$ and $(\lambda_i)_{i \in N} \in \mathbb{R}^n_+$ such that $\displaystyle\sum_{i=1}^n \lambda_i = 1$ and $p^* = \displaystyle\sum_{i=1}^n \lambda_i p_i$. Then, there exists $i_0 \in N$ such that

$$\int u(f)dp_{i_0} > u(x) > \max_{p \in D_0} \int u(f)dp,$$

[28] We have already used this argument several times. See, for example, the Conclusion step of the proof of Theorem 1.

which implies

$$\max_{p\in D_{i_0}} \int u(f)dp > u(x) > \max_{p\in D_0} \int u(f)dp.$$

Because $x \succ_0 f$, as shown in the right hand side of this inequality, the left hand part of it implies, by caution for incomparability, $\min_{p\in C_{i_0}} \int u(f)dp > u(x)$, *i.e.* $f \succ_{i_0} x$. Let $\bar{x} \in X$ such that $f \bowtie_{i_0} \bar{x}$.[29] By caution for incomparability, $f \bowtie_0 \bar{x}$. However, by definition of $\bar{x}$, $u(\bar{x}) \geq \min_{p\in C_{i_0}} \int u(f)dp > u(x) > \max_{p\in D_0} \int u(f)dp$, *i.e.* $\bar{x} \succ_0 f$; a contradiction. Therefore, $D \subseteq D_0$. Similarly, one gets $C \subseteq C_0$.

We now prove the converse implication. Assume that $D \subseteq D_0$ and $C \subseteq C_0$. Let $f \in \mathcal{F}$ and $x \in X$ such that $f \bowtie_i x$ for some $i \in N$. This is equivalent to

$$\max_{p\in D_i} \int u(f)dp \geq u(x) \geq \min_{p\in C_i} \int u(f)dp.$$

Since $C_i \subseteq C$ and $D_i \subseteq D$, $C_i \subseteq C_0$ and $D_i \subseteq D_0$. Therefore,

$$\max_{p\in D_0} \int u(f)dp \geq \max_{p\in D_i} \int u(f)dp \geq u(x) \geq \min_{p\in C_i} \int u(f)dp \geq \min_{p\in C_0} \int u(f)dp,$$

that is, $f \bowtie_0 x$.

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[29]Such a constant act always exists.

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