

Quasilinear time for computing all eccentricities of median graphs

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Abstract

Contributions:

- Compute all eccentricities of median graphs in quasilinear time
- Oracle for finding distances in $O(\sqrt{n})$
- Distance profile for diameter-bounded median graphs ?

Directions of research:

- Distance labeling scheme of polylogarithmic size

1 Introduction

ECCENTRICITIES

Input: A median graph $G = (V, E)$.

Output: All labels $\text{ecc}(u \mid G) = \max\{d(u, v) : v \in V\}$ for each vertex $u \in V$.

WEIGHTED ECCENTRICITIES

Input: A weighted median graph $G = (V, E, \omega)$, with $\omega : V \rightarrow \mathbb{N}$.

Output: All labels $\text{ecc}(u \mid (G, \omega)) = \max\{d(u, v) + \omega(v) : v \in V\}$ for each $u \in V$.

2 Preliminaries

We begin with a reminder of some notions of graphs, and more particularly related to median graphs. We emphasize on a very important tool in this area: Θ -classes, which are equivalences classes over the edge set, and especially the orthogonality between two Θ -classes.

2.1 Graph properties

All graphs $G = (V, E)$ considered in this paper are undirected, simple, finite and connected. To avoid confusions when several graphs are considered, we denote by $V(G)$ (resp. $E(G)$) the vertex set (resp. edge set) of G . Let $N(u)$ be the *open neighborhood* of $u \in V$, i.e. the set of vertices adjacent to u in G . We extend it naturally: for any set $A \subseteq V$, the neighborhood $N(A)$ of A is the set of vertices outside A adjacent to some $u \in A$.

A subgraph G' of G is a graph $G' = (V', E')$, where $V' \subseteq V$ and $E' \subseteq E \cap (V' \times V')$. For any $U \subseteq V$, we denote by $E[U]$ the set of edges of G with two endpoints in U . We denote by $G[U]$ the subgraph of G induced by U : $G[U] = (U, E[U])$.

Given two vertices $u, v \in V$, let $d(u, v)$ be the *distance* between u and v , i.e. the length of the shortest (u, v) -path. The *eccentricity* $\text{ecc}(u)$ of a vertex $u \in V$ is the length of the longest shortest path starting from u . Put formally, $\text{ecc}(u)$ is the maximum value $d(u, v)$ for all $v \in V$: $\text{ecc}(u) = \max_{v \in V} d(u, v)$. When the context is not clear enough, we might notice, as a subscript, in which graph these distance are considered, e.g. $d_G(u, v)$, $\text{ecc}_G(u)$. The diameter of graph G is the maximum distance between two of its vertices: $\text{diam}(G) = \max_{u \in V} \text{ecc}_G(u)$.

We denote by $I(u, v)$ the *interval* of pair u, v . It contains exactly the vertices which are metrically between u and v : $I(u, v) = \{x \in V : d(u, x) + d(x, v) = d(u, v)\}$. The vertices of $I(u, v)$ are lying on at least one shortest (u, v) -path.

Definition 1 (Convex and gated sets). *We say that a set $H \subseteq V$ (or the induced subgraph $G[H]$) is convex if $I(u, v) \subseteq H$ for any pair $u, v \in H$. Moreover, we say that H is gated if any vertex $v \notin H$ admits a H -gate $g_H(v) \in H$, i.e. a unique vertex that belongs to all intervals $I(v, x)$, $x \in H$. For any $x \in H$, we have $d(v, g_H(v)) + d(g_H(v), x) = d(v, x)$.*

Gated sets are convex by definition. Indeed, by contradiction, for a gated set H , if a shortest path from $x \in H$ to $y \in H$ was containing a section outside H , it would imply the existence of two H -gates for the vertices of this section. Conversely, convex sets are not necessarily gated in general (e.g. any pair of vertices in the K_3). But, we will see in the remainder that it is true on median graphs. A well-known property is that the intersection of two gated sets is itself gated.

Lemma 1 (Intersection of gated subgraphs [2]). *Given two gated sets H_1, H_2 of a graph G , the set $H_1 \cap H_2$ is gated.*

We naturally focus on the set of vertices which admit a given vertex as a gate.

Definition 2 (Fibers). *Given a gated set H of G and a vertex $x \in H$, the fiber $F_H[x]$ is the set of vertices which admit $x \in H$ as a gate for H . As each vertex in H is its own gate, the fibers $F_H[x]$, $x \in H$ form a partition of $V(G)$.*

A fiber is thus a set of vertices which all admit the same gate for a given H . Observe that, except x itself, $F_H[x]$ contains vertices outside H : $F_H[x] \setminus \{x\} \subseteq V(G) \setminus H$. In the following sections, we often manipulate this set $F_H[x] \setminus \{x\}$ that we call the *open fiber*.

Definition 3 (Open fibers). *Given a gated set H of G and a vertex $x \in H$, we define the open fiber as $F_H(x) = F_H[x] \setminus \{x\}$.*

There is a natural linear-time algorithm (in fact, a very slight variation of BFS) for computing the fibers (and open fibers) of a given gated set H . To keep this paper as self-contained as possible, we recall a sketch of this algorithm, proposed in [12]. The classical BFS uses a queue to store the visited vertices of the graph. Here, we initialize this queue by putting into it all the vertices of H (instead of a single starting vertex). Each vertex $v \in V$ is labeled by three variables: $d(v)$ (distance to the gate), $f(v)$ (parent of v in the BFS tree) and $\text{fib}(v)$ (H -gate of v). All vertices $x \in H$ are initialized with $d(x) = 0$, $f(x) = \text{NULL}$, and $\text{fib}(x) = x$. Then, we proceed the search as follows. Once a vertex v is at the head of the queue, all not yet discovered neighbors w of v are inserted into the queue, and we fix $d(w) = d(v) + 1$, $f(w) = v$, $\text{fib}(w) = \text{fib}(v)$. At the end of the execution, each vertex v is labeled by its gate $\text{fib}(v) = g_H(v)$ in H . For the correctness of this BFS traversal, see Lemma 16 and Corollary 6 from [11], the open access version of [12].

Lemma 2 ([11, 12]). *For any gated set H of a graph G , one can compute all fibers $F_H(x)$, with $x \in H$, in linear time $O(m)$.*

In this article, we say a *weighted graph* is a generalization $(G, \omega) = (V, E, \omega)$ where the vertices are labeled with nonnegative integer weights: $\omega : V \rightarrow \mathbb{N}$. Observe that the definitions of the structural notions above are independent from any weight consideration.

Now, considering a weighted graph (G, ω) , the distance is naturally generalized to a *weighted distance* which is not symmetrical anymore: $d_\omega(u, v) = d(u, v) + \omega(v)$. When there is some ambiguity on the graph considered, we may write this weighted distance $d_{(G, \omega)}(u, v)$. The *weighted eccentricity* of a vertex $u \in V$ is the maximum weighted distance from this vertex, *i.e.* $\text{ecc}(u \mid (G, \omega)) = \max_{v \in V} d_\omega(u, v)$.

However, notions of interval, gate, convex and gated sets, fiber remain independent from the weight function ω . Their definition is only based on the unweighted distance.

2.2 Median graphs and Θ -classes

From now on, we focus on the family of graphs which is studied in this article: median graphs.

Definition 4 (Median graph). *A graph is median if, for any triplet x, y, z of distinct vertices, the set $I(x, y) \cap I(y, z) \cap I(z, x)$ contains exactly one vertex $m(x, y, z)$ called the median of x, y, z .*

Observe that certain well-known families of graphs are median: trees, grids, square-graphs [4], and hypercubes Q_k . Median graphs are bipartite and do not contain an induced $K_{2,3}$ [2, 14, 19]. They can be obtained by Mulder's convex expansion [19, 20] starting from a single vertex.

Given an integer $k \geq 1$, the hypercube of dimension k , Q_k , is a graph representing all the subsets of $\{1, \dots, k\}$ as the vertex set. An edge connects two subsets if one is included into the other and they differ by only one element. Hypercube Q_2 is a *square* and Q_3 is a 3-cube.

Now, we define a parameter which has a strong influence on the study of median graphs. The dimension $d = \dim(G)$ of a median graph G is the dimension of the largest hypercube contained in G as an induced subgraph. In other words, G admits Q_d as an induced subgraph, but not Q_{d+1} . Median graphs with $d = 1$ are exactly the trees. Median graphs with $d \leq 2$ are called *cube-free* median graphs.

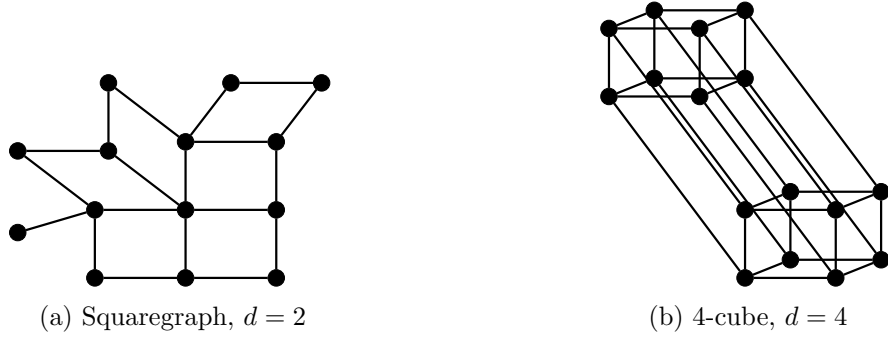


Figure 1: Examples of median graphs

Figure 1 presents three examples of median graphs. (a) is a tree: $d = 1$. (b) is a cube-free median graph: it has dimension $d = 2$. To be more precise, it is a squaregraph [4], which is a sub-family of cube-free median graphs. The last one (c) is a 4-cube: it has dimension $d = 4$.

We provide a list of properties satisfied by median graphs. In particular, we define the notion of Θ -classes which is a key ingredient of several existing algorithms [7, 13, 15].

In general graphs, all gated subgraphs are convex. The reverse is true in median graphs.

Lemma 3 (Convex \Leftrightarrow Gated [2, 7]). *Any convex subgraph of a median graph is gated.*

In fact, fibers are also gated.

Lemma 4 ([10, 12]). *For any gated set H of a graph G and any vertex $x \in H$, set $F_H(x)$ is gated.*

To improve readability, edges $(u, v) \in E$ are sometimes denoted by uv . We remind the notion of Θ -class, which is well explained in [7], and enumerate some properties related to it. We say that the edges uv and xy are in relation Θ_0 if they form a square $uvyx$, where uv and xy are opposite edges. Then, Θ refers to the reflexive and transitive closure of relation Θ_0 . Let q be the number of equivalence classes obtained with this relation. The classes of the equivalence relation Θ are denoted by E_1, \dots, E_q . Concretely, two edges uv and $u'v'$ belong to the same Θ -class if there is a sequence $uv = u_0v_0, u_1v_1, \dots, u_rv_r = u'v'$ such that u_iv_i and $u_{i+1}v_{i+1}$ are opposite edges of a square. We denote by $\mathcal{E}(G)$ the set of Θ -classes of G : $\mathcal{E}(G) = \{E_1, \dots, E_q\}$. To avoid confusions, let us highlight that parameter q is different from the dimension d : for example, on trees, $d = 1$ whereas $q = n - 1$. Moreover, the dimension d is at most $\lfloor \log n \rfloor$ in general.

Lemma 5 (Θ -classes in linear time [7]). *There exists an algorithm which computes the Θ -classes E_1, \dots, E_q of a median graph in linear time $O(|E|) = O(dn)$.*

In median graphs, each class E_i , $1 \leq i \leq q$, is a perfect matching cutset and its two sides H'_i and H''_i verify nice properties, that are presented below.

Lemma 6 (Halfspaces of E_i [7, 13, 20]). *For any $1 \leq i \leq q$, the graph G deprived of edges of E_i , i.e. $G \setminus E_i = (V, E \setminus E_i)$, has two connected components H'_i and H''_i , called halfspaces. Edges of E_i form a matching: they have no endpoint in common. Halfspaces satisfy the following properties.*

- Both H'_i and H''_i are convex/gated.
- If uv is an edge of E_i with $u \in H'_i$ and $v \in H''_i$, then $H'_i = W(u, v) = \{x \in V : d(x, u) < d(x, v)\}$ and $H''_i = W(v, u) = \{x \in V : d(x, v) < d(x, u)\}$.

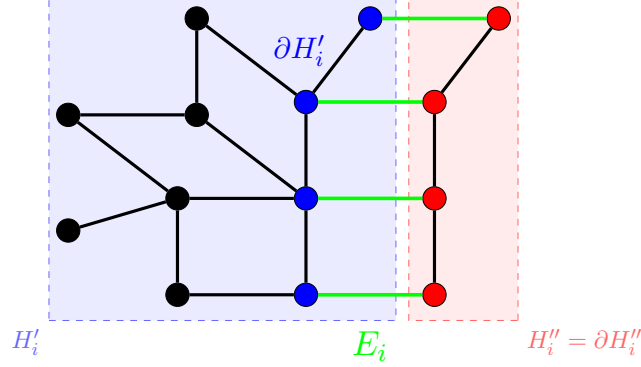


Figure 2: A class E_i with sets $H'_i, H''_i, \partial H'_i, \partial H''_i$

We say a *minority halfspace* is a halfspace, say H'_i w.l.o.g, such that $|H'_i| < |H''_i|$. The opposite halfspace H''_i is thus naturally called a *majority halfspace*. Moreover, we say that H'_i and H''_i are *egalitarian halfspaces* if $|H'_i| = |H''_i|$. In summary, a pair (H'_i, H''_i) is made up of either a minority-majority configuration or two egalitarian halfspaces.

We denote by $\partial H'_i$ the subset of H'_i containing the vertices which are adjacent to a vertex in H''_i : $\partial H'_i = N(H''_i)$. Put differently, the set $\partial H'_i$ is made up of vertices of H'_i which are endpoints of edges in E_i . Symmetrically, set $\partial H''_i$ contains the vertices of H''_i which are adjacent to H'_i . We say these sets are the *boundaries* of halfspaces H'_i and H''_i respectively. A halfspace satisfying $H'_i = \partial H'_i$ is called a *peripheral halfspace* and its associated Θ -class is also called a *peripheral Θ -class*. Observe that any median graph necessarily admits at least one peripheral Θ -class. Figure 2 illustrates the notions of Θ -class, halfspace and boundary on a small example. In this particular case, E_i is a peripheral Θ -class. The vertices of $\partial H'_i$ are colored in blue.

Lemma 7 (Boundaries [7, 13, 20]). *Both $\partial H'_i$ and $\partial H''_i$ are convex/gated. Moreover, the edges of E_i define an isomorphism between $\partial H'_i$ and $\partial H''_i$.*

As a consequence, suppose uv and $u'v'$ belong to E_i : if uu' is an edge and belongs to class E_j , then vv' is an edge too and it belongs to E_j . We terminate this list of lemmas with a last property dealing with the orientation of edges from a canonical basepoint $v_0 \in V$. The v_0 -orientation of the edges of G according to v_0 is such that, for any edge uv , the orientation is \vec{uv} if $d(v_0, u) < d(v_0, v)$. Indeed, we cannot have $d(v_0, u) = d(v_0, v)$ as G is bipartite. The v_0 -orientation is acyclic.

Lemma 8 (Orientation [7]). *All edges can be oriented according to any canonical basepoint v_0 .*

From now on, we suppose that an arbitrary basepoint $v_0 \in V$ has been selected and we refer automatically to the v_0 -orientation when we mention incoming or outgoing edges. For

each class E_i , we say that the halfspace containing v_0 is H'_i . Given two vertices $u, v \in V$, we define the set which contains the Θ -classes separating u from v .

Definition 5 (Signature $\sigma_{u,v}$ [8]). *We say that the signature of the pair of vertices u, v , denoted by $\sigma_{u,v}$, is the set of classes E_i such that u and v are separated in $G \setminus E_i$. In other words, u and v are in different halfspaces of E_i .*

The signature of two vertices provide us with the composition of any shortest (u, v) -path. Indeed, all shortest (u, v) -paths contain exactly one edge for each class in $\sigma_{u,v}$.

Lemma 9 ([9]). *For any shortest (u, v) -path P , the edges in P belong to classes in $\sigma_{u,v}$ and, for any $E_i \in \sigma_{u,v}$, there is exactly one edge of E_i in path P . Conversely, a path containing at most one edge of each Θ -class is a shortest path between its departure and its arrival.*

This lemma is a consequence of the convexity of halfspaces. Indeed, a shortest path that would pass through two edges of some Θ -class E_i would escape temporarily from an halfspace. This is not possible since any halfspace is convex (Lemma 6).

2.3 Orthogonal Θ -classes and hypercubes

We present now another important notion on median graphs: *orthogonality*. In [16], Kovse studied a relationship between *splits* which refer to the halfspaces of Θ -classes. It says that two splits $\{H'_i, H''_i\}$ and $\{H'_j, H''_j\}$ are *incompatible* if the four sets $H'_i \cap H'_j$, $H''_i \cap H'_j$, $H'_i \cap H''_j$, and $H''_i \cap H''_j$ are nonempty. Another definition was proven equivalent to this one.

Definition 6 (Orthogonal Θ -classes). *We say that classes E_i and E_j are orthogonal ($E_i \perp E_j$) if there is a square $uvyx$ in G , where $uv, xy \in E_i$ and $ux, vy \in E_j$.*

Indeed, classes E_i and E_j are orthogonal if and only if the splits produced by their halfspaces are incompatible.

Lemma 10 (Orthogonal \Leftrightarrow Incompatible [9]). *Given two Θ -classes E_i and E_j of a median graph G , the following statements are equivalent:*

- *Classes E_i and E_j are orthogonal,*
- *Splits $\{H'_i, H''_i\}$ and $\{H'_j, H''_j\}$ are incompatible,*
- *The four sets $\partial H'_i \cap \partial H'_j$, $\partial H''_i \cap \partial H'_j$, $\partial H'_i \cap \partial H''_j$, and $\partial H''_i \cap \partial H''_j$ are nonempty.*

The concept of *orthogonality* is sometimes described with different words in the literature depending on the context: *incompatible*, *concurrent* or *crossing*. We say that E_i and E_j are *parallel* if they are not orthogonal, that is $H_i \subseteq H_j$ for some $H_i \in \{H'_i, H''_i\}$ and $H_j \in \{H'_j, H''_j\}$.

We pursue with a property on orthogonal Θ -classes: if two edges of two orthogonal classes E_i and E_j are incident, they belong to a common square.

Lemma 11 (Squares [6, 9]). *Let $xu \in E_i$ and $uy \in E_j$. If E_i and E_j are orthogonal, then there is a vertex v such that $uyvx$ is a square.*

Pairwise orthogonal families. We focus on the set of Θ -classes which are pairwise orthogonal.

Definition 7 (Pairwise Orthogonal Family). *We say that a set of classes $X \subseteq \mathcal{E}$ is a Pairwise Orthogonal Family (POF for short) if for any pair $E_j, E_h \in X$, we have $E_j \perp E_h$.*

The empty set is considered as a POF. We denote by \mathcal{L} the set of POFs of the median graph G . The notion of POF is strongly related to the induced hypercubes in median graphs. First, observe that all Θ -classes of a median graph form a POF if and only if the graph is a hypercube of dimension $\log n$ [16, 18]. Secondly, the next lemma precises the relationship between POFs and hypercubes.

Lemma 12 (POFs adjacent to a vertex [9]). *Let X be a POF, $v \in V$, and assume that for each $E_i \in X$, there is an edge of E_i adjacent to v . There exists a hypercube Q containing vertex v and all edges of X adjacent to v . Moreover, the Θ -classes of the edges of Q are the classes of X .*

There is a natural bijection between the vertices of a median graph and the POFs. The next lemma exhibits this relationship.

Lemma 13 (POFs and hypercubes [3, 5, 16]). *Consider an arbitrary canonical basepoint $v_0 \in V$ and the v_0 -orientation for the median graph G . Given a vertex $v \in V$, let $N^-(v)$ be the set of edges going into v according to the v_0 -orientation. Let $\mathcal{E}^-(v)$ be the classes of the edges in $N^-(v)$. The following propositions are true:*

- For any vertex $v \in V$, $\mathcal{E}^-(v)$ is a POF. Moreover, vertex v and the edges of $N^-(v)$ belong to an induced hypercube formed by the classes $\mathcal{E}^-(v)$. Hence, $|\mathcal{E}^-(v)| = |N^-(v)| \leq d$.
- For any POF X , there is an unique vertex v_X such that $\mathcal{E}^-(v_X) = X$. Vertex v_X is the closest-to- v_0 vertex v such that $X \subseteq \mathcal{E}^-(v)$.
- The number of POFs in G is equal to the number n of vertices: $n = |\mathcal{L}|$.

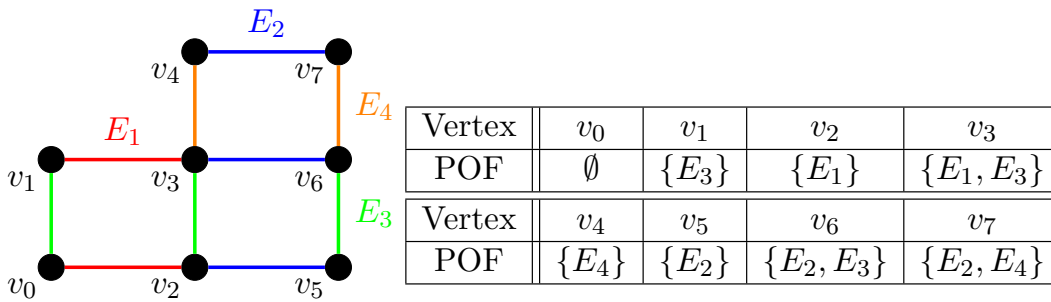
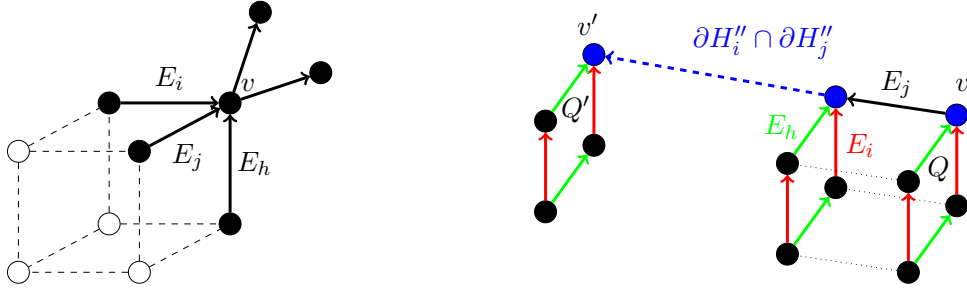


Figure 3: Illustration of the bijection between V and the set of POFs.

An example is given in Figure 3 with a small median graph of dimension $d = 2$. v_0 is the canonical basepoint and edges are colored according to their Θ -class. For example, $v_1 v_3 \in E_1$. We associate with any POF X of G the vertex v_X satisfying $\mathcal{E}^-(v_X) = X$ with

the v_0 -orientation. Obviously, the empty POF is associated with v_0 which has no incoming edges.

A straightforward consequence of this bijection is that parameter q , the number of Θ -classes, is less than the number of vertices n . But it can be used less trivially to enumerate the POFs of a median graph in linear time [5, 16]. Given a basepoint v_0 , we say that the *basis* (resp. *anti-basis*) of an induced hypercube Q is the single vertex v such that all edges of the hypercube adjacent to v are outgoing from (resp. incoming into) v . Said differently, the basis of Q is its closest-to- v_0 vertex and its anti-basis is its farthest-to- v_0 vertex. What Lemma 13 states is also that we can associate with any POF X a hypercube Q_X which contains exactly the classes X and admits v_X as its anti-basis. This observation implies that the number of POFs is less than the number of hypercubes in G . Moreover, the hypercube Q_X is the closest-to- v_0 hypercube formed with the classes in X . Figure 4a shows a vertex v with its incoming and outgoing edges with the v_0 -orientation. The dashed edges represent the hypercube with anti-basis v and POF $\mathcal{E}^-(v)$.



(a) The hypercube “induced” by the edges incoming into a vertex (its antibasis). (b) A POF signing at least two hypercubes Q and Q' is not maximal.

Figure 4: Properties of POFs

2.4 Median set and majority rule

Bénéteau *et al.* [7] recently proposed a linear-time algorithm which computes the median set of median graphs. We recall here some key observations of this article but also previous works that will be useful for us. We begin with the definition of a median set.

Definition 8 (Median vertex and set). *Given a graph G , a median vertex of G is a vertex u which minimizes $F(u) = \sum_{v \in V} d(u, v)$. The median set $\text{Med}(G)$ is the set containing all median vertices.*

In median graphs, the median set admits an interesting characterization related to the Θ -classes. Indeed, any vertex which belongs to at least one minority halfspace is not median.

Lemma 14 (Majority rule [1]). *$\text{Med}(G)$ is the intersection of all majority halfspaces. It coincides with the interval of a diametral pair of its vertices.*

The majority rule is a key tool for the algorithm presented in [7]. The first part consists in computing the cardinality of each halfspace of the input median graph G . This is based on a *peripheral peeling* that we describe briefly. Consider G where all vertex weights are fixed to 1.

Pick up some peripheral Θ -class E_i : first retrieve the cardinality of its peripheral halfspace (say $H'_i = \partial H'_i$) by summing up all weights of H'_i , second transfer the weights of vertices in H'_i to their E_i -neighbors. Remove H'_i from the current graph and restart the process on another peripheral Θ -class, etc.

Lemma 15 (Halfspaces sizes [7]). *Given a median graph G , there is a combinatorial algorithm computing in linear time $O(m)$ all triplets $(E_i, |H'_i|, |H''_i|)$ where $E_i \in \mathcal{E}(G)$.*

Once the cardinality of each halfspace is known, the majority rule allows them to retrieve the median set $\text{Med}(G)$. The second part consists simply in orienting edges uv (say $uv \in E_j$ and $u \in H'_j$ w.l.o.g) such that v is the head of the arc if and only if (iff) $|H'_j| > |H''_j|$. From Lemma 14, the median set coincides with the sinks of this partially directed graph.

3 Exploiting halfspaces for the computation of eccentricities

3.1 Balanced Θ -classes

As stated in Section 2, the Θ -classes are natural separators for a median graph. Consequently, the ratio between the size of the two halfspaces is a potential tool to design divide-and-conquer procedures on this family of graphs. We begin with the definition of f -balanced Θ -classes.

Definition 9 (Balanced and unbalanced Θ -classes). *Let $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ be the set of positive integers. Given some median graph G and a function $f : \mathbb{N}^* \rightarrow \mathbb{N}^*$, a Θ -class E_i of G is f -balanced if:*

$$\min\{|H'_i|, |H''_i|\} \geq \frac{|V(G)|}{f(|V(G)|)}. \quad (1)$$

Conversely, a Θ -class is f -unbalanced if it is not f -balanced, so formally either $|H'_i| < \frac{|V(G)|}{f(|V(G)|)}$ or $|H''_i| < \frac{|V(G)|}{f(|V(G)|)}$.

The halfspaces of some Θ -class form a bipartition of $V(G)$. Hence, being f -unbalanced means that one halfspace has a small size less than $\frac{n}{f(n)}$, with $n = |V(G)|$, while the other halfspace is large with cardinality at least $n(1 - \frac{1}{f(n)})$. Observe that the case f being a constant function (for example, $f(n) = 3$ for all integers n) is the classical sense given to a *balanced separator* in graphs, as defined originally by [17]. In the remainder of this article, we will focus more particularly on f -unbalanced Θ -classes, where f is a logarithmic function.

Certain median graphs do not admit any f -balanced Θ -classes, for some given function f .

Definition 10 (Unbalanced median graphs). *Let \mathcal{U}_f be the family of all median graphs G satisfying the following property: all Θ -classes of G are f -unbalanced.*

From now on, we fix a specific function f and the notion of *balanced/unbalanced* Θ -class will be naturally associated with this function : let $f = 2 \log$, i.e. the function $f : n \rightarrow 2 \log(n)$. We denote by $\mathcal{U}_{2 \log}$ the family of median graphs which do not admit any f -balanced Θ -classes, with $f = 2 \log$. Star graphs are certainly the most natural examples of median graphs without balanced Θ -classes: for each Θ -class, its minority halfspace has size 1.

Checking whether a median graph admits (or not) a balanced Θ -class can be achieved in linear time $O(m)$, according to Lemma 15.

As a consequence, we are able in linear time either to pick up a balanced Θ -class (for any balanceability criterion f) or answer that the input median graph contains only unbalanced ones. It consists simply in applying the algorithm of Lemma 15 and returning a balanced Θ -class when Equation (1) is satisfied.

The idea of our future recursive scheme is to identify whether a balanced Θ -class of the input median graph G exists. If it is the case, we compute recursively the weighted eccentricities of its halfspaces in order to retrieve the weighted eccentricities of the whole graph. Otherwise, we fall into the “base case”, which corresponds to the fact that $G \in \mathcal{U}_{2\log}$. This base case will be treated in Section 4.

We state an analytical result on some integer sequences which will be a key win-win argument for this global recursive process. The consequence of the following lemma is that the depth of the recursive tree, built upon balanced Θ -classes separation, is at most poly-logarithmic.

Lemma 16. *Let s_n be a sequence of integers such that s_0 is a positive integer, and let $\lambda \geq 2$ be a positive real number such that:*

$$s_{n+1} = \begin{cases} \lfloor s_n \left(1 - \frac{1}{\lambda \log(s_n)}\right) \rfloor & \text{if } s_n > 2 \\ s_n & \text{if } s_n \leq 2 \end{cases}$$

The total stopping time τ of sequence s_n , i.e. the minimum positive integer τ such that $s_\tau = s_{\tau+1}$, verifies $\tau \leq \lambda(\log(s_0))^2$. Moreover, $s_\tau \in \{1, 2\}$.

Proof. Consider a finite subsequence s_0, s_1, \dots, s_n and assume that $s_n > 2$. By definition, it is monotonically decreasing since $\lambda \log(s_i) > 1$ for any $0 \leq i \leq n$.

Then, $s_n \leq s_0 \prod_{i=0}^{n-1} \left(1 - \frac{1}{\lambda \log(s_i)}\right) \leq s_0 \left(1 - \frac{1}{\lambda \log(s_n)}\right)^n$. Assume by contradiction that $n \geq \lambda(\log(s_0))^2$, then by exploiting the fact that $(1 - \frac{1}{x})^x \leq \frac{1}{e}$ for $x \geq 2$,

$$s_n \leq s_0 \left(1 - \frac{1}{\lambda \log(s_n)}\right)^{\lambda(\log(s_0))^2} \leq s_0 \left(1 - \frac{1}{\lambda \log(s_n)}\right)^{\lambda \log(s_n) \log(s_0)} \leq s_0 e^{-\log(s_0)} = 1.$$

This yields a contradiction since we assumed $s_n > 2$. Hence, for $n \geq \lambda(\log(s_0))^2$, $s_n \leq 2$, so the total stopping time verifies $\tau \leq \lambda(\log(s_0))^2$. Now, we verify that $s_\tau > 0$. As $s_{\tau-1} \geq 3$, we have $s_\tau = \lfloor s_{\tau-1} \left(1 - \frac{1}{\lambda \log(s_{\tau-1})}\right) \rfloor \geq \lfloor 3(1 - \frac{1}{2\log(3)}) \rfloor \geq 1$. Therefore, $s_\tau \in \{1, 2\}$. \square

3.2 Retrieving all eccentricities thanks to Θ -classes

The following theorem is crucial for our algorithm: it states that, given a Θ -class and the weighted eccentricity of each vertex inside its induced halfspace, we can re-assemble all weighted eccentricities of G in linear time.

Theorem 1. *Let $G = (V, E, \omega)$ be a weighted median graph and E_i one of its Θ -classes. Assume that:*

- *all weighted eccentricities of $G[H'_i]$ are known,*
- *all weighted eccentricities of $G[H''_i]$ are known.*

Then, one can compute all weighted eccentricities of G in linear time $O(|E(G)|) = O(m)$.

Proof. At the beginning of our computation, every vertex of $V(G)$ is labeled with a weighted distance: the vertices $u' \in H'_i$ are labeled with their weighted eccentricity in $G[H'_i]$, formally $\text{ecc}(u' \mid (H'_i, \omega))$. Similarly, each vertex $u'' \in H''_i$ is labeled with $\text{ecc}(u'' \mid (H''_i, \omega))$.

For any vertex $u' \in H'_i$, we determine its H''_i -gate $g(u') \in H''_i$ and, conversely, for any vertex $u'' \in H''_i$, we determine its H'_i -gate $g(u'')$. This operation can be achieved in total $O(m)$ time, as recalled in Lemma 2. We compute, for any $v' \in \partial H'_i$ (resp. $v'' \in \partial H''_i$) its open fiber $F_{H'_i}(v')$ in H''_i (resp. $F_{H''_i}(v'')$ in H'_i). Through this BFS, we store the unweighted distance from any vertex to its gate: we obtain all values $d(u', g(u'))$ (resp. $d(u'', g(u''))$) also in linear time.

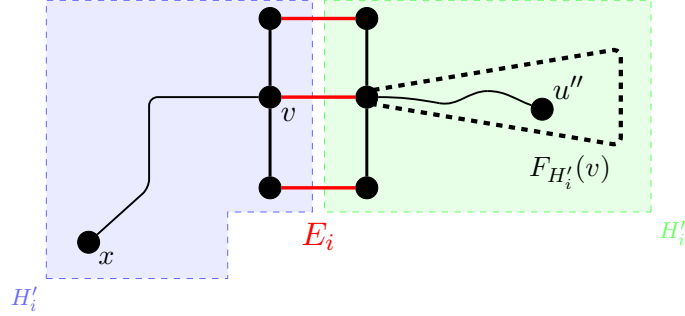


Figure 5: The largest weighted distance from $u'' \in H''_i$ to some vertex in H'_i is given by the (unweighted) distance from u'' to its gate $v = g_{H'_i}(u'')$ in addition with the weighted eccentricity of v : $\text{ecc}(v \mid (H'_i, \omega)) = d(v, x)$.

One can retrieve at this moment the weighted eccentricity of $u' \in H'_i$ (resp. $u'' \in H''_i$ with the same arguments). Indeed, the farthest vertex from u' (in the weighted sense) is either in H'_i or in H''_i . If it belongs to H'_i , as this halfspace is convex, the weighted eccentricity of u' is its label $\text{ecc}(u' \mid (H'_i, \omega))$. Else, as H''_i is gated, for any vertex $v'' \in H''_i$, there is a shortest path from u' to v'' passing through the H''_i -gate $g(u')$. Conversely, any shortest path induced in H''_i from $g(u')$ to some v'' , concatenated with a shortest $(u', g(u'))$ -path, produces a shortest (u', v'') -path, since $d(u', g(u')) + d(g(u'), v'') = d(u', v'')$. Hence, the weighted eccentricity of u' can be decomposed as the sum of $d(u', g(u'))$ (distance to the gate) with the weighted eccentricity of $g(u')$ in H''_i . Formally,

$$\text{ecc}(u' \mid (G, \omega)) = \max \{ \text{ecc}(u' \mid (H'_i, \omega)), d(u', g(u')) + \text{ecc}(g(u') \mid (H''_i, \omega)) \}.$$

Figure 5 illustrates this formula with a vertex $u'' \in H''_i$ instead of $u' \in H'_i$.

The whole procedure, consisting first in computing the gates $g(u)$ and second in applying the latter formula, takes linear time $O(m)$. \square

When our input graph G has a balanced Θ -class E_i , one can recursively compute the weighted eccentricities of the two induced halfspaces $G[H'_i]$ and $G[H''_i]$ and then retrieve the weighted eccentricities of G with an extra linear time. With some “ideal” instance, we could always find such a balanced Θ -class and use this recursive scheme until falling into the trivial case of a single weighted vertex. Observe that, with such a utopian situation, the total running time would be quasilinear: the depth of the recursive tree is poly-logarithmic thanks

to Lemma 16. Indeed, fixing $s_0 = n = |V(G)|$ and $\lambda = 2$, value s_n gives an upper bound of the number of vertices remaining in each branch of the recursive tree after n calls.

Nevertheless, we might find at some moment an induced subgraph of G which has no balanced Θ -class, in other words which belongs to $\mathcal{U}_{2\log}$. Obviously, even the input G could belong to this family. As a conclusion of this section, we observe that, to pursue the description of our recursive algorithm, we need to focus on the case of weighted median graphs which admit only unbalanced Θ -classes.

4 Computation of eccentricities on median graphs without balanced Θ -classes

The study of median graphs with only unbalanced Θ -classes is of interest since, as we prove it now, the existence of quasilinear-time algorithm finding all weighted eccentricities for the subfamily $\mathcal{U}_{2\log}$ of median graphs would imply the same result for the whole family of median graphs. From now on, we fix $(G, \omega) \in \mathcal{U}_{2\log}$. Notation G naturally refers to the same graph without any weight consideration.

Pierre: gérer les cas G et (G, ω)

4.1 Notation and preliminaries

We pursue now with notations concerning Θ -classes. As all Θ -classes E_i of $G \in \mathcal{U}_{2\log}$ are unbalanced, all of them admit both a minority halfspace denoted from now on by H'_i (the halfspace with the smaller size), and a majority halfspace denoted by H''_i . From Definition 9, we have $|H'_i| \leq \frac{n}{2\log(n)}$ with $n = |V(G)|$.

No graph of $\mathcal{U}_{2\log}$ contains an egalitarian halfspace since all its Θ -classes are unbalanced. Consequently, these graphs admit a unique median vertex.

Lemma 17 (Unique median vertex in $G \in \mathcal{U}_f$). *For any $G \in \mathcal{U}_f$ where $f(n) \geq 2$ for any integer $n \geq 2$, there exists a unique vertex v_0 such that, for any Θ -class E_i this vertex v_0 belongs to the majority halfspace of E_i .*

Proof. From Lemma 14, the median set of a graph $G \in \mathcal{U}_f$ is the intersection of all majority halfspaces. Moreover, there exists $u, v \in V(G)$ such that $\text{Med}(G) = I(u, v)$. Assume that $u \neq v$. As $\text{Med}(G)$ is connected, consider two adjacent vertices $x, y \in \text{Med}(G)$ and let $xy \in E_i$. The Θ -class E_i admits two egalitarian halfspaces since, otherwise, either x or y would not be a median vertex. We have a contradiction as E_i is clearly balanced. \square

From now on, this unique median vertex v_0 will be taken as the basis of the orientation of G . We define $L(G)$ as a ladder set $L_{v_0, u_{\max}}$ between v_0 and one farthest-to- v_0 vertex u_{\max} .

Definition 11. Let u_{\max} be a vertex such that $d_\omega(v_0, u_{\max}) = \text{ecc}(v_0 \mid (G, \omega))$. We denote by $L(G)$ the wide ladder of G , which is the POF containing the Θ -classes E_i adjacent to v_0 such that u_{\max} belongs to their minority halfspace, i.e. $u_{\max} \in H'_i$. Formally, $L(G) = L_{v_0, u_{\max}}$ and $\ell = |L(G)|$ is its size.

In order to simplify our notation, we modify the indices of the Θ -classes such that:

$$L(G) = \{E_1, E_2, \dots, E_\ell\}.$$

Observe that this change has no impact on the previous results since the indices of Θ -classes were not considered yet.

Our technique to handle median graphs without balanced Θ -classes relies on the following crucial observation. All vertices v of G such that $L_{v_0,v} \cap L(G) = \emptyset$ form, due to the unbalancedness of the Θ -classes, a large set of vertices. Moreover, we can directly deduce their weighted eccentricity since u is their extremal vertex. As a consequence, the eccentricity of at least half of the vertices of the graph are already known. We will explain how to handle the remaining vertices afterwards, in Section 4.2.

Lemma 18. *Let $G \in \mathcal{U}_{2\log}$ and v_0 its median vertex. Any vertex v such that $L_{v_0,v} \cap L(G) = \emptyset$ has a weighted eccentricity $\text{ecc}(v \mid (G, \omega)) = d(v, v_0) + d_\omega(v_0, u_{\max})$.*

Proof. We show that $v_0 \in I(v, u_{\max})$. Let P be a (v, u_{\max}) -path consisting in the concatenation of a shortest (v, v_0) -path (denoted by Q) with a shortest (v_0, u_{\max}) -path (denoted by R). It suffices to show that P is made up of edges belonging to pairwise different Θ -classes (Lemma 9). Assume by contradiction that there exists a Θ -class $E_j \in I(v_0, v) \cap I(v_0, u_{\max})$ and that there is no other Θ -class $E_k \in I(v_0, v) \cap I(v_0, u_{\max})$ such that:

- the edge of E_k on path Q is closer to v_0 than the edge of E_j on Q ,
- the edge of E_k on path R is closer to v_0 than the edge of E_j on R .

In brief, if we assume $I(v_0, v) \cap I(v_0, u_{\max})$ nonempty, we fix E_j as a Θ -class of this set which is minimal by distance towards the median vertex v_0 . Such a minimal class necessarily exists.

By definition of E_j , there is no Θ -class that appears twice in the path P_j , defined as the section of P containing v_0 obtained after removing edges of E_j . Hence, P_j is a shortest path. Moreover, its endpoints both belong to the same boundary $\partial H_j''$ of E_j . As $\partial H_j''$ is convex (Lemma 7) and $v_0 \in P_j$, then $v_0 \in \partial H_j''$. This yields a contradiction since E_j is adjacent to v_0 and should be part of $L(G)$. \square

Definition 12 (Large sets). *We define recursively a finite sequence $(G_i)_{0 \leq i \leq \ell}$ of induced subgraphs of G called large sets. Let $G_0 = G$ and, for any integer $0 \leq i \leq \ell - 1$,*

$$G_{i+1} = G_i \setminus H'_{i+1}.$$

In other words, G_{i+1} is incrementally obtained from G_i by removing all vertices in the minority halfspace of $E_{i+1} \in L(G)$.

A first trivial remark is that the number of vertices of the graphs of the sequence is decreasing: $|V(G_{i+1})| \leq |V(G_i)|$. A second fact is that each G_i is a gated subgraph of G .

Lemma 19. *For each $0 \leq i \leq \ell$, G_i is a gated subgraph of G .*

Proof. We proceed by induction. Obviously, as the base case, $G_0 = G$ is trivially gated.

We assume that G_i is a gated subgraph of G : $V(G_i)$ is gated in G . Moreover, the halfspace H'_{i+1} is also gated in G (Lemma 6). As we know from Definition 12, $V(G_{i+1}) = V(G_i) \setminus H'_{i+1} = V(G_i) \cap H''_{i+1}$. The intersection of two gated sets is gated (Lemma 1). Hence, $V(G_{i+1})$ is gated and our induction step holds. \square

In fact, we can rewrite, for any $1 \leq i \leq \ell$, the definition of G_i as the following one:

$$G_i = G[H_1'' \cap H_2'' \cap \dots \cap H_i'']. \quad (2)$$

As all large sets G_i are gated subgraphs of G , large sets are median graphs. So, they admit Θ -classes which can in fact be retrieved from the Θ -classes of the original graph G .

Lemma 20 (Θ -classes of isometric subgraphs [8]). *Let G be a median graph. If H is an isometric subgraph of G , then the Θ -classes of H are exactly the nonempty subsets among $E_i \cap E(H)$, for $E_i \in \mathcal{E}(G)$.*

Since gated subgraphs are also isometric (the converse is false), then the Θ -classes of G_i are inherited from the Θ -classes of G .

We define *slices* as the subgraphs withdrawn from each transition between G_i and G_{i+1} .

Definition 13 (Slices). *We define a finite sequence $(S_i)_{0 \leq i \leq \ell-1}$ of graphs called slices:*

$$S_i = G_i \setminus V(G_{i+1}).$$

The vertex set of G_i can be thus bi-partitioned with S_i and $V(G_{i+1})$.

Pierre: dessin slices et large sets

Slices S_i can be rewritten as the intersection of the minority halfspace H'_{i+1} with the current vertex set $V(G_i)$: $S_i = G_i[H'_{i+1}]$. In fact,

$$S_i = G[H'_{i+1} \setminus (H'_1 \cup H'_2 \cup \dots H'_i)]. \quad (3)$$

Lemma 21. *For each $0 \leq i \leq \ell - 1$, S_i is a gated subgraph of G*

Proof. Graph G_i is gated subgraph of G (Lemma 19). Consequently, S_i is gated since it is the intersection between H'_{i+1} and $V(G_i)$ (Lemma 1). \square

Slices are pairwise disjoint. In terms of cardinality, slices are thus small, $|V(S_i)| \leq |H'_{i+1}| \leq \frac{n}{2 \log(n)}$, while even the last large set G_ℓ contains at least half of the vertices.

Lemma 22 (Cardinality of large sets). *For any $0 \leq i \leq \ell$, $|V(G_i)| \geq n(1 - \frac{i}{2 \log(n)})$.*

Proof. By induction: $i = 0$ trivially holds. The large set G_{i+1} corresponds to G_i after withdrawing the slice S_i , which contains at most $\frac{n}{2 \log n}$ vertices. Therefore,

$$|V(G_{i+1})| \geq |V(G_i)| - \frac{n}{2 \log n} = n \left(1 - \frac{i}{2 \log(n)} \right) - \frac{n}{2 \log n} = n \left(1 - \frac{i+1}{2 \log(n)} \right).$$

We obtain the announced inequality for all integers $0 \leq i \leq \ell$. \square

Indeed, as $\ell \leq d \leq \log n$, we have $|V(G_\ell)| \geq n(1 - \frac{\ell}{2 \log(n)}) \geq \frac{n}{2}$. From Equation (2), set $V(G_\ell)$ contains exactly the vertices which does not belong to any H'_1, \dots, H'_ℓ , in other words whose ladder set has no intersection with $L(G)$. As stated in Lemma 18, the weighted eccentricities of all vertices v of $V(G_\ell)$ can be directly deduced. On the other hand, any vertex v with a ladder set $L_{v_0, v}$ intersecting $L(G)$ belongs to one of the slices $S_0, \dots, S_{\ell-1}$ (Equation (3)).

4.2 Peeling the slices

We show that, assuming we know all the weighted eccentricities on slices, which are small-sized induced subgraphs of the input graph $(G, \omega) \in \mathcal{U}_{2\log}$, we can deduce all weighted eccentricities of G . We begin with a description of how to modify weights on slices so that our recursive calls will enable us to retrieve all eccentricities.

Definition 14. Let $(G, \omega) \in \mathcal{U}_{2\log}$ and an integer $0 \leq i \leq \ell - 1$. The weight function ω_i^* on slice S_i is defined as:

- for any $x \in \partial H'_{i+1} \cap V(G_i)$, $\omega_i^*(x)$ is equal to the maximum distance $d_\omega(x, z)$ where z belongs to the fiber $F_{H'_{i+1}}(x)$ of G_i .¹
- for any $y \in V(G_i) \setminus \partial H'_{i+1}$, $\omega_i^*(y) = \omega(y)$.

We state now a theorem which introduce our recursive process to compute the weighted eccentricities of $(G, \omega) \in \mathcal{U}_{2\log}$. As we already observed it, the eccentricity of the vertices of the large set G_ℓ are already known. Therefore, we focus on computing the eccentricities of the vertices of the slices.

We define a labeling \mathcal{B}_i , for any $0 \leq i \leq \ell - 1$ on the vertices of G_i . This labeling gives:

- for any vertex $v \in V(G_\ell)$, the maximum distance $d_\omega(v, z)$, where z belongs to some slice $S_i \cup S_{i+1} \cup \dots \cup S_{\ell-1}$,
- for any vertex $x \in S_i \cup S_{i+1} \cup \dots \cup S_{\ell-1}$, its weighted eccentricity on the graph (G_i, ω) .

The label of \mathcal{B}_i for some vertex $u \in G_i$ is denoted by $\beta_i(u)$. Observe that labeling \mathcal{B}_0 would allow us to determine all eccentricities of (G, ω) : for vertices in the slices, the labels give their eccentricity while for vertices in G_ℓ , we use Lemma 18.

Our proposition is the following. Assume that we already computed some labeling \mathcal{B}_{i+1} and our intention is to obtain \mathcal{B}_i . Our idea to achieve this recursive step consists in launching two recursive calls, in order to compute the weighted eccentricities on graph S_i but with two different weight functions ω and ω_i^* . We prove that given labeling \mathcal{B}_{i+1} and the results of these calls, we can retrieve labeling \mathcal{B}_i .

Theorem 2. Let $0 \leq i \leq \ell - 1$. Assume that the following values are known:

- all weighted eccentricities of (S_i, ω) ,
- all weighted eccentricities of (S_i, ω_i^*) ,
- all labels of \mathcal{B}_{i+1} .

One can compute in linear time $O(|E(G_i)|)$ all labels of \mathcal{B}_i .

4.3 Summary and analysis for the eccentricities computation

Theorem 3. Let $G \in \mathcal{U}_{2\log}$. Assume that $L = \{E_1, \dots, E_r\}$ and that the eccentricities ... *Pierre: theoreme final pour les unbalanced, même forme que le theoreme final de 2.2*

¹Here, we abuse notation: H'_{i+1} refers to the intersection of $V(G_i)$ with H'_{i+1} , which is a halfspace of the Θ -class $E_{i+1} \cap E(G_i)$ (Lemma 20).

5 Distance profiles in diameter-bounded median graphs

Let $G = (V, E)$ be a median graph with n vertices, and let $D \geq 0$ to be fixed later in the proof.

5.1 Reduction to median graphs with low diameter

Let us recall the following results from our previous paper:

Lemma 23. *Let $G = (V, E)$ be an n -vertex median graph, let E_1, E_2, \dots, E_q be some of its Θ -classes, and let G_1, G_2, \dots, G_p denote its isometric subgraphs obtained from the removal of all Θ -classes E_j with $1 \leq j \leq q$. If for every i such that $1 \leq i \leq p$, the eccentricities of G_i can be computed in $\tilde{O}(T(n_i))$ time, where $n_i := |V(G_i)|$, then we can compute all eccentricities of G in $\tilde{O}(\sum_{i=1}^p T(n_i) + qn)$ time.*

Corollary 1. *Let $G = (V, E)$ be an n -vertex median graph, and, for any $D \geq 0$, let G_1, G_2, \dots, G_p denote its isometric subgraphs obtained from the removal of all Θ -classes with at least D edges. If for every i such that $1 \leq i \leq p$, the eccentricities of G_i can be computed in $\tilde{O}(T(n_i))$ time, where $n_i := |V(G_i)|$, then we can compute all eccentricities of G in $\tilde{O}(\sum_{i=1}^p T(n_i) + \frac{n^2}{D})$ time.*

In what follows, we may only consider median graphs such that every Θ -class contains at most D edges. We prove the following new decomposition scheme for such median graphs:

Lemma 24. *Let $G = (V, E)$ be an n -vertex median graph such that every Θ -class contains at most D edges. There exists a subset of Θ -classes E_1, E_2, \dots, E_q such that $q = \tilde{O}(n/D)$, and every of the isometric subgraphs G_1, G_2, \dots, G_p obtained from the removal of all these Θ -classes has diameter in $\tilde{O}(D)$. Furthermore, the Θ -classes E_1, E_2, \dots, E_q can be computed in $\tilde{O}(n^2/D)$ time.*

Note that the lemma holds with high probability for any random sample of $\tilde{O}(n/D)$ Θ -classes. By contrast, our computation in what follows is deterministic.

Proof. Pick an arbitrary basepoint u and execute a BFS from u . Doing so, we compute its eccentricity $e(u)$. If $e(u) \leq D$, then G has diameter at most $2D$, and so we are done. Otherwise, let v be a furthest from u vertex. Pick an arbitrary shortest uv -path, and select the edge xy on it such that $d(x, v) > d(y, v) = D$ (such an edge must exist because we assume $d(u, v) = e(u) > D$). Let E_1 be the Θ -class of xy . By removing E_1 , we get isometric subgraphs H_1, G_1 such that $u \in H_1$ and $v \in G_1$. We claim that G_1 has its diameter in $O(D)$. Indeed, let w be an arbitrary vertex of G_1 . By construction of G_1 we have $d(u, w) \geq d(u, E_1) + d(E_1, w)$. Since G is a median graph, the boundaries are connected, and so, $d(u, E_1) \geq d(u, y) - |E_1| \geq d(u, y) - D = e(u) - 2D$. Furthermore, $d(u, w) \geq e(u)$, which implies $d(w, E_1) \leq 2D$. For every $w' \in G_1$, we get $d(w, w') \leq d(w, E_1) + |E_1| + d(E_1, w') \leq 2D + D + 2D = 5D$, thus proving the claim. Finally, we apply the process recursively to H_1 . Since G_1 contains at least D vertices, namely, those on a shortest yv -path, this above process can only result in at most n/D subgraphs G_1, G_2, \dots, G_p . \square

5.2 Computation of the distance vectors

Hence, we are left with computing all eccentricities in a median graph $G = (V, E)$ of diameter at most D . For that, we are considering a seemingly more difficult problem, presented next:

- Every vertex v is assigned a $(D + 1)$ -dimensional vector $\vec{\pi}_G(v)$;
- For every vertex v , we want to compute a $(D + 1)$ -dimensional vector $\vec{\delta}_G(v)$ such that for every $0 \leq i \leq D$, $\delta_G(v, i) = \sum_{j=0}^i \sum \{\pi_G(u, j) : d(v, u) = i - j\}$.

Initially, for every vertex v we set $\pi_G(v, 0) = 1$ and $\pi_G(v, i) = 0$ for every $1 \leq i \leq D$. Then, $\delta_G(v, i)$ counts for the number of vertices at distance i from v . In particular, we can compute its eccentricity $e(v)$ from $\vec{\delta}_G(v)$, using the straightforward relation $e(v) = \max\{i : \delta_G(v, i) > 0\}$.

Definition 15. Let H'_1, H''_1 be complementary halfspaces of median graph $G = (V, E)$ such that H'_1 is a peripheral halfspace. For every vertex v' of H'_1 , let $\vec{\pi}_{H'_1}(v') := \vec{\pi}_G(v')$. For every vertex v'' of H''_1 , let $\vec{\pi}_{H''_1}(v'')$ be as follows:

1. If v'' is not in the boundary, then $\vec{\pi}_{H''_1}(v'') = \vec{\pi}_G(v'')$;
2. Otherwise, let v' be the unique neighbor of v'' in H'_1 , we have:

$$\begin{cases} \pi_{H''_1}(v'', 0) = \pi_G(v'', 0) \\ \pi_{H''_1}(v'', i) = \pi_G(v'', i) + \pi_G(v', i - 1) \text{ for every } 1 \leq i \leq D \end{cases}$$

The following relation is the cornerstone of our new algorithm:

Lemma 25. Let H'_1, H''_1 be complementary halfspaces of median graph $G = (V, E)$, with respect to some Θ -class E_1 , such that H'_1 is a peripheral halfspace. The following properties hold:

1. For every vertex v'' of H''_1 , $\vec{\delta}_G(v'') = \vec{\delta}_{H''_1}(v'')$;
2. For every vertex v' of H'_1 , with v'' its unique neighbor in H''_1 :

$$\begin{cases} \delta_G(v', 0) = \delta_{H'_1}(v', 0) = \pi_G(v', 0) \\ \delta_G(v', 1) = \delta_{H'_1}(v', 1) + \pi_G(v'', 0) \\ \delta_G(v', i) = \delta_{H'_1}(v', i) + \delta_{H''_1}(v'', i - 1) - \delta_{H'_1}(v', i - 2) \text{ for every } 2 \leq i \leq D \end{cases}$$

Proof. Let v'' be an arbitrary vertex of H''_1 . Clearly, $\delta_G(v'', 0) = \pi_G(v'', 0) = \pi_{H''_1}(v'', 0) =$

$\delta_{H_1''}(v'', 0)$. For every $1 \leq i \leq D$, we have:

$$\begin{aligned}
\delta_G(v'', i) &= \sum_{j=0}^i \sum \{\pi_G(u, j) : d(v'', u) = i - j\} \\
&= \sum_{j=0}^i \sum \{\pi_G(u', j) : u' \in H_1', d(u', v'') = i - j\} + \sum_{j=0}^i \sum \{\pi_G(u'', j) : u'' \in H_1'', d(u'', v'') = i - j\} \\
&= \sum_{j=0}^{i-1} \sum \{\pi_G(u', j) : u''u' \in E_1, d(u'', v'') = i - j - 1\} + \sum_{j=0}^i \sum \{\pi_G(u'', j) : u'' \in H_1'', d(u'', v'') = i - j\} \\
&= \sum_{j=1}^i \sum \{\pi_G(u', j-1) : u''u' \in E_1, d(u'', v'') = i - j\} + \sum_{j=0}^i \sum \{\pi_G(u'', j) : u'' \in H_1'', d(u'', v'') = i - j\} \\
&= \sum \{\pi_G(u'', 0) : u'' \in H_1'', d(u'', v'') = i\} + \sum_{j=1}^i \sum \{\pi_G(u'', j) : u'' \in H_1'' \setminus \partial H_1'', d(u'', v'') = i - j\} \\
&\quad + \sum_{j=1}^i \sum \{\pi_G(u'', j) + \pi_G(u', j-1) : u''u' \in E_1, d(u'', v'') = i - j\} \\
&= \sum_{j=0}^i \sum \{\pi_{H_1''}(u'', j) : u'' \in H_1'', d(u'', v'') = i - j\} \\
&= \delta_{H_1''}(v'', i)
\end{aligned}$$

Now, let v' be an arbitrary vertex of H_1' , with v'' its unique neighbor in H_1'' . Again, we clearly have $\delta_G(v', 0) = \pi_G(v', 0) = \pi_{H_1'}(v, 0) = \delta_{H_1'}(v, 0)$. Since v'' is the unique neighbor of v' in H_1'' , and $\vec{\pi}_{H_1'}$ is the restriction of $\vec{\pi}_G$ to H_1' , we also have $\delta_G(v', 1) = \delta_{H_1'}(v', 1) + \pi_G(v'', 0)$.

Then, for every $2 \leq i \leq D$:

$$\begin{aligned}
\delta_G(v', i) &= \sum_{j=0}^i \sum \{\pi_G(u, j) : d(u, v') = i - j\} \\
&= \sum_{j=0}^i \sum \{\pi_G(u', j) : u' \in H'_1, d(u', v') = i - j\} + \sum_{j=0}^i \sum \{\pi_G(u'', j) : u' \in H''_1, d(u'', v') = i - j\} \\
&= \sum_{j=0}^i \sum \{\pi_{H'_1}(u', j) : u' \in H'_1, d(u', v') = i - j\} + \sum_{j=0}^{i-1} \sum \{\pi_G(u'', j) : u' \in H''_1, d(u'', v'') = i - j - 1\} \\
&= \delta_{H'_1}(v', i) + \delta_G(v'', i - 1) - \sum_{j=0}^{i-1} \sum \{\pi_G(u', j) : u' \in H'_1, d(u', v'') = i - j - 1\} \\
&= \delta_{H'_1}(v', i) + \delta_{H''_1}(v'', i - 1) - \sum_{j=0}^{i-2} \sum \{\pi_G(u', j) : u' \in H'_1, d(u', v') = i - j - 2\} \\
&= \delta_{H'_1}(v', i) + \delta_{H''_1}(v'', i - 1) - \sum_{j=0}^{i-2} \sum \{\pi_{H'_1}(u', j) : u' \in H'_1, d(u', v') = i - j - 2\} \\
&= \delta_{H'_1}(v', i) + \delta_{H''_1}(v'', i - 1) - \delta_{H'_1}(v', i - 2)
\end{aligned}$$

□

By using the above lemma, we can solve our problem by traversing backward and forward a peeling ordering E_1, E_2, \dots, E_q of the Θ -classes of G , and calling our algorithm recursively on vertex-disjoint subgraphs of G representing the successive boundaries. The total cost, excluding recursive calls, is in $O(Dm')$, where m' is the number of edges to process (an edge of E_i is processed if it is still present in the subgraph G_i obtained from the successive contractions of E_1, E_2, \dots, E_{i-1}). Since the subsets of edges processed at every recursion stage are pairwise disjoint, the final runtime is in $O(Dm) = \tilde{O}(Dn)$.

Finally, the following result is obtained by setting $D = \sqrt{n}$:

Theorem 4. *We can compute all eccentricities in a median graph in $\tilde{O}(n^{3/2})$ time.*

5.3 Comments (not visible)

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