Mathematics CS1001

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Timetable

Weekly Schedule:

- We have 33 scheduled contact hours.
- There are three sessions when we are all timetabled together per week:
  - Monday 9am to 10am, room LB04 Lloyd Institute
  - Tuesday 3pm to 5pm, room LB08 Lloyd Institute
- In some weeks there will be separate class group tutorials and in others there will be none.
  - Monday 12pm to 1pm, Goldsmith Hall – CS only
  - Monday 4pm to 5pm, LB0.4, LLoyd Building – CSB, CSL

  I will let you know when we will be using this sessions (e.g. there will be no tutorial sessions today, 23rd September 2013).
- Course webpage: http://mymodule.tcd.ie.

Attendance will be taken!

Content of CS1001

- Linear Algebra
  - Linear systems
  - Matrices
  - Finding eigenvalues and eigenvectors
- Integration
- Taylor Series
- The Newton-Raphson Method

Content of CS1001

- The course is about both continuous and discrete mathematics, and is focused on those elements of mathematics at the foundations of many real-world applications in Computer Science, Engineering and the Social Sciences.

- The course is illustrated with applications.

- Objectives:
  - derive, formulate and apply solutions for linear systems;
  - devlop Taylor Series expansions and recognise their limitations;
  - discriminate between, and calculate, a variety of integral.

Assessment

- There will be five homeworks, one ever second week. Each is worth 2% of your mark for this module.
  - These are to be handed to the main computer science office before the time indicated. Late submissions will not be accepted. We will do our best to mark these and return them to you promptly. This will depend on the demonstrator support made available by the School.
- These will be an in-class test in the last week of term, worth 10% of your mark for this module.
- The final exam is in the Examination Period (April/May).
- Exam regulations... make sure you familiarize yourself with these.

Reading materials

General

- Course Webpage , `http://mymodule.tcd.ie`

- Mathematical Methods for Scientists and Engineers
  D. A. McQuarrie, University Science Book, 2003.

- Engineering Mathematics through Applications
  K. Singh, Palgrave Macmillan, 2003.

Linear algebra

- Linear Algebra
  J. Hefferon, Online textbook: `http://joshua.smcvt.edu/linearalgebra/`
- Elementary Linear Algebra
  K. R. Matthews, chapter 1, Online textbook: `http://www.numbertheory.org/book/`
- Elementary Linear Algebra
  Howard Anton, Wiley (any edition will do)

Linear Algebra – Some Basics I

Definition:

A linear function f is a mathematical function in which the variables appear only in the first degree, are multiplied by constants, and are combined only by addition and subtraction. A linear equation is of the form $f(x, y, \cdots) = 0$ with f linear.

Tell me if those following functions or equations are linear. If not, do you know what they are called ?

A.

1. $f(x) = 3x - 4$
2. $f(x) = 3x^2 - 4$
3. $f(x) = \frac{3}{x} - 4$
4. $f(x, y) = 3x + y - 2$

B.

1. $f(x, y) = 3xy$
2. $x^4 + 3x^2 - 4 = 0$
3. $\frac{3}{x} - 4 = 0$
4. $3x + y - 2 = 0$

Linear Algebra – Some Basics II

Definition:

The root (or zero) of a function f is a value that makes the function equal to zero.

In geometric terms, a root of a function is where the graph of the function crosses the x-axis.

Definition:

An equation is written in the form:

$$f(x) = 0$$

A solution to an equation is a value for x that make the equation true.

For example:
The equation $3x - 4 = 0$ has solution $x = \frac{4}{3}$.

Linear Algebra – Some Basics III

Solving Linear equations:

- Find x such that: $3x = 4$.

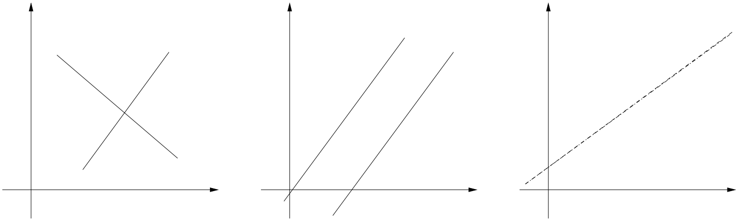
- Find (x, y) such that:

$$\begin{cases} x - 2y - 3 = 0 \\ 3x + y = -1 \end{cases}$$

Linear Systems in Two Unknowns I

- One equation: $ax + by = c$ represents a line in $\mathbb{R}^2$. There are infinitely many solutions of this equation (one for each point on the line). (Here a, b and c are real numbers, and a and b are not both zero).
- Two equations: Solutions of linear equations in two variables correspond to intersection points of lines.

Three ways in which lines can intersect:



Intersect at unique point — Parallel lines - no intersection point — Lines coincide

Linear Systems in Two Unknowns II

- Three possibilities for solutions of a Linear System in two unknowns:
  - (1) A **unique** solution
  - (0) **No** solution
  - (∞) **Infinitely many** solutions

  We call a linear system with no solutions **INCONSISTENT** or **INCOMPATIBLE**.

  We call a linear system with infinitely many solutions **INDETERMINATE**.

- Solve:
  1. $$\begin{cases} x+y=1 \\ 2x-y=2 \end{cases}$$
  2. $$\begin{cases} 2x-y=-1 \\ 2x-y=0 \end{cases}$$
  3. $$\begin{cases} y-x=2 \\ 3y-3x=6 \end{cases}$$

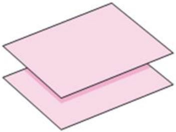
Linear Systems in Two Unknowns III

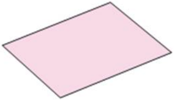
Simultaneous linear equations in two unknowns are solved with the following steps:

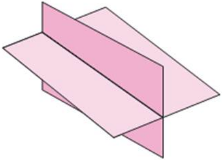
1. Write both equations in the form $ax + by = k$
2. Make the coefficients of one of the variables the same in both equations.
3. Add or substract (depending on signs) on equation from the other to form a new equation in one variable.
4. Solve the new equation obtained in step 3.
5. Put the value obtained in step 4 into one of the given equations to find the corresponding value of the other variables.

Linear Systems in Three Unknowns I

- One equation: $ax + by + cz = d$ represents a line in $\mathbb{R}^3$. There are infinitely many solutions of this equation (one for each point in the plane). (Here a, b, $c$and d are real numbers, and a, b and c are not both zero).
- Two equations: Solutions of two simultaneous linear equations in three unknown variables correspond to intersection points of two planes in $\mathbb{R}^3$.





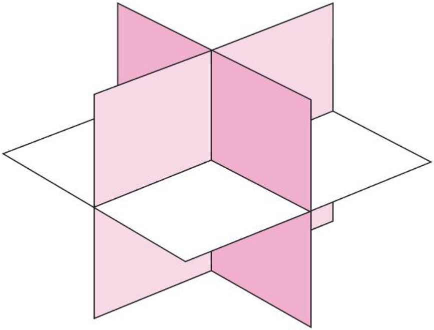


The first image above corresponds to no solution while the second two images correspond to an infinite number of solutions (where the planes either coincide (lie on top of each other) or the planes intersect in a line.

Linear Systems in Three Unknowns II

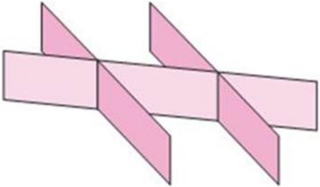
Three equations:

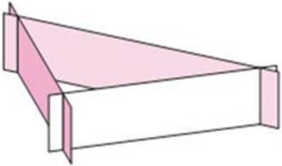
- A unique solution when the three planes meet at one point.



Linear Systems in Three Unknowns III

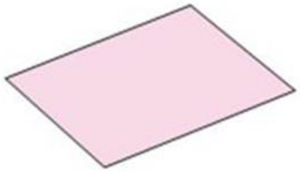
- No solution — when the planes are parallel or when the three planes form a triangular prism.

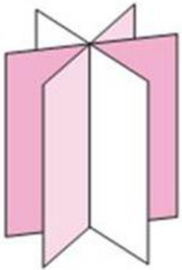




Linear Systems in Three Unknowns IV

- Infinitely many solutions, when the planes intersect either in a plane or in a line.





Matrices – Some Terminology I

Definition:

A matrix is a rectangular table of (real or complex) numbers. The order of a matrix gives the number of rows and columns it has. A matrix with m rows and n columns has order $m \times n$.

Write down the order of the following matrices:

1. $A = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$
2. $B = \begin{pmatrix} 1 & -1 \\ 2 & 5 \end{pmatrix}$
3. $C = \begin{pmatrix} -5 & 8 \end{pmatrix}$

Matrices – Some Terminology II

Definition:

A square matrix has the same number of rows and columns (order of the form $(m \times m)$). For instance, $A = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ is not square, but $B = \begin{pmatrix} 1 & -1 \\ 2 & 5 \end{pmatrix}$ is.

Definition:

The transpose of a matrix M, written M^T is obtained by interchanging the rows and columns. For example,

$$\textit{If } M = \begin{pmatrix} -1 & 5 \\ 3 & -2 \end{pmatrix}, \textit{ then } M^T = \begin{pmatrix} -1 & 3 \\ 5 & -2 \end{pmatrix}$$

Matrices – Some Terminology III

Compute

1. $(M^T)^T$ with $M = \begin{pmatrix} -1 & 5 \\ 3 & -2 \end{pmatrix}$
2. M^T with
$$M = \begin{pmatrix} 8 & 2 & 4 \\ 3 & 1 & 2 \end{pmatrix}$$

Matrix Addition I

Definitions:

- Matrices can only be added, or substracted, if they are of the same order. Simply add or substract corresponding elements. This gives a matrix of the same order.
- To multiply a matrix by a scalar (a number), multiply each element of the matrix by the number.

Example

$$2\begin{pmatrix} 8 & 2 & 4 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 16 & 4 & 8 \\ 6 & 2 & 4 \end{pmatrix}.$$

Matrix Addition II

Example

$$\begin{pmatrix} 3 & 2 & -1 \\ 6 & 2 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 2 & 1 \\ 0 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 8 & 4 & 0 \\ 6 & 1 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 8 & 4 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 1 & 6 \end{pmatrix} \quad \text{is } \textbf{not} \text{ defined.}$$

With $A = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 5 \end{pmatrix}$ and $C = \begin{pmatrix} -5 & 8 \end{pmatrix}$, compute (if you can):

1. $A + B$
2. $A + C^T$
3. $3B$

Matrix Addition III

Definitions:

- Matrix addition is commutative. Let A and B be two matrices of the same order then:

$$A + B = B + A$$

- Matrix addition is associative. Let A, B and C be three matrices of the same order then:

$$A + (B + C) = (A + B) + C$$

Matrix Multiplication I

Multiply each row of the first matrix by each column of the second matrix.
Two matrices can only be multiplied if the number of columns in the first matrix equals the number of rows in the second.

Memory aid: **Row by Column**

Example

$$\begin{pmatrix} 1 & 1 \\ \boxed{0 \quad 1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 6 & \boxed{\begin{matrix} 8 \\ 9 \end{matrix}} & 2 \\ 5 & 7 & & 3 \end{pmatrix} = \begin{pmatrix} 9 & 13 & 17 & 5 \\ 5 & 7 & \boxed{9} & 3 \\ 4 & 6 & 8 & 2 \end{pmatrix}$$

The entry “9” in the new matrix was found by calculating $(0 \times 8) + (1 \times 9)$ to get $0 + 9 = 9$.

Matrix Multiplication II

Let

$$A = \begin{pmatrix} 2 & 1 & -5 \\ 2 & 8 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -8 & 4 & 0 \\ 6 & 5 & 1 & 2 \\ 2 & 9 & 0 & 0 \end{pmatrix}$$

1. Find AB and BA, if possible.
2. Is the multiplication of matrices commutative?

The Determinant of a Matrix I

Definition:

The determinant of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the number $ad - bc = \det(A)$.

If $\det(A) = 0$, A is called a singular matrix.

Compute the determinant of the matrices:

1. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.
2. $B = \begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}$

The Identity Matrix & Inverse Matrices I

Definition:

A square matrix with 1s on the diagonal and 0s elsewhere is called the identity matrix and denoted by I.

Definition:

If A is a square matrix, and there exists a matrix B so that

$$AB = BA = I,$$

we say that A is invertible, and B is called an inverse of A. It can be shown that an invertible matrix has exactly one inverse, so we refer to the inverse of an invertible matrix.

Show that if A is not square, it cannot have an inverse.

The Identity Matrix & Inverse Matrices II

- If A is invertible, we denote the inverse of A by A^{-1}, so

$$A^{-1}A = AA^{-1} = I.$$

- A^{-1} plays the rôle of the reciprocal or multiplicative inverse in ordinary multiplication, since

$$aa^{-1} = a^{-1}a = 1,$$

for non–zero $a \in \mathbb{R}$.

The Identity Matrix & Inverse Matrices III

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

1. Exchange the elements of the diagonal.
2. Change the signs of the elements of the other diagonal.
3. Multiply by $\frac{1}{\det(A)}$.

Prove the above result (i.e. compute AA^{-1} and $A^{-1}A$, and check they are equal to I).

Working with Matrices I

We can:

- Add, subtract and multiply matrices.
- Form an identity matrix and a zero matrix (i.e. a matrix where all entries are 0).
- Find an inverse of a matrix using the determinant.

So we have all the tools we need to carry out calculations using matrices:

Let $M = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 7 & 10 \\ 21 & 23 \end{pmatrix}$.

1. Compute $M^{-1}A$.

2. If $MB = 2M + A$, express B in matrix form.

Working with Matrices II

If $M = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}$ and $N = \begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix}$,

1. Compute $(MN)^{-1}$.
2. Compute $N^{-1}M^{-1}$
3. Any comment on $(MN)^{-1}$ and $N^{-1}M^{-1}$?

Working with Matrices III

Let's come back to solving a system of linear equations:

$$\begin{cases} x - 2y - 3 = 0 \\ 3x + y = -1 \end{cases}$$

Solve this system by matrix methods.

Working with Matrices IV

Definitions:

A vector (or column vector) is a matrix with a single column. A matrix with a single row is a row vector. The entries of a vector are its components.

Solving simultaneous equations of the form:

$$A\mathbf{x} = \mathbf{b}$$

with A a square matrix and $\mathbf{x}$ unknown vector and $\mathbf{b}$ vector of constants, is given by

$$\mathbf{x} = A^{-1}\mathbf{b}$$

when A is invertible.

Working with Matrices V

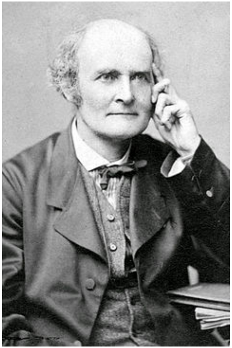
Solve

$$\begin{cases} 2x + 3y = 4 \\ 10x + 4y = 9 \end{cases}$$

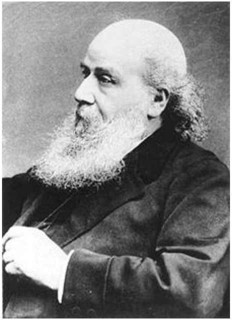
Working with Matrices VI

How old are matrices?

- Matrices were invented by the British mathematician Arthur Cayley (1821-1895).
- Cayley presented a paper giving the rule for matrix operations and the conditions under which a matrix has an inverse to the Royal Society in 1858.
- Cayley's friend, James Joseph Sylvester (1814-1897), was the person who first used the term "matrix" in 1850.



Arthur Cayley (1821-1895)



James Joseph Sylvester (1814-1897)

Linear Systems I

Definition:

A linear equation in n unknowns $x_1, x_2, \cdots, x_n$ is an equation of the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \cdots, a_n, b$ are given real numbers.

Definition:

A system of m linear equations in n unknowns $x_1, x_2, \cdots, x_n$ is a family of linear equations:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{aligned}$$

Linear Systems II

Definition:

A system of m linear equations in n unknowns is said to be:

- consistent if it has (at least) one solution,
- inconsistent otherwise.

Each equation of a system can be rewritten for $i = 1, 2, \cdots, m$:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

and the system can be written $A\mathbf{x} = \mathbf{b}$ considering the matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

Augmented Matrix

Definition:

The augmented matrix of the system is the concatenated matrix $[A\mathbf{b}]$ or:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

Write the augmented matrix of the following system of linear equations:

$$\begin{cases} 6x_3 + 2x_4 - 4x_5 - 8x_6 = 8 \\ 3x_3 + x_4 - 2x_5 - 4x_6 = 4 \\ 2x_1 - 3x_2 + x_3 + 4x_4 - 7x_5 + x_6 = 2 \\ 6x_1 - 9x_2 + 11x_4 - 19x_5 + 3x_6 = 1 \end{cases}$$

Row Echelon Form

Definition:

A matrix is in row-echelon form if

1. all zero rows (if any) are at the bottom of the matrix and
2. if two successive rows are non-zero, the second row starts with more zeros than the first (moving from left to right).

Example of row-echelon form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is not a row echelon form:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Reduced Row Echelon Form

Definition:

A matrix is in reduced row-echelon form if

1. it is in row-echelon form,
2. the leading (leftmost non-zero) entry in each non-zero row is 1,
3. all other elements of the column in which the leading entry 1 occurs are zeros.

Reduced row-echelon matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Non-reduced row-echelon matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Elementary Row Operations

Definition:

There are three types of elementary row operations that can be performed on matrices:

1. Interchanging two rows: $\mathbf{r_i} \leftrightarrow \mathbf{r_j}$ interchanges rows i and j.
2. Multiplying a row by a non-zero scalar: $\mathbf{r_i} \to \alpha\mathbf{r_i}$ multiplies row i by the non-zero scalar α.
3. Adding a multiple of one row to another row: $\mathbf{r_j} \to \mathbf{r_j} + \alpha\mathbf{r_i}$ adds α times row i to row j.

Row Equivalent Matrices

Definition:

A matrix A is row-equivalent to a matrix B if B is obtained from A by a sequence of elementary row operations.

If A and B are row equivalent augmented matrices of two systems of linear equations, then the two systems have the same solution sets (i.e. a solution of one system is a solution of the other).

Compare the systems:

$$\begin{cases} x_1 + 2x_2 = 0 \\ 2x_1 + x_2 = 1 \\ x_1 - x_2 = 2 \end{cases} \quad \text{and} \quad \begin{cases} 2x_1 + 4x_2 = 0 \\ x_1 - x_2 = 2 \\ 4x_1 - x_2 = 5 \end{cases}$$

Gauss-Jordan Elimination I

Gauss-Jordan Algorithm:

This is a process for computing a matrix in reduced row-echelon form B starting from a given matrix A. It is performed using the elementary row operations.

Here's how we implement the algorithm, step by step:

Label the rows in the matrix: **row 1** as **R1**, **row 2** as **R2** etc.

STEP 1:

Interchange the top row with another (if necessary) so as to have a **non-zero entry** as **far left as possible** in the first row.

STEP 2:

Multiply the top row by a suitable **constant** to get a **leading entry of 1**.

STEP 3:

Add multiples of the top row to each row below so that **all entries below the leading 1 are zero**.

Gauss-Jordan Elimination II

We have now dealt with the first non-zero element in the first row (it is a “1” and all the entries below it are zero), we turn our attention to the second row:

STEP 4:
Ignore the top row and move on to the second row.

STEP 5:
If necessary, **interchange** the second row with another row **below** it so as to get a **non-zero entry** as **far to the left as possible** in the second row.

STEP 6:
If necessary, **multiply** the second row by a **constant** to get a **leading entry of 1**.

STEP 7:
Add multiples of the second row to each row above and below so that **all entries above and below the leading 1 in the second row are zero**.

Gauss-Jordan Elimination III

We have now dealt with the first non-zero element in the first row and the first non-zero entry in the second row, we turn our attention to the third row:

STEP 8:
Move onto the third row and **repeat steps 5, 6 and 7** (but **this time looking at the third row, rather than the second row**). Keep going until you have dealt with all rows in the matrix.

STEP 9:
Stop when the matrix is in **reduced row echelon form**. Remember if there is a row that consists of all 0's then these rows should come at the bottom of the matrix. If your matrix represents a linear system of equations then you can read off your solution from this final matrix.

Gauss-Jordan Elimination in Operation I

Use Gauss-Jordan Elimination to transform the following matrix into reduced row echelon form:

$$A = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \\ 2 & 2 & -2 & 5 \end{pmatrix}$$

1. The first entry in the first row is zero, we see that the first entry in both of the rows below it is non-zero. So we interchange row 1 with one of these rows.

$$A \text{ with } \quad \mathbf{r_1} \leftrightarrow \mathbf{r_3} \text{ becomes } \quad \begin{pmatrix} 2 & 2 & -2 & 5 \\ 5 & 5 & -1 & 5 \\ 0 & 0 & 4 & 0 \end{pmatrix}$$

Gauss-Jordan Elimination in Operation II

2. We want the first entry in row 1 to be 1, so we divide by 2 i.e. $\mathbf{r_1} \to \frac{\mathbf{r_1}}{2}$

$$\begin{pmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 5 & 5 & -1 & 5 \\ 0 & 0 & 4 & 0 \end{pmatrix}$$

3. We use the first row to "eliminate" the 5 at the start of the second row: $\mathbf{r_2} \to \mathbf{r_2} - 5\mathbf{r_1}$. Note the third row already starts with zero so we don't need to do anything with it.

$$\begin{pmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 4 & \frac{-15}{2} \\ 0 & 0 & 4 & 0 \end{pmatrix}$$

Gauss-Jordan Elimination in Operation III

4. We have completed steps 1, 2 and 3 of the Gauss-Jordan Algorithm. We move our focus to the second row in step 4. Step 5 requires us to get a "1" as far to the left as we can in the second row: $\mathbf{r_2} \to \frac{\mathbf{r_2}}{4}$

$$\begin{pmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 1 & \frac{-15}{8} \\ 0 & 0 & 4 & 0 \end{pmatrix}$$

5. We now eliminate the terms above and below this 1: $\mathbf{r_1} \to \mathbf{r_1} + \mathbf{r_2}$ and $\mathbf{r_3} \to \mathbf{r_3} - 4\mathbf{r_2}$

$$\begin{pmatrix} 1 & 1 & 0 & \frac{5}{8} \\ 0 & 0 & 1 & -\frac{15}{8} \\ 0 & 0 & 0 & \frac{15}{2} \end{pmatrix}$$

6. We now shift our focus to the last row and we want the first non-zero entry in it to be a "1": $\mathbf{r_3} \to \mathbf{r_3} \times \frac{2}{15}$

$$\begin{pmatrix} 1 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Gauss-Jordan Elimination in Operation IV

7. To finish $\mathbf{r_1} \to \mathbf{r_1} - \frac{5}{8}\mathbf{r_3}$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This final matrix is now in reduced row echelon form since

- the first non-zero entry in each row is a 1,
- the entries above and below these 1's are zero,
- the leading 1's move progressively to the right as we step down though the rows of the matrix.

Aside: If the matrix above represents a linear system, what might this last row tell us about the solution of that system?

Gauss-Jordan Elimination in Operation V

Using Gauss-Jordan Algorithm to find solutions to Linear Systems :

Put in matrix form, to solve the system $A\mathbf{x} = \mathbf{b}$:

1. Create the augmented matrix $(A\mathbf{b})$,
2. Compute its reduced row echelon form B,
3. We can break B up, separating off the last column $B = (B'\mathbf{b}')$. The original linear system is equivalent to the following set of equations:

$$B'\mathbf{x} = \mathbf{b}'$$

Gauss-Jordan Elimination in Operation VI

Let's go back to the matrix we've just worked with

$$A = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \\ 2 & 2 & -2 & 5 \end{pmatrix}$$

supposing it represents the system of equations

$$\begin{cases} 4x_3 = 0 \\ 5x_1 + 5x_2 - x_3 = 5 \\ 2x_1 + 2x_2 - 2x_3 = 5 \end{cases}$$

We found the reduced row echelon form of the matrix to be:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Gauss-Jordan Elimination in Operation VII

this can be rewritten as the equations:

$$\begin{cases} x_1 + x_2 = 0 \\ x_3 = 0 \\ 0 = 1 \end{cases}$$

The solution (or lack thereof) of this system of equations is the same as that of the original equations.
The last equation shows explicitly that this system is inconsistent i.e. that the equations have no solution as clearly we can't have 0=1.

Gauss-Jordan Elimination in Operation VIII

Historical context:

- The method of Gaussian elimination is mentioned in the Chinese mathematical text “Jiuzhang suanshu“ or “The Nine Chapters on the Mathematical Art“. It dates to between 150BC and 179AD.
- In Europe it appears in the notes of Isaac Newton(1642 -1727) in 1670, where he commented that all the algebra books he knew of lacked a lesson for solving simultaneous equations – which he then supplied.
- Cambridge University published his notes as “Arithmetica Universalis” in 1707.
- By the end of the 18th century the method was a standard lesson in algebra textbooks.
- In 1810 Carl Friedrich Gauss devised a notation for symmetric elimination that was adopted in the 19th century by professional hand computers to solve the normal equations of least-squares problems (a methodology used for curve fitting to a set of points) .
- The algorithm we know was named after Gauss only in the 1950s as a result of confusion over the history of the subject

Linear Systems with an Infinite Number of Solutions I

The examples we've considered so far all have **unique** solutions or no solution. Can the procedure we have developed account for the possibility of **infinitely many** solutions?

Yes! Modifying the **Back Substitution** stage slightly shows how infinitely many solutions can arise **AND** determines structure of the solutions.

Let's see how it's done.....

Linear Systems with an Infinite Number of Solutions II

EXAMPLE 1: Find all solutions of the linear system

$$\begin{aligned} 2x_1 + 2x_2 \quad & -6x_3 \quad & +6x_5 &= 2 \\ 4x_1 + 4x_2 \quad & -11x_3 + 2x_4 \quad & +11x_5 &= 6 \\ x_1 + x_2 \quad & +x_3 + 8x_4 \quad & +x_5 &= 11 \\ -3x_1 - 3x_2 \quad & +11x_3 + 4x_4 \quad & &= 12. \end{aligned}$$

(I know this is written a bit oddly, but I was trying to get all the terms involving x_1 alligned, all the terms involving x_2 alligned etc – that way we can form the augmented matrix more easily)

SOLUTION: The augmented matrix of this linear system is

$$\begin{pmatrix} 2 & 2 & -6 & 0 & 6 & 2 \\ 4 & 4 & -11 & 2 & 11 & 6 \\ 1 & 1 & 1 & 8 & 1 & 11 \\ -3 & -3 & 11 & 4 & 0 & 12 \end{pmatrix}.$$

Linear Systems with an Infinite Number of Solutions III

Converting this to row echelon form:

$$\begin{array}{cl} \begin{pmatrix} 2 & 2 & -6 & 0 & 6 & 2 \\ 4 & 4 & -11 & 2 & 11 & 6 \\ 1 & 1 & 1 & 8 & 1 & 11 \\ -3 & -3 & 11 & 4 & 0 & 12 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \\ \text{R4} \end{array} \end{array}$$

$$\underset{\longrightarrow}{R1 \to 1/2 \times R1} \quad \begin{array}{cl} \begin{pmatrix} 1 & 1 & -3 & 0 & 3 & 1 \\ 4 & 4 & -11 & 2 & 11 & 6 \\ 1 & 1 & 1 & 8 & 1 & 11 \\ -3 & -3 & 11 & 4 & 0 & 12 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \\ \text{R4} \end{array} \end{array}$$

$$\underset{\longrightarrow}{\begin{array}{c} R2 \to R2 - 4 \times R1 \\ R3 \to R3 - R1 \\ R4 \to R4 + 3 \times R1 \end{array}} \quad \begin{array}{cl} \begin{pmatrix} 1 & 1 & -3 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 4 & 8 & -2 & 10 \\ 0 & 0 & 2 & 4 & 9 & 15 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \\ \text{R4} \end{array} \end{array}$$

Linear Systems with an Infinite Number of Solutions IV

$$\xrightarrow[\substack{R3 \to R3 - 4 \times R1 \\ R4 \to R4 - 2 \times R2}]{R1 \to R1 + 3R2} \begin{array}{cc} \begin{pmatrix} 1 & 1 & 0 & 6 & 0 & 7 \\ 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 11 & 11 \end{pmatrix} & \begin{array}{c} \text{R1} \\ \text{R2} \\ \text{R3} \\ \text{R4} \end{array} \end{array}$$

$$\xrightarrow[R3 \to 1/2 \times R3]{} \begin{array}{cc} \begin{pmatrix} 1 & 1 & 0 & 6 & 0 & 7 \\ 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 11 & 11 \end{pmatrix} & \begin{array}{c} \text{R1} \\ \text{R2} \\ \text{R3} \\ \text{R4} \end{array} \end{array}$$

$$\xrightarrow[R4 \to R4 - 11 \times R3]{R2 + R3} \begin{array}{cc} \begin{pmatrix} 1 & 1 & 0 & 6 & 0 & 7 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} \text{R1} \\ \text{R2} \\ \text{R3} \\ \text{R4} \end{array} \end{array}$$

The final matrix is in **reduced row echelon form**.

Linear Systems with an Infinite Number of Solutions V

- The linear system corresponding to this matrix is

$$\begin{aligned} x_1 + x_2 - 0x_3 + \qquad 6x_4 &= 7 \qquad \text{(E1)} \\ x_3 + 2x_4 &= 3 \qquad \text{(E2)} \\ x_5 &= 1 \qquad \text{(E3)} \end{aligned}$$

- **N.B.**: The **last row** of the row echelon matrix has equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 0 \quad \text{i.e.} 0 = 0,$$

  which imparts no additional information.
- Solve from bottom up: First, (E3) gives $x_5 = 1$.

Linear Systems with an Infinite Number of Solutions VI

A DIFFICULTY: Next variable is x_4. But (E2) is written so as to determine x_3 given x_4 and x_5. Moreover, there is **no equation** involving x_4 **in terms of** x_5, **NOR** one for x_2 **in terms of** x_3, x_4, x_5.

SOURCE OF OUR DIFFICULTY: We need to solve for **five unknowns**

$$x_1, x_2, \ldots x_5$$

in terms of **three** pieces of relevant information (**equations** (E1)–(E3))

However, we cannot determine $x_1, x_2, \ldots x_5$ **UNIQUELY**.

Linear Systems with an Infinite Number of Solutions VII

RESOLUTION: Assigning **values** to x_4, x_2 determines x_1, x_3, x_5. Values can be assigned in **infinitely many ways**. So there are infinitely many solutions to this linear system.

The values assigned to x_4, x_2 are called **PARAMETERS**. For x_4 we express this mathematically by letting

$$x_4 = t, \qquad t \in \mathbb{R}$$

where t is the **parameter**. (We are assuming that all our equations involve real numbers, so we note that $t \in \mathbb{R}$).

- Substitute expressions for x_4, x_5 into (E2) to obtain x_3:

$$x_3 = 3 - 2t.$$

- There is **no equation** for x_2 in terms of x_3, x_4, x_5, so we **introduce** another parameter for x_2. Mathematically, we let

$$x_2 = s, \qquad s \in \mathbb{R}$$

Linear Systems with an Infinite Number of Solutions VIII

- Substituting expressions for $x_2, \ldots x_5$ into (E1), we get x_1:

$$x_1 = 7 - 6x_4 - x_2 = 7 - 6t - s.$$

ANSWER: $x_1 = 7 - 6t - s$, $x_2 = s$, $x_3 = 3 - 2t$, $x_4 = t$, $x_5 = 1$ for any $s, t \in \mathbb{R}$.

Linear Systems with an Infinite Number of Solutions IX

Remarks:

- As there are **infinitely many** choices for s, t, so there are **infinitely many** solutions to system.
- Once s, t are **fixed**, the solution is determined. An **arbitrary** set of 5 values is **NOT** a solution: solutions have a particular form.

Introducing Parameters:

- Introduce parameters whenever the **reduced row echelon form** has **fewer equations than unknowns**.
- Introduce a **new parameter** for each **variable** that **does NOT appear** as the **first unknown** in an equation e.g., x_2, x_4 above.
- When we introduction of one or more parameters, it automatically means we have **infinitely many** solutions.

Linear Systems with No Solutions I

- Linear systems which have **NO SOLUTION** (called **INCONSISTENT** linear systems) can be detected by Gaussian Elimination by making some minor changes to the **Back Substitution** stage.

- Consider the linear system

$$\begin{aligned} 2x_1 + x_2 &= 1 \\ 4x_1 + 2x_2 &= 0. \end{aligned}$$

- This can be re-written as

$$\begin{aligned} x_1 + \frac{1}{2}x_2 &= \frac{1}{2} \\ x_1 + \frac{1}{2}x_2 &= 0, \end{aligned}$$

so clearly no pair of numbers x_1, x_2 can satisfy these equations.

Linear Systems with No Solutions II

- Subtracting the first equation from the second gives

$$\begin{aligned} x_1 + 1/2 \quad x_2 &= 1/2 \\ 0 &= -1/2 \end{aligned}$$

- Thus the **signature** of an **INCONSISTENT** linear system is a **contradictory** expression such as $0 = -1/2$.

Detecting Inconsistent Linear Systems with Gaussian Elimination

- First simplify the augmented matrix of the linear system using **elementary row operations**.
- If one of the **equations** of the linear system associated with the **simplified matrix** is **contradictory** then the system has **NO** solution.

Linear Systems with No Solutions III

- Using Gaussian Elimination, we show that the above example has no solution: performing row operations gives

$$\begin{array}{cl} \begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 0 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \end{array} \end{array}$$

$$\xrightarrow{R1 \to 1/2 \times R1} \quad \begin{array}{cl} \begin{pmatrix} 1 & 1/2 & 1/2 \\ 4 & 2 & 0 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \end{array} \end{array}$$

$$\xrightarrow{R2 \to R2 - 4 \times R1} \quad \begin{array}{cl} \begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 0 & -2 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \end{array} \end{array}$$

- Last row of final matrix tells us:
  $0x_1 + 0x_2 = -2$, or $0 = -2$, a **contradiction**, so the system has **no solution**.

A More Challenging Example I

Before we move on from using Gaussian elimination, let's have a look at a harder example:

EXAMPLE: For which values of the constant a will the following linear system have:

(i) No solutions

(ii) Infinitely many solutions

(iii) A unique solution

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 4 \\ 5x_1 + 3x_2 - x_3 &= 10 \\ 9x_1 + 4x_2 + (a^2 - 15)x_3 &= a + 12. \end{aligned}$$

A More Challenging Example II

SOLUTION:

- Simplify the augmented matrix using elementary row operations:

$$
\left(\begin{array}{cccc} 1 & 2 & -3 & 4 \\ 5 & 3 & -1 & 10 \\ 9 & 4 & a^2-15 & a+12 \end{array}\right)
\begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array}
$$

$$
\xrightarrow[R3 \to R3 - 9 \times R1]{R2 \to R2 - 5 \times R1}
\left(\begin{array}{cccc} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -14 & a^2+12 & a-24 \end{array}\right)
\begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array}
$$

$$
\xrightarrow{R2 \to R2 - 1/7 \times R1}
\left(\begin{array}{cccc} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & -14 & a^2+12 & a-24 \end{array}\right)
\begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array}
$$

$$
\xrightarrow[R3 \to R3 + 14 \times R2]{R1 \to R1 - 2R2}
\left(\begin{array}{cccc} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{array}\right)
\begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array}
$$

A More Challenging Example III

- Rewrite as a linear system.

$$
\begin{aligned}
x_1 - x_3 &= \frac{8}{7} && (1)\\
x_2 - 2x_3 &= \frac{10}{7} && (2)\\
(a^2 - 16)x_3 &= a - 4 && (3)
\end{aligned}
$$

(i) No solution:

- The only possible inconsistency is in (3) (a doesn't appear elsewhere).
- For the system to be **inconsistent**, (3) must have form

$$0x_3 = c \quad \text{where} \quad c \neq 0,$$

so that (3) reads $0 = c$ (**contradictory**).

A More Challenging Example IV

So there are no solutions if:

$$a^2 - 16 = 0 \quad \text{and} \quad a - 4 \neq 0,$$

Solving $a^2 - 16 = 0$ gives $a = +4$ or $a = -4$.
However $a - 4 \neq 0$ tells us $a \neq 4$.
Putting these two pieces of information together tells us that the only acceptable value is $a = -4$.

- **ANSWER**: If $a = -4$, there are **no solutions**.

A More Challenging Example V

(ii) Infinitely many solutions:

- There are infinitely many solutions when we find ourselves **introducing parameters**.
- Here this can only happen if a parameter is introduced for x_3, so that there is **no equation** in which x_3 is the **first unknown**
  So (3) must read $0x_3 = 0$.
  This means there are infinitely many solutions if $a^2 - 16 = 0$ and $a - 4 = 0$. Using similar calculations and reasoning to (i) above, we find that $a = 4$.
- **ANSWER**: If $a = 4$, there are **infinitely many solutions**.

A More Challenging Example VI

(iii) Unique solution:

- If $a^2 - 16 \neq 0$, we can divide (3) by $a^2 - 16$ so (3) now reads

$$x_3 = \frac{a-4}{a^2-16}.$$

- We can substitute this expression into (2) to solve for x_2, and then substitute the expressions for x_2, x_3 into (1) to obtain x_1
  So there is a unique solution if $a^2 - 16 \neq 0$.
- **ANSWER**: If $a \neq 4$ or $a \neq -4$, there is a **unique solution**.