Mathematics CS1001

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Integration by Substitution I

GUIDELINES

- **Q:** What is the best choice for u?
- Let $u =$ the "most complicated expression", particularly if that expression is the **argument** of **another function**. This is the **same choice** of u that one would use if **differentiating** using the **Chain Rule** (e.g., choosing $u = x^3$ above).
- Let $u = f(x)$ if $f'(x)\,dx$ is also present under the integral.
- If the **INTEGRAND** (function to be differentiated) is a **quotient**, let $u =$**denominator** (the bottom).
- These hints may be contradictory, and they are not foolproof! However, they are sufficient to tackle a large class of integrals, and with enough practice, one develops a feel for the correct strategy.

EXAMPLES

(1.) $$\int \frac{x^4}{x^5+2}\,dx.$$

SOLUTION: Quotient: let

$$u = x^5 + 2, \quad \text{so} \quad \frac{du}{dx} = 5x^4, \text{hence } \; du = 5x^4\,dx.$$

Replacing the x's by u's:

$$\begin{aligned}
\int \frac{x^4}{x^5+2}\,dx &= \int \frac{x^4\,dx}{x^5+2} \\
&= \int \frac{1/5\,du}{u} = \frac{1}{5}\int u^{-1}\,du \\
&= \frac{1}{5}\log u = \frac{1}{5}\log(x^5+2).
\end{aligned}$$

Always write the final answer in terms of the **original variable** (in this case x).

Integration by Substitution III

(2.) $\displaystyle\int x^3\sqrt{16+x^4}\,dx.$

SOLUTION: The expression which is the **argument** of the square root is a good candidate for **substitution**: let

$$u = 16 + x^4, \quad \text{so} \quad \frac{du}{dx} = 4x^3 \text{ hence } du = 4x^3\,dx.$$

Look to arrange $4x^3\,dx(=du)$ in one group in the integral:

$$\begin{aligned}\int x^3\sqrt{16+x^4}\,dx &= \int \sqrt{16+x^4}.x^3\,dx \\ &= \int \sqrt{u}.\frac{1}{4}\,du = \frac{1}{4}\int u^{1/2}\,du \\ &= \frac{1}{4}\frac{u^{1+1/2}}{1+1/2} = \frac{u^{3/2}}{6} = \frac{(16+x^4)^{3/2}}{6}.\end{aligned}$$

A check: The **derivative** of the **final answer** should equal the **integrand**.

Integration by Substitution IV

(3.) $\displaystyle\int \sin 6x\,dx.$

<u>SOLUTION:</u> Let $u = 6x$, the **argument** of $\sin$:

$$\frac{du}{dx} = 6 \quad \text{hence} \quad du = 6\,dx.$$

$$\begin{aligned}
\int \sin 6x\,dx &= \int \sin u \cdot \frac{1}{6}\,du = \frac{1}{6}\int \sin u\,du \\
&= \frac{1}{6} \cdot -\cos u = -\frac{1}{6}\cos 6x.
\end{aligned}$$

Integration by Substitution V

$$\textbf{(4.)} \quad \int \frac{\log x}{x}\, dx.$$

<u>SOLUTION:</u> If $f(x) = \log x$, then $f'(x)\, dx = \frac{1}{x}\, dx$ is present in the integrand. Thus put

$$u = \log x, \qquad \frac{du}{dx} = \frac{1}{x}, \qquad du = \frac{dx}{x}.$$

$$\int \frac{\log x}{x}\, dx = \int \log x \cdot \frac{dx}{x} = \int u\, du = \frac{u^2}{2} = \frac{1}{2}(\log x)^2.$$

Integration by Substitution VI

(5.) $\displaystyle \int \frac{\sin x}{2+\cos x}\,dx.$

<u>SOLUTION:</u> Quotient: put $u = 2+\cos x$, so $du = -\sin x\,dx$. Therefore, we have

$$\begin{aligned}\int \frac{\sin x}{2+\cos x}\,dx &= \int \frac{\sin x\,dx}{2+\cos x} = \int \frac{-du}{u} \\ &= -\log u = -\log(2+\cos x).\end{aligned}$$

Integration by Substitution VII

$$\textbf{(6.)} \quad \int x^3 (x^4+4)^5 \, dx.$$

<u>SOLUTION:</u> Put $u = x^4 + 4$, the **argument** of the fifth power: Then

$$\frac{du}{dx} = 4x^3 \quad \text{hence} \quad du = 4x^3 \, dx.$$

Group $4x^3 \, dx (= du)$ together and substitute:

$$\begin{aligned} \int x^3 (x^4+4)^5 \, dx &= \int (x^4+4)^5 . x^3 \, dx \\ &= \int u^5 . \frac{1}{4} \, du = \frac{1}{4} \int u^5 \, du = \frac{1}{4} \frac{u^6}{6} = \frac{(x^4+4)^6}{24}. \end{aligned}$$

Integration by Substitution VIII

$$\textbf{(7.)} \quad \int \sin x \cos^3 x \, dx.$$

<u>SOLUTION:</u> Put $u = \cos x$, the **argument** of the cube: Then

$$\frac{du}{dx} = -\sin x \quad \text{hence} \quad du = -\sin x \, dx.$$

Group $-\sin x \, dx (= du)$ together and substitute:

$$\begin{aligned} \int \sin x \cos^3 x \, dx &= \int \cos^3 x . \sin x \, dx = \int u^3 . -du \\ &= -\int u^3 \, du = -\frac{u^4}{4} = -\frac{\cos^4 x}{4}. \end{aligned}$$

Integration by Substitution IX

Sometimes, our **substitution**

$$u = f(x)$$

does not entirely eliminate the original variable in the integral, and in these cases, we must **find x in terms of u**

$$\textbf{(8.)} \quad \int \frac{x}{\sqrt{2x+2}}\, dx.$$

<u>SOLUTION:</u> Put $u = 2x + 2$, the **argument** of the square root. Thus

$$\frac{du}{dx} = 2 \quad \text{hence} \quad du = 2\, dx.$$

Now, we replace x (and dx) by u (and du):

$$\int \frac{x}{\sqrt{2x+2}}\, dx = \int \frac{x}{\sqrt{u}}\, \frac{1}{2}\, du = \frac{1}{2} \int \frac{x}{\sqrt{u}}\, du.$$

Integration by Substitution X

We now replace x by expressing it in terms of u: $x = \frac{u-2}{2}$, so

$$\begin{aligned}
\int \frac{x}{\sqrt{2x+2}}\,dx &= \frac{1}{2}\int \frac{1}{2}\,\frac{u-2}{\sqrt{u}}\,du \\
&= \frac{1}{4}\int \left(\sqrt{u} - \frac{2}{\sqrt{u}}\right) du = \frac{1}{4}\left(\frac{u^{3/2}}{3/2} - 2\frac{u^{1/2}}{1/2}\right) \\
&= \frac{1}{6}u^{3/2} - u^{1/2} = \frac{1}{6}(2x+2)^{3/2} - (2x+2)^{1/2}.
\end{aligned}$$

Integration by Substitution XI

$$\textbf{(9.)} \quad \int \frac{\tan^5 x}{\cos^2 x}\, dx.$$

<u>SOLUTION:</u> Notice that

$$\frac{d}{dx}\tan x = \sec^2 x, \quad \text{so put } u = \tan x \text{ to obtain}$$

$$du = \sec^2 x\, dx = \frac{dx}{\cos^2 x}.$$

$$\int \frac{\tan^5 x}{\cos^2 x}\, dx = \int u^5\, du = \frac{u^6}{6} = \frac{\tan^6 x}{6}.$$

Integration by Substitution XII

Sometimes, our strategy requires some persistence in **trying different substitutions:**

$$\textbf{(10.)} \quad \int \cot x \log(\sin x)\, dx.$$

<u>SOLUTION:</u> Ultimately, we try

$$u = \log(\sin x) \quad \text{hence} \quad \frac{du}{dx} = \frac{1}{\sin x}\cos x = \cot x,$$

so $du = \cot x\, dx$, which gives

$$\int \cot x \log(\sin x)\, dx = \int u\, du = \frac{u^2}{2} = \frac{1}{2}\log^2(\sin x).$$

Integration by Substitution XIII
HARDER SUBSTITUTIONS

TYPE 1: Substituting $u = f(x)$ does not entirely eliminate the original variable (x).

Write x in terms of u, i.e., $x = f^{-1}(u)$.

DETAILS

1: Put $u =$"**most complicated bit**" (same substitution as if differentiating using the **Chain Rule**), and find dx in terms of x and du.

2: Substitute u for the "complicated expression" and replace dx in the integral. **Rewrite** any x's left in the integral **in terms of** u.

3: Calculate the integral with respect to the new variable u.

- **REMEMBER:** If there is a du at the end of the integral, **everything** under the integral must be written in terms of u.

Integration by Substitution XIV

EXAMPLE 1: Find

$$\int x^5\sqrt{1+x^3}\,dx.$$

SOLUTION:

STEP 1: Let $u = 1 + x^3$. Then

$$\frac{du}{dx} = 3x^2 \quad \text{so} \quad du = 3x^2\,dx \quad \text{hence} \quad dx = \frac{du}{3x^2}.$$

STEP 2:

$$\int x^5\sqrt{1+x^3}\,dx = \int x^5\sqrt{u}\,\frac{du}{3x^2} = \frac{1}{3}\int u^{\frac{1}{2}}x^3\,du.$$

Now write x^3 in terms of u: $\quad x^3 = u - 1$

STEP 3:

$$\int x^5\sqrt{1+x^3}\,dx = \frac{1}{3}\int \sqrt{u}\,\underbrace{(u-1)}_{=x^3}\,du$$

$$= \frac{1}{3}\int u^{3/2} - u^{1/2}\,du = \frac{1}{3}\left(\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right)$$

$$= \frac{2u^{\frac{5}{2}}}{15} - \frac{2u^{\frac{3}{2}}}{9} = \frac{2}{15}(1+x^3)^{\frac{5}{2}} - \frac{2}{9}(1+x^3)^{\frac{3}{2}}.$$

EXAMPLE 2: Find

$$\int \frac{x^9 - x^4}{(x^5 - 2)^{3/2}}\, dx.$$

SOLUTION:

STEP 1: Let $u = x^5 - 2$. Then

$$\frac{du}{dx} = 5x^4 \quad \text{so} \quad du = 5x^4\, dx \quad \text{hence} \quad dx = \frac{du}{5x^4}.$$

STEP 2:

$$\int \frac{x^9 - x^4}{(x^5 - 2)^{3/2}}\, dx = \int \frac{x^9 - x^4}{u^{3/2}} \cdot \frac{du}{5x^4} = \frac{1}{5} \int \frac{x^5 - 1}{u^{3/2}}\, du.$$

Now write x^5 **in terms of** u: $\quad x^5 = u + 2$

STEP 3:

$$\begin{aligned}
\int \frac{x^9 - x^4}{(x^5 - 2)^{3/2}}\, dx &= \frac{1}{5} \int \frac{(u+2) - 1}{u^{3/2}}\, du \\
&= \frac{1}{5} \int u^{-3/2} + u^{-1/2}\, du = \frac{1}{5} \left(\frac{u^{-1/2}}{-1/2} + \frac{u^{1/2}}{1/2} \right) \\
&= -\frac{2}{5} u^{-\frac{1}{2}} + \frac{2}{5} u^{\frac{1}{2}} = \frac{-2}{5\sqrt{x^5 - 2}} + \frac{2}{5}(x^5 - 2)^{\frac{1}{2}}.
\end{aligned}$$

Integration by Substitution XVI

TYPE 2: If the integrand contains $\sqrt{a^2 - x^2}$, it is often worthwhile to

Let $x = a \sin u$. Then $\sqrt{a^2 - x^2} = a \cos u$.

DETAILS

1: Put $x = a \sin u$. Then

$$\begin{aligned} \sqrt{a^2 - x^2} &= \sqrt{a^2(1 - \sin^2 u)} \\ &= \sqrt{a^2 \cos^2 u} = a \cos u \end{aligned}$$

and $$\frac{dx}{du} = a \cos u, \quad \text{we have} \quad dx = a \cos u \, du.$$

2: Replace all the x's and dx and calculate the integral with respect to u.

3: At the end, write u **in terms of** x:

$$u = \sin^{-1} \frac{x}{a}.$$

Integration by Substitution XVII

EXAMPLE 3: Find

$$\int \frac{1}{\sqrt{9-x^2}}\,dx.$$

SOLUTION:

STEP 1: Let $x = 3\sin u$. Then $\sqrt{9-x^2} = 3\cos u$ and

$$\frac{dx}{du} = 3\cos u \quad \text{hence} \quad dx = 3\cos u\,du.$$

STEP 2:

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{1}{3\cos u} \cdot \underbrace{3\cos u\,du}_{=dx} = u.$$

STEP 3: **Write u in terms of x** . . .

$$x = 3\sin u, \quad \text{so} \quad \sin u = \frac{x}{3}, \quad \text{hence} \quad u = \sin^{-1}\frac{x}{3}$$

. . . and find the integral **in terms of x**

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \sin^{-1}\frac{x}{3}.$$

Integration by Substitution XVIII

EXAMPLE 4: Find

$$\int \sqrt{25-4x^2}\,dx.$$

SOLUTION: This time we don't quite have a term in the form $\sqrt{a^2-x^2}$. However, notice that

$$\sqrt{25-4x^2} = \sqrt{5^2-(2x)^2}.$$

STEP 1: IDEA: let $2x$ **play the rôle** of x and set $2x = 5\sin u$. Then we have

$$\begin{aligned}\sqrt{25-4x^2} &= \sqrt{5^2-(2x)^2}\\ &= \sqrt{5^2-(5\sin u)^2} = 5\cos u\end{aligned}$$

and since $x = 5/2\sin u$,

$$\frac{dx}{du} = \frac{5}{2}\cos u \quad \text{so} \quad dx = \frac{5}{2}\cos u\,du.$$

STEP 2:

$$\int \sqrt{25-4x^2}\,dx = \int 5\cos u \cdot \frac{5}{2}\cos u\,du = \frac{25}{2}\int \cos^2 u\,du.$$

Integration by Substitution XIX

STEP 3: To calculate this integral we have recourse to the **Tables Book** (p.42)

$$\int \cos^2 x\,dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right),$$

so

$$\int \sqrt{25-4x^2}\,dx = \frac{25}{2}\int \cos^2 u\,du$$

$$= \frac{25}{2}\left(\frac{1}{2}\left(u + \frac{1}{2}\sin 2u\right)\right) = \frac{25}{4}u + \frac{25}{8}\sin 2u.$$

Write u in terms of x: $x = 5/2\sin u$ implies

$$\sin u = \frac{2x}{5} \quad \text{and} \quad u = \sin^{-1}\frac{2x}{5}.$$

Next, we express $\sin 2u$ in terms of x: to do this, recall that

$$\begin{aligned}
\sin 2u &= 2\cos u \sin u && \ldots \textbf{Tables p.10} \\
&= 2\sqrt{1-\sin^2 u}\,\sin u && \ldots \cos^2 u = 1-\sin^2 u \\
&= 2\sqrt{1-\left(\frac{2x}{5}\right)^2}\cdot\frac{2x}{5}.
\end{aligned}$$

Integration by Substitution XX

Thus

$$\begin{aligned}
\int \sqrt{25-4x^2}\,dx &= \frac{25}{4}u + \frac{25}{8}\sin 2u \\
&= \frac{25}{4}\cdot\sin^{-1}\frac{2x}{5} + \frac{25}{8}\cdot 2\sqrt{1-\left(\frac{2x}{5}\right)^2}\cdot\frac{2x}{5} \\
&= \frac{25}{4}\sin^{-1}\frac{2x}{5} + \frac{5x}{2}\sqrt{1-\frac{4x^2}{25}}.
\end{aligned}$$

Integration by Substitution XXI

TYPE ∞: Neat tricks, particular to the given problem. The following is a good example, and a tough question.

EXAMPLE 5: Calculate

$$\int \frac{1}{x^{1/4} + x^{3/4}}\, dx.$$

SOLUTION: The **trick** here is to let $u = x^{1/4}$, so

$$x = u^4 \text{ and } \frac{dx}{du} = 4u^3 \quad \text{so} \quad dx = 4u^3\, du. \text{ Thus}$$

$$\int \frac{1}{x^{1/4} + x^{3/4}}\, dx = \int \frac{1}{u + u^3} \cdot 4u^3\, du = \int \frac{4u^2}{1 + u^2}\, du.$$

By **long division**, we obtain

$$\frac{u^2}{1 + u^2} = 1 - \frac{1}{1 + u^2}$$

so

$$\begin{aligned} \int \frac{1}{x^{1/4} + x^{3/4}}\, dx &= 4 \int 1 - \frac{1}{1 + u^2}\, du \\ &= 4u - 4 \int \frac{du}{1 + u^2}. \end{aligned}$$

Integration by Substitution XXII

The **Log tables** again come to our aid: on p.41 we have

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x,$$

so

$$\begin{aligned} \int \frac{1}{x^{1/4}+x^{3/4}}\,dx &= 4u - 4\tan^{-1} u \\ &= 4x^{1/4} - 4\tan^{-1} x^{1/4}. \end{aligned}$$