Mathematics CS1001

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Cramer's Rule I

Cramer's Rule uses determinants to solve a system of linear equations that have a solution. Such a system has a solution if the determinant of the coefficients is not zero.

Consider a system of n linear equations in n unknowns:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\vdots \\
&a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n
\end{aligned}$$

The determinant of the coefficients, D, is

$$\det(D) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Cramer's Rule II

If the determinant, D, of the coefficients of a system of n linear equations in n unknowns is not zero ($D \neq 0$) then the equations have an unique solution. Each unknown may be expressed as a fraction of 2 determinants, with denoniminator (the bottom) the determinant, D, and with numerator (the top) obtained from D by replacing the column of coefficients of the unknown in question by the constants, $b_1, b_2, \ldots, b_n$.

$$D_{x_1} = \begin{vmatrix} b_1 & a_{12} & \ldots & a_{1n} \\ b_2 & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_n & a_{n2} & \ldots & a_{nn} \end{vmatrix}, \ldots, D_{x_k} = \begin{array}{c} \begin{matrix} \phantom{a_{11}} & & k^{th} col & & \phantom{a_{1n}} \end{matrix} \\ \begin{vmatrix} a_{11} & \ldots & b_1 & \ldots & a_{1n} \\ a_{21} & \ldots & b_2 & \ldots & a_{2n} \\ \vdots & \ldots & \vdots & \vdots & \vdots \\ a_{n1} & \ldots & b_n & \ldots & a_{nn} \end{vmatrix} \end{array}$$

The unknown, x_k, can be expressed as

$$x_k = \frac{D_{x_k}}{D}$$

Cramer's Rule III

More formally:

Cramer's Theorem:

Given a system of linear equations represented in matrix form as $Ax = b$ where A is an invertible square matrix and the vector x is the column vector of the variables. Then

$$x_i = \frac{\det(A_i)}{\det(A)} \qquad i = 1, \ldots, n$$

where A_i is the matrix formed by replacing the i –th column of A by the column vector b.

Note: We don't prove this rule here – wikipedia has an easy to follow proof if you are interested.

Cramer's Rule IV

Example: Assume the three points $(-1, 8)$, $(2, -1)$, $(4, 3)$ lie on the quadratic curve $y = ax^2 + bx + c$, find the values of a, b, and c,

Since $(-1, 8)$ lies on the quadratic curve we have

$$8 = a(-1)^2 + b(-1) + c \qquad \text{or} \qquad a - b + c = 8$$

Similarly for point $(2, -1)$ we get

$$4a + 2b + c = -1$$

and for point $(4, 3)$ we get

$$16a + 4b + c = 3$$

Cramer's Rule V

From this system of linear equations we get the determinant of the coefficients:

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{vmatrix} = -30$$

We find a, b, and c, using Cramer's Rule:

$$a = \frac{\begin{vmatrix} \mathbf{8} & -1 & 1 \\ \mathbf{-1} & 2 & 1 \\ \mathbf{3} & 4 & 1 \end{vmatrix}}{-30} = \frac{-30}{-30} = 1, \qquad b = \frac{\begin{vmatrix} 1 & \mathbf{8} & 1 \\ 4 & \mathbf{-1} & 1 \\ 16 & \mathbf{3} & 1 \end{vmatrix}}{-30} = \frac{120}{-30} = -4$$

$$c = \frac{\begin{vmatrix} 1 & -1 & \mathbf{8} \\ 4 & 2 & \mathbf{-1} \\ 16 & 4 & \mathbf{3} \end{vmatrix}}{-30} = \frac{-90}{-30} = 3$$

The quadratic curve is $y = x^2 - 4x + 3$. (Check for yourself that the three points $(-1, 8)$, $(2, -1)$, $(4, 3)$ line on this curve).