Mathematics CS1001

Meriel Huggard

Room 110 Lloyd Institute
School of Computer Science and Statistics
Trinity College Dublin, IRELAND
http://mymodule.tcd.ie
Meriel.Huggard@tcd.ie

17 October 2012

Integration by Parts I

- We have seen how **integration by substitution** corresponds to the **chain rule** for differentiation.
- **REASONABLE Q:** Is there a rule of integration corresponding to the **product rule**?
- **A:** Yes! **INTEGRATION BY PARTS** is useful for calculating the **integrals of products**.

INTEGRATION BY PARTS FORMULA

$$\int u\,dv = uv - \int v\,du \qquad \textbf{(IBP)}$$

EXAMPLE 1: Find

$$\int xe^{2x}\,dx.$$

Integration by Parts II

SOLUTION:

- We will **choose** $u = x$, $dv = e^{2x}\,dx$ so that

$$\int xe^{2x}\,dx = \int u\,dv \quad \text{gives L.H.S of (IBP).}$$

- Find terms on R.H.S of (IBP). **Need:** du and v

$$u = x, \quad \text{so} \quad \frac{du}{dx} = 1, \quad \text{hence} \quad du = dx$$

DIFFERENTIATE u to GET du.

$$dv = e^{2x}\,dx, \quad \text{so} \quad v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}.$$

INTEGRATE dv to GET v.

- Substitute u, v, du, dv into (IBP):

$$\int xe^{2x}\,dx = \int u\,dv = uv - \int v\,du$$

$$= \quad x.\frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\cdot\frac{1}{2}e^{2x}.$$

Integration by Parts III

EXAMPLE 2: Find $\int x \cos x \, dx$.

SOLUTION:

- **Choose** $u = x$, $dv = \cos x \, dx$, so

$$\int x \cos x \, dx = \int u \, dv \quad \text{gives L.H.S of (IBP).}$$

- Find terms on R.H.S of (IBP). **Need:** du and v

$$u = x \quad \text{gives} \quad \frac{du}{dx} = 1, \quad \text{hence} \quad du = dx$$

$$dv = \cos x \, dx \quad \text{gives} \quad v = \int \cos x \, dx = \sin x.$$

- Substitute u, v, du, dv into (IBP):

$$\int x \cos x \, dx = \int u \, dv = uv - \int v \, du$$

$$= \quad x.\sin x - \int \sin x \, dx = x \sin x + \cos x.$$

EXERCISE: Check by differentiating the answer.

Integration by Parts IV

CHOOSING u AND dv

EXAMPLE 3: Find

$$\int x^2 \log x \, dx.$$

SOLUTION: In this question, we could choose

$$u = x^2 \quad \text{or} \quad u = \log x.$$

Which should one try?

$\underline{u = x^2}$: Then $dv = \log x \, dx$, so

$$v = \int \log x \, dx.$$

This isn't in the tables book. Abandon this line.

$\underline{u = \log x}$: Then $dv = x^2 \, dx$, so

$$v = \int x^2 \, dx = \frac{1}{3}x^3, \text{ and } \frac{du}{dx} = \frac{1}{x}, \text{ so } du = \frac{1}{x} \, dx.$$

Integration by Parts V

Substitute into the (IBP) formula:

$$\int x^2 \log x \, dx = \int u \, dv = uv - \int v \, du$$

$$= \log x \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cdot \frac{1}{x} \, dx$$

$$= \frac{1}{3}x^3 \log x - \frac{1}{3}\int x^2 \, dx = \frac{1}{3}x^3 \log x - \frac{1}{9}x^3.$$

EXERCISE: Check this by differentiating.

- **QUESTION:** How should u be chosen in general?

GUIDELINE: Choose u in the order

L – **l**ogarithm

A – **a**lgebraic: powers of x, e.g. x, x^3

T – **t**rigonometric

E – **e**xponential

Integration by Parts VI

The guideline has been used in all the **EXAMPLES**:

1. $\int xe^{2x}\,dx \quad x$ – **A**, e^{2x} – **E**

LAT**E**: $u = x$ (**A**), so $dv = e^{2x}\,dx$.

2. $\int x\cos x\,dx \quad x$ – **A**, $\cos x$ – **T**

LATE: $u = x$ (**A**), so $dv = \cos x\,dx$.

3. $\int x^2\log x\,dx \quad x^2$ – **A**, $\log x$ – **L**

LATE: $u = \log x$ (**L**), so $dv = x^2\,dx$.

This rule gives **very reliable** guidance, but occasionally doesn't work.

Integration by Parts VII

Sometimes we must integrate by parts **successively** to get an answer:

EXAMPLE 4: Find

$$\int x^2 \cos x \, dx.$$

SOLUTION:

- **LATE**: $u = x^2$ (**A**), so $dv = \cos x \, dx$ (**T**).
- **Differentiate** u, **integrate dv**:

$$\frac{du}{dx} = 2x, \quad v = \int \cos x \, dx = \sin x.$$

- **Substitute** u, v, du, dv into (IBP):

$$\begin{aligned} \int x^2 \cos x \, dx &= \int u \, dv = uv - \int v \, du \qquad (1) \\ &= x^2 . \sin x - \int \sin x . 2x \, dx = x^2 \sin x - 2 \int x \sin x \, dx. \end{aligned}$$

- Can't calculate the last integral directly: use IBP again.
- For

$$\int x \sin x \, dx \quad \textbf{LATE}: \quad u = x(\textbf{A}), \text{ so } dv = \sin x \, dx \, (\textbf{T}).$$

Integration by Parts VIII

- **Differentiate** u, **integrate** dv:

$$\frac{du}{dx} = 1, \quad v = \int \sin x \, dx = -\cos x.$$

- Substituting into (IBP):

$$\int x \sin x \, dx = -x \cos x - \int -\cos x \, dx = -x \cos x + \sin x.$$

- Replacing this in (1) gives

$$\begin{aligned} \int x^2 \cos x \, dx &= x^2 \sin x - 2(-x \cos x + \sin x) \\ &= x^2 \sin x + 2x \cos x - 2 \sin x. \end{aligned}$$

Integration by Parts IX

EXAMPLE 5: Find
$$\int e^{2x}\cos x\,dx.$$

SOLUTION:

• e^{2x} – **E**, $\cos x$ – **T**, so by LATE $u = \cos x$, so $dv = e^{2x}\,dx$. Thus
$$\frac{du}{dx} = -\sin x, \quad v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}.$$

• **Substitute** u, v, du, dv into (IBP):
$$\begin{aligned}\int e^{2x}\cos x\,dx = \int u\,dv &= uv - \int v\,du \\ &= \cos x.\frac{1}{2}e^{2z} - \int \frac{1}{2}e^{2x}. - \sin x\,dx \\ &= \frac{1}{2}e^{2x}\cos x + \frac{1}{2}\int e^{2x}\sin x\,dx. \quad (2)\end{aligned}$$

• No improvement: the last integral looks as hard as the first one. Try **integrating by parts** again, and see what happens.

Integration by Parts X

- Use LATE again: $u = \sin x$, $dv = e^{2x}\,dx$

$$\frac{du}{dx} = \cos x, \quad v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}.$$

- Using (IBP) gives

$$\begin{aligned} \int e^{2x}\sin x\,dx &= \sin x\frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\cos x\,dx \\ &= \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x\,dx. \quad (*) \end{aligned}$$

Call the desired integral I, and **substitute** back into (2) . . .

$$I = \frac{1}{2}e^{2x}\cos x + \frac{1}{2}\underbrace{\left(\frac{1}{2}e^{2x}\sin x - \frac{1}{2}I\right)}_{=(*)}$$

. . . and now tidy up and solve for I:

$$I = \frac{1}{2}e^{2x}\cos x + \frac{1}{4}e^{2x}\sin x - \frac{1}{4}I$$

$$I = \frac{2}{5}e^{2x}\cos x + \frac{1}{5}e^{2x}\sin x.$$

EXERCISE: Check this by differentiating.