Mathematics CS1001

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Integration using Partial Fractions I

- A **RATIONAL FUNCTION** is just a ratio of two polynomials:

$$\text{e.g.,}\quad \frac{x(x+2)}{(x-1)^2(x+1)}.$$

- To integrate such functions, we first **decompose** the whole fractional term into two **partial fractions**. These fractions will be simpler rational functions which are easier to integrate.
- We illustrate the method by an example.

EXAMPLE 1: Find

$$\int \frac{x(x+2)}{(x-1)^2(x+1)}\,dx.$$

Integration using Partial Fractions II

PARTIAL FRACTION DECOMPOSITION

- We first must decompose
$$\frac{x(x+2)}{(x-1)^2(x+1)}.$$
- The **two** distinct roots in the denominator ($x = 1$, $x = -1$), result in **two** partial fractions.
- **Denominator of P.F.** is a **factor** of the **denominator** which has the same roots. Here, partial fractions have denom. $x+1$ and $(x-1)^2$.
- **Numerator of P.F.** is an expression in x which has highest power in x **one less** than that in the denominator. Thus the partial fractions are
$$\frac{A}{x+1}, \quad \frac{Bx+C}{(x-1)^2}$$
where A, B, C are undetermined constants.

Integration using Partial Fractions III

- Therefore, for some A, B, C we have

$$\frac{x(x+2)}{(x-1)^2(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{(x-1)^2}. \tag{1}$$

- **PROBLEM:** How do we find A, B, C?
- **STEP 1:** Take the R.H.S. of (1) and bring it to a **common denominator**. By **collecting like powers of x**, simplify the expression until it has the same form as the L.H.S.

$$\begin{aligned}
\frac{A}{x+1} + \frac{Bx+C}{(x-1)^2} &= \frac{A(x-1)^2 + (Bx+C)(x+1)}{(x+1)(x-1)^2} \\
&= \frac{A(x^2-2x+1) + (Bx^2+Bx+Cx+C)}{(x+1)(x-1)^2} \\
&= \frac{(A+B)x^2 + (-2A+B+C)x + (A+C)}{(x+1)(x-1)^2}.
\end{aligned}$$

The L.H.S. is given by

$$\frac{x(x+2)}{(x-1)^2(x+1)} = \frac{x^2+2x}{(x-1)^2(x+1)}.$$

Integration using Partial Fractions IV

• **STEP 2:** Both expressions have the same denominator, so they must have the same numerator . . .

$$\underbrace{x^2 + 2x}_{\text{numerator LHS}} = \underbrace{(A+B)x^2 + (-2A+B+C)x + (A+C)}_{\text{numerator RHS}}$$

. . . so **the coefficients of like powers of x in the numerators are equal**:

$$\begin{aligned} x^2: &\quad 1 = A + B && \text{(I)} \\ x: &\quad 2 = -2A + B + C && \text{(II)} \\ \text{const.}: &\quad 0 = A + C && \text{(III)} \end{aligned}$$

• **STEP 3:** Solve these equations by inspection:

From (III), $C = -A$, and from (I) $B = 1 - A$. Substituting these expressions into (II) gives

$$2 = -2A + (1 - A) + -A \quad \text{hence} \quad 2 = -4A + 1.$$

Thus $A = -1/4$. Putting this result into (I), (III) gives $B = 1 - A = 5/4$, $C = -A = 1/4$.

Integration using Partial Fractions V

- Therefore the decomposition is

$$\frac{x(x+2)}{(x-1)^2(x+1)} = -\frac{1}{4}\frac{1}{x+1} + \frac{1}{4}\frac{5x+1}{(x-1)^2}.$$

- Now we can integrate to get:

$$\int \frac{x(x+2)\,dx}{(x-1)^2(x+1)} = -\frac{1}{4}\int \frac{1}{x+1}\,dx + \frac{1}{4}\int \frac{5x+1}{(x-1)^2}\,dx. \tag{2}$$

- **1st integral in (2):**

$$-\frac{1}{4}\int \frac{1}{x+1}\,dx$$

Let $u = x+1$, so $du = dx$ and thus

$$-\frac{1}{4}\int \frac{1}{x+1}\,dx = -\frac{1}{4}\int \frac{1}{u}\,du = -\frac{1}{4}\log(x+1).$$

- **2nd integral in (2):**

$$\frac{1}{4}\int \frac{5x+1}{(x-1)^2}\,dx$$

Integration using Partial Fractions VI

(*) Let $u = x - 1$, so $du = dx$, and thus

$$\frac{1}{4}\int \frac{5x+1}{(x-1)^2}\,dx = \frac{1}{4}\int \frac{5x+1}{u^2}\,du.$$

(*) Now write x **in terms of** u, replace x in the integral, and then integrate: we have $u = x - 1$, so $x = u + 1$, and thus

$$\begin{aligned}\frac{1}{4}\int \frac{5x+1}{(x-1)^2}\,dx &= \frac{1}{4}\int \frac{5(u+1)+1}{u^2}\,du = \frac{1}{4}\int \frac{5u+6}{u^2}\,du \\ &= \frac{5}{4}\int \frac{1}{u}\,du + \frac{6}{4}\int u^{-2}\,du = \frac{5}{4}\log u + \frac{6}{4}\cdot\frac{u^{-1}}{-1} \\ &= \frac{5}{4}\log u - \frac{6}{4u} = \frac{5}{4}\log(x-1) - \frac{6}{4(x-1)}\end{aligned}$$

- Substituting this into (2) gives

$$\int \frac{x(x+2)\,dx}{(x-1)^2(x+1)} = \frac{5}{4}\log(x-1) - \frac{6}{4(x-1)} - \frac{1}{4}\log(x+1).$$

Integration using Partial Fractions VII

EXAMPLE 2: Find

$$\int \frac{x^2+14}{(x+4)^2(x-2)}\,dx.$$

SOLUTION:

- **STEP 0:** Decomposition is

$$\frac{x^2+14}{(x+4)^2(x-2)} = \frac{A}{x-2} + \frac{Bx+C}{(x+4)^2}.$$

- **STEP 1:** Bring the RHS to a **common denominator** and gather like powers of x together in the numerator:

$$\begin{aligned}
\frac{x^2+14}{(x+4)^2(x-2)} &= \frac{A}{x-2} + \frac{Bx+C}{(x+4)^2} \\
&= \frac{A(x+4)^2 + (Bx+C)(x-2)}{(x-2)(x+4)^2} \\
&= \frac{A(x^2+8x+16) + Bx^2 - 2Bx + Cx - 2C}{(x-2)(x+4)^2} \\
&= \frac{(A+B)x^2 + (8A-2B+C)x + (16A-2C)}{(x-2)(x+4)^2}.
\end{aligned}$$

Integration using Partial Fractions VIII

• **STEP 2:** Now the **coefficients of like powers of x in the numerator of the left and right hand side are equal**, so that

$$\begin{aligned} x^2: &\quad 1 = A + B && \text{(I)} \\ x: &\quad 0 = 8A - 2B + C && \text{(II)} \\ \text{const.}: &\quad 14 = 16A - 2C && \text{(III)} \end{aligned}$$

• **STEP 3:** Solve these equations by inspection:

From (III), $2C = 16A - 14$, so that $C = 8A - 7$. Using (I), we have $B = 1 - A$. Substituting these expressions into (II) gives

$$0 = 8A - 2(1 - A) + 8A - 7 \quad \text{so } 0 = 18A - 9.$$

Thus $A = 1/2$. Putting this result into (I), (III) gives $B = 1 - A = 1/2$ and $C = 8A - 7 = -3$.

• Integrating this decomposition gives

$$\int \frac{x^2 + 14}{(x+4)^2(x-2)}\,dx = \int \frac{1/2}{x-2}\,dx + \int \frac{\frac{1}{2}x - 3}{(x+4)^2}\,dx. \tag{3}$$

Integration using Partial Fractions IX

- **1st integral in (3):**

$$\frac{1}{2}\int \frac{1}{x-2}\,dx$$

Let $u = x - 2$, so $du = dx$ and thus

$$\frac{1}{2}\int \frac{1}{x-2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\log u = \frac{1}{2}\log(x-2).$$

- **2nd integral in (3):**

$$\int \frac{\frac{1}{2}x-3}{(x+4)^2}\,dx$$

(*) Let $u = x + 4$, so $du = dx$, and thus

$$\int \frac{\frac{1}{2}x-3}{(x+4)^2}\,dx = \int \frac{\frac{1}{2}x-3}{u^2}\,du.$$

(*) Now write x **in terms of** u and replace x in the integral: we have $u = x + 4$, so $x = u - 4$.

$$\int \frac{\frac{1}{2}x-3}{(x+4)^2}\,dx = \int \frac{\frac{1}{2}(u-4)-3}{u^2}\,du.$$

Integration using Partial Fractions X

(*) Integrate

$$\begin{aligned}
\int \frac{\frac{1}{2}x-3}{(x+4)^2}\,dx &= \int \frac{\frac{1}{2}u-5}{u^2}\,du. \\
&= \frac{1}{2}\int \frac{1}{u}\,du - 5\int u^{-2}\,du = \frac{1}{2}\log u - 5\frac{u^{-1}}{-1} \\
&= \frac{1}{2}\log u + 5\frac{1}{u} = \frac{1}{2}\log(x+4) + \frac{5}{x+4}.
\end{aligned}$$

- Substituting this into (3) gives

$$\int \frac{x^2+14}{(x+4)^2(x-2)}\,dx = \frac{1}{2}\log(x-2) + \frac{1}{2}\log(x+4) + \frac{5}{x+4}.$$