Mathematics CS1001

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Eigenvalues and Eigenvectors I

Considering a matrix A, we want to calculate

- its eigenvalues
- its eigenvectors

Definition:

We want to find real numbers, λ, and non-zero vectors, $\mathbf{v}$; where they exist; such that $\mathbf{v}$ and $A\mathbf{v}$ are scalar multiples of each other:

$$A\mathbf{v} = \lambda\mathbf{v}$$

- λ is called an eigenvalue of A.
- $\mathbf{v}$ is the eigenvector of A corresponding to λ.

Eigenvalues and Eigenvectors II

Our equation

$$A\mathbf{v} = \lambda\mathbf{v}$$

may be rewritten as:

$$A\mathbf{v} - \lambda\mathbf{v} = 0$$

or, using the identity matrix I:

$$(A - \lambda I)\mathbf{v} = 0$$

To find when this has a non-trivial solution we need to find when

$$\det(A - \lambda I) = 0$$

This is called the characteristic equation of A.

When we expand this we obtain the characteristic polynomial of A.

Eigenvalues and Eigenvectors III

EXAMPLE : Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix}.$$

Eigenvalues and Eigenvectors IV

SOLUTION: We found that

$$\det(A - \lambda I) = -\lambda^3 + 2\lambda^2 + 15\lambda - 36.$$

To find solutions to $\det(A - \lambda I) = 0$ i.e., to solve

$$\lambda^3 - 2\lambda^2 - 15\lambda + 36 = 0.$$

- Find integer valued solutions. Such solutions divide the constant term (36). Possibilities: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36$.
- $\lambda = 3$: $3^3 - 2.3^2 - 15.3 + 36 = 0$.
- Now factor out $\lambda - 3$:

$$(\lambda - 3)(\lambda^2 + \lambda - 12) = \lambda^3 - 2\lambda^2 - 15\lambda + 36.$$

Eigenvalues and Eigenvectors V

- Solve $\lambda^2 + \lambda - 12 = 0$ by formula:

$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4.1. - 12}}{2} = \frac{-1 \pm 7}{2}$$

  Thus $\lambda = -4$ or 3.

- So

$$\begin{aligned} \det(A - \lambda I) &= -\lambda^3 + 2\lambda^2 + 15\lambda - 36 \\ &= (\lambda - 3)(\lambda - 3)(\lambda + 4) \end{aligned}$$

The eigenvalues of A are $\lambda = -4, 3$. Note that $\lambda = 3$ is a repeated root of the characteristic equation.

Eigenvalues and Eigenvectors VI

Once the eigenvalues of a matrix A have been found, we can find the eigenvectors by Gaussian Elimination.

1. For each eigenvalue λ, we have
$$(A - \lambda I)\mathbf{x} = \mathbf{0},$$
where $\mathbf{x}$ is the eigenvector associated with eigenvalue λ.
2. Find $\mathbf{x}$ by Gaussian elimination. That is, convert the augmented matrix
$$\left(A - \lambda I \,\vdots\, \mathbf{0} \right)$$
to reduced row echelon form, and solve the resulting linear system.

Eigenvalues and Eigenvectors VII

EXAMPLE: Find the eigenvectors of

$$A = \begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{pmatrix}.$$

we know that the eigenvalues of A are $\lambda = -4, 3$ with 3 being a repeated root (twice).

Eigenvalues and Eigenvectors VIII

SOLUTION:

- **Case 1:** $\lambda = -4$

  We must find vectors $\mathbf{x}$ which satisfy $(A - \lambda I)\mathbf{x} = \mathbf{0}$:

$$\lambda = -4 \Rightarrow A - \lambda I = \begin{pmatrix} 9 & 6 & 2 \\ 0 & 3 & -8 \\ 1 & 0 & 2 \end{pmatrix}.$$

Eigenvalues and Eigenvectors IX

- Construct the augmented matrix $\left(A - \lambda I \,\vdots\, \mathbf{0}\right)$ and convert it to row echelon form

$$\begin{array}{cc} \begin{pmatrix} 9 & 6 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array} \xrightarrow{R1 \leftrightarrow R3} \begin{array}{cc} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 9 & 6 & 2 & 0 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

$$\xrightarrow{R3 \rightarrow R3 - 9 \times R1} \begin{array}{cc} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & -8 & 0 \\ 0 & 6 & -16 & 0 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

$$\xrightarrow{R2 \rightarrow 1/3 \times R2} \begin{array}{cc} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -8/3 & 0 \\ 0 & 6 & -16 & 0 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

$$\xrightarrow{R3 \rightarrow R3 - 6 \times R2} \begin{array}{cc} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -8/3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

Eigenvalues and Eigenvectors X

- Rewrite as a linear system

$$\begin{aligned} x_1 \quad + 2x_3 &= 0 \\ x_2 - 8/3x_3 &= 0 \end{aligned}$$

  or, introducing parameters

$$\begin{aligned} x_1 &= -2t \\ x_2 &= 8/3t \\ x_3 &= t \end{aligned}$$

- Thus

$$\mathbf{x} = \begin{pmatrix} -2t \\ 8/3t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 8/3 \\ 1 \end{pmatrix} \quad \text{for any } t \in \mathbb{R}$$

  are eigenvectors of A associated with the eigenvalue $\lambda = -4$.

Eigenvalues and Eigenvectors XI

- **Case 2**: $\lambda = 3$

  We seek vectors x for which $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

$$\lambda = 3 \Rightarrow A - \lambda I = \begin{pmatrix} 2 & 6 & 2 \\ 0 & -4 & -8 \\ 1 & 0 & -5 \end{pmatrix}.$$

Eigenvalues and Eigenvectors XII

- Construct the augmented matrix $\left(A - \lambda I \,\vdots\, \mathbf{0}\right)$ and reduce it to row echelon form.

$$\begin{array}{cc} \left(\begin{array}{cccc} 2 & 6 & 2 & 0 \\ 0 & -4 & -8 & 0 \\ 1 & 0 & -5 & 0 \end{array}\right) & \begin{array}{c} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

$$\xrightarrow{R1 \leftrightarrow R3} \quad \begin{array}{cc} \left(\begin{array}{cccc} 1 & 0 & -5 & 0 \\ 0 & -4 & -8 & 0 \\ 2 & 6 & 2 & 0 \end{array}\right) & \begin{array}{c} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

$$\xrightarrow{R3 \to R3 - 2 \times R1} \quad \begin{array}{cc} \left(\begin{array}{cccc} 1 & 0 & -5 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 6 & 12 & 0 \end{array}\right) & \begin{array}{c} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

$$\xrightarrow{R2 \to -1/4 \times R2} \quad \begin{array}{cc} \left(\begin{array}{cccc} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 6 & 12 & 0 \end{array}\right) & \begin{array}{c} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

$$\xrightarrow{R3 \to R3 - 6 \times R2} \quad \begin{array}{cc} \left(\begin{array}{cccc} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) & \begin{array}{c} \text{R1} \\ \text{R2} \\ \text{R3} \end{array} \end{array}$$

Eigenvalues and Eigenvectors XIII

- Rewrite this as a linear system:

$$\begin{aligned} x_1 \qquad - 5x_3 &= 0 \\ x_2 + 2x_3 &= 0. \end{aligned}$$

  or, introducing parameter t,

$$\begin{aligned} x_1 &= 5t \\ x_2 &= -2t \\ x_3 &= t \end{aligned}$$

- Thus

$$\mathbf{x} = \begin{pmatrix} 5t \\ -2t \\ t \end{pmatrix} = t \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \quad \text{for any } t \in \mathbb{R}$$

  are eigenvectors associated with eigenvalue $\lambda = 3$.

Eigenvalues and Eigenvectors XIV

Conclusions:

- The solution of the linear system will involve at least one parameter, so there are infinitely many eigenvectors.
- However, all eigenvectors have a very specific form.

Geometric interpretation of eigenvalues and eigenvectors I

To compute the eigenvalues of a square matrix A:

1. Compute the matrix $A - \lambda I$.
2. Compute the characteristic equation $\det(A - \lambda I) = 0$.
3. Compute all the eigenvalues as the roots of the characteristic equation.

To compute the eigenvectors for each eigenvalue λ:

1. compute the solution $\mathbf{x}$ of the linear system $(A - \lambda I)\mathbf{x} = 0$.

Geometric interpretation of eigenvalues and eigenvectors II

The following exercise should convince you that you have an infinite number of eigenvectors associated to each eigenvalue.

Exercise:

1. Show that if $\mathbf{x}$ is an eigenvector of a matrix A with eigenvalue λ, then $\mathbf{x}' = \alpha\mathbf{x}$ is also an eigenvector associated with the same eigenvalue for any real number α.

2. Give a geometric interpretation of the previous question.

Definition:

An eigenspace is the set of eigenvectors with a common eigenvalue

Geometric interpretation of eigenvalues and eigenvectors III

Definition:

The length of a n dimensional vector $\mathbf{v}$ is:

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Example:

The length of $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is:

$$\|\mathbf{v}\| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$$

Geometric interpretation of eigenvalues and eigenvectors IV

Definition:

The dot product (or *inner product* or *scalar product*) of two n component column vectors is:

$$\mathbf{u} \bullet \mathbf{v} = \mathbf{u}^T \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Example:

Consider $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ then:

$$\mathbf{u} \bullet \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = (1\ 0\ 2) \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 1 \times 1 + 0 \times 2 + 2 \times (-1) = -1$$

Geometric interpretation of eigenvalues and eigenvectors V

Definition:

The angle between two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ is

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\| \times \|\mathbf{v}\|}\right)$$

We can redefine the scalar product:

$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\| \times \|\mathbf{v}\| \cos(\theta)$$

Geometric interpretation of eigenvalues and eigenvectors VI

Example:

Consider $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ the angle is:

$$\theta = \cos^{-1}\left(\frac{1 \times 2 + 0 \times 0 + 2 \times (-1)}{\sqrt{1^2 + 2^2}\sqrt{2^2 + (-1)^2}}\right) = \cos^{-1} 0 = \frac{\pi}{2}$$

Geometric interpretation of eigenvalues and eigenvectors VII

Definitions:

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\theta = \frac{\pi}{2}$ or $\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos(\theta) = 0$.

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are collinear if $\theta = 0$ or $\theta = \pi$, or if it exist a number α such that $\mathbf{u} = \alpha\mathbf{v}$.

Draw in a 3-D space the vectors $\mathbf{u}$ and $\mathbf{v}$ as defined in the previous slide.

Geometric interpretation of eigenvalues and eigenvectors VIII

Previously we computed the eigenvalues of the matrix:

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

We found:

- eigenvalue $\lambda = 4$, and the associated eigenvectors are:

$$\mathbf{x} = x_3 \times \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

Geometric interpretation of eigenvalues and eigenvectors IX

- a repeated eigenvalue $\lambda = -2$. In this case the eigenvectors $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ have to satisfy only one equation: $x_1 = x_3 - x_2$. So the set of eigenvectors associated with $\lambda = -2$ can be written:

$$\mathbf{x} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

  - $\mathbf{e_a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is found by choosing $x_2 = 0$ and $x_3 = 1$.
  - $\mathbf{e_b} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is found by choosing $x_3 = 0$ and $x_2 = 1$.
  - **Warning:** when we choose the second eigenvector $\mathbf{e_b}$, we have to make sure that it is not collinear to the first one $\mathbf{e_a}$.

Geometric interpretation of eigenvalues and eigenvectors X

Now, I would like to choose a second eigenvector $\mathbf{e_b}' = (e'_{b1}\ e'_{b2}\ e'_{b3})$ such that it is orthogonal to $\mathbf{e_a}$.

- $\mathbf{e_b}'$ is an eigenvector or A associated to the eigenvalue $\lambda = -2$. So its components satisfy the equation:

$$e'_{b1} = e'_{b3} - e'_{b2}$$

- we want $\mathbf{e_b}'$ orthogonal to $\mathbf{e_a}$, so :

$$\mathbf{e_b}' \bullet \mathbf{e_a} = e'_{b1} \times 1 + e'_{b2} \times 0 + e'_{b3} \times 1 = 0$$

So we need to solve the following system:

$$\begin{cases} e'_{b1} = e'_{b3} - e'_{b2} \\ e'_{b1} + e'_{b3} = 0 \end{cases}$$

which is easily reduced to

$$\begin{cases} e'_{b1} = -e'_{b3} \\ e'_{b2} = 2e'_{b3} \end{cases}$$

Geometric interpretation of eigenvalues and eigenvectors XI

This is a system with again an infinite number of solutions. So the new eigenvector is expressed as

$$\mathbf{e_b}' = e'_{b3} \times \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Draw in a 3D space

1. the eigenvectors associated to $\lambda = 4$ and $\lambda = -2$.
2. verify that the eigenvectors associated to $\lambda = 4$ are not collinear to the ones associated to $\lambda = -2$.