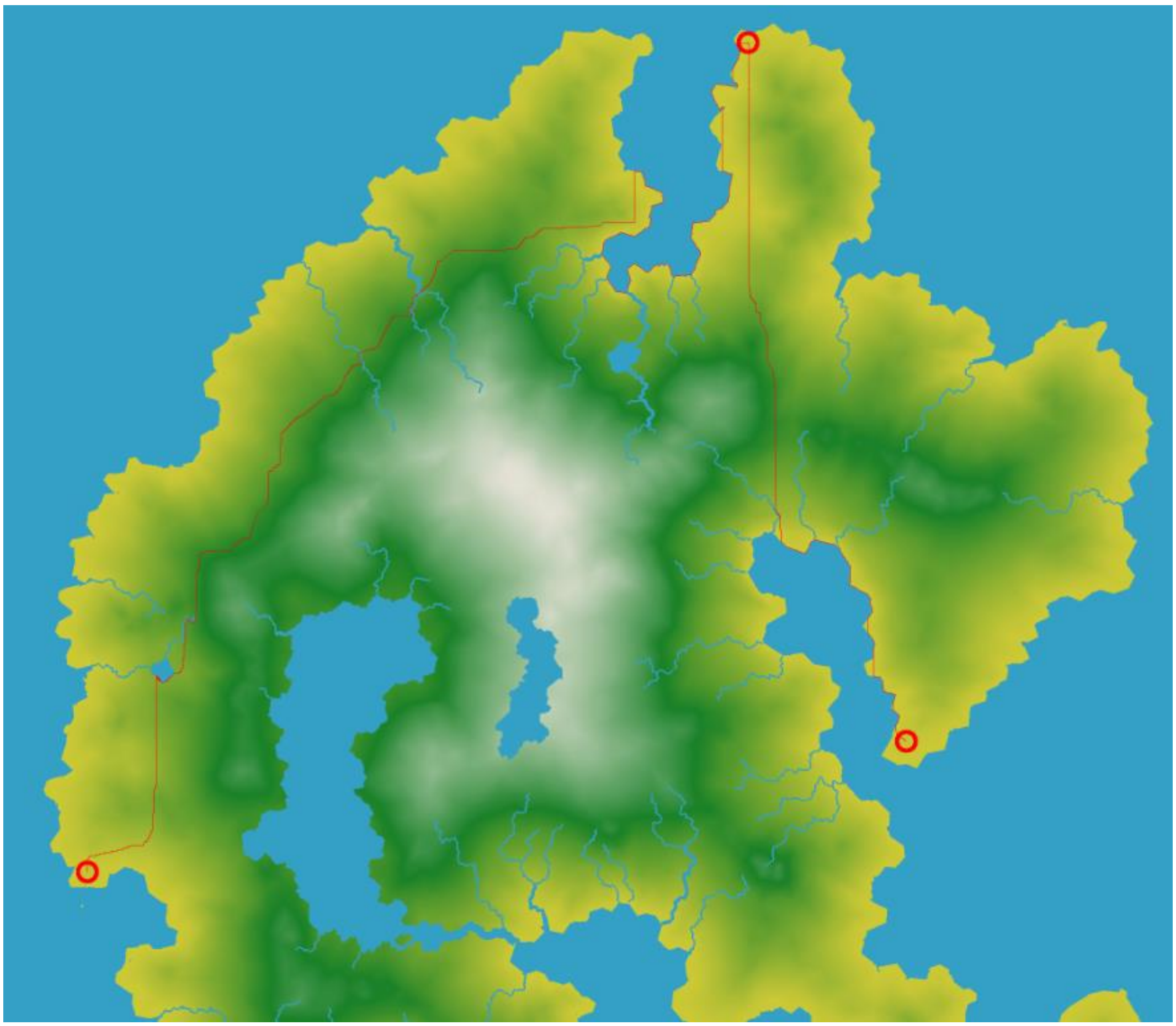
The Bachelor problem
Solution presentation



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I-1 Notations

In this solution presentation document, I will use the notations relatives to the Affine Space algebraic structures. For more information about the Affine Spaces and the notations I wrote the section “Additional”.

- We denote by $\{C(t)\} = \{O_{C(t)}, \boldsymbol{B_{C(t)}}\}$ the mobile direct orthonormal reference frame attached to the car at the time t with $\boldsymbol{B_{C(t)}} = [e_x(t)|e_y(t)|e_z(t)]$ where e_x is directed toward the front of the vehicle, $e_y(t)$ is directed toward the left of the vehicle and $e_z(t)$ toward the up.

- We denote by $\{W\} = \{O_W, \boldsymbol{B_W}\}$ the fixed direct orthonormal reference frame attached to the island. With $\boldsymbol{B_W} = [f_x(t)|f_y(t)|f_z(t)]$

A point $X \in \mathbb{A}^3$ can be expressed in $\{C(t)\}$ or $\{W\}$ as a vector of $\mathbb{R}^3$ with the following reference frame changes:

$$X_{O_W,\boldsymbol{B_W}} = P_{\boldsymbol{B_W} \to \boldsymbol{B_{C(t)}}} * X_{O_{C(t)},\boldsymbol{B_{C(t)}}} + \left(O_{C(t)} - O_W\right)_{\boldsymbol{B_W}} \ (\mathbf{1})$$

Since both $\{W\}$ and $\{C(t)\}$ are direct orthonormal reference frame, the change of basis matrix $P_{\boldsymbol{B_W} \to \boldsymbol{B_{C(t)}}}$ is an direct orthogonal matrix, i.g a rotation matrix. In the next sections of this document we will write (1) using:

$$X_w = R(t) * X_C + T(t) \ (\mathbf{2})$$

Using an Euler Angle parametrization of the $SO(3)$ lie-group manifold:

$$R(t) = R_z(rz) * R_y(ry) * R_x(rx) \ (\mathbf{3})$$

I-2 Modelisation of the problem

I-2-A Definition of the manifold

The position $T(t)$ of the car is lying on a 2-dimensional manifold explicitly parametrized as follow:

$$\mathbb{S} \subset \mathbb{R}^3 = \{S(x,y) = \begin{bmatrix} x \\ y \\ Z(x,y) \end{bmatrix}, \ (x,y) \in D \subset \mathbb{R}^2\} \ (4)$$

With $Z: \mathbb{R}^2 \to \mathbb{R}$ is called the elevation map function.

I-2-B Promote it to a Riemannian manifold by crafting a space-based metric

We will now equip this manifold with a metric in order to promote it as a Riemannian manifold. In a first stance we will create a metric based on the length that the car has made.

We derive this metric by considering the tangent space of the $\mathbb{S}$ manifold and defining a dot product canonically induced from the dot product of $\mathbb{R}^3$:

Let’s $I(x,y) \in M_{2,2}(\mathbb{R})$ be the first fundamental form:

$$I(x,y) = \begin{bmatrix} <\frac{\partial s}{\partial x}, \frac{\partial s}{\partial x}> & <\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}> \\ <\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}> & <\frac{\partial s}{\partial y}, \frac{\partial s}{\partial y}> \end{bmatrix} \text{(5)}$$

Then, for a small displacement dx, dy, the displacement linked to the manifold will be

$$dl^2 = I(1,1) * dx^2 + 2 * I(2,1) * dxdy + I(2,2) * dy^2 \text{ (6)}$$

In our specific case, $\frac{\partial s}{\partial x} = [1 \quad 0 \quad \frac{\partial Z}{\partial x}]^T$ and $\frac{\partial s}{\partial y} = [0 \quad 1 \quad \frac{\partial Z}{\partial y}]^T$ and then:

$$I(x,y) = \begin{bmatrix} 1 + \frac{\partial Z^2}{\partial x} & \frac{\partial Z}{\partial x} * \frac{\partial Z}{\partial y} \\ \frac{\partial Z}{\partial x} * \frac{\partial Z}{\partial y} & 1 + \frac{\partial Z^2}{\partial y} \end{bmatrix} \text{(7)}$$

And finally,

$$dl^2 = \left(1 + \frac{\partial Z^2}{\partial x}\right) * dx^2 + 2 * \frac{\partial Z}{\partial x} * \frac{\partial Z}{\partial y} * dxdy + \left(1 + \frac{\partial Z^2}{\partial y}\right) * dy^2 \text{ (8)}$$

I-2-C Adapt the space-based metric to craft a time-based metric

In our case, it is not the total length that the car has made that is important. We want our future groom to be on time at his wedding. Hence, we should transform the space-based metric we just crafted in a time-based metric. To do that, we should modelize the velocity of the car and then incorporate it in the metric.

Let's apply Newton's second law of dynamic (we don't care about the momentum):

$$m * \vec{A} = \vec{P} + \vec{R} + \overrightarrow{Engine} + \overrightarrow{Friction} \text{ (9)}$$

With $\vec{P}$ being the weight of the car (directed along f_z), $\vec{R}$ is the ground reaction, directed along the normal of the $\mathbb{S}$ manifold. $\overrightarrow{Engine} = Fe_x$ is the propulsion of the car engine, directed along e_x and $\overrightarrow{Friction} = -V(t) * e_x$ is the friction directed along $-e_x$.

Let's express the orientation of the car at the point (x, y). We note $U = [dx, dy]^T$:

$$e_x = \begin{bmatrix} dx \\ dy \\ U^T * \nabla Z \end{bmatrix} \text{(10)}$$

Where $U^T * \nabla Z$ is the directional differentiation of Z along the U direction:

$$ez = \frac{\partial s}{\partial x} \wedge \frac{\partial s}{\partial x} \text{(11)}$$

Where $\wedge$ is the vector product. e_y is obtained since $R(t) = [e_x(t)|e_y(t)|e_z(t)]$ is an orthonormal matrix.

We now express the Newton's dynamic law along the e_x vector:

$$m * \dot{v} = F - \sin(\theta) P - kv \text{ (12)}$$

Hence, assuming $v(t=0)=0$,

$$v(t) = \frac{\sin(\theta)P - F}{k} * \exp\left(-\frac{k}{m}t\right) + \frac{F - \sin(\theta)P}{k} \quad (13)$$

Once the vehicle reaches its full velocity, we have:

$$v_{max} = \frac{F - \sin(\theta)P}{k} \quad (14)$$

And what is important for us is the relation between the limit velocity and the angle between the ground and the vehicle moving direction

$$v_{max}(\theta) = v_0 - K * \sin(\theta) \quad (14)$$

Finally, our time-based metric will be, assuming the vehicle is always moving at $v_{max}(\theta)$ and ignoring the transitional phase while the vehicle is stabilizing its velocity:

$$dt^2 = \frac{dl^2}{v_{max}{}^2} \quad (15)$$

Hence,

$$\boxed{dt(x,y)^2 = \frac{\left(1 + \frac{\partial Z(x,y)}{\partial x}^2\right) * dx^2 + 2 * \frac{\partial Z(x,y)}{\partial x} * \frac{\partial Z(x,y)}{\partial y} * dxdy + \left(1 + \frac{\partial Z(x,y)}{\partial y}^2\right) * dy^2}{[v_0 - K * \sin(\theta(x,y))]^2}}$$

When crossing a river, the vehicle is affected by a velocity reduction which is a multiplication by a factor between 0 and 1.

We could also study the impact of the lack of oxygen on the engine power when the vehicle is in high altitude.

II Solution of the problem

What we want now is to find the shortest path = the geodesic of our Riemannian manifold between the two interest points of the island. There are two paradigms to study:

- The vehicle has the full elevation map available
-The vehicle discovers the elevation map and only have a local knowledge of the it

II-1 Knowing the full elevation map: Calculus of variations approach

If the entire elevation map is available, we can compute the geodesic between the two points using an iterative approach: calculus of variations.

Let's note we will build a sequence of function $\{\gamma_n\}$:

$$\gamma_n : \begin{array}{ccc} [0,1] & \to & \mathbb{R}^2 \\ u & \to & \gamma_n(u) \end{array} \quad (17)$$

With:

$$\gamma_0 : \begin{array}{ccc} [0,1] & \to & \mathbb{R}^2 \\ u & \to & (1-u) * P_0 + u * P_1 \end{array} \quad (18)$$

With P_0 and P_1 the interest starting (resp ending) points. Finally, we want the sequence to converge toward the solution geodesic. We denote by γ^* the limit, γ^* must satisfies:

$$\gamma^* = \underset{\gamma}{argmin}\, L(\gamma) \quad (19)$$

With:

$$L(\gamma) = \int_{u=0}^{u=1} \frac{\left(1 + \frac{\partial Z(\gamma(u))}{\partial x}^2\right) * dx(\gamma(u))^2 + 2 * \frac{\partial Z(\gamma(u))}{\partial x} * \frac{\partial Z(\gamma(u))}{\partial y} * dxdy + \left(1 + \frac{\partial Z(\gamma(u))}{\partial y}^2\right) * dy(\gamma(u))^2}{[v_0 - K * \sin(\theta(\gamma(u)))]^2} du$$

A solution of (19) must satisfies the Euler-Lagrange equation:

$$\frac{\partial L}{\partial \gamma} - \frac{d}{du}\left(\frac{\partial L}{\partial \gamma'}\right) = 0 \quad (20)$$

The iterative algorithm is similar to the gradient descent one:

$$\gamma_{n+1} = \gamma_n - \eta * [\frac{\partial L}{\partial \gamma_n} - \frac{d}{du}\left(\frac{\partial L}{\partial \gamma_n{}'}\right)] \quad (21)$$

Implementing this algorithm would have ask too much time, that is why I decided to choose a heuristic method that will give a reasonable short path but not the shortest one.

II-2 Knowing only local elevation map: Heuristic approach

We can image that the car has only a local elevation map. For example, an elevation map estimated / measured using a system composed of multi-camera and lidar. Once the local map is estimated, the car has to take a decision about the direction that must be taken. The best choice, without knowledges of what will come next is to take the direction that will minimize the time-based crafter metric. Of course, we must add to this metric a component so that the car will go toward the objective point when it is possible.

We craft:

$$dt_2^2(dir) = dt\big(\gamma(u)\big)^2 - \frac{<P_1 - \gamma(u), dir>}{\|P_1 - \gamma(u)\| * \|dir\|} \quad (22)$$

And minimize it to choose the right direction.

<u>Implementation hints:</u>

-I decided to use an 8-Neighborhood connectivity, meaning that the directions the car can take is a discrete set. It results that sometimes the car seems to not go along the shortest path because the correct direction would be somewhere between two discrete direction set

- When the vehicle encounters a water area, it follows the water area banks. I could have implemented an optimization step to avoid the strict following of the water area and take the shortest path. But this would have broken the local-knowledge paradigm and I was lacking of time.

II-3 Knowing the full elevation map: Heuristic approach (the one implemented)

What have been implemented is a heuristic approach with a full knowledge of the elevation map. This implementation is for practical reasons only. We can expect this algorithm to provides a reasonable short path between the two interest points.

To do that we "virtually" move the car like we would have done it with the II-2 approach and using the (22) metric. When the car encounter water we move the car along the both sides of the river / water banks until the car is inblocked. Once the car is unblock, we go back in time and launch an optimization process between a certain point and the unblocking point.

Additional- Affine Space

Affine space:

Let $\vec{E}$ be a vector space associated with the field $\mathbb{K}$. An affine space $\mathbb{A}$ of direction $\vec{E}$ is a none empty set E provided with an application $\psi: E \times E \rightarrow \vec{E}$ which associates each couple $(A, B) \in E \times E$ an element of $\vec{E}$ (a vector) noted $\overrightarrow{AB}$ that satisfies:

$$(i) \forall (A, B, C) \in E^3, \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$(ii) \forall A \in E, \forall \vec{u} \in \vec{E}, \exists! B \in E, \overrightarrow{AB} = \vec{u}$$

About the algebraic structure of the E set:

Let $\mathbb{A} = (E, \vec{E}, \psi)$ be an affine space. If we fix an origin point O of $\mathbb{A}$, then it exists (by definition) $\psi_O: E \rightarrow \vec{E}$ which associates to a point $P \epsilon E$ a vector $\overrightarrow{OM} \in \vec{E}$. From (ii), it is clear that the application ψ_O is a vector space isomorphism. Hence, using an origin O of the affine space, we can provide to the space E a vector space structure isomorphic to $\vec{E}$.

Conversely, every vector space $\vec{E}$ is canonically provided with an affine space structure using:

$$\psi: \begin{array}{ccc} \vec{E} \times \vec{E} & \rightarrow & E \\ (u, v) & \rightarrow & u - v \end{array}$$

That is why it is not necessary to distinguish between $\vec{E}$ and E since they are isomorphic. We will sometimes make the distinction to be clearer.

Representation of a point using a reference frame:

Let $X \in \mathbb{A}$ a point of a $\mathbb{K}$ finite dimension affine space of dimension N. Given an origin point O of the affine space and a basis $\boldsymbol{B} = \{\boldsymbol{e_i}\}_{i \in [|1,N|]}$ of the direction vector space, we can provide an extrinsic representation of X:

$$X_{O,\boldsymbol{B}} = (X - O)_{\boldsymbol{B}}$$

Where $(X - O)_{\boldsymbol{B}}$ is the $\mathbb{K}^N$ vector $[\lambda_1, \dots, \lambda_N]^T$ such as $X = O + \sum_{i=1}^{N} \lambda_i * \boldsymbol{e_i}$. In other words, we represent the point X using the coordinates of the vector $(X - O)$ expressed in the basis $\boldsymbol{B}$

Reference frame changes:

Let $X \in \mathbb{A}$ a point of a finite dimension affine space and $(O_1, \boldsymbol{B}_1); (O_2, \boldsymbol{B}_2)$ two references frame of $\mathbb{A}$. We denote by $P_{\boldsymbol{B_1} \rightarrow \boldsymbol{B_2}}$ the change of basis matrix from $\boldsymbol{B_1}$ to $\boldsymbol{B_2}$. We have two extrinsic representation of X using the two references frames linked by the expression:

$$P_{\boldsymbol{B_1}\to\boldsymbol{B_2}} * X_{O_2,\boldsymbol{B_2}} + (O_2 - O_1)_{\boldsymbol{B_1}} = X_{O_1,\boldsymbol{B_1}}$$

Euclidean affine space:
$\mathbb{A}$ is called an Euclidean affine space if its direction vector space is a Euclidean vector space:
i.e, $\vec{A}$ is a $\mathbb{R}$-vector space of finite dimension provided with a scalar product:

$$<.,.>: \begin{array}{ccc} \vec{A}\times\vec{A} & \to & \mathbb{R} \\ (u,v) & \to & <u;\ v> \end{array}$$

With $<.;.>$ being bilinear, symmetric, definite and positive application from $\vec{A}$ to $\mathbb{R}$.

With the introduction of a scalar product come notion as norm, distance, angles, orthogonality, …

Direct orthonormal references frames:
Let $\mathbb{A}$ be an Euclidean affine space with its scalar product $<.;.>$. A reference frame $(O,\boldsymbol{B})$ of $\mathbb{A}$ is said to be orthonormal if $\forall e_i, \forall e_j \in \boldsymbol{B}, i \neq j, < e_i, e_j > = 0\ and < e_i, e_i > = 1$.
An endomorphism f of $end(\vec{A})$ is said to be an orthogonal automorphism if:

$$\forall u, \forall v \in \vec{A}, < f(u), f(v) > = < u, v >$$

If $\boldsymbol{B}$ is an orthonormal basis for $<.;.>$, $S = Mat(<.;.>)_{\boldsymbol{B}}$ and $M = Mat(f)_{\boldsymbol{B}}$ then:

$S = Id$ and $U^T S V = U^T V = (MU)^T (MV) = U^T M^T M V$

This being true for all $(u,v) \in \vec{A}\times\vec{A}$ we have:

$$M^T M = Id$$

M is said to be an orthonormal matrix; it represents an orthonormal automorphism in an orthonormal basis.
In $\mathbb{R}^3$, the set of orthonormal matrix is a non-connected manifold constitute of :
- Rotation matrix with det(.) = 1
- Symmetry matrix with det(.) = -1

When a reference frame change is proceeded from a orthonormal basis $\boldsymbol{B_1}$ to another orthonormal basis $\boldsymbol{B_2}$ then the change basis matrix $P_{\boldsymbol{B_1}\to\boldsymbol{B_2}}$ is an orthonormal matrix.