Pricing generic TRS using risk-neutral expectation methodology.

I – Pricing equation

In order to determine today's value of the TRS we need to evaluate the corresponding expectation of the discounted future cash-flows, hence each payment which takes place at times $T_{m+1}, \dots, T_n$ needs to be discounted back to today, assuming here that financing flows are based on fixed rate:

$$\frac{\mathrm{TRS}_{\mathrm{m,n}}(\mathrm{t}_0)}{M(\mathrm{t}_0)} = \mathbb{E}^{\mathbb{Q}}\left(\sum_{\mathrm{k=m+1}}^{\mathrm{n}} \frac{1}{M(\mathrm{T_k})} \cdot \mathrm{N_{k\text{-}1}}\left(\frac{P(\mathrm{T_k}) - P(\mathrm{T_{k\text{-}1}})}{P(\mathrm{T_{k\text{-}1}})} - \mathrm{r_{TRS}} \cdot \tau_{\mathrm{k}}\right)\right)$$

$$\Leftrightarrow \mathrm{TRS}_{\mathrm{m,n}}(\mathrm{t}_0) = \mathbb{E}^{\mathbb{Q}}\left(\sum_{\mathrm{k=m+1}}^{\mathrm{n}} \frac{M(\mathrm{t}_0)\cdot \mathrm{N_{k\text{-}1}}}{M(\mathrm{T_k})} \cdot \left(\frac{P(\mathrm{T_k}) - P(\mathrm{T_{k\text{-}1}})}{P(\mathrm{T_{k\text{-}1}})}\right)\right) - \mathrm{r_{TRS}} \cdot \mathbb{E}^{\mathbb{Q}}\left(\sum_{\mathrm{k=m+1}}^{\mathrm{n}} \frac{M(\mathrm{t}_0)\cdot \mathrm{N_{k\text{-}1}}}{M(\mathrm{T_k})}\tau_{\mathrm{k}}\right)$$

$$\Leftrightarrow \mathrm{TRS}_{\mathrm{m,n}}(\mathrm{t}_0) = \mathbb{E}^{\mathbb{Q}}\left(\sum_{\mathrm{k=m+1}}^{\mathrm{n}} \frac{M(\mathrm{t}_0)\cdot \mathrm{N_{k\text{-}1}}}{M(\mathrm{T_k})} \cdot \left(\frac{P(\mathrm{T_k}) - P(\mathrm{T_{k\text{-}1}})}{P(\mathrm{T_{k\text{-}1}})}\right)\right) - \mathrm{r_{TRS}} \cdot \sum_{\mathrm{k=m+1}}^{\mathrm{n}} \mathrm{N_{k\text{-}1}} \cdot \tau_{\mathrm{k}} \cdot \mathbb{E}^{\mathbb{Q}}\left(\frac{M(\mathrm{t}_0)}{M(\mathrm{T_k})}\right)$$

Assuming $\mathrm{N_{k\text{-}1}}$ is constant and scaled to 1, we reach the following formula:

$$\Leftrightarrow \mathrm{TRS}_{\mathrm{m,n}}(\mathrm{t}_0) = \mathbb{E}^{\mathbb{Q}}\left(\sum_{\mathrm{k=m+1}}^{\mathrm{n}} \frac{M(\mathrm{t}_0)\cdot \mathrm{N_{k\text{-}1}}}{M(\mathrm{T_k})} \cdot \left(\frac{P(\mathrm{T_k}) - P(\mathrm{T_{k\text{-}1}})}{P(\mathrm{T_{k\text{-}1}})}\right)\right) - \mathrm{r_{TRS}} \cdot \mathrm{A_{m,n}}(\mathrm{t}_0)$$

With the classical annuity factor:

$$\mathrm{A_{m,n}}(\mathrm{t}_0) = \sum_{\mathrm{k=m+1}}^{\mathrm{n}} \tau_{\mathrm{k}} \cdot \mathrm{ZC}(\mathrm{t}_0, \mathrm{T_k})$$

As fair $\mathrm{r_{TRS}}$ solves the TRS price equation $\mathrm{TRS}_{\mathrm{m,n}}(\mathrm{t}_0) = 0$, it follows, taking the notation for fait TRS rate : $\mathrm{r_{m,n}}(\mathrm{t}_0)$

$$\mathrm{r_{m,n}}(\mathrm{t}_0) = \frac{\mathbb{E}^{\mathbb{Q}}\left(\sum_{\mathrm{k=m+1}}^{\mathrm{n}} \frac{M(\mathrm{t}_0)\cdot \mathrm{N_{k\text{-}1}}}{M(\mathrm{T_k})} \cdot \left(\frac{P(\mathrm{T_k}) - P(\mathrm{T_{k\text{-}1}})}{P(\mathrm{T_{k\text{-}1}})}\right)\right)}{\mathrm{A_{m,n}}(\mathrm{t}_0)}$$

we can express the value of the TRS with similar formula than IRS:

$$\Leftrightarrow \mathrm{TRS}_{\mathrm{m,n}}(\mathrm{t}_0) = \mathrm{A_{m,n}}(\mathrm{t}_0) \cdot (\mathrm{r_{m,n}}(\mathrm{t}_0) - \mathrm{r_{TRS}})$$

II – Assumptions and model discussion

Starting from less impacting to most impacting ones:

- 1/The fixed/floating rate assumption can be "easily" relaxed through the dedicated formula in case of up-front floating rate:

$$\frac{\mathrm{TRS}_{\mathrm{m,n}}(\mathrm{t}_0)}{M(\mathrm{t}_0)} = \mathbb{E}^{\mathbb{Q}}\left(\sum_{\mathrm{k=m+1}}^{\mathrm{n}} \frac{1}{M(\mathrm{T_k})} \cdot \mathrm{N_{k\text{-}1}}\left(\frac{P(\mathrm{T_k}) - P(\mathrm{T_{k\text{-}1}})}{P(\mathrm{T_{k\text{-}1}})} - \left(l(\mathrm{T_{k\text{-}1}}, \mathrm{T_{k\text{-}1}}, \mathrm{T_k}) + \mathrm{s_{TRS}}\right) \cdot \tau_{\mathrm{k}}\right)\right)$$

  And in that case, after measure change from $\mathbb{E}^{\mathbb{Q}} \rightarrow \mathbb{E}^{\mathbb{T}_{k+1}}$ for summation libor components, we can reach similar mechanism than classical IRS for this part.

- 2/The fixed/floating rate assumption can also be relaxed in case of RFR rate, with the following formula:

$$\frac{\mathrm{TRS}_{\mathrm{m,n}}(\mathrm{t}_0)}{M(\mathrm{t}_0)} = \mathbb{E}^{\mathbb{Q}}\left(\sum_{\mathrm{k = m+1}}^{\mathrm{n}} \frac{1}{M(\mathrm{T_k})} \cdot \mathrm{N_{k\text{-}1}}\left(\frac{P(\mathrm{T_k}) - P(\mathrm{T_{k\text{-}1}})}{P(\mathrm{T_{k\text{-}1}})} - (\mathrm{RFR}(\mathrm{T_{k\text{-}1}}, \mathrm{T_{k\text{-}1}}, \mathrm{T_{k)}} + \mathrm{s_{TRS}}) \cdot \tau_{\mathrm{k}}\right)\right)$$

Except that in case the measure change should be done according to extended zero coupon numeraire and no more with classical zero coupon

- 3/The constant nominal is clearly a pain point and a major drawback of the previous construction, as:
  - It’s not the market practice at all.
  - The nominal is in fact related to risk factor, here prices (the nominal is fixed according to reset prices), it’s then stochastic (and it can also, for exotics use case, be linked to floating rate itself, another risk factor in the “balance” then).
  - For those complex use cases, a simple and elegant formula can’t be easily derived, as, especially in floating case (libor or RFR), measure change won’t help.

- 4/But unfortunately the most impacting drawback lies in the end formula itself:

$$\Leftrightarrow \mathrm{TRS}_{\mathrm{m,n}}(\mathrm{t}_0) = \mathrm{A}_{\mathrm{m,n}}(\mathrm{t}_0) \cdot (\boldsymbol{r_{m,n}(t_0)} - \mathrm{r_{TRS}})$$

Contrary to liquid IRS which highlight market swap rate, TRS remains OTC and no liquid enough in order to take the assumption that TRS price can also be solved through market instruments + market datas.

But and exit door may exist repo rate curve.

It can be proved that $\mathrm{r}_{\mathrm{m,n}}(\mathrm{t}_0) \approx -\,\mathrm{repo\ rate}$ (cf. Total return futures pricing methodology), then if we can rely on repo curve itself, we can, with approximations, consider that even this $\mathrm{r}_{\mathrm{m,n}}(\mathrm{t}_0)$ can be derived from market data, and hence all the TRS pricing will rely, such as IRS, on market observables

III – Research axes and numerical results target (program of study)

1. Capture repo rates market date (through existing and liquid enough products which embeds repo rate: repo themselves, Futures, TRF). In first approach the analysis should be done using macro-data’s, aka with index products.
2. Construct the repo curve if not available directly in the market.
3. Test the $\mathrm{r}_{\mathrm{m,n}}(\mathrm{t}_0) \approx -\,\mathrm{repo\ rate}$ assumption by pricing TRS using theoretical formula (Monte-Carlo or PDE, or both) VS $\mathrm{TRS}_{\mathrm{m,n}}(\mathrm{t}_0) = \mathrm{A}_{\mathrm{m,n}}(\mathrm{t}_0) \cdot (\mathrm{r}_{\mathrm{m,n}}(\mathrm{t}_0) - \mathrm{r_{TRS}})$
4. Data capture and algorithm should be done in first approach with python and then using C++.