Callable/putable Total return swap

I – Introduction

This paper aims to highlight a pricing model about callable total return swap, like classical callable interest rate swap. Key distinctions between the former and the latter are risk factors (floating rate for IRS, asset price and possibly floating rate for the TRS) and liquidity, as IRS are very liquid and quoted, contrary to TRS. The first difference is a calculation issue, that we'll may solve, the second one is a structuring difference, that we can't easily bypass, as we'll explained. For all sections we'll took the TRS performance receiver view (aka TRS payer view), and then we'll, strictly speaking, highlight callable TRS. But the reader will easily infer the putable TRS twin mechanism.

II – TRS Pricing framework and notation

In all section, for sake of simplicity, we'll consider that performance and financing payments have the same schedule.

We'll also assume that Nominal (trade balance) is scaled to 1.

Several ***OTC*** TRS flavours exists, depending on financing rate characteristic, which lead to following pricing formulas:

$$\begin{cases} Float/fixed : \dfrac{TRS_{m,n}(t_0)}{M(t_0)} = \mathbb{E}^{\mathbb{Q}}\left(\displaystyle\sum_{k=m+1}^{n} \frac{1}{M(T_k)} \cdot \left(P(T_k) - P(T_{k-1}) - \boldsymbol{s_{TRS}} \cdot \tau_k\right)\right) \\ Float/Up-front\ float : \dfrac{TRS_{m,n}(t_0)}{M(t_0)} = \mathbb{E}^{\mathbb{Q}}\left(\displaystyle\sum_{k=m+1}^{n} \frac{1}{M(T_k)} \cdot \left(P(T_k) - P(T_{k-1}) - (\boldsymbol{l(T_{k-1}, T_{k-1}, T_k)} + \boldsymbol{m_{TRS}}) \cdot \tau_k\right)\right) \\ Float/In\ arrears\ float : \dfrac{TRS_{m,n}(t_0)}{M(t_0)} = \mathbb{E}^{\mathbb{Q}}\left(\displaystyle\sum_{k=m+1}^{n} \frac{1}{M(T_k)} \cdot \left(P(T_k) - P(T_{k-1}) - (\boldsymbol{RFR(T_{k-1}, T_{k-1}, T_k)} + \boldsymbol{m_{TRS}}) \cdot \tau_k\right)\right) \end{cases}$$

With following notations:

- t_0 the spot date, T_m the inception date (the TRS can be forward start)
- T_{m+1} the first payment date, T_n the last payment date (also equal to TRS maturity T)
- $TRS_{m,n}(t)$ the TRS market value at time t
- $M(t) = e^{\int_{t_0}^{t} r_s ds}$ the bank account at time t, considering stochastic or at least time-dependent interest rate
- $P(t)$ the underlying asset price at time t
- τ_k the tenor $T_k - T_{k-1}$
- s_{TRS} the fixed TRS spread.
- m_{TRS} the floating TRS margin.
- $l(t, T_{k-1}, T_k)$ the libor-like rate, fixed at T_{k-1} and paid at T_k , seen at time t.
- $RFR(t, T_{k-1}, T_k)$ the RFR-like rate, starting accrued at T_{k-1} and paid at T_k , seen at time t.

In addition, a very specific kind of ***standardized*** TRS now exists on the market:

$$\frac{STRS_{m,n}(t_0)}{M(t_0)} = \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{M(T)} \cdot \left(P(T) - P^{STRS}(T_m)\right)\right) - \mathbb{E}^{\mathbb{Q}}\left(\sum_{k=m+1}^{n} \frac{1}{M(T_k)} \cdot RFR(T_{k-1}, T_{k-1}, T_k) \cdot \tau_k\right)$$

With following characteristics:

- Contrary to previous OTC TRS whose "price" (quote) was s_{TRS}/m_{TRS}, the quoted metric for STRS is the settlement price $P^{STRS}(T_m)$
- Performance is bullet.
- Financing payment schedule follows IMM dates.

All those kinds of trades will be further detailed in dedicated parts of this paper, but their differences don't have a global impact on how the callable TRS should be modelled, let's dive in.

III – How to price a callable TRS?

This part will highlight general ideas about the model, calculation details will be highlighted in next parts, flavours after flavours.

The pricing model will be very similar to the one used for callable interest rate swap and first let's define TRSwaption product:

$$TRSwaption_{m,n}(T_m) \; = \; max\big(TRS_{m,n}(T_m), 0\big)$$

$$\Rightarrow \frac{TRSwaption_{m,n}(t_0)}{M(t_0)} \; = \; \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{M(T_m)} max\big(TRS_{m,n}(T_m), 0\big)\right)$$

We'll stop it for now about the TRSwaption product itself, as pricing derivation will be highlighted later, and let's define (after all those preliminaries!) callable TRS:

A callable TRS is a regular TRS where one side has the right but not the obligation to terminate all future payments during the life of the TRS.

There are two styles of callable TRS, the difference between them being in when the TRS can be called.

- European style callable TRS (also known as a cancellable TRS), the TRS can only be cancelled on one specific date.
- Bermudan style of callable TRS, the TRS can be cancelled on any one of several predefined dates.

In addition, there are two types of this TRS, depending on who has the right to termination of the underlying TRS:

- If the TRS payer (aka performance receiver) has the right to terminate the TRS on several dates, it is known as a callable TRS. This is a combination of a vanilla payer TRS and a receiver TRSwaption.
- If the TRS receiver (aka performance payer) has the right to terminate the TRS on several dates, it is known as a putable TRS. This is a combination of a vanilla receiver TRS and a payer TRSwaption

For our callable TRS target, the generic pricing formula is then, assuming the TRS is payer and the TRSwaption is receiver:

$$CallTRS_{m,n}(t_0) \; = \; TRS_{m,n}(t_0) + TRSwaption_{m,n}(t_0)$$

IV – Fixed rate callable Total return swap

1 – TRS pricing

Let's first consider the fixed total return swap:

$$\frac{TRS_{m,n}(t_0)}{M(t_0)} = \mathbb{E}^{\mathbb{Q}}\left(\sum_{k=m+1}^{n}\frac{1}{M(T_k)}\cdot\left(P(T_k) - P(T_{k-1}) - s_{TRS}\cdot\tau_k\right)\right)$$

$$\Leftrightarrow TRS_{m,n}(t_0) = \mathbb{E}^{\mathbb{Q}}\left(\sum_{k=m+1}^{n}\frac{M(t_0)}{M(T_k)}\cdot\left(P(T_k) - P(T_{k-1})\right)\right) - s_{TRS}\cdot\mathbb{E}^{\mathbb{Q}}\left(\sum_{k=m+1}^{n}\frac{M(t_0)}{M(T_k)}\tau_k\right)$$

$$\Leftrightarrow TRS_{m,n}(t_0) = \mathbb{E}^{\mathbb{Q}}\left(\sum_{k=m+1}^{n}\frac{M(t_0)}{M(T_k)}\cdot\left(P(T_k) - P(T_{k-1})\right)\right) - s_{TRS}\cdot\sum_{k=m+1}^{n}\mathbb{E}^{\mathbb{Q}}\left(\frac{M(t_0)}{M(T_k)}\right)\tau_k$$

$$\Leftrightarrow TRS_{m,n}(t_0) = \mathbb{E}^{\mathbb{Q}}\left(\sum_{k=m+1}^{n}\frac{M(t_0)}{M(T_k)}\cdot\left(P(T_k) - P(T_{k-1})\right)\right) - s_{TRS}\cdot A_{m,n}(t_0)$$

With the annuity factor: $A_{m,n}(t_0) = \sum_{k=m+1}^{n}\tau_k\cdot ZC(t_0,T_k)$, calculated through zero-coupon Bond $ZC(t_0,T_k) = \mathbb{E}^{\mathbb{Q}}\left(\frac{M(t_0)}{M(T_k)}\right)$

As discounted asset is a martingale under risk-neutral $\mathbb{Q}$ measure, we can simplify the expression:

$$TRS_{m,n}(t_0) = \sum_{k=m+1}^{n}M(t_0)\mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_k)}{M(T_k)}\right) - \sum_{k=m+1}^{n}M(t_0)\mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right) - s_{TRS}\cdot A_{m,n}(t_0)$$

$$\Leftrightarrow TRS_{m,n}(t_0) = \sum_{k=m+1}^{n}P(t_0) - \sum_{k=m+1}^{n}M(t_0)\mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_{k-1})e^{\int_{T_{k-1}}^{T_k}r_s ds}}\right) - s_{TRS}\cdot A_{m,n}(t_0)$$

As a side note, we can "toy a bit" with the second summation using two assumptions/tricks:

- Assuming constant or at least deterministic interest rate, in that case $\sum_{k=m+1}^{n}M(t_0)\mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_{k-1})e^{\int_{T_{k-1}}^{T_k}r_s ds}}\right) = \sum_{k=m+1}^{n}P(t_0)e^{-\int_{T_{k-1}}^{T_k}r_s ds}$
- Assuming asset P doesn't pay income , we can perform a measure change $\mathbb{Q}\to\mathbb{Q}^P$, with P as numéraire, and the reader can verify that in that case $\sum_{k=m+1}^{n}M(t_0)\mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_{k-1})e^{\int_{T_{k-1}}^{T_k}r_s ds}}\right) = \sum_{k=m+1}^{n}P(t_0)\mathbb{E}^{\mathbb{Q}^P}\left(e^{-\int_{T_{k-1}}^{T_k}r_s ds}\right)$, and the reader can note that if we in addition we assume constant/deterministic interest rate, we once again reach the previous formula

But those two assumptions are far too restrictive, and we won't use resulting formulas, the TRS pricing formula then remains:

$$TRS_{m,n}(t_0) = P(t_0)\cdot(n - m - 1) - M(t_0)\sum_{k=m+1}^{n}\mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right) - s_{TRS}\cdot A_{m,n}(t_0)$$

Fair TRS involves $TRS_{m,n}(t_0) = 0$, which leads, taking the notation $r_{m,n}^{TRS}(t_0)$ for the fair TRS spread:

$$r_{m,n}^{TRS}(t_0) = \frac{P(t_0)\cdot(n - m - 1) - M(t_0)\sum_{k=m+1}^{n}\mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right)}{A_{m,n}(t_0)}$$

which also leads to TRS market value formula rearrangement:

$$TRS_{m,n}(t_0) = A_{m,n}(t_0)(r_{m,n}^{TRS}(t_0) \ - s_{TRS})$$

We should be extremely about this formula to price the TRS itself:

- Contrary to IRS, TRS are not liquid enough to highlight, as market data, spread quote.
- Then in first approach this $r_{m,n}^{TRS}(t_0)$ should be **calculated** and not **implied** from TRS spread market data (which doesn't exist), using:
  - Available spot asset price and zero-coupon prices (probably implied from IRS themselves)
  - Monte-Carlo simulation for the risk-neutral summation part, with two stochastic dynamics:
    - Dynamic for asset underlying (range choice from vanilla black Scholes to more complex Bates model, but as we don't really care about implied volatilities in this context, the simple the best), with special attention about the kind of underlying (single equity, equity index etc..) and if the underlying is a Bond, please refer to the next point.
    - Dynamic for short-term interest rate, in first approach using a fitting model like Hull and White, which is used for bank-account dynamic but also for the underlying itself if this one is a Bond.
- Another approach would be, if the underlying is "holistic" enough (embedding large market sectors, like equity indices, Iboxx etc…), using repo rate as proxy for this $r_{m,n}^{TRS}(t_0)$. The reader interested in this approach may refer to https://github.com/PierreMoureaux/Research-and-development/tree/main/5%20-%20Repo%20rate%20curve%20-%20building. This proxy methods relies, this time, on market data too.

2 – TRSwaption pricing

The receiver TRSwaption follows next formulas, assuming strike is equal to ongoing vanilla TRS agreed spread, as the key functional point is obviously the comparison between this agreed financing spread and possible "new" one available on the market (we'll discuss this TRS spread "market availability" on final paragraph of this part, let's assume for now it exists):

$$\frac{TRSwaption_{m,n}(t_0)}{M(t_0)} = \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{M(T_m)} max\big(TRS_{m,n}(T_m), 0\big)\right)$$

$$\Leftrightarrow \frac{TRSwaption_{m,n}(t_0)}{M(t_0)} = \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{M(T_m)} max\big(A_{m,n}(T_m)(s_{TRS} - r_{m,n}^{TRS}(T_m)), 0\big)\right)$$

$$\Leftrightarrow TRSwaption_{m,n}(t_0) = \mathbb{E}^{\mathbb{Q}}\left(\frac{M(t_0) \cdot A_{m,n}(T_m)}{M(T_m)} max\big(s_{TRS} - r_{m,n}^{TRS}(T_m), 0\big)\right)$$

For next move we'll use the classical measure change, also used for classical swaption pricing: $A_{m,n}(T_m)$ is nothing more than a combination of tradable asset without income (zero-coupon Bonds), we can use it as numéraire and use the measure change $\mathbb{Q} \rightarrow \mathbb{Q}^{m,n}$. Using this new measure, pricing formula becomes:

$$TRSwaption_{m,n}(t_0) = \mathbb{E}^{\mathbb{Q}^{m,n}}\left(\frac{M(t_0) \cdot A_{m,n}(T_m)}{M(T_m)} \frac{A_{m,n}(t_0)}{A_{m,n}(T_m)} \frac{M(T_m)}{M(t_0)} max\big(s_{TRS} - r_{m,n}^{TRS}(T_m), 0\big)\right)$$

$$\Rightarrow TRSwaption_{m,n}(t_0) = A_{m,n}(t_0)\mathbb{E}^{\mathbb{Q}^{m,n}}\left(max\big(s_{TRS} - r_{m,n}^{TRS}(T_m), 0\big)\right)$$

The key point is now the $r_{m,n}^{TRS}(t)$ dynamic under annuity measure.

To price this product, we'll focus on three methods:

<u>2/a – "Black-Scholes" model</u>

The most straightforward approach, once again following classical swaption pricing literature, is assuming that $r_{m,n}^{TRS}(t)$ is a martingale under $\mathbb{Q}^{m,n}$ (the author insists it's an assumption), then driftless. Adding the lognormality assumption, to catch-up with BS model, the TRS fair spread follows the next dynamic.

$$\frac{dr_{m,n}^{TRS}(t)}{r_{m,n}^{TRS}(t)} = \sigma_{m,n}^{TRS} dW_{m,n}(t)$$

With this assumption, pricing derivation is classic and leads to for our receiver TRSwaption:

$$\begin{cases} TRSwaption_{m,n}(t_0) = A_{m,n}(t_0)\left[s_{TRS}\mathcal{N}(-d_2) - r_{m,n}^{TRS}(t_0)\mathcal{N}(-d_1)\right] \\ d_1 = \dfrac{log\left(\dfrac{r_{m,n}^{TRS}(t_0)}{s_{TRS}}\right) + \dfrac{1}{2}(\sigma_{m,n}^{TRS})^2(T_m - t_0)}{\sigma_{m,n}^{TRS}\sqrt{T_m - t_0}} \\ d_2 = d_1 - \sigma_{m,n}^{TRS}\sqrt{T_m - t_0} \end{cases}$$

But of course, the most impacting drawback of this model is, in this OTC TRS context, the choice of volatility $\sigma_{m,n}^{TRS}$ and then its implied calculation, because as an exotic feature on illiquid trade, the reader may be sure that TRSwaption market doesn't exist and then implying volatility is, practically speaking, impossible.

Assuming the $r_{m,n}^{TRS}$ is equal to -repo rate (still following the previously mentioned page about repo curve building), another way would be relying on optional products on repo rate. Unfortunately, those kinds of derivatives don't exist or are extremely OTC for now, waiting for full repo market derivative emerging.

This model, despite its simplicity, is then not so useful if practitioner intends to have a fitting-model and leads to kind of equilibrium parametrization of the volatility, with so called expertise approach instead of fitting/implied approach.

We won't be able to produce a fitting model, but we can have a different overview about fair TRS spread dynamic, as we'll see in next paragraph.

<u>2/b – Hull and white - Fair TRS spread dynamic under risk-neutral measure – Global diffusion processes</u>

Instead of the previous closed-form formula, we can step back a bit and focus on risk-neutral dynamic of the $r_{m,n}^{TRS}$, using the previously mentioned formulas:

$$\begin{cases} TRSwaption_{m,n}(t_0) = \mathbb{E}^{\mathbb{Q}}\left(\dfrac{M(t_0)\cdot A_{m,n}(T_m)}{M(T_m)} max\left(s_{TRS} - r_{m,n}^{TRS}(T_m),0\right)\right) \\ r_{m,n}^{TRS}(t) = \dfrac{P(t)\cdot(n - m - 1) - M(t)\sum_{k=m+1}^{n}\mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right)}{A_{m,n}(t)} \end{cases}$$

Then instead of looking for a closed-formula, we can also price the TRSwaption using a Monte-Carlo solver, but several dynamics should be defined:

- Interest rate dynamic, to model *M*, $A_{m,n}$ and eventually *P* too if this later is a Bond or Bond index.
- Specific P dynamic if this one is a single equity or equity index.

Several model choices can be highlighted at this step, let's choose fitting short-rate Hull and White diffusion process for the interest rate and let's assume for now the asset is a single equity: the latter will follow classical log-normal diffusion process. Both of course under risk-neutral measure $\mathbb{Q}$.

We have then:

$$\begin{cases} dr(t) = \lambda\big(\theta(t) - r(t)\big)dt + \eta dW_t^r \\ dP(t) = P(t)(r(t)dt + \sigma dW_t^P) \\ d\langle W_t^r, W_t^P\rangle = 0 \end{cases}$$

Please note that we assumed there are no correlation between Brownian motions, which of course doesn't mean there are independent (the asset price depends on interest rate level). Please also note that we considered time-dependent $\theta(t)$ factor for fitting purposes.

Under this HW interest rate dynamic, zero-coupon Bond formula are well-known and follow:

$$
\begin{cases}
ZC(t,T_k) = \; e^{A(t,T_k)+B(t,T_k)r(t)} \\
A(t,T_k) = -\dfrac{\eta^2}{4\lambda^3}\left(3 + e^{-2\lambda(T_k-t)} - 4e^{-\lambda(T_k-t)} - 2\lambda(T_k - t)\right) + \lambda \displaystyle\int_t^{T_k} \theta(s)B(s,T_k)ds \\
B(t,T_k) = -\dfrac{1}{\lambda}(1 - e^{-\lambda(T_k-t)})
\end{cases}
$$

It then leads to:

$$\frac{dZC(t,T_k)}{ZC(t,T_k)} = \left(\frac{dA(t,T_k)}{dt} + \frac{dB(t,T_k)}{dt} r(t)\right) dt + B(t,T_k)dr(t)$$

$$\Rightarrow dA_{m,n}(t) = d\left(\sum_{k\,=\,m+1}^{n} \tau_k \cdot ZC(t,T_k)\right)$$

$$\Rightarrow dA_{m,n}(t) = \sum_{k\,=\,m+1}^{n} \tau_k \cdot dZC(t,T_k)$$

$$\Rightarrow dA_{m,n}(t) = \sum_{k\,=\,m+1}^{n} \tau_k \cdot \; e^{A(t,T_k)+B(t,T_k)r(t)} \left[\left(\frac{dA(t,T_k)}{dt} + \frac{dB(t,T_k)}{dt} r(t)\right) dt + B(t,T_k)dr(t)\right]$$

And the bank-account dynamic follow next diffusion process:

$$dM(t) = M(t)r(t)dt$$

Itô formula applied on $r_{m,n}^{TRS}$ leads to:

$$dr_{m,n}^{TRS}(t) = \frac{1}{A_{m,n}(t)} d\left(P(t) \cdot (n \, - \, m - 1) - M(t) \sum_{k\,=\,m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right)\right)$$

$$-\frac{P(t) \cdot (n \, - \, m - 1) - M(t) \sum_{k\,=\,m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right)}{A_{m,n}(t)^2} dA_{m,n}(t)$$

$$+2\frac{P(t) \cdot (n \, - \, m - 1) - M(t) \sum_{k\,=\,m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right)}{A_{m,n}(t)^3} d\langle A_{m,n}(t)\rangle$$

$$\Rightarrow dr_{m,n}^{TRS}(t) = \frac{1}{A_{m,n}(t)}\left(dP(t) \cdot (n \, - \, m - 1) - dM(t) \sum_{k\,=\,m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right)\right)$$

$$-\frac{P(t) \cdot (n \, - \, m - 1) - M(t) \sum_{k\,=\,m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right)}{A_{m,n}(t)^2} dA_{m,n}(t)$$

$$+2\frac{P(t) \cdot (n \, - \, m - 1) - M(t) \sum_{k\,=\,m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right)}{A_{m,n}(t)^3} d\langle A_{m,n}(t)\rangle$$

We already highlighted dynamics of $P(t)$, $A_{m,n}(t)$ and $M(t)$, the $d\langle A_{m,n}(t)\rangle$ factor computation is slightly more complex but let's detail it:

Using usual rules of thumbs $dt^2 \ll dt$ and $dtdr(t) \approx dtdW_t^r = 0$ and multiplying terms per terms, the reader can verify than we reach following formulas:

$$d\langle A_{m,n}(t)\rangle = \left[\sum_{k=m+1}^{n}\left(\tau_k \cdot e^{A(t,T_k)+B(t,T_k)r(t)}B(t,T_k)\eta\right)^2\right]dt + \left[\sum_{\substack{k=m+1\\ j=m+1\\ k\neq j}}^{n} \tau_k\tau_j \cdot e^{A(t,T_k)+A(t,T_j)+(B(t,T_k)+B(t,T_j))r(t)}B(t,T_k)B(t,T_j)\eta^2\right]dt$$

We reached the fair TRS spread diffusion:

$$dr_{m,n}^{TRS}(t) = f(dP(t), dr(t), dt)$$

And we can use it to price our TRSwaption using Monte-Carlo method:

$$TRSwaption_{m,n}(t_0) = \mathbb{E}^{\mathbb{Q}}\left(\frac{M(t_0)\cdot A_{m,n}(T_m)}{M(T_m)} max\left(s_{TRS} - r_{m,n}^{TRS}(T_m), 0\right)\right)$$

The watchful reader may note that in fact we'll need two Monte-Carlo simulation, an inner one to calculate the $\sum_{k=m+1}^{n}\mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right)$ component and an outer one for the pricing itself.

<u>2/c – Hull and white - Fair TRS spread dynamic under risk-neutral measure – Exact processes</u>

The previous model highlighted method based on diffusion for each metrics/risk factors, which can (and will) be very costly. We can take another way, following the next algorithm:

- Generate $r(t)$ paths, still following Hull and White model.
- If the underlying asset is "well-behaving" enough (like single equity, equity index, Bond far from its maturity…), we can assume log-normality diffusion and then take the exact formula for $P(t)$:

$$P(T_m) = P(t_0)e^{\int_{t_0}^{T_m} r(s)ds - \frac{\sigma^2}{2}(T_m - t_0) + \sigma W_{T_m - t_0}^P}$$

- The bank-account remains:

$$M(T_m) = e^{\int_{t_0}^{T_m} r(s)ds}$$

- And the annuity factor is equal to:

$$A_{m,n}(T_m) = \sum_{k=m+1}^{n}\tau_k \cdot ZC(T_m, T_k) = \sum_{k=m+1}^{n}\tau_k\, e^{A(T_m,T_k)+B(T_m,T_k)r(T_m)}$$

- With all those quantities, TRSwaption Pay-off can be calculated, and we loop again to generate enough paths, and we'll take the mean.

3 – Summary and disclaimers

Model	**Pros**	**Cons**
BS	Extremely easy to implement	Impossible to really catch the volatility
HW MC - Global diffusion processes	Works with large range of processes' choices	Cumbersome and costly
HW MC - Exact processes	Easy to implement	Strong assumption about asset process

V – Floating rate callable Total return swap

For the whole chapter, we especially rely on following papers:

https://repository.tudelft.nl/islandora/object/uuid%3A13246c89-4707-499e-a8fd-22f760acbcc2

https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3482132

1 – TRS pricing

Let's now focus on following TRS flavours:

$$\begin{cases} \dfrac{TRS_{m,n}^{libor}(t_0)}{M(t_0)} = \mathbb{E}^{\mathbb{Q}}\left(\displaystyle\sum_{k=m+1}^{n} \dfrac{P(T_k) - P(T_{k-1}) - (l(T_{k-1},T_{k-1},T_k) + m_{TRS}) \cdot \tau_k}{M(T_k)}\right) \\ \dfrac{TRS_{m,n}^{RFR}(t_0)}{M(t_0)} = \mathbb{E}^{\mathbb{Q}}\left(\displaystyle\sum_{k=m+1}^{n} \dfrac{P(T_k) - P(T_{k-1}) - (RFR(T_{k-1},T_{k-1},T_k) + m_{TRS}) \cdot \tau_k}{M(T_k)}\right) \end{cases}$$

Please note that the second formula makes sense **if and only if** the RFR **is not secured**, unless RFR itself is a repo rate and the functional concept of TRS spread equivalent to -repo rate become meaningless. The interesting case of secured RFR will be highlighted when we'll focus on standardized TRS.

To remove ambiguities for watchful reader, here m_{TRS} highlights the **delta** (which can be negative or positive) between unsecured floating interest rate and theoretical secured repo rate, and not the repo rate itself just like the previous s_{TRS}.

We won't again derive all pricing steps but will focus on key steps:

$$\begin{cases} TRS_{m,n}^{libor}(t_0) = P(t_0)(n - m - 1) - M(t_0) \displaystyle\sum_{k=m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\dfrac{P(T_{k-1})}{M(T_k)}\right) - \displaystyle\sum_{k=m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\dfrac{M(t_0)}{M(T_k)} l(T_{k-1},T_{k-1},T_k)\tau_k\right) - m_{TRS} \cdot A_{m,n}(t_0) \\ TRS_{m,n}^{RFR}(t_0) = P(t_0)(n - m - 1) - M(t_0) \displaystyle\sum_{k=m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\dfrac{P(T_{k-1})}{M(T_k)}\right) - \displaystyle\sum_{k=m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\dfrac{M(t_0)}{M(T_k)} RFR(T_{k-1},T_{k-1},T_k)\tau_k\right) - m_{TRS} \cdot A_{m,n}(t_0) \end{cases}$$

At this step we'll use following quantities to simply formulas, first let's highlight the concept of extended zero-coupon Bond:

$$\overline{ZC(t,T_k)} = \begin{cases} ZC(t,T_k) \; if \; t \le T_k \\ \dfrac{M(t)}{M(T_k)} = e^{\int_{T_k}^{t} r_s ds} if \; t > T_k \end{cases}$$

And let's remind libor and RFR rates definitions:

$$\begin{cases} l(t,T_{k-1},T_k) ZC(t,T_k) = \dfrac{1}{\tau_k}(ZC(t,T_{k-1}) - ZC(t,T_k)) \\ RFR(t,T_{k-1},T_k)\overline{ZC(t,T_k)} = \dfrac{1}{\tau_k}(\overline{ZC(t,T_{k-1})} - \overline{ZC(t,T_k)}) \end{cases}$$

Now let's change measures from risk-neutral to forward, respectively $\mathbb{Q} \to \mathbb{Q}^{T_k}$ and $\mathbb{Q} \to \overline{\mathbb{Q}^{T_k}}$,TRS pricing formulas become:

$$\begin{cases} TRS_{m,n}^{libor}(t_0) = P(t_0)(n - m - 1) - M(t_0) \displaystyle\sum_{k=m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\dfrac{P(T_{k-1})}{M(T_k)}\right) - \displaystyle\sum_{k=m+1}^{n} \tau_k ZC(t_0,T_k) \mathbb{E}^{\mathbb{Q}^{T_k}}(l(T_{k-1},T_{k-1},T_k)) - m_{TRS} \cdot A_{m,n}(t_0) \\ TRS_{m,n}^{RFR}(t_0) = P(t_0)(n - m - 1) - M(t_0) \displaystyle\sum_{k=m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\dfrac{P(T_{k-1})}{M(T_k)}\right) - \displaystyle\sum_{k=m+1}^{n} \tau_k \overline{ZC(t_0,T_k)} \mathbb{E}^{\overline{\mathbb{Q}^{T_k}}}(RFR(T_{k-1},T_{k-1},T_k)) - m_{TRS} \cdot A_{m,n}(t_0) \end{cases}$$

And we'll use classical results that l and RFR are martingales under respectively $\mathbb{Q}^{T_k}$ and $\overline{\mathbb{Q}^{T_k}}$:

$$\begin{cases} TRS^{libor}_{m,n}(t_0) = P(t_0)(n - m - 1) - M(t_0) \sum_{k = m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right) - \sum_{k=m+1}^{n} \tau_k ZC(t_0, T_k) l(t_0, T_{k-1}, T_k) - m_{TRS} \cdot A_{m,n}(t_0) \\ TRS^{RFR}_{m,n}(t_0) = P(t_0)(n - m - 1) - M(t_0) \sum_{k = m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right) - \sum_{k=m+1}^{n} \tau_k \overline{ZC(t_0, T_k)} RFR(t_0, T_{k-1}, T_k) - m_{TRS} \cdot A_{m,n}(t_0) \end{cases}$$

If all $l(t_0, T_{k-1}, T_k)$ and $RFR(t_0, T_{k-1}, T_k)$ are market observables, those TRS can then be prices using similar mechanisms than fixed financing rate case. If set of ZC and extended ZC had already been derived, the formula can be simplified again using libor and RFR definition:

$$\begin{cases} TRS^{libor}_{m,n}(t_0) = P(t_0)(n - m - 1) - M(t_0) \sum_{k = m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right) - \sum_{k=m+1}^{n} (ZC(t_0, T_{k-1}) - ZC(t_0, T_k)) - m_{TRS} \cdot A_{m,n}(t_0) \\ TRS^{RFR}_{m,n}(t_0) = P(t_0)(n - m - 1) - M(t_0) \sum_{k = m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right) - \sum_{k=m+1}^{n} (\overline{ZC(t_0, T_{k-1})} - \overline{ZC(t_0, T_k)}) - m_{TRS} \cdot A_{m,n}(t_0) \end{cases}$$

$$\Rightarrow \begin{cases} TRS^{libor}_{m,n}(t_0) = P(t_0)(n - m - 1) - M(t_0) \sum_{k = m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right) - (ZC(t_0, T_m) - ZC(t_0, T_n)) - m_{TRS} \cdot A_{m,n}(t_0) \\ TRS^{RFR}_{m,n}(t_0) = P(t_0)(n - m - 1) - M(t_0) \sum_{k = m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right) - (\overline{ZC(t_0, T_m)} - \overline{ZC(t_0, T_n)}) - m_{TRS} \cdot A_{m,n}(t_0) \end{cases}$$

Fair TRS margins derivation leads then to:

$$\begin{cases} r^{TRS,libor}_{m,n}(t_0) = \dfrac{P(t_0) \cdot (n - m - 1) - M(t_0) \sum_{k = m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right) - (ZC(t_0, T_m) - ZC(t_0, T_n))}{A_{m,n}(t_0)} \\ r^{TRS,RFR}_{m,n}(t_0) = \dfrac{P(t_0) \cdot (n - m - 1) - M(t_0) \sum_{k = m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right) - (\overline{ZC(t_0, T_m)} - \overline{ZC(t_0, T_n)})}{A_{m,n}(t_0)} \end{cases}$$

We reach similar formulas than before, with exact same disclaimers and assumptions:

$$\begin{cases} TRS^{libor}_{m,n}(t_0) = A_{m,n}(t_0)(r^{TRS,libor}_{m,n}(t_0) - m_{TRS}) \\ TRS^{RFR}_{m,n}(t_0) = A_{m,n}(t_0)(r^{TRS,RFR}_{m,n}(t_0) - m_{TRS}) \end{cases}$$

2 – TRSwaption pricing

Foe this part, we won't try again to model the TRSwaption pricing through BS model (the reader can try as an exercise following previous chapter steps), as we know that the volatility puzzle can't be really solved, and we won't focus again on this approach. The pricing problem remains then, for x = libor/RFR:

$$\frac{TRSwaption_{m,n}(t_0)}{M(t_0)} = \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{M(T_m)} max\left(TRS^{x}_{m,n}(T_m), 0\right)\right)$$

$$\Leftrightarrow TRSwaption_{m,n}(t_0) = \mathbb{E}^{\mathbb{Q}}\left(\frac{M(t_0) \cdot A_{m,n}(T_m)}{M(T_m)} max\left(m_{TRS} - r^{TRS,x}_{m,n}(T_m), 0\right)\right)$$

Instead, we'll highlight three approaches once again based on Monte-Carlo simulation:

<u>2/a – Hull and white - Fair TRS spread dynamic under risk-neutral measure – Exact processes</u>

$$\begin{cases} r^{TRS,libor}_{m,n}(T_m) = \dfrac{P(T_m) \cdot (n - m - 1) - M(T_m) \sum_{k = m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right) - (1 - ZC(T_m, T_n))}{A_{m,n}(t_0)} \\ r^{TRS,RFR}_{m,n}(T_m) = \dfrac{P(T_m) \cdot (n - m - 1) - M(T_m) \sum_{k = m+1}^{n} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right) - (1 - \overline{ZC(T_m, T_n)})}{A_{m,n}(t_0)} \end{cases}$$

We can exactly mimic what had been already highlighted using Hull and White, simply by adding $ZC(T_m, T_n)/\overline{ZC(T_m, T_n)}$ in the loop, which doesn't add complexity, as those quantities are either classical zero-coupon Bond or bank-account, and both are already defined. This model is obviously "incorrect" as:

- We first took the assumption, to price the TRS, that $ZC(T_m, T_n)/\overline{ZC(T_m, T_n)}$ are strongly linked to libor/RFR definition.
- Then at final pricing step, we change the dynamic for those quantities, putting aside libor/RFR and choose Hull and White dynamic for the short-term interest rate.

The reader can then rightfully think it's quite incoherent, despite its simplicity. This fact leads us to focus on next models instead.

2/b – Forward market model

We'll surprisingly start by the RFR case, which is in fact the simplest one because, contrary to libor case we'll highlight later, all dynamics can be done under risk-neutral measure.

The TRSwaption Pay-off is still:

$$r_{m,n}^{TRS,RFR}(T_m) = \frac{P(T_m)\cdot(n - m - 1) - M(T_m)\sum_{k=m+1}^{n}\mathbb{E}^{\mathbb{Q}}\left(\frac{P(T_{k-1})}{M(T_k)}\right) - (1 - \overline{ZC(T_m, T_n)})}{A_{m,n}(T_m)}$$

But we need this time highlight zero-coupon and bank-account formulas using specific RFR rate dynamic. Following papers, we highlighted up-front, key formulas are, all under risk-neutral measure:

$$\begin{cases} dRFR(t, T_{k-1}, T_k) = \sigma_k(t)\gamma_k(t)\displaystyle\sum_{i=1}^{k}\rho_{i,k}\frac{\tau_i\sigma_i(t)\gamma_i(t)}{1+\tau_i RFR(t, T_{i-1}, T_i)}dt + \sigma_k(t)\gamma_k(t)dW_k(t) \\ ZC(t, T_k) = ZC\left(t, T_{\eta(t)}\right)\displaystyle\prod_{i=\eta(t)+1}^{\eta(T_k)-1}\frac{1}{1+\tau_i RFR(t, T_{i-1}, T_i)}\cdot\frac{ZC(t, T_k)}{ZC\left(t, T_{\eta(T_k)-1}\right)} \\ M(T_k) = ZC\left(t, T_{\eta(t)}\right)\displaystyle\prod_{i=1}^{\eta(t)}(1+\tau_i RFR(t, T_{i-1}, T_i)) \end{cases}$$

With following notations:

- $\eta(t) = \min_k T_k > t$
- $\sigma_k(t)$ the volatility process for the RFR
- $\gamma_k(t)$ = 1 for $t < T_{k-1}$, decreasing for the tenor $[T_{k-1}, T_k]$, and equal to 0 for $t > T_k$
- $d\langle W_i, W_k\rangle = \rho_{i,k}dt$
- $ZC\left(t, T_{\eta(t)}\right)$ the so-called front-stub zero-coupon
- $\frac{ZC(t,T_k)}{ZC\left(t,T_{\eta(T_k)-1}\right)}$ the so-called back-stub zero-coupon

This paper won't aim to dig into computation details, which are quite complex, and the interested reader can refer to mentioned papers.

For our TRSwaption pricing matter, and taking the additional assumption that interest rate embedded in $P(T_m)$ formula also follow RFR dynamic (considered as applicable risk-free rate by construction), the Monte-Carlo algorithm will use $P(T_m)$, $M(T_m)$, $A_{m,n}(T_m)$, $\overline{ZC(T_m, T_n)}$, all based on several paths generation for $dRFR(t, T_{k-1}, T_k)$.

We also consider that all quantities embedded in RFR dynamics are already calibrated through liquid transaction like classical swaption and caplets, as once again we can't calibrate those datas using this OTC TRSwaption, considered as exotic.

2/c – Libor market model

Even if the LMM sounds very similar than the previous one (dynamic process for $dl(t, T_{k-1}, T_k)$), it highlights several differences which makes the pricing more complex:

- Libor can't be considered as risk-free rate and then contrary to previous FMM, we can't easily derive bank-account dynamic for discounting, we'll need to discount step by step using $\frac{1}{1+\tau_i l(t, T_{i-1}, T_i)}$, from one period to the next.
- The $dl(t, T_{k-1}, T_k)$ dynamic, contrary to $dRFR(t, T_{k-1}, T_k)$, is not under risk-neutral measure but under specific spot measure, and then we can directly use our TRSwaption pricing formula:
  - We should change measure from risk-neutral to spot measure for the pricing formula itself.
  - We should derive new asset dynamic under spot measure.

As this model implies several difficulties but most important as libor rates will be in short/medium term all transitioned, the author estimates it's useless to dig into computation details, the reader can do if she/he (theoretically) is interested in.

3 – Summary and disclaimers

Model	Pros	Cons
HW MC - Exact processes	Easy to implement	Functionally/mathematically Incoherent
FMM (for RFR-linked TRS)	Coherent with other derivatives	Complex but not so once all quantities are already calibrated using liquid interest rate derivatives
LMM (for Libor-linked TRS)	Coherent with other derivatives	Measure modification Not direct discounting metric Useless in short/medium term due to Libor transition

VI – Standardized Total return swap

1 – STRS pricing

Let's focus on STRS pricing formula:

$$\frac{STRS_{m,n}(t_0)}{M(t_0)} = \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{M(T)}\cdot(P(T) - P^{STRS}(T_m))\right) - \mathbb{E}^{\mathbb{Q}}\left(\sum_{k=m+1}^{n}\frac{1}{M(T_k)}\cdot RFR(T_{k-1},T_{k-1},T_k)\cdot\tau_k\right)$$

The STRS is quite particular as contrary to OTC TRS, the quoted metric, then fixed and agreed, is the initial price $P(T_m)$. The functional mechanism is the following one:

- Financing leg embeds RFR rate, and this one, in first approach and following market practice, is a secured one (like SOFR, SARON). It's then equivalent to repo rate.
- Traders then quote the initial price in a very similar manner than equity futures: it's considered as a strike, and this quote reflects the "amount" of repo embedded in financing leg.
- The longer the STRS maturity is, the higher the repo is, which explains the decrease of settlement price according to maturities (and once again following the classical Futures mechanism: the higher the repo rate is, the lower the Future price is)

Saying that, the pricing formula become, following already explained RFR scheme:

$$STRS_{m,n}(t_0) = M(t_0)\mathbb{E}^{\mathbb{Q}}\left(\frac{P(T)}{M(T)}\right) - M(t_0)P^{STRS}(T_m)\mathbb{E}^{\mathbb{Q}}\left(\frac{1}{M(T)}\right) - \sum_{k=m+1}^{n}\tau_k\overline{ZC(t_0,T_k)}RFR(t_0,T_{k-1},T_k)$$

$$\Leftrightarrow STRS_{m,n}(t_0) = P(t_0) - P^{STRS}(T_m) - (\overline{ZC(t_0,T_m)} - \overline{ZC(t_0,T_n)})$$

The reader may be concerns about the m=0 case, which **seems** to lead to weird pricing involving null performance. But please take care that P and P^{STRS} are very different metrics:

- P is the underlying asset price.
- P^{STRS} is the STRS quote (strike), which is a totally different thing (it's linked to asset price, it's not the asset price), it's a quantity which belong to STRS.

Once again, the fair STRS price involves $STRS_{m,n}(t_0) = 0$, which leads to, taking the notation $P_{m,n}^{STRS}(t_0)$:

$$P_{m,n}^{STRS}(t_0) = P(t_0) - (\overline{ZC(t_0,T_m)} - \overline{ZC(t_0,T_n)})$$

$$\Rightarrow STRS_{m,n}(t_0) = P_{m,n}^{STRS}(t_0) - P^{STRS}(T_m)$$

2 – STRSwaption pricing

With previous STRS pricing formula, the STRSwaption pricing formula is:

$$\frac{STRSwaption_{m,n}(t_0)}{M(t_0)} = \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{M(T_m)}max\left(STRS_{m,n}(T_m),0\right)\right)$$

$$\Leftrightarrow \frac{STRSwaption_{m,n}(t_0)}{M(t_0)} = \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{M(T_m)}max\left(P^{STRS}(T_m) - P_{m,n}^{STRS}(T_m)),0\right)\right)$$

$$\Leftrightarrow \frac{STRSwaption_{m,n}(t_0)}{M(t_0)} = \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{M(T_m)}max\left(P^{STRS}(T_m) - P(T_m) + ((\overline{ZC(T_m,T_m)} - \overline{ZC(T_m,T_n)})),0\right)\right)$$

$$\Leftrightarrow STRSwaption_{m,n}(t_0) = \mathbb{E}^{\mathbb{Q}}\left(\frac{M(t_0)}{M(T_m)}max\left(P^{STRS}(T_m) + 1 - P(T_m) - \overline{ZC(T_m,T_n)}),0\right)\right)$$

2/a – Forward market model

Clearly the most straightforward approach is once again the FMM which, thanks to RFR dynamic, will allow us to calculate $P(T_m)$ and $\overline{ZC(T_m, T_n)}$, all under risk-neutral measure. Generating several paths for this RFR, and then using Monte-Carlo algorithm then taking the mean will directly lead to STRSwaption price.

<u>2/b – "Black-Scholes" model</u>

Taking following notations:

$$\begin{cases} K^{STRS} = 1 + P^{STRS}(T_m) \\ composite\ underlying : \ S = P + \overline{ZC} \end{cases}$$

We can simplify the STRSwaption formula:

$$STRSwaption_{m,n}(t_0) = \mathbb{E}^{\mathbb{Q}}\left(\frac{M(t_0)}{M(T_m)} max(K^{STRS} - S(T_m)),0)\right)$$

And we recognize the famous Black-Scholes formula, which leads to closed-form formula:

$$\begin{cases} STRSwaption_{m,n}(t_0) = -S(t_0)\mathcal{N}(-d_1) + K^{STRS} ZC(t_0, T_m)\mathcal{N}(-d_2) \\ d_1 = \dfrac{log\left(\frac{S(t_0)}{K^{STRS}}\right) + \frac{1}{2}(\sigma_{m,n}^{STRS})^2(T_m - t_0)}{\sigma_{m,n}^{STRS}\sqrt{(T_m - t_0)}} \\ d_2 = d_1 - \sigma_{m,n}^{STRS}\sqrt{(T_m - t_0)} \end{cases}$$

But it's not for free:

- The choice of dynamics for sub-components P and $\overline{ZC}$ is key as complex choices can lead to complex S dynamic processes.
- The previous remark leads to definition of $\sigma_{m,n}^{STRS}$, which, according to dynamic choices, can be a simple uncorrelated sum of volatilities or on the contrary complex combination between initial underlying volatility and RFR volatilities.

3 – Summary and disclaimers

Model	**Pros**	**Cons**
FMM	Coherent for both STRS and OTC TRS	Complex but not so once all quantities are already calibrated using liquid interest rate derivatives
BS	Closed-form formula	Very model-based and volatility calculation may be cumbersome regarding dynamic choices

VII – Final remarks and next research axis

Let's focus on strong assumptions and gaps:

- The assumption of scaled nominal to 1 is a very strong one, which impacts pricing formulas derivation id this assumption is relaxed (typically with nominal based on reset prices, then linked to price risk factors and eventually other inflation/amortization risk factors in Bond underlying case)
- The paper focused on european callable/putable TRS, the bermudean callable/putable needs to be modelled too, following for example http://www.columbia.edu/~mh2078/market_models.pdf
- X-ccy TRS trades had not been analysed (for example composite cases with underlying prices conversion) and adding this risk factor complexifies pricing.
- Underlying were assumed to be relatively vanilla, embedding exotic Bonds for example, with additional inflation/amortization factors, once again leads to more complex pricing.