Pricing vanilla TRSs with known interest rate using Black Scholes

Let's S to mean the underlying asset price, the maturity date is T. To introduce the ideas behind pricing TRS let's start by assuming that:

- interest rates are deterministic for the life of the TRS.
- Financing rate r_0 is fixed at inception (old way "Libor" use case).
- Performance and financing legs are bullet.

Each of those assumptions will be relaxed in next documentations but for sake of simplicity, we'll focus on this paper on the most vanilla use case.

Since the TRS value depends on the price of that asset we have:

$$\mathbf{TRS = TRS(S, t)}$$

The contract value depends on an asset price and on the time to maturity. Repeating the Black–Scholes analysis, with a portfolio consisting of one TRS and −Δ assets, we find that the change in the value of the portfolio is (please note that at this step we don't introduce delta one results for derivatives, this concept will be seen at the end):

$$\mathbf{d\Pi = \frac{\partial TRS}{\partial t}dt + \frac{\partial TRS}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 TRS}{\partial S^2}dt - \Delta dS}$$

Classical choice for Δ in order to eliminate risk from this portfolio:

$$\mathbf{\Delta = \frac{\partial TRS}{\partial S}}$$

The return on this risk-free portfolio is at most that from a bank deposit and so:

$$\mathbf{\frac{\partial TRS}{\partial t}dt + (r - C_{yield})S\frac{\partial TRS}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 TRS}{\partial S^2}dt - rTRS = 0}$$

With C_{yield} is for income yield here, only relevant for indexes (CAC40, Iboxx etc…). As we'll see in next section, discrete incomes imply jump condition.

This inequality is the basic Black–Scholes inequality. Scaling quantity to 1 and assuming a receiver performance TRS, the final condition is:

$$\mathbf{TRS(S(T), T) = (S(T) - S(t_0)) - r_0.S(t_0).DCF(t_0 \to T)}$$

Incomes can be paid discretely so we have the jump condition across each income date:

$$\mathbf{TRS(S, t_c^-) = TRS(S, t_c^+) + C_{discrete}}$$

where $C_{discrete}$ is the amount of the discrete coupon paid on date t_c.

In order to summarize, the full model is then:

$$\boxed{\begin{gathered}\mathbf{\frac{\partial TRS}{\partial t}dt + (r - C_{yield})S\frac{\partial TRS}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 TRS}{\partial S^2}dt - rTRS = 0} \\ \mathbf{TRS(S(T), T) = (S(T) - S(t_0)) - r_0.S(t_0).DCF(t_0 \to T)} \\ \mathbf{\forall t_c, TRS(S, t_c^-) = TRS(S, t_c^+) + C_{discrete}}\end{gathered}}$$