Value at risk and Expected shortfall for option position

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I – Introduction

This paper aims to introduce value at risk (VaR) and expected shortfall (ES) risk metrics for a very simple portfolio based on a unique equity call option derivative. This paper doesn't have the ambition to highlight new financial features or "revolutionary" topics of those already very well-known quantities, but modestly aims to explain/popularize, using financial-mathematical tools and python experiments.

The paper will be structured in following parts:

- Value at risk and expected shortfall explanations/calculations.
- Numerical results
- Conclusion
- Sources
- Annexes

Following notations will be used:

- α: the confidence level, equal to 99,9%, 99%, 95% etc.…
- $VaR_{\delta t,\alpha}$: Value at risk for a time interval δt and a confidence level α
- $ES_{\delta t,\alpha}$: Expected shortfall for a time interval δt and a confidence level α
- S_t : Asset value at time t
- μ : Real-world asset's drift
- r : Risk-free interest rate
- σ : Asset's actual volatility
- φ : standard normal distribution
- φ^2 = χ^2: chi-squared distribution
- $\Pi(S_t, t)$: portfolio (which depends on underlying S_t) value at time t
- $V(S_t, t)$: Option (on underlying S_t) value at time t, with maturity T
- $\tilde{\sigma}$: Black-Scholes (BS) implied volatility
- $\Delta_t(\tilde{\sigma}) = \frac{\delta V(S_t,t)}{\delta S_t}$: BS option delta
- $\theta_t(\tilde{\sigma}) = \frac{\delta V(S_t,t)}{\delta t}$: BS option theta
- $\Gamma_t(\tilde{\sigma}) = \frac{\delta^2 V(S_t,t)}{\delta {S_t}^2}$: BS option gamma
- F_X : the cumulative distribution of specific random variable X
- $\mathcal{N}^{-1}$: the inverse normal law
- ${\chi^2}^{-1}$: the inverse chi-squared law

For the whole paper, we'll use discrete settings instead of continuous settings, which drives next notation δx instead of classical dx, and using Taylor expansion instead of Ito's lemma for diffusions equations.

Convention comments: as there are several ways to define value at risk/expected shortfall, three conventions will be followed for the paper:

- The confidence level α is indeed defined "up": 99,9%, 99%, 95% etc.… In the literature we can often see the "reverse" point of view: c = 99,9%, 99%, 95% etc.… = (1 - α)
- Value at risk/expected shortfall are considered non-negative quantities.
- When we'll mention return it will mean absolute return (aka $\delta\Pi(S_t, t)$, aka P&L) and note relative return (aka $\frac{\delta\Pi(S_t,t)}{\Pi(S_t,t)}$). As $\Pi(S_t, t) = V(S_t, t)$ is a constant (basically the already paid/received premium), the analysis without this scaling factor will facilitate the reasoning, and changes nothing.

II – Value at risk and expected shortfall

1 – Functional overview

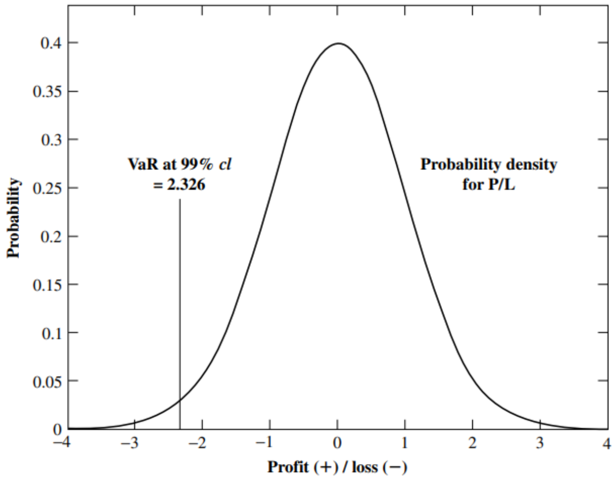
1/a – General overview

The value at risk metric aims to highlight the loss amount that an operator can lose, at most, for a specific investment (in our case an option position) during a given period (in our case one business day).

The best way to understand this concept and how it's calculated is considering it as a quantile associated with the potential loss. For example, let's assume we managed to obtain our portfolio's distribution. The portfolio can generate profit or losses, and the value at risk metric especially focuses on losses side of the distribution, and can be defined as:

$$\mathbb{P}\left(\delta\Pi(S_t, t) \le -VaR_{\delta t,\alpha}\right) = 1 - \alpha$$

Looking at next picture, the quantile characteristic is very clear:



Source: An introduction to market risk measurement, Kevin Dowd, page 21

One of the main practical drawbacks of the value at risk is that it doesn't provide an information about what will possibly happen if losses will exceed this threshold: how much will we possibly lose in that case? the value at risk indicator is once again only a threshold, and risk managers need a metric which can indicates the possible amount of losses beyond the value at risk level.

The expected shortfall metric tries to capture this information and it's defined by the following conditional expectation (or average):

$$ES_{\delta t,\alpha} = \mathbb{E}\left(\delta\Pi(S_t, t) \middle| \delta\Pi(S_t, t) \le -VaR_{\delta t,\alpha}\right) = -\frac{1}{1-\alpha}\int_{-\infty}^{-VaR_{\delta t,\alpha}} x f_{\delta\Pi(S_t,t)}(x)\, dx$$

with $f_{\delta\Pi(S_t,t)}(x)$ the probability density of $\delta\Pi(S_t, t)$.

In addition, expected shortfall is a what is defined as coherent risk measure, and highlights a useful feature about diversification financial idea: if our portfolio is built for example with two components A and B, we have:

$$ES_{\delta t,\alpha}(A+B) \le ES_{\delta t,\alpha}(A) + ES_{\delta t,\alpha}(B)$$

The value at risk metric, on the contrary, doesn't have this feature, which basically means that a diversified portfolio can have a "risk" (here equal to value at risk proxy) superior to the risk of independent components.

We can also notice that by design the expected shortfall will be more conservative than the value at risk: as only losses superior to value at risk are embedded in the average calculation, the expected shortfall quantity can't be inferior to value at risk.

1/b – Value at risk/expected shortfall "calculation flavors."

There are several ways to define/calculate value at risk and expected shortfall:

- Parametric VaR: Under the assumption of normally distributed returns the estimation of the portfolio values is computed. Please note that this closed-form formula method only helps for VaR, as the ES doesn't have a closed-form formulation in this parametric framework.

- Monte-Carlo Simulation: portfolio value follows a specific diffusion equation, and several paths are simulated to create a sample distribution, then quantile is captured, and specific average is calculated to produce ES.

The paper will analyze in detail those two first methods. Saying that, two other methods which relies on historical data exists and are also used in production (but we won't focus on those methods):

- Historical VaR/ES: Historical data (running window) is used to assess the distribution of the portfolio.
- Stressed VaR/ES: A stressed period of historical data is used to estimate the distribution of potential portfolio losses.

2 - Parametric Value at Risk and Expected shortfall

2/a – General framework

First, let's derive diffusion equation for both stock underlying and our portfolio based on naked option position. We'll use Black-Scholes discrete settings for derivation:

$$\begin{cases} \delta S_t = \mu S_t \delta t + \sigma S_t \varphi \sqrt{\delta t} \\ \delta \Pi(S_t, t) = \delta V(S_t, t) = \dfrac{\delta V(S_t, t)}{\delta t} \delta t + \dfrac{\delta V(S_t, t)}{\delta S_t} \delta S_t + \dfrac{1}{2} \dfrac{\delta^2 V(S_t, t)}{\delta {S_t}^2} (\delta S_t)^2 \end{cases}$$

$$\Rightarrow \begin{cases} \delta S_t = \mu S_t \delta t + \sigma S_t \varphi \sqrt{\delta t} \\ \delta V(S_t, t) = \theta_t(\tilde{\sigma}) \delta t + \Delta_t(\tilde{\sigma}) \left[\mu S_t \delta t + \sigma S_t \varphi \sqrt{\delta t} \right] + \dfrac{1}{2} \Gamma_t(\tilde{\sigma}) (\sigma S_t)^2 \varphi^2 \delta t \end{cases}$$

$$\Rightarrow \begin{cases} \delta S_t = \mu S_t \delta t + \sigma S_t \varphi \sqrt{\delta t} \\ \delta V(S_t, t) = \left[\theta_t(\tilde{\sigma}) + \Delta_t(\tilde{\sigma}) \mu S_t + \dfrac{1}{2} \Gamma_t(\tilde{\sigma}) (\sigma S_t)^2 \varphi^2 \right] \delta t + \Delta_t(\tilde{\sigma}) \sigma S_t \varphi \sqrt{\delta t} \end{cases}$$

At this step, a quick (but important) comment: the reader with Black-Scholes framework knowledge may be surprised about the final equation for option diffusion, as it embeds Black-Scholes greeks (theta, gamma, delta), all based on risk-free rate (risk-neutral framework), and real-world drift. It sounds incoherent, and the author indeed thinks it is: assuming on one side risk-neutral quantities and on the other side a real-world quantity can't be true. But we'll see in next delta approximation section that we can bypass this "debate", and the hedged position section will also highlight a "coherent" derivation.

A less important but connected topic is the choice of volatility: once again for BS greeks implied volatility should be used but actual volatility (the real one of the assets, but which can't be captured by financial products, only through backward observation) should be used for the diffusion itself. We can argue that both should be, theoretically speaking, equal, and then we won't dig deeper in that direction, and we'll assume that equality for next numerical experiments.

In a nutshell: the derivation is a bit incoherent, but we can argue that ***<u>quantities used for derivation (greeks)</u>*** should be risk-neutral ones (because we need metrics based on pricing of the derivative) but as we're interested in ***<u>risk</u>*** in this value at risk context the diffusion ***<u>itself</u>*** may embed real-world drift of the stock risk-factor, because the goal is risk-management (and eventually real losses).

With previous remarks in mind, let's focus on all risk-factors embedded in the option return diffusion:

- The underlying spot value.
- The actual/implied volatility but assumed to be constant with our BS choice. Relaxing this assumption, for example with stochastic volatility, will lead to a more complex $\delta V(S_t, t)$, "à la" Heston.
- The risk-free interest rate, but once again assume to be constant. Relaxing this assumption will lead to hybrid derivation, following classical BS-Hull & White for example.
- The real-world underlying drift, not only considered as constant but simply assumed to be known!

Please also note that even if we don't "enhance" the model with additional stochasticity, we can even calculate "shadow greeks" for our simple risk model by bumping volatility and/or risk-free rate (or even real-world drift), calculate again the new return and eventually new value at risk/expected shortfall, and then highlight impacts of those risk factors modification. The reader with habits about vega/rho in BS context won't be surprised by this "tweak".

After all those long (but necessary) digressions, let's go back to our parametric VaR/ES, starting with the simplest example, the delta approximation.

2/b - Delta approximation

The delta approximation takes three assumptions:

1. Short-term overview: we ignore the "view" of the underlying itself, then only Brownian part is kept.
2. Stock movements are considered as small.
3. And the most important: $\delta V(S_t, t)$ is assumed to follow a normal distribution (strong assumption)

The equation system then becomes:

$$\begin{cases} \delta S_t = \sigma S_t \varphi \sqrt{\delta t} \\ \delta V(S_t, t) = \dfrac{\delta V(S_t,t)}{\delta S_t} \delta S_t = \Delta_t(\tilde{\sigma}) \sigma S_t \varphi \sqrt{\delta t} \end{cases}$$

All components with δt are excluded using this approximation, assuming (theoretical) short-term and hence $\sqrt{\delta t} \gg \delta t$. As discussed in previous section, the delta approximation, assuming short-term and tiny stock movements, also gets rid of the real-world drift quantity. Saying that, it leads to:

$$\Rightarrow \begin{cases} \mathbb{E}(\delta S_t) = 0, \mathbb{V}(\delta S_t) = (\sigma S_t)^2 \delta t \\ \mathbb{E}\big(\delta V(S_t, t)\big) = 0, \mathbb{V}\big(\delta V(S_t, t)\big) = (\Delta_t(\tilde{\sigma}) \sigma S_t)^2 \delta t \end{cases}$$

Assuming $\delta V(S_t, t)$ follows a normal distribution, we can define our $VaR_{\delta t,\alpha}$:

$$\begin{cases} \delta V(S_t, t) \sim \mathcal{N}(0, 2(\Delta_t(\tilde{\sigma}) \sigma S_t)^2 \delta t\,) \\ \mathbb{P}\big(\delta V(S_t, t) \ \leq -VaR_{\delta t,\alpha}\big) = 1 - \alpha \end{cases}$$

$$\Rightarrow \mathbb{P}\left(\varphi \ \leq \frac{-VaR_{\delta t,\alpha}}{\Delta_t(\tilde{\sigma}) \sigma S_t \sqrt{\delta t}}\right) = 1 - \alpha$$

$$\Rightarrow \frac{-VaR_{\delta t,\alpha}}{\Delta_t(\tilde{\sigma}) \sigma S_t \sqrt{\delta t}} = \mathcal{N}^{-1}(1 - \alpha)$$

$$\Rightarrow VaR_{\delta t,\alpha} = -\Delta_t(\tilde{\sigma}) \sigma S_t \sqrt{\delta t} \mathcal{N}^{-1}(1 - \alpha)$$

As an example, and following the previous focus on risk factors impacts, we can also derive the impact of underlying spot risk factor on value at risk quantity:

$$\frac{\delta VaR_{\delta t,\alpha}}{\delta S_t} = -\left[\frac{\delta V \Delta_t(\tilde{\sigma})}{\delta S_t} S_t + \Delta_t(\tilde{\sigma})\right] \sigma \sqrt{\delta t} \mathcal{N}^{-1}(1 - \alpha)$$

$$\Rightarrow \frac{\delta VaR_{\delta t,\alpha}}{\delta S_t} = -[\Gamma_t(\tilde{\sigma}) S_t + \Delta_t(\tilde{\sigma})] \sigma \sqrt{\delta t} \mathcal{N}^{-1}(1 - \alpha)$$

The interested reader may look at annex #1 for other value at risk “greeks” and please note that those derivations won’t be done for next delta-theta-gamma approximation and let as exercise.

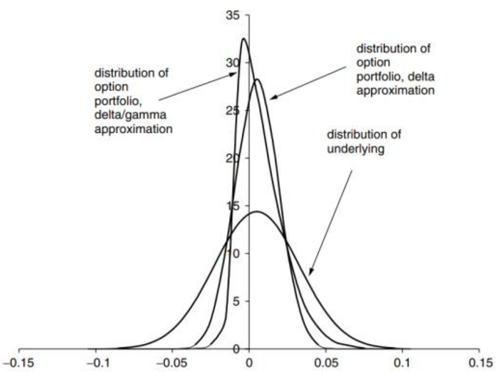
2/c - Delta-Theta-Gamma approximation

Now let’s relaxed previous delta-approximations assumptions and let’s remember general equations system:

$$\begin{cases} \delta S_t = \mu S_t \delta t + \sigma S_t \varphi \sqrt{\delta t} \\ \delta V(S_t, t) = \left[\theta_t(\tilde{\sigma}) + \Delta_t(\tilde{\sigma}) \mu S_t + \dfrac{1}{2} \Gamma_t(\tilde{\sigma}) (\sigma S_t)^2 \boldsymbol{\varphi^2}\right] \delta t + \Delta_t(\tilde{\sigma}) \sigma S_t \boldsymbol{\varphi} \sqrt{\delta t} \end{cases}$$

Key point is that, contrary to previous delta approximation, $\delta V(S_t, t)$ can’t be considered as gaussian anymore, due to additional φ^2, which in that case follows a χ^2 distribution with 1 degree of freedom. The distribution of $\delta V(S_t, t)$ is then a mixture between gaussian and chi-squared distributions, and unfortunately this kind of mixture doesn’t have standard or well-known distributions form. Then we have basically three solutions:

- “Force” gaussian behavior, by assuming $\delta V(S_t, t)$ still follows a normal distribution with more complex parameters. As the writer considers this approach as a “tweak” (the resulting distribution will be still symmetric, which is basically not the behavior we want to observe, especially if we’re interested in loss risk), the interested reader may look at annex #2 and should not consider this approach as the correct one. The qualitative behavior of different distributions can be checked below:



Source : Paul Wilmott on quantitative finance, second edition, page 337

- By assuming a (very) long-term overview, we'll reach the $\sqrt{\delta t} \ll \delta t$ case, which is equivalent to mention that the "drift" (in commas as this quantity still embed a stochastic chi-squared element) part of $\delta V(S_t, t)$ become preeminent, compared to the Brownian part. In that case the $\delta V(S_t, t)$ distribution will be a chi-squared one (and in fact degenerates to deterministic quantity, as highlighted in the annex). As this formulation is only for long-term, which is not the framework/purpose of this document, the interested reader may look at annex #3.
- Using numerical method, basically simulation. Here lies the previously spotted Monte-Carlo value at risk, which will be discussed in next chapter.

2/d - Hedged position

Until this section, we analyzed what we can define as "naked" option position, considering the derivative is stand-alone. Let's now look at portfolio based on an option and its dynamic hedge. The framework is a little bit different:

$$\begin{cases} \delta S_t = \mu S_t \delta t + \sigma S_t \varphi \sqrt{\delta t} \\ \delta \Pi(S_t, t) = \delta V(S_t, t) - \Delta_t(\tilde{\sigma}) \delta S_t = \dfrac{\delta V(S_t, t)}{\delta t} \delta t + \dfrac{\delta V(S_t, t)}{\delta S_t} \delta S_t + \dfrac{1}{2} \dfrac{\delta^2 V(S_t, t)}{\delta {S_t}^2} (\delta S_t)^2 - \Delta_t(\tilde{\sigma}) \delta S_t \end{cases}$$

$$\Rightarrow \begin{cases} \delta S_t = \mu S_t \delta t + \sigma S_t \varphi \sqrt{\delta t} \\ \delta \Pi(S_t, t) = \theta_t(\tilde{\sigma}) \delta t + \dfrac{1}{2} \Gamma_t(\tilde{\sigma}) (\sigma S_t)^2 \varphi^2 \delta t \end{cases}$$

No surprises: the real-world drift vanished (just like in BS pricing framework) and we retrieve the well-known BS P&L formulation. An interesting feature appears: contrary to previous naked option position using delta-theta-gamma approximation, we have the possibility to derive a closed-form formula for the value at risk, because $\delta \Pi(S_t, t)$ is only linked to φ^2 distribution and not the previous (φ^2, φ). The simulation solution is still doable, but let's first focus on the parametric value at risk formula:

$$\Rightarrow \mathbb{P}\left(\delta \Pi(S_t, t) \leq -VaR_{\delta t, \alpha}\right) = 1 - \alpha$$

$$\Leftrightarrow \mathbb{P}\left(\varphi^2 \leq \frac{-VaR_{\delta t, \alpha} - \theta_t(\tilde{\sigma}) \delta t}{\frac{1}{2} \Gamma_t(\tilde{\sigma}) (\sigma S_t)^2 \delta t}\right) = 1 - \alpha$$

$$\Rightarrow \frac{-VaR_{\delta t, \alpha} - \theta_t(\tilde{\sigma}) \delta t}{\frac{1}{2} \Gamma_t(\tilde{\sigma}) (\sigma S_t)^2 \delta t} = {\chi^2}^{-1}(1 - \alpha)$$

$$\Rightarrow VaR_{\delta t, \alpha} = -\frac{1}{2} \Gamma_t(\tilde{\sigma}) (\sigma S_t)^2 \delta t {\chi^2}^{-1}(1 - \alpha) - \theta_t(\tilde{\sigma}) \delta t$$

3 – Monte-Carlo Value at Risk and Expected shortfall

3/a – General framework

Instead of finding a closed-form formula for the value at risk, the Monte-Carlo methods relies on simulation of N possible paths for $\delta \Pi(S_t, t)$, which will give a distribution sample, after sorting results. The next step is quite straightforward: considering this distribution, we just must capture the dedicated quantile for our value at risk calculation.

The calculation of expected shortfall using this method is also quite simple: once the previous quantile is calculated, we simply must calculate the mean of all values of the distribution which ranks are inferior to the quantile.

3/b – Algorithms

As mathematical frameworks are identical to previous ones, we'll just highlight three different algorithms, which basically share the same pattern:

<u>Monte-Carlo Value at Risk – delta approximation</u>

Start
Initialize/calculate $(\mu, S_t, \delta t, \sigma, \tilde{\sigma}, \Delta_t(\tilde{\sigma}))$ Generate N normal random numbers $(\varphi_1, \dots, \varphi_N)$ Initialize a result vector v (length N) Loop on $(\varphi_1, \dots, \varphi_N)$ v[i] = $\Delta_t(\tilde{\sigma}) \sigma S_t \varphi_i \sqrt{\delta t}$ Sort v Return the dedicated quantile, according to confidence level α → VaR Return the v average until –VAR → ES End

Monte-Carlo Value at Risk – delta/gamma/theta approximation

Start

Initialize/calculate $\left(\mu, S_t, \delta t, \sigma, \tilde{\sigma}, \theta_t(\tilde{\sigma}), \Delta_t(\tilde{\sigma}), \Gamma_t(\tilde{\sigma})\right)$
Generate N normal random numbers $(\varphi_1, \dots, \varphi_N)$
Initialize a result vector v (length N)
Loop on $(\varphi_1, \dots, \varphi_N)$
    v[i] = $\left[\theta_t(\tilde{\sigma}) + \Delta_t(\tilde{\sigma})\mu S_t + \frac{1}{2}\Gamma_t(\tilde{\sigma})(\sigma S_t)^2 {\varphi_i}^2\right]\delta t + \Delta_t(\tilde{\sigma})\sigma S_t \varphi_i \sqrt{\delta t}$
Sort v
Return the dedicated quantile, according to confidence level α→ VaR
Return the v average until –VAR → ES

End

Monte-Carlo Value at Risk – delta/gamma/theta approximation for hedged position

Start

Initialize/calculate $\left(S_t, \delta t, \sigma, \tilde{\sigma}, \theta_t(\tilde{\sigma}), \Gamma_t(\tilde{\sigma})\right)$
Generate N normal random numbers $(\varphi_1, \dots, \varphi_N)$
Initialize a result vector v (length N)
Loop on $(\varphi_1, \dots, \varphi_N)$
    v[i] = $\theta_t(\tilde{\sigma})\delta t + \frac{1}{2}\Gamma_t(\tilde{\sigma})(\sigma S_t)^2 {\varphi_i}^2 \delta t$
Sort v
Return the dedicated quantile, according to confidence level α→ VaR
Return the v average until –VAR → ES

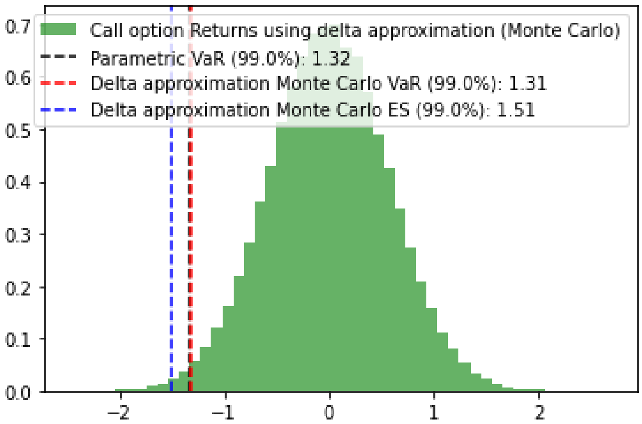
End

III – Numerical results

For all numerical experiments, following parameters set is used (as also highlighted in the python code) and the option quantity is assumed to be 1:

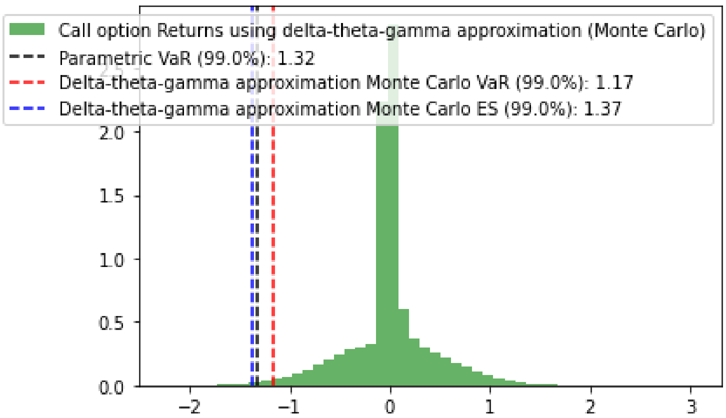
```
# Parameters
S0 = 100  # Current stock price
K = 100   # Strike price
DCF = 365 # Day-count fraction
T = 1/DCF # VaR period of time
T_opt = 0.1 # Option maturity (in years)
r = 0.05  # Risk-free rate
mu = 0.05 # Real-world stock drift
sigma = 0.2  # Actual Volatility
sigma_implied = 0.2 # Implied Volatility

# Confidence level
alpha = 0.99

# Number of simulation paths
nbSimul = 100000
```

1 – Parametric VaR VS MC VaR/ES using delta approximation



As expected, the expected shortfall is higher (more conservative) than the value at risk. In addition, with relatively high number of simulation paths, parametric value at risk and Monte-Carlo value at risk are quite close, which is obviously expected.

2 – Parametric VaR VS MC VaR/ES using delta-theta-gamma approximation

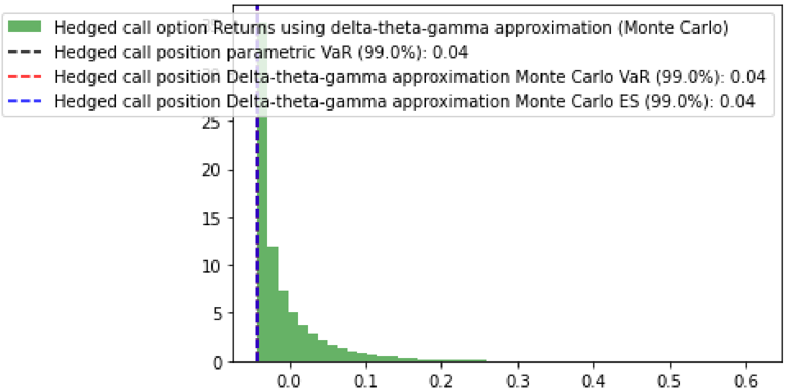


We can observe that the sample distribution is highly leptokurtic, compared to previous quasi-normal sample distribution using the delta approximation, and calculus indeed highlight this characteristic:

```
The skew of delta approximation MC call return is : -0.02
The kurtosis of delta approximation MC call return is : 0.01
The skew of delta-theta-gamma approximation MC call return is : -0.02
The kurtosis of delta-theta-gamma approximation MC call return is : 2.93
```

But far more interesting, we can observe that VaR/ES using this approximation are ***less*** conservative than previous results. The convexity and time-effect effects are magnified by the relatively high level of volatility, here 20%. Please note that if we strongly decrease the volatility level, results become similar between two approximation methods, as expected.

3 – Parametric VaR VS MC VaR using delta-theta-gamma approximation with hedged call position



As expected and due to additional hedging, the risk strongly decreased. It's not a surprise, before being a tool for risk-neutral pricing, the hedge is a risk-management tool!

4 – Results summary

Metric definition	Metric value
Parametric VaR (99.0%)	1.324979
Delta approximation Monte Carlo VaR (99.0%)	1.314369
Delta approximation Monte Carlo ES (99.0%)	1.505449
Delta-theta-gamma approximation Monte Carlo VaR (99.0%)	1.173567
Delta-theta-gamma approximation Monte Carlo ES (99.0%)	1.374101
Hedged call position parametric VaR (99.0%)	0.041420
Hedged call position Delta-theta-gamma approximation Monte Carlo VaR (99.0%)	0.041420
Hedged call position Delta-theta-gamma approximation Monte Carlo ES (99.0%)	0.041424

In a nutshell:

- Methods per methods, expected shortfall is always higher than value at risk.
- Delta-theta-gamma approximation leads to less conservative risk metrics than delta approximation method.
- Hedging drastically decreases the risk, evaluated as value at risk or expected shortfall.

IV - Conclusion

The author hopes that the paper highlighted value at risk and expected shortfall in clear manners. The target use case was a simple equity call position and of course several “add-ons” arise if we’ll start to focus on complex portfolio:

- Risk factors list will be modified if we add interest rate (linear or non-linear) / credit / fx / commodities etc… products.
- Even if we still focus on equity products, a portfolio based on derivatives with several underlying will lead to additional correlation factors (notoriously difficult to capture and instable, even with available and liquid correlation products like CDOs etc…), in addition of volatilities.

Apart this “technical” comment, and as open areas, the reader can be interested by usage of value at risk in other contexts than risk management. This metric is indeed also useful in pricing context for margin value adjustment (MVA) and in collateral context (repurchase agreement haircut models).

V - Sources

1 - Value at risk/Expected shortfall methodologies

An introduction to market risk measurement, Kevin Dowd

Paul Wilmott on quantitative finance, second edition

2 - Technical ressources

https://github.com/LechGrzelak, Lech Grzelak

https://www.finmath.net/finmath-lib/, Christian Fries

VI – Annexes

Annex #1 – $VaR_{\delta t,\alpha}$ greeks for Delta-Gamma-Theta approximation

Risk factor	Greek
S_t	$\frac{\delta VaR_{\delta t,\alpha}}{\delta S_t} = -[\Gamma_t(\tilde{\sigma})S_t + \Delta_t(\tilde{\sigma})]\sigma\sqrt{\delta t}\mathcal{N}^{-1}(1-\alpha)$
σ, assuming $\sigma = \tilde{\sigma}$	$\frac{\delta VaR_{\delta t,\alpha}}{\delta \sigma} = -\left[\frac{\delta V\Delta_t(\sigma)}{\delta\sigma}\sigma + \Delta_t(\sigma)\right] S_t\sqrt{\delta t}\mathcal{N}^{-1}(1-\alpha)$
r	$\frac{\delta VaR_{\delta t,\alpha}}{\delta r} = -\frac{\delta V\Delta_t(\sigma)}{\delta r}\sigma S_t\sqrt{\delta t}\mathcal{N}^{-1}(1-\alpha)$

Annex #2 – $\delta V(S_t, t)$ Gaussian assumption for Delta-Gamma-Theta approximation

Let's remember diffusions' equations:

$$\begin{cases} \delta S_t = \mu S_t \delta t + \sigma S_t \varphi\sqrt{\delta t} \\ \delta V(S_t,t) = \left[\theta_t(\tilde{\sigma}) + \Delta_t(\tilde{\sigma})\mu S_t + \frac{1}{2}\Gamma_t(\tilde{\sigma})(\sigma S_t)^2\varphi^2\right]\delta t + \Delta_t(\tilde{\sigma})\sigma S_t\varphi\sqrt{\delta t} \end{cases}$$

Due to additional φ^2 element, which is nothing more than a χ^2 distribution with 1 degree of freedom, the variance derivation of $\delta V(S_t, t)$ is a bit more complicated but still achievable:

$$\mathbb{V}(\delta V(S_t,t)) = \left(\frac{1}{2}\Gamma_t(\tilde{\sigma})(\sigma S_t)^2\delta t\right)^2 \mathbb{V}(\varphi^2) + \left(\Delta_t(\tilde{\sigma})\sigma S_t\sqrt{\delta t}\right)^2\mathbb{V}(\varphi) + 2\left(\frac{1}{2}\Gamma_t(\tilde{\sigma})(\sigma S_t)^2\delta t\right)\left(\Delta_t(\tilde{\sigma})\sigma S_t\sqrt{\delta t}\right)Cov(\varphi^2,\varphi)$$

$$\Rightarrow \mathbb{V}(\delta V(S_t,t)) = 2\left(\frac{1}{2}\Gamma_t(\tilde{\sigma})(\sigma S_t)^2\delta t\right)^2 + \left(\Delta_t(\tilde{\sigma})\sigma S_t\sqrt{\delta t}\right)^2 + 2\left(\frac{1}{2}\Gamma_t(\tilde{\sigma})(\sigma S_t)^2\delta t\right)\left(\Delta_t(\tilde{\sigma})\sigma S_t\sqrt{\delta t}\right)\mathbb{E}(\varphi^3 - \varphi)$$

$$\xRightarrow{\mathbb{E}(\varphi^3-\varphi)=0} \mathbb{V}(\delta V(S_t,t)) = 2\left(\frac{1}{2}\Gamma_t(\tilde{\sigma})(\sigma S_t)^2\delta t\right)^2 + \left(\Delta_t(\tilde{\sigma})\sigma S_t\sqrt{\delta t}\right)^2$$

which eventually leads to :

$$\begin{cases} \mathbb{E}(\delta S_t) = \mu S_t\delta t, \mathbb{V}(\delta S_t) = (\sigma S_t)^2\delta t \\ \mathbb{E}(\delta V(S_t,t)) = \left[\theta_t(\tilde{\sigma}) + \Delta_t(\tilde{\sigma})\mu S_t + \frac{1}{2}\Gamma_t(\tilde{\sigma})(\sigma S_t)^2\right]\delta t \\ \mathbb{V}(\delta V(S_t,t)) = 2\left(\frac{1}{2}\Gamma_t(\tilde{\sigma})(\sigma S_t)^2\delta t\right)^2 + \left(\Delta_t(\tilde{\sigma})\sigma S_t\sqrt{\delta t}\right)^2 \end{cases}$$

$$\Rightarrow \delta V(S_t,t) \sim \mathcal{N}\left(\left[\theta_t(\tilde{\sigma}) + \Delta_t(\tilde{\sigma})\mu S_t + \frac{1}{2}\Gamma_t(\tilde{\sigma})(\sigma S_t)^2\right]\delta t, 2\left(\frac{1}{2}\Gamma_t(\tilde{\sigma})(\sigma S_t)^2\delta t\right)^2 + \left(\Delta_t(\tilde{\sigma})\sigma S_t\sqrt{\delta t}\right)^2\right)$$

Assuming the normality of $\delta V(S_t, t)$, the $VaR_{\delta t,\alpha}$ derivation can be calculated with same methods than delta approximation (gaussian quantile).

Annex #3 – $\delta V(S_t, t)$ long-term assumption for Delta-Gamma-Theta approximation

- The watchful reader may rightfully notice that if we indeed take the long-term assumption for $\delta V(S_t, t)$'s diffusion, same should be done for the underlying itself, which leads to following equations:

  $$\begin{cases} \delta S_t = \mu S_t\delta t \\ \delta V(S_t,t) = [\theta_t(\tilde{\sigma}) + \Delta_t(\tilde{\sigma})\mu S_t]\delta t \end{cases}$$

  Everything become deterministic (which is a classical "result" of long-term assumption in finance and economy: the author insists it's ***very*** theoretical) and in that case the $\mathbb{P}(\delta V(S_t,t) \leq -VaR_{\delta t,\alpha}) = 1-\alpha$ formula become meaningless, as the left part is equal to 0 or 1, and for sure (no more stochasticity, it's a simple comparison between constants)

- If we still want a "meaningful" value at risk quantity, we can consider the following experiment (the author considers it as incoherent, "splitting" long-term assumption between underlying and derivative): we can keep the Brownian stochasticity of the underlying (but still "ignoring" the option's one), and as we already highlighted in hedge position section, we can derive the following value at risk:

  $$\begin{cases} \delta S_t = \mu S_t\delta t + \sigma S_t\varphi\sqrt{\delta t} \\ \delta V(S_t,t) = \left[\theta_t(\tilde{\sigma}) + \Delta_t(\tilde{\sigma})\mu S_t + \frac{1}{2}\Gamma_t(\tilde{\sigma})(\sigma S_t)^2\varphi^2\right]\delta t \end{cases}$$

  $$\Rightarrow \mathbb{P}(\delta V(S_t,t) \leq -VaR_{\delta t,\alpha}) = 1-\alpha$$

$$\Leftrightarrow \mathbb{P}\left(\varphi^2 \ \leq \frac{-VaR_{\delta t,\alpha} - \delta t(\theta_t(\tilde{\sigma}) + \Delta_t(\tilde{\sigma})\mu S_t)}{\frac{1}{2}\Gamma_t(\tilde{\sigma})(\sigma S_t)^2 \delta t}\right) = 1 - \alpha$$

$$\Rightarrow \frac{-VaR_{\delta t,\alpha} - \delta t(\theta_t(\tilde{\sigma}) + \Delta_t(\tilde{\sigma})\mu S_t)}{\frac{1}{2}\Gamma_t(\tilde{\sigma})(\sigma S_t)^2 \delta t} = \chi^{2^{-1}}(1 - \alpha)$$

$$\Rightarrow VaR_{\delta t,\alpha} = -\frac{1}{2}\Gamma_t(\tilde{\sigma})(\sigma S_t)^2 \delta t \chi^{2^{-1}}(1 - \alpha) - \delta t(\theta_t(\tilde{\sigma}) + \Delta_t(\tilde{\sigma})\mu S_t)$$