TRS on Bonds – Explicit difference finite method

I – Introduction

Total return swap on debt products; including Bonds, loans, or index like IBoxxes, are, functionally speaking, very similar than their “brothers” Total return swap on equities. It shares exact same mechanisms (payer/receiver performance against receiver/payer financing) but their pricing in Black-Scholes finite difference context highlights a second order complexity due to Bond’s underlying, and this paper aims to highlight a model which takes this subtlety into account.

Following models assume the next list of assumptions, which will be discussed in final part of this paper:

- TRS is a bullet one, with fixed financing rate., assumed to be a performance receiver.
- Bond’s underlying is a coupon-bearing one, without default risk.
- Interest rate will follow an equilibrium model.
- Interest rate can be negative.

II – Bond pricing model

It’s well-known and especially highlighted in Paul Wilmott’s book that Bond product, contrary to real equity *__underlying__*, can be in fact seen as a *__derivative__* on the interest rate, and this latter is the real underlying. As interest rate is not a tradable quantity, a famous scheme involving risk-neutrality is invoked (we don’t highlight it as it’s a “classic” black Scholes derivation than reader can directly, if he’s interested in, find in the book) to find the Bond’s pricing equation:

$$Bond\ pricing\ scheme:\begin{cases}\dfrac{\partial V}{\partial t}+\dfrac{1}{2}\omega^2\dfrac{\partial^2 V}{\partial r^2}+(u-\lambda\omega)\dfrac{\partial V}{\partial r}-rV=0\\ dr=(u-\lambda\omega)dt+\ \omega dW\end{cases}$$

$$Bond\ pricing\ conditions:\begin{cases}V(r,T,T)=1+c\\ V(r,t_c^-,T)=\ V(r,t_c^+,T)+c\end{cases}$$

With following notations:

- V the Bond’s value.
- ω the volatility of interest rate dynamic
- u the real-world drift of interest rate dynamic
- λ the interest rate market price of risk
- Hence $(u-\lambda\omega)$ the risk-neutral drift of interest rate dynamic
- c coupon payments
- T the Bond’s maturity
- W a Brownian motion

To find Bond’s price, the classical backward resolution (here through explicit Euler scheme) should be achieved using two-dimensional grid $\{t\in[T^{TRS},T],r\in[r^{min},r^{max}]\}$. A specific attention point about jump condition before/after coupon payment.

III – TRS pricing model

The TRS pricing scheme is very similar to equity compound options/bond options schemes, with second-order feature:

- The previous bond's scheme is solved (still backward) from T to T^{TRS}, which gives a set of prices $(V^{i}_{T^{TRS}})_{i\in[r^{min},r^{max}]}$.
- Then the outer scheme is solved following equations, as the TRS is also a derivative on interest rate in this context, through its dependency to Bond's price:

$$TRS\ pricing\ scheme:\begin{cases}\dfrac{\partial TRS}{\partial t}+\dfrac{1}{2}\omega^2\dfrac{\partial^2 TRS}{\partial r^2}+(u-\lambda\omega)\dfrac{\partial TRS}{\partial r}-rTRS=0\\ dr=(u-\lambda\omega)dt+\ \omega dW\end{cases}$$

$$TRS\ pricing\ conditions:\begin{cases}TRS(r,T^{TRS})=V_{T^{TRS}}-K-\ s_{TRS}T^{TRS}\\ TRS(r,t_c^-)=\ TRS(r,t_c^+)+c\end{cases}$$

With additional notations:

- TRS the TRS market value
- K the TRS settlement price (aka strike)
- s_{TRS} the fixed and agreed TRS financing spread.
- T^{TRS} the TRS maturity

To find TRS market value, once again the classical backward resolution is done using two-dimensional grid $\{t\in[0,T^{TRS}],r\in[r^{min},r^{max}]\}$. We should again point the jump condition after Bond's coupon payment, which is part of the performance.

IV – Finite difference algorithm

As we especially excluded non-negativity assumption for interest rate (to tackle possible negative interest), the "natural" (natural choice for an equilibrium model, as please remember that we took this assumption) choice for risk-neural interest rate dynamic is Vasicek model:

$$dr_t = \alpha(\beta - r_t)dt + \sigma dW_t$$

With following notations:

- α the interest rate speed reversion to long-term mean
- β the interest rate long-term mean.

Pricing schemes then became, respectively for Bond and for TRS:

$$pricing\ schemes : \begin{cases} \dfrac{\partial V_t}{\partial t} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 V_t}{\partial r^2} + \alpha(\beta - r_t)\dfrac{\partial V_t}{\partial r} - r_t V_t = 0 \\ \dfrac{\partial TRS_t}{\partial t} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 TRS_t}{\partial r^2} + \alpha(\beta - r_t)\dfrac{\partial TRS_t}{\partial r} - r_t TRS_t = 0 \end{cases}$$

$$\xrightarrow{Euler\ discretization} \begin{cases} \dfrac{V_t^k - V_{t+1}^k}{\delta t} + \dfrac{1}{2}\sigma^2 \dfrac{V_t^{k+1} - 2V_t^k + V_t^{k-1}}{(\delta r)^2} + \alpha(\beta - r^k)\dfrac{V_t^{k+1} - V_t^{k-1}}{2\delta r} - r^k V_t^k = 0 \\ \dfrac{TRS_t^k - TRS_{t+1}^k}{\delta t} + \dfrac{1}{2}\sigma^2 \dfrac{TRS_t^{k+1} - 2TRS_t^k + TRS_t^{k-1}}{(\delta r)^2} + \alpha(\beta - r^k)\dfrac{TRS_t^{k+1} - TRS_t^{k-1}}{2\delta r} - r^k TRS_t^k = 0 \end{cases}$$

$$\Rightarrow \begin{cases} V_{t+1}^k = a^k V_t^{k-1} + b^k V_t^k + c^k V_t^{k+1} \\ TRS_{t+1}^k = a^k TRS_t^{k-1} + b^k TRS_t^k + c^k TRS_t^{k+1} \end{cases}$$

With:

$$(\boldsymbol{inner\ grid\ points}) \begin{cases} a^k = \dfrac{\delta t}{2\delta r}\left(\dfrac{\sigma^2}{\delta r} - \alpha(\beta - r^k)\right) \\ b^k = -\dfrac{\sigma^2 \delta t}{(\delta r)^2} - r^k \delta t + 1 \\ c^k = \dfrac{\delta t}{2\delta r}\left(\dfrac{\sigma^2}{\delta r} + \alpha(\beta - r^k)\right) \end{cases}$$

This discretized scheme will be followed for both Bond (inner scheme) and eventually TRS (outer scheme), taking into respective final and jump conditions.

The previous scheme should be used for inner grid points, but we need to modify a bit the discretization for lower/upper interest rate of the grid:

<u>Lower limit case:</u>

- The first order derivative, initially calculated using central finite difference, should be calculated using forward formulation: $\frac{V_t^{k+1} - V_t^k}{\delta r}$
- The second order derivative should be also modified using forward formulation. We can't rely on V_t^k / V_t^{k+2} as second order derivatives offset (the reader can do the Taylor math if he/she want) and we should choose : $\frac{-\frac{2}{3}V_t^k + V_t^{k+1} - \frac{1}{3}V_t^{k+3}}{(\delta r)^2}$

The scheme is then:

$$\begin{cases} \dfrac{V_t^k - V_{t+1}^k}{\delta t} + \dfrac{1}{2}\sigma^2 \dfrac{-\frac{2}{3}V_t^k + V_t^{k+1} - \frac{1}{3}V_t^{k+3}}{(\delta r)^2} + \alpha(\beta - \ r^k)\dfrac{V_t^{k+1} - V_t^k}{\delta r} - r^k V_t^k = 0 \\ \dfrac{TRS_t^k - TRS_{t+1}^k}{\delta t} + \dfrac{1}{2}\sigma^2 \dfrac{-\frac{2}{3}TRS_t^k + TRS_t^{k+1} - \frac{1}{3}TRS_t^{k+3}}{(\delta r)^2} + \alpha(\beta - \ r^k)\dfrac{TRS_t^{k+1} - TRS_t^k}{\delta r} - r^k TRS_t^k = 0 \end{cases}$$

$$\Rightarrow \begin{cases} V_{t+1}^k = a^k V_t^k + b^k V_t^{k+1} + c^k V_t^{k+3} \\ TRS_{t+1}^k = a^k TRS_t^k + b^k TRS_t^{k+1} + c^k TRS_t^{k+3} \end{cases}$$

With:

$$(\boldsymbol{lower\ grid\ points}) \begin{cases} a^k = \delta t\left(-\dfrac{\sigma^2}{3(\delta r)^2} - \dfrac{1}{\delta r}\alpha(\beta - \ r^k) - r^k\right) + 1 \\ b^k = \dfrac{\delta t}{\delta r}\left(\dfrac{\sigma^2}{2\delta r} + \alpha(\beta - \ r^k)\right) \\ c^k = -\dfrac{\delta t \sigma^2}{6(\delta r)^2} \end{cases}$$

<u>Upper limit case:</u>

- The first order derivative, initially calculated using central finite difference, should be calculated using backward formulation: $\frac{V_t^k - V_t^{k-1}}{\delta r}$
- The second order derivative should be also modified using backward formulation, and this time we can rely on V_t^{k-2} and not V_t^{k-3} (same thing the reader can prove it by simple Taylor expansion), but for "symmetry" reason we'll continue with following scheme : $\frac{\frac{2}{3}V_t^k - V_t^{k-1} + \frac{1}{3}V_t^{k-3}}{(\delta r)^2}$

The scheme is then:

$$\begin{cases} \dfrac{V_t^k - V_{t+1}^k}{\delta t} + \dfrac{1}{2}\sigma^2 \dfrac{\frac{2}{3}V_t^k - V_t^{k-1} + \frac{1}{3}V_t^{k-3}}{(\delta r)^2} + \alpha(\beta - \ r^k)\dfrac{V_t^k - V_t^{k-1}}{\delta r} - r^k V_t^k = 0 \\ \dfrac{TRS_t^k - TRS_{t+1}^k}{\delta t} + \dfrac{1}{2}\sigma^2 \dfrac{\frac{2}{3}TRS_t^k - TRS_t^{k-1} + \frac{1}{3}TRS_t^{k-3}}{(\delta r)^2} + \alpha(\beta - \ r^k)\dfrac{TRS_t^k - TRS_t^{k-1}}{\delta r} - r^k TRS_t^k = 0 \end{cases}$$

$$\Rightarrow \begin{cases} V_{t+1}^k = a^k V_t^k + b^k V_t^{k-1} + c^k V_t^{k-3} \\ TRS_{t+1}^k = a^k TRS_t^k + b^k TRS_t^{k-1} + c^k TRS_t^{k-3} \end{cases}$$

With:

$$(\boldsymbol{upper\ grid\ points}) \begin{cases} a^k = \delta t\left(\dfrac{\sigma^2}{3(\delta r)^2} + \dfrac{1}{\delta r}\alpha(\beta - \ r^k) - r^k\right) + 1 \\ b^k = \dfrac{\delta t}{\delta r}\left(-\dfrac{\sigma^2}{2\delta r} - \alpha(\beta - \ r^k)\right) \\ c^k = \dfrac{\delta t \sigma^2}{6(\delta r)^2} \end{cases}$$

V – Finite difference algorithm – Stability and convergence

As explicit finite difference model, results are extremely sensitive to model parameters, and for stability purpose, following links between quantities should be set:

$$\begin{cases} (\mathbf{1})\ -r^k \ \leq 0 \\ (\mathbf{2})\ \delta t \ \leq \dfrac{(\delta r)^2}{\sigma^2} \\ (\mathbf{3})\ \delta r \ \leq \dfrac{\sigma^2}{|\alpha(\beta - \ r^k)|} \end{cases}$$

- $(\mathbf{1})$ is sustained if and only if interest rates are considered as non-negative.
- $(\mathbf{3})$ should be sustained taking into $(\alpha, \beta, r^k, \sigma)$ as inputs, with careful calculation handled by the algorithm itself.
- Then $(\mathbf{2})$ should be sustained too, once again the algorithm should choose a dedicated δt

VI – Model assumptions, discussion, and next axis

- As second-order pricing model, it suffers exact same drawbacks than Bond option: Bond(s price at TRS maturity are evaluated ***theoretically***, but the TRS will eventually use ***market*** price. The Bond's model accuracy is then critical, as every errors in inner loop will be magnified in the outer loop (TRS).
- We assumed to choose a very simple Vasicek model (equilibrium without curve fitting), which is not the market practice, and the model should eventually be enhanced with short-rate dynamic in HJM framework.
- Relaxing TRS bullet feature will have several impacts:
    - Bond price calculation should be done until first reset TRS date and no more only until TRS maturity.
    - Jump condition should be incorporated in the model for each TRS payment date.
- Relaxing TRS fixed rate feature from floating rate (Libor-like or RFR) leads to a third dimension, which will be discussed in VII-Two-factor explicit, with both stochastic asset and interest rate.
- Relaxing risk-free assumption for Bond, by incorporating default mechanism, and assuming this latter is also stochastic, leads to a modification of the inner Bond pricing scheme:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\omega^2 \dfrac{\partial^2 V}{\partial r^2} + (u - \lambda\omega)\dfrac{\partial V}{\partial r} - (r - \lambda_D p)V = 0 \\ dr = (u - \lambda\omega)dt + \omega dW \\ dp = \gamma dt + \delta dX \end{cases}$$

With additional notations:

- p the instantenous probability of default (aka as hazard rate)
- λ_D the market price of default risk

It introduces a third dimension in the inner pricing scheme for Bond.