TRS on Bonds – Explicit difference finite method

I – Introduction

Total return swap on debt products; including Bonds, loans, or index like IBoxxes, are, functionally speaking, very similar than their "brothers" Total return swap on equities. It shares exact same mechanisms (payer/receiver performance against receiver/payer financing) but their pricing in Black-Scholes finite difference context highlights a second order complexity due to Bond's underlying, and this paper aims to highlight a model which takes this subtlety into account.

Following models assume the next list of assumptions, which will be discussed in final part of this paper:

- TRS is a bullet one, with fixed financing rate., assumed to be a performance receiver.
- Bond's underlying is a coupon-bearing one, without default risk.
- Interest rate will follow an equilibrium model.
- Interest rate can be negative.

II – Bond pricing model

It's well-known and especially highlighted in Paul Wilmott's book that Bond product, contrary to real equity ***underlying***, can be in fact seen as a ***derivative*** on the interest rate, and this latter is the real underlying. As interest rate is not a tradable quantity, a famous scheme involving risk-neutrality is invoked (we don't highlight it as it's a "classic" black Scholes derivation than reader can directly, if he's interested in, find in the book) to find the Bond's pricing equation:

$$Bond\ pricing\ scheme : \begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\omega^2 \dfrac{\partial^2 V}{\partial r^2} + (u - \lambda\omega)\dfrac{\partial V}{\partial r} - rV = 0 \\ dr = (u - \lambda\omega)dt + \ \omega dW \end{cases}$$

$$Bond\ pricing\ conditions : \begin{cases} V(r, T, T) = 1 + c \\ V(r, t_c^-, T) = \ V(r, t_c^+, T) + c \end{cases}$$

With following notations:

- V the Bond's value.
- ω the volatility of interest rate dynamic
- u the real-world drift of interest rate dynamic
- λ the interest rate market price of risk
- Hence $(u - \lambda\omega)$ the risk-neutral drift of interest rate dynamic
- c coupon payments
- T the Bond's maturity
- W a Brownian motion

To find Bond's price, the classical backward resolution (here through explicit Euler scheme) should be achieved using two-dimensional grid $\{t \in [T^{TRS}, T], r \in [r^{min}, r^{max}]\}$. A specific attention point about jump condition before/after coupon payment.

III – TRS pricing model

The TRS pricing scheme is very similar to equity compound options/bond options schemes, with second-order feature:

- The previous bond's scheme is solved (still backward) from T to T^{TRS}, which gives a set of prices $(V^i_{T^{TRS}})_{i\in[r^{min},r^{max}]}$.
- Then the outer scheme is solved following equations, as the TRS is also a derivative on interest rate in this context, through its dependency to Bond's price:

$$TRS\ pricing\ scheme : \begin{cases} \dfrac{\partial TRS}{\partial t} + \dfrac{1}{2}\omega^2 \dfrac{\partial^2 TRS}{\partial r^2} + (u - \lambda\omega)\dfrac{\partial TRS}{\partial r} - rTRS = 0 \\ dr = (u - \lambda\omega)dt + \omega dW \end{cases}$$

$$TRS\ pricing\ conditions : \begin{cases} TRS(r, T^{TRS}) = V_{T^{TRS}} - K - s_{TRS}T^{TRS} \\ TRS(r, t_c^-) = TRS(r, t_c^+) + c \end{cases}$$

With additional notations:

- TRS the TRS market value
- K the TRS settlement price (aka strike)
- s_{TRS} the fixed and agreed TRS financing spread.
- T^{TRS} the TRS maturity

To find TRS market value, once again the classical backward resolution is done using two-dimensional grid $\{t \in [0, T^{TRS}], r \in [r^{min}, r^{max}]\}$. We should again point the jump condition after Bond's coupon payment, which is part of the performance.

IV – Finite difference algorithm

As we especially excluded non-negativity assumption for interest rate (to tackle possible negative interest), the "natural" (natural choice for an equilibrium model, as please remember that we took this assumption) choice for risk-neural interest rate dynamic is Vasicek model:

$$dr_t = \alpha(\beta - r_t)dt + \sigma dW_t$$

With following notations:

- α the interest rate speed reversion to long-term mean
- β the interest rate long-term mean.

Pricing schemes then became, respectively for Bond and for TRS:

$$pricing\ schemes : \begin{cases} \dfrac{\partial V_t}{\partial t} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 V_t}{\partial r^2} + \alpha(\beta - r_t)\dfrac{\partial V_t}{\partial r} - r_t V_t = 0 \\ \dfrac{\partial TRS_t}{\partial t} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 TRS_t}{\partial r^2} + \alpha(\beta - r_t)\dfrac{\partial TRS_t}{\partial r} - r_t TRS_t = 0 \end{cases}$$

$$\xRightarrow{Euler\ discretization} \begin{cases} \dfrac{V_{t+1}^k - V_t^k}{\delta t} + \dfrac{1}{2}\sigma^2 \dfrac{V_t^{k+1} - 2V_t^k + V_t^{k-1}}{(\delta r)^2} + \alpha(\beta - r^k)\dfrac{V_t^{k+1} - V_t^{k-1}}{2\delta r} - r^k V_t^k = 0 \\ \dfrac{TRS_{t+1}^k - TRS_t^k}{\delta t} + \dfrac{1}{2}\sigma^2 \dfrac{TRS_t^{k+1} - 2TRS_t^k + TRS_t^{k-1}}{(\delta r)^2} + \alpha(\beta - r^k)\dfrac{TRS_t^{k+1} - TRS_t^{k-1}}{2\delta r} - r^k TRS_t^k = 0 \end{cases}$$

$$\Rightarrow \begin{cases} V_{t+1}^k = a^k V_t^{k-1} + b^k V_t^k + c^k V_t^{k+1} \\ TRS_{t+1}^k = a^k TRS_t^{k-1} + b^k TRS_t^k + c^k TRS_t^{k+1} \end{cases}$$

With:

$$\begin{cases} a^k = -\dfrac{\delta t}{2\delta r}\left(\dfrac{\sigma^2}{\delta r} + \alpha(\beta - r^k)\right) \\ b^k = \dfrac{\sigma^2 \delta t}{(\delta r)^2} + r^k \delta t + 1 \\ c^k = -\dfrac{\delta t}{2\delta r}\left(\dfrac{\sigma^2}{\delta r} + \alpha(\beta - r^k)\right) \end{cases}$$

This discretized scheme will be followed for both Bond (inner scheme) and eventually TRS (outer scheme), taking into respective final and jump conditions.

V – Model assumptions, discussion, and next axis

- As second-order pricing model, it suffers exact same drawbacks than Bond option: Bond(s price at TRS maturity are evaluated ***theoretically***, but the TRS will eventually use ***market*** price. The Bond's model accuracy is then critical, as every errors in inner loop will be magnified in the outer loop (TRS).
- We assumed to choose a very simple Vasicek model (equilibrium without curve fitting), which is not the market practice, and the model should eventually be enhanced with short-rate dynamic in HJM framework.
- Relaxing TRS bullet feature will have several impacts:
  - Bond price calculation should be done until first reset TRS date and no more only until TRS maturity.
  - Jump condition should be incorporated in the model for each TRS payment date.
- Relaxing TRS fixed rate feature from floating rate (Libor-like or RFR) leads to a third dimension, which will be discussed in VII-Two-factor explicit, with both stochastic asset and interest rate.
- Relaxing risk-free assumption for Bond, by incorporating default mechanism, and assuming this latter is also stochastic, leads to a modification of the inner Bond pricing scheme:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\omega^2 \dfrac{\partial^2 V}{\partial r^2} + (u - \lambda\omega)\dfrac{\partial V}{\partial r} - (r - \lambda_D p)V = 0 \\ dr = (u - \lambda\omega)dt + \omega dW \\ dp = \gamma dt + \delta dX \end{cases}$$

With additional notations:

- p the instantenous probability of default (aka as hazard rate)
- λ_D the market price of default risk

It introduces a third dimension in the inner pricing scheme for Bond.