Pricing vanilla TRSs with known interest rate using PDE.

I – The model #1

Let's S to mean the underlying asset price, the maturity date is T. To introduce the ideas behind pricing TRS let's start by assuming that:

- Underlying is assumed to be an equity (or equity index) and hence follow the classical log-normal distribution.
- interest rates are deterministic for the life of the TRS.
- Financing rate r_0 is fixed at inception.
- Performance and financing legs are bullet.
- No income, no income yield

Each of those assumptions are impacting and then be relaxed gradually as we'll move on program's study.

Since the TRS value depends on the price of that asset we have:

$$\mathbf{TRS} = \mathbf{TRS}(\mathbf{S}, \mathbf{t})$$

The contract value depends on an asset price and on the time to maturity. Repeating the Black–Scholes analysis, with a portfolio consisting of one TRS and −Δ assets, we find that the change in the value of the portfolio is (please note that at this step we don't introduce delta one results for derivatives, we don't need it right now, it will be used for numerical steps):

$$\mathbf{d\Pi} = \frac{\partial \mathbf{TRS}}{\partial \mathbf{t}}\mathbf{dt} + \frac{\partial \mathbf{TRS}}{\partial \mathbf{S}}\mathbf{dS} + \frac{1}{2}\sigma^2\mathbf{S}^2\frac{\partial^2 \mathbf{TRS}}{\partial \mathbf{S}^2}\mathbf{dt} - \mathbf{\Delta dS}$$

Classical choice for Δ in order to eliminate risk from this portfolio:

$$\mathbf{\Delta} = \frac{\partial \mathbf{TRS}}{\partial \mathbf{S}}$$

The return on this risk-free portfolio is at most that from a bank deposit and so:

$$\frac{\partial \mathbf{TRS}}{\partial \mathbf{t}} + (\mathbf{r} - \mathbf{C}_{\mathbf{yield}})\mathbf{S}\frac{\partial \mathbf{TRS}}{\partial \mathbf{S}} + \frac{1}{2}\sigma^2\mathbf{S}^2\frac{\partial^2 \mathbf{TRS}}{\partial \mathbf{S}^2} - \mathbf{rTRS} = \mathbf{0}$$

With $\boldsymbol{C_{yield}}$ is for income yield here, only relevant for indexes (CAC40, Iboxx etc…). As we'll see in next section, discrete incomes imply jump condition.

This inequality is the basic Black–Scholes inequality. Scaling quantity to 1 and assuming a receiver performance TRS, the final condition is:

$$\mathbf{TRS}(\mathbf{S}(\mathbf{T}), \mathbf{T}) = (\mathbf{S}(\mathbf{T}) - \mathbf{S}(\mathbf{t_0})) - \mathbf{r_0}.\mathbf{S}(\mathbf{t_0}).\mathbf{DCF}(\mathbf{t_0} \to \mathbf{T})$$

In order to summarize, the full model is then:

Pricing equation: $\frac{\partial \mathbf{TRS}}{\partial \mathbf{t}} + \mathbf{rS}\frac{\partial \mathbf{TRS}}{\partial \mathbf{S}} + \frac{1}{2}\sigma^2\mathbf{S}^2\frac{\partial^2 \mathbf{TRS}}{\partial \mathbf{S}^2} - \mathbf{rTRS} = \mathbf{0}$

Final condition: $\mathbf{TRS}(\mathbf{S}(\mathbf{T}), \mathbf{T}) = (\mathbf{S}(\mathbf{T}) - \mathbf{S}(\mathbf{t_0})) - \mathbf{r_0}.\mathbf{S}(\mathbf{t_0}).\mathbf{DCF}(\mathbf{t_0} \to \mathbf{T})$

II – The model #2

With this model we'll relax following assumptions, keeping one-factor characteristics (only underlying is seen as random source):

- Performance and financing legs are bullet.
- No income, no income yield

In that case we can rely on following models (slight difference between then due to discrete/continuous incomes):

- **a/Continuous dividend**
  This model is useful for index underlying, for example CAC40, Nasdaq etc…
  Due to intermediate performance/financing interest payments, following jump condition should be added:

$$\mathbf{TRS}(\mathbf{S},\mathbf{t}_{\mathbf{pay}}^{-}) = \mathbf{TRS}(\mathbf{S},\mathbf{t}_{\mathbf{pay}}^{+}) + (\mathbf{S}(t_{pay}) - \mathbf{S}(t_{prev\,pay})) - \mathbf{r_0}.\mathbf{S}(t_{prev\,pay}).\mathbf{DCF}(\mathbf{t}_{t_{prev\,pay}} \rightarrow t_{pay})$$

Due to income yield, pricing equation should be slightly modified, following classical drift modification.

The full model is then:

Pricing equation: $\frac{\partial \mathbf{TRS}}{\partial \mathbf{t}} + (\mathbf{r} - \mathbf{C_{yield}})\mathbf{S}\frac{\partial \mathbf{TRS}}{\partial \mathbf{S}} + \frac{1}{2}\boldsymbol{\sigma}^2\mathbf{S}^2\frac{\partial^2 \mathbf{TRS}}{\partial \mathbf{S}^2} - \mathbf{rTRS} = \mathbf{0}$

Jump condition: $\mathbf{TRS}(\mathbf{S},\mathbf{t}_{\mathbf{pay}}^{-}) = \mathbf{TRS}(\mathbf{S},\mathbf{t}_{\mathbf{pay}}^{+}) + (\mathbf{S}(t_{pay}) - \mathbf{S}(t_{prev\,pay})) - \mathbf{r_0}.\mathbf{S}(t_{prev\,pay}).\mathbf{DCF}(\mathbf{t}_{t_{prev\,pay}} \rightarrow t_{pay})$

Final condition: $\mathbf{TRS}(\mathbf{S}(\mathbf{T}),\mathbf{T}) = (\mathbf{S}(\mathbf{T}) - \mathbf{S}(t_{prev\,pay})) - \mathbf{r_0}.\mathbf{S}(t_{prev\,pay}).\mathbf{DCF}(t_{prev\,pay} \rightarrow \mathbf{T})$

- **b/Discrete dividend**
  This model is useful for single equity.
  Incomes can be paid discretely so we have the jump condition across each income date:

$$\mathbf{TRS}(\mathbf{S},\mathbf{t_c^-}) = \mathbf{TRS}(\mathbf{S},\mathbf{t_c^+}) + \mathbf{C_{discrete}}$$

where $\boldsymbol{C_{discrete}}$ is the amount of the discrete coupon paid on date t_c.

The full model is then:

Pricing equation: $\frac{\partial \mathbf{TRS}}{\partial \mathbf{t}} + \mathbf{rS}\frac{\partial \mathbf{TRS}}{\partial \mathbf{S}} + \frac{1}{2}\boldsymbol{\sigma}^2\mathbf{S}^2\frac{\partial^2 \mathbf{TRS}}{\partial \mathbf{S}^2} - \mathbf{rTRS} = \mathbf{0}$

Jump condition #1: $\mathbf{TRS}(\mathbf{S},\mathbf{t}_{\mathbf{pay}}^{-}) = \mathbf{TRS}(\mathbf{S},\mathbf{t}_{\mathbf{pay}}^{+}) + (\mathbf{S}(t_{pay}) - \mathbf{S}(t_{prev\,pay})) - \mathbf{r_0}.\mathbf{S}(t_{prev\,pay}).\mathbf{DCF}(\mathbf{t}_{t_{prev\,pay}} \rightarrow t_{pay})$

Jump condition #2:$\mathbf{TRS}(\mathbf{S},\mathbf{t_c^-}) = \mathbf{TRS}(\mathbf{S},\mathbf{t_c^+}) + \mathbf{C_{discrete}}$

Final condition: $\mathbf{TRS}(\mathbf{S}(\mathbf{T}),\mathbf{T}) = (\mathbf{S}(\mathbf{T}) - \mathbf{S}(t_{prev\,pay})) - \mathbf{r_0}.\mathbf{S}(t_{prev\,pay}).\mathbf{DCF}(t_{prev\,pay} \rightarrow \mathbf{T})$

III – Program of study – numerical results

- Each model will be developed through python and C++ through finite difference method using explicit scheme.
- In order to do that as simple as possible, and as those simple models don't imply embedded option, following pricing equations will be used, considering delta one features (first and second-order derivatives are simplified):

$$\frac{\partial \mathrm{TRS}}{\partial \mathrm{t}} + (\mathrm{r} - \mathrm{C_{yield}})\mathrm{S} - \mathrm{rTRS} = 0$$

$$\xRightarrow[\text{discrete}]{} \frac{TRS_{t+1}^{k} - TRS_{t}^{k}}{\delta t} + (r - C_{yield})S^{k} - rTRS_{t}^{k} = 0$$

$$\Leftrightarrow \frac{TRS_{t+1}^{k} - TRS_{t}^{k}}{\delta t} + (r - C_{yield})S^{k} - rTRS_{t}^{k} = 0$$

$$\Leftrightarrow TRS_{t+1}^{k} = \delta t(r - C_{yield})S^{k} + (1 - r\delta t)TRS_{t}^{k}$$

$$\Leftrightarrow TRS_{t+1}^{k} = a^{k} + b_{t}^{k}TRS_{t}^{k}$$