TRS – Bates model

Author: Pierre Moureaux

I – Introduction

This paper aims to highlight Monte-Carlo methodology to price total return swap which underlying follows Bates model, embedding stochastic volatility and jump process.

II – Pricing model

In this paper, we took following assumptions for TRS pricing:

- The underling S_t is assumed to be single equity without dividend.
- Tenors/payment dates are assumed to be the same for both performance and financing leg.
- Financing rate s_{TRS} is fixed and assumed to be agreed at inception. We won't use here fair rate concept and this quantity can take any values.
- Equity underlying follows Bates model based on Heston volatility term v_t + jump process.
- Interest rate is assumed to be constant, and discount factors/zero-coupon Bonds are calculated with this constant quantity.

Total return swap market value $TRS(t_0)$, scaling the quantity to 1, is then equal to (for notation simplification $\mathcal{F}_{t_0}$ is assumed to be today's filtration and won't be indicated in formulas):

$$TRS(t_0) = \sum_{k=1}^{N} \mathbb{E}^{\mathbb{Q}}\left(\frac{S(t_0,t_k) - S(t_0,t_{k-1})}{B(t_0,t_k)}\right) - s_{TRS} \cdot \sum_{k=1}^{N} \mathbb{E}^{\mathbb{Q}}\left(\frac{S(t_0,t_{k-1})}{B(t_0,t_k)}\right) \cdot DCF(t_{k-1},t_k)$$

Following Bates model, underlying dynamic is following the next scheme under risk-neutral measure:

$$\begin{cases} \dfrac{dS_t}{S_t} = \left(r - \lambda \mathbb{E}(e^J - 1)\right)dt + \sqrt{v_t} dW_t^S + (e^J - 1)dN_t^S \\ dv_t = \kappa(\theta - v_t)dt + \gamma\sqrt{v_t} dW_t^v, with\ d\langle W_t^S, W_t^v\rangle = \rho_{S,v} dt \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{dS_t}{S_t} = \left(r - \lambda \mathbb{E}(e^J - 1)\right)dt + \sqrt{v_t} dW_t^S + (e^J - 1)dN_t^S \\ dv_t = \kappa(\theta - v_t)dt + \gamma\sqrt{v_t}\left(\rho_{S,v} dW_t^S + \sqrt{1 - \rho_{S,v}{}^2} dZ_t\right) \end{cases}$$

With following notations:

- κ the volatility speed reversion to long-term mean.
- θ the volatility long-term mean.
- γ the volatility of volatility.
- $\rho_{S,v}$ the correlation between underlying and volatility.
- N_t^S a Poisson process with in tensity λ
- Normally distributed jump sizes $J \sim \mathcal{N}(\mu_J, \sigma_J^2)$
- (W_t^S, Z_t, N_t^S, J) mutually independents.

III – Algorithm

First, let's discretize stochastic processes:

$$\begin{cases} S_{k+1} = S_k\left(\left(r-\lambda\left(e^{\mu_J+\frac{\sigma_J^2}{2}}-1\right)\right)\delta t + \sqrt{\nu_k \delta t}\varphi^S + \left(e^{\varphi^J}-1\right)\mathcal{P}(\lambda dt)\right) + S_k \\ \nu_{k+1} = \kappa(\theta-\nu_k)\delta t + \gamma\sqrt{\nu_k}\left(\rho_{S,\nu}\sqrt{\delta t}\varphi^S + \sqrt{1-\rho_{S,\nu}{}^2}\sqrt{\delta t}\tilde{\varphi}\right) + \nu_k \end{cases}$$

with:

- $(\varphi^S, \tilde{\varphi})$ independent $\mathcal{N}(0,1)$ distributions.
- $\mathcal{P}$ Poisson distribution

Now let's detail TRS market value components for components:

- $S(t_0, t_k)$ will be calculated with $(S_k)_k$ process, itself based on $(\nu_k)_k$ process.
- $B(t_0, t_k)$ will be calculated using:
$$B(t_0, t_k) = e^{r(t_k - t_0)}$$
- s_{TRS} is considered as a constant input.
- $DCF(t_{k-1}, t_k) = t_k - t_{k-1}$, with $t_N = T$ the TRS maturity

And now let's detail the Monte-Carlo algorithm:

- A number n_{simul} of $\left(S_k^j\right)_{k,j}$,$\left(\nu_k^j\right)_{k,j}$ processes will be simulated. Under script means date, superscript means the j-th simulated path.
- For each payment dates t_k, $k \in [1, N]$, following quantities will be calculated:
  - $\alpha_k \approx \frac{1}{n_{simul}} \sum_{j=1}^{n_{simul}} \frac{S_k^j - S_{k-1}^j}{e^{rk}}$
  - $\beta_k \approx \frac{1}{n_{simul}} \sum_{j=1}^{n_{simul}} \frac{S_{k-1}^j}{e^{rk}}$
- Thanks to those two quantities, TRS market value can be calculated following:

$$TRS(t_0) = \sum_{k=1}^{N} \alpha_k - s_{TRS} \cdot \sum_{k=1}^{N} \beta_k \cdot (\, t_k - t_{k-1})$$

IV – Model assumptions and next axis

We need to discuss about previous assumptions and set next approaches:

- The normality choice for jump size is arbitrary and several other choices may exist, for example parametric non-symmetric double exponential, following Kou's model about the distribution:

$$\begin{cases} f_J(x) = p_1\alpha_1 e^{-\alpha_1 x}\mathbb{1}_{x\geq 0} + p_2\alpha_2 e^{\alpha_2 x}\mathbb{1}_{x<0} \\ p_1 + p_2 = 1 \\ \alpha_1 > 1, \alpha_1 > 0 \end{cases}$$

- In previous Bates model we didn't highlight exact closed-form formula for the asset itself, relying on dynamics and its discretization for MC numerical goal. Another approach can be followed using Merton's jump diffusion model with compound Poisson process and, this time, constant volatility:

$$\begin{cases} X_t = log(S_t) \\ X_t = X_0 + \left(r - \frac{1}{2}\sigma^2 - \lambda\mathbb{E}(e^J - 1)\right)t + \sigma W_t^X + \sum_{k=1}^{N(t)} J_k \end{cases}$$

  with Poisson process $N(t)$, and then $\mathbb{E}(N(t)) = \lambda t$

- The constant interest rate assumption can be relaxed, for example using a mean-reversion stochastic diffusion. The MC model won't dramatically change but with additional computation cost.
- The model can be generalized to basket of equities, with additional correlation step computation cost and probably issues about correlation factor catching (classical issue).