TRS on Bonds – Monte Carlo method

I – Introduction

This paper aims to highlight Monte-Carlo methodology in order to price total return swap which underlying is a debt product, here a risky coupon-bearing Bond.

II – Pricing model

In this paper, we took following assumptions for TRS pricing:

- The underling P_t is assumed to be a single risky Bond with fixed coupon payment.
- Tenors/payment dates are assumed to be the same for both performance and financing leg.
- Financing rate s_{TRS} is fixed and assumed to be agreed at inception. We won't use here fair rate concept and this quantity can take any values.
- Discounting, and then zero-coupon Bonds ZC, are assumed to follow the same short-term rate r_t risk-neutral equilibrium model than Bond's underlying.
- The LMN model will be used for Bond's pricing, and we assume that interest rate, hazard rate and liquidity factor are uncorrelated.

Some of those assumptions will be discussed in last section of this paper.

Total return swap market value $TRS(t_0)$, scaling the quantity to 1, is then equal to:

$$TRS(t_0) = \sum_{k=1}^{N} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(t_0,t_k) - P(t_0,t_{k-1})}{B(t_0,t_k)}\right) + \sum_{t_c \in [t_0,t_N]} \mathbb{E}^{\mathbb{Q}}\left(\frac{c}{B(t_0,t_c)}\right) - s_{TRS} \cdot \sum_{k=1}^{N} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(t_0,t_{k-1})}{B(t_0,t_k)}\right) \cdot DCF(t_{k-1},t_k)$$

$$\Rightarrow TRS(t_0) = \sum_{k=1}^{N} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(t_0,t_k) - P(t_0,t_{k-1})}{B(t_0,t_k)}\right) + c \cdot \sum_{t_c \in [t_0,t_N]} \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{B(t_0,t_c)}\right) - s_{TRS} \cdot \sum_{k=1}^{N} \mathbb{E}^{\mathbb{Q}}\left(\frac{P(t_0,t_{k-1})}{B(t_0,t_k)}\right) \cdot DCF(t_{k-1},t_k)$$

The LMN model will be used for Bond's price, which requires several factors:

- λ_t the hazard rate.
- R the recovery rate.
- γ_t the liquidity factor.
- Bank account $B(t_0,t_k) = e^{\int_{t_0}^{t_k} r_s ds}$
- Survival probability $Q(t_0,t_k) = e^{-\int_{t_0}^{t_k} \lambda_s ds}$
- Liquidity cumulative impact $L(t_0,t_k) = e^{-\int_{t_0}^{t_k} \gamma_s ds}$

The underlying Bond $P(t_0,t_k)$ is following next scheme formula:

$$P(t_0,t_k) = \mathbb{E}^{\mathbb{Q}}\left(\sum_{t_c > t_k}^{T_P} \frac{c \cdot e^{-\int_{t_k}^{t_c} \lambda_s ds} \cdot e^{-\int_{t_k}^{t_c} \gamma_s ds}}{e^{\int_{t_k}^{t_c} r_s ds}} + \frac{e^{-\int_{t_k}^{T_P} \lambda_s ds} \cdot e^{-\int_{t_k}^{t_c} \gamma_s ds}}{e^{\int_{t_k}^{T_P} r_s ds}} + (1-R) \cdot \sum_{t=t_k+1}^{T_P} \lambda_t \frac{e^{-\int_{t_k}^{t} \lambda_s ds} \cdot e^{-\int_{t_k}^{t} \gamma_s ds}}{e^{\int_{t_k}^{t} r_s ds}}\right)$$

$$\Rightarrow P(t_0,t_k) = \mathbb{E}^{\mathbb{Q}}\left(c \cdot \sum_{t_c > t_k}^{T_P} e^{-\int_{t_k}^{t_c}(r_s+\lambda_s+\gamma_s)ds} + e^{-\int_{t_k}^{T_P}(r_s+\lambda_s+\gamma_s)ds} + (1-R) \cdot \sum_{t=t_k+1}^{T_P} \lambda_t \, e^{-\int_{t_k}^{t}(r_s+\lambda_s+\gamma_s)ds}\right)$$

LMN factors follow next stochastic processes:

$$\begin{cases} dr_t = \xi(\eta - r_t)dt + \vartheta\sqrt{r_t}dW_t^r, r_0 \\ d\lambda_t = \beta(\alpha - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t^\lambda, \lambda_0 \\ d\gamma_t = \omega dW_t^\gamma, \gamma_0 \end{cases}$$

With following notations:

- η the interest rate long-term mean.
- ξ the interest rate speed reversion to long-term mean.

- ϑ the volatility of interest rate.
- α the hazard rate long-term mean.
- β the hazard rate speed reversion to long-term mean.
- σ the hazard rate volatility.
- ω the liquidity volatility.
- $(W_t^r, W_t^\lambda, W_t^\gamma)$ independent Brownian motions.

III – Algorithm

First, let's discretize stochastic processes:

$$\xrightarrow{Euler\ scheme}\begin{cases} r_{k+1} = \xi(\eta - r_t)\delta t + \vartheta\sqrt{r_k\delta t}\varphi^r + r_k \\ \lambda_{k+1} = \beta(\alpha - \lambda_t)\delta t + \sigma\sqrt{\lambda_k\delta t}\varphi^\lambda + \lambda_k \\ \gamma_{k+1} = \omega\sqrt{\delta t}\varphi^\gamma + \gamma_k \end{cases}$$

With $(\varphi^r, \varphi^\lambda, \varphi^\gamma)$ independent $\mathcal{N}(0,1)$ distributions.

Let's also discretize Bond's prices:

$$P(t_0, t_k) \approx \mathbb{E}^{\mathbb{Q}}\left(c \cdot \sum_{t_c > t_k}^{T_P} e^{-\sum_{k=t_k}^{t_c}(r_k+\lambda_k+\gamma_k)} + e^{-\sum_{k=t_k}^{T_P}(r_k+\lambda_k+\gamma_k)} + (1-R)\cdot\sum_{t=t_k+1}^{T_P}\lambda_t\, e^{-\sum_{k=t_k}^{t}(r_k+\lambda_k+\gamma_k)}\right)$$

The Monte-Carlo algorithm's path is:

- Each P_k^j is calculated following:
    - One path i is generated for each process $(r_k^i)_{i,k}$, $(\lambda_k^i)_{i,k}$ and $(\gamma_k^i)_{i,k}$.
    - Intermediate cumulative quantities are calculated:
        - $c1 = \sum_{k=t_k}^{t_c}(r_k + \lambda_k + \gamma_k)$
        - $c2 = \sum_{k=t_k}^{T_P}(r_k + \lambda_k + \gamma_k)$
        - $c3 = \sum_{k=t_k}^{t}(r_k + \lambda_k + \gamma_k)$
    - One nested path is generated:

$$nestedPath(i) = c \cdot \sum_{t_c > t_k}^{T_P} e^{-c1} + e^{-c2} + (1-R)\cdot\sum_{t=t_k+1}^{T_P}\lambda_t^i\, e^{-c3}$$

    - Then the price is calculated through:

$$P_k^j \approx \frac{1}{m_{simul}}\sum_{i=1}^{m_{simul}} nestedPath(i)$$

- For each payment dates t_k, $k \in [1, N]$, following quantities will be calculated:
    - $\alpha_k \approx \frac{1}{n_{simul}}\sum_{j=1}^{n_{simul}}\frac{P_k^j - P_{k-1}^j}{e^{\sum_{s=0}^{k} r_s^j}}$
    - $\beta_k \approx \frac{1}{n_{simul}}\sum_{j=1}^{n_{simul}}\frac{P_{k-1}^j}{e^{\sum_{s=0}^{k} r_s^j}}$
- For each coupon payment dates $t_c \in [t_0, t_N]$, following quantity will be calculated:
    - $c_{t_c} \approx \frac{1}{n_{simul}}\sum_{j=1}^{n_{simul}}\frac{1}{e^{\sum_{s=0}^{k} r_s^j}}$
- Thanks to those two quantities, TRS market value can be calculated following:

$$TRS(t_0) = \sum_{k=1}^{N}\alpha_k + c \cdot \sum_{t_c \in [t_0, t_N]} c_{t_c} - s_{TRS}\cdot\sum_{k=1}^{N}\beta_k \cdot (t_k - t_{k-1})$$

IV – Model assumptions and next axis

- Due to Bond's price calculation itself, the model embeds nested Monte-Carlo algorithm in global one used for TRS market value calculation. This model is then by design time-consuming and need to be optimized with following ideas:
    - We perhaps don't need to recalculate each Bond's forward price at each Monte-Carlo launching, as we can use $P(t_0, t_k)$, calculated for one payment date, and use it again for $P(t_0, t_{k-1})$ for the next payment date. It will decrease computation cost.
    - We can also use the same $(r_k^i)_{i,k}$ path for both Bond's price calculation and discounting for TRS flows. Once again it will avoid possible useless interest rate computations.
- The LMN model used for Bond's forward price calculation embeds several factors and parameters that need to be calibrated. Of course, we can't use TRS, as pure OTC and then no liquid at all, in order to calibrate those parameters. Strictly speaking, the highlighted model is then not complete, as we should also add the calibration steps:
    - Calibration of hazard rate through liquid CDS
    - Calibration of liquidity factor through liquid govies
- In addition, the interest rate is assumed to follow an equilibrium model, the market practice is more relying on short rate in HJM framework for fitting purpose, and then calibration step will also occur in that case.
- Interest rate and hazard rate are for part correlated and relaxing the uncorrelated Brownian assumption should be done.