Vanilla TRS – Monte Carlo method

I – Introduction

This paper aims to highlight Monte-Carlo methodology in order to price vanilla total return swap. That's the first milestone and the "corner stone" of next papers which will highlight another TRS flavours.

II – Pricing model

In this paper, we took following assumptions for TRS pricing:

- The underling S_t is assumed to be single equity without dividend.
- Tenors/payment dates are assumed to be the same for both performance and financing leg.
- Financing rate s_{TRS} is fixed and assumed to be agreed at inception. We won't use here fair rate concept and this quantity can take any values.
- Discounting, and then zero-coupon Bonds ZC, are assumed to follow a short-term rate r_t risk-neutral equilibrium model.
- Equity underlying follows hybrid model based on Heston volatility term ν_t + short term rate r_t.
- Volatility and interest rate are assumed to be not correlated.

Some of those assumptions will be discussed in last section of this paper.

Total return swap market value $TRS(t_0)$, scaling the quantity to 1, is then equal to (for notation simplification $\mathcal{F}_{t_0}$is assumed to be today's filtration and won't be indicated in formulas):

$$TRS(t_0) = \sum_{k=1}^{N} \mathbb{E}^{\mathbb{Q}}\left(\frac{S(t_0,t_k) - S(t_0,t_{k-1})}{B(t_0,t_k)}\right) - s_{TRS} \cdot \sum_{k=1}^{N} \mathbb{E}^{\mathbb{Q}}\left(\frac{S(t_0,t_{k-1})}{B(t_0,t_k)}\right) \cdot DCF(t_{k-1},t_k)$$

Underlying dynamic is following the next scheme:

$$\begin{cases} \dfrac{dS_t}{S_t} = r_t dt + \sqrt{\nu_t} dW_t^S \\ d\nu_t = \kappa(\theta - \nu_t)dt + \gamma\sqrt{\nu_t} dW_t^{\nu}, with\ d\langle W_t^S, W_t^{\nu}\rangle = \rho_{S,\nu} dt \\ dr_t = (u - \lambda\omega)dt + \omega dW_t^r, with\ d\langle W_t^S, W_t^r\rangle = \rho_{S,r} dt \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{dS_t}{S_t} = r_t dt + \sqrt{\nu_t} dW_t^S \\ d\nu_t = \kappa(\theta - \nu_t)dt + \gamma\sqrt{\nu_t}\left(\rho_{S,\nu} dW_t^S + \sqrt{1 - {\rho_{S,\nu}}^2} dZ_t\right) \\ dr_t = (u - \lambda\omega)dt + \omega\left(\rho_{S,r} dW_t^S + \sqrt{1 - {\rho_{S,r}}^2} dX_t\right) \end{cases}$$

With following notations:

- κ the volatility speed reversion to long-term mean.
- θ the volatility long-term mean.
- γ the volatility of volatility.
- $\rho_{S,v}$ the correlation between underlying and volatility.
- $\rho_{S,r}$ the volatility between underlying and interest rate.
- u the real-world interest rate drift.
- ω the interest rate volatility.
- λ the interest rate market price of risk.
- (W_t^S, Z_t, X_t) independent Brownian motions.

In addition, bank account dynamic is:

$$B(t_0, t_k) = e^{\int_{t_0}^{t_k} r_s ds}$$

We must choose specific risk-neutral drift and volatility for interest rate dynamic. As this paper doesn't aim to highlight a yield curve fitting (which will be one of the purposes of the analysis based on HJM framework), we took the one of most simple models, the CIR process.

Underlying dynamic is then:

$$\begin{cases} \dfrac{dS_t}{S_t} = r_t dt + \sqrt{v_t} dW_t^S \\ dv_t = \kappa(\theta - v_t)dt + \gamma\sqrt{v_t}\left(\rho_{S,v} dW_t^S + \sqrt{1 - {\rho_{S,v}}^2} dZ_t\right) \\ dr_t = (\eta - \xi r_t)dt + \sqrt{\alpha r_t}\left(\rho_{S,r} dW_t^S + \sqrt{1 - {\rho_{S,r}}^2} dX_t\right) \end{cases}$$

With additional notations:

- ξ the interest rate speed reversion to long-term mean
- $\frac{\eta}{\xi}$ the interest rate long-term mean
- $\sqrt{\alpha}$ the volatility of interest rate, with positive rate constraint $\alpha < 2\eta$

III – Algorithm

First, let's discretize stochastic processes:

$$\xRightarrow{Euler\ scheme} \begin{cases} S_{k+1} = S_k(r_k \delta t + \sqrt{v_k \delta t} \varphi^S) + S_k \\ v_{k+1} = \kappa(\theta - v_k)\delta t + \gamma\sqrt{v_k}\left(\rho_{S,v}\sqrt{\delta t}\varphi^S + \sqrt{1 - {\rho_{S,v}}^2}\sqrt{\delta t}\tilde{\varphi}\right) + v_k \\ r_{k+1} = (\eta - \xi r_k)\delta t + \sqrt{\alpha r_k}\left(\rho_{S,r}\sqrt{\delta t}\varphi^S + \sqrt{1 - {\rho_{S,r}}^2}\sqrt{\delta t}\hat{\varphi}\right) + r_k \end{cases}$$

With $(\varphi^S, \tilde{\varphi}, \hat{\varphi})$ independent $\mathcal{N}(0,1)$ distributions.

Now let's detail TRS market value components for components:

- $S(t_0, t_k)$ will be calculated with $(\mathrm{S_k})_\mathrm{k}$ process, itself based on both $(\nu_k)_\mathrm{k}$ and $(r_k)_\mathrm{k}$ processes.
- $B(t_0, t_k)$ will be calculated with $(r_k)_\mathrm{k}$ process.
- s_{TRS} is considered as a constant input.
- $DCF(t_{k-1}, t_k) = t_k - t_{k-1}$, with $t_N = T$ the TRS maturity

And now let's detail the Monte-Carlo algorithm:

- A number $\mathrm{n_{simul}}$ of $\left(\mathrm{S_k^j}\right)_{\mathrm{k,j}}$,$\left(\nu_\mathrm{k}^\mathrm{j}\right)_{\mathrm{k,j}}$ and $\left(\mathrm{r_k^j}\right)_{\mathrm{k,j}}$ processes will be simulated. Under script means date, superscript means the j-th simulated path.
- For each payment dates t_k, $k \in [1, N]$, following quantities will be calculated:
    - $\alpha_\mathrm{k} \approx \frac{1}{n_{simul}} \sum_{j=1}^{n_{simul}} \frac{\mathrm{S_k^j - S_{k-1}^j}}{e^{\sum_{s=0}^{k} r_s^j}}$
    - $\beta_\mathrm{k} \approx \frac{1}{n_{simul}} \sum_{j=1}^{n_{simul}} \frac{\mathrm{S_{k-1}^j}}{e^{\sum_{s=0}^{k} r_s^j}}$
- Thanks to those two quantities, TRS market value can be calculated following:

$$TRS(t_0) = \sum_{k=1}^{N} \alpha_\mathrm{k} - s_{TRS} \cdot \sum_{k=1}^{N} \beta_\mathrm{k} \cdot (\, t_k - t_{k-1})$$

IV – Model assumptions and next axis

As previously mentioned, we need to discuss about previous assumptions and set next approaches:

- 1/Underlying was assumed to be a single equity without dividend. This assumption can be relaxed with:
    - Adding dividend yield d_t (more for index equity then), possibly also stochastic. In that case model is slightly modified:

$$\frac{dS_t}{S_t} = (r_t - d_t)dt + \sqrt{\nu_t} dW_t^S$$

    - Adding discrete ***and known*** dividend list $(d_1 ... d_D)$, paid on dates $[\mathrm{t_{d_1}} ... \mathrm{t_{d_D}}]$. In that case several modifications should be done for the model:

$$\begin{cases} S_{k+1} = S_k(r_k \delta t + \sqrt{\nu_k \delta t} \varphi^S) + S_k - d_{t_{d_k}} \mathbb{1}_{k \in [t_{d_1} \dots t_{d_D}]} \\ TRS(t_0) = \sum_{k=1}^{N} \mathbb{E}^{\mathbb{Q}} \left(\frac{S(t_0, t_k) - S(t_0, t_{k-1})}{B(t_0, t_k)} \right) + \sum_{k=1}^{D} \mathbb{E}^{\mathbb{Q}} \left(\frac{d_{t_{d_k}}}{B\left(t_0, t_{d_k}\right)} \right) - s_{TRS} \cdot \sum_{k=1}^{N} \mathbb{E}^{\mathbb{Q}} \left(\frac{S(t_0, t_{k-1})}{B(t_0, t_k)} \right) \cdot DCF(t_{k-1}, t_k) \end{cases}$$

$$\Rightarrow \begin{cases} S_{k+1} = S_k(r_k \delta t + \sqrt{\nu_k \delta t} \varphi^S) + S_k - d \mathbb{1}_{k \in [t_{d_1} \dots t_{d_D}]} \\ TRS(t_0) = \sum_{k=1}^{N} \mathbb{E}^{\mathbb{Q}} \left(\frac{S(t_0, t_k) - S(t_0, t_{k-1})}{B(t_0, t_k)} \right) + \sum_{k=1}^{D} d_{t_{d_k}} ZC(t_0, t_{d_k}) - s_{TRS} \cdot \sum_{k=1}^{N} \mathbb{E}^{\mathbb{Q}} \left(\frac{S(t_0, t_{k-1})}{B(t_0, t_k)} \right) \cdot DCF(t_{k-1}, t_k) \end{cases}$$

    - Another paper will focus on basket of equities, with additional correlation subtleties.

    - Another paper will focus on Bond as underlying.
- Relaxing same schedules constraint for both performance and financing leg is just a matter of algorithm complexity, with additional dates and additional quantities calculations, but without important modifications.
- The fixed financing rate assumption is on the contrary quite impacting, as relaxing it leads to two main model modifications:
    - Switch from fixed rate to libor-like financing rate: in that case Libor market model should be used for financing leg.
    - Switch from fixed rate to RFR financing rate: in that case Forward market model should be used for financing leg.
    - Next papers will focus on those two additional computations methods.
- In this current paper we chose short rate equilibrium model for its simplicity, but it's clearly not the market practice for years. Relaxing this assumption is quite linked to the previous one, as the best approach is auto-fitting short rate model issued from HJM framework. This kind of specific short rate dynamic will be used in another paper.