TRS on equity basket – Monte Carlo method

I – Introduction

This paper, which can be considered as a follow-up of the previous one which highlighted vanilla TRS pricing, aims to highlight Monte-Carlo methodology in order to price total return swap which performance leg is based on basket of equities. As we'll see in next sections and contrary to single equity case, this TRS flavour implies correlations between equities.

II – Pricing model

In this paper, we took following assumptions for TRS pricing:

- Performance is based on basket of equities $(S^1, \dots, S^M)$
- Tenors/payment dates are assumed to be the same for both performance and financing leg.
- Financing rate s_{TRS} is fixed and assumed to be agreed at inception. We won't use here fair rate concept and this quantity can take any values.
- Discounting, and then zero-coupon Bonds ZC, are assumed to follow a short-term rate r_t risk-neutral equilibrium model, chosen here as CIR.
- Each equities components follow hybrid model based on Heston volatility term ν_t^i + short term rate r_t.
- Volatilities and interest rate are assumed to be not correlated.

Total return swap market value $TRS(t_0)$ is then equal to:

$$TRS(t_0) = \sum_{k=1}^{N} \mathbb{E}^{\mathbb{Q}}\left(\frac{\sum_{i=1}^{M} \omega_i \cdot S^i(t_0, t_k) - \sum_{i=1}^{M} \omega_i \cdot S^i(t_0, t_{k-1})}{B(t_0, t_k)}\right) - s_{TRS} \cdot \sum_{k=1}^{N} \mathbb{E}^{\mathbb{Q}}\left(\frac{\sum_{i=1}^{M} \omega_i \cdot S^i(t_0, t_{k-1})}{B(t_0, t_k)}\right) \cdot DCF(t_{k-1}, t_k)$$

Each underlying dynamics is following the next scheme:

$$\begin{cases} \dfrac{dS_t^i}{S_t^i} = r_t dt + \sqrt{\nu_t^i} dW_t^{S^i}, with\ d\left\langle W_t^{S^i}, dW_t^{S^j}\right\rangle = \rho_{i,j} dt \\ d\nu_t^i = \kappa^i(\theta^i - \nu_t^i)dt + \gamma^i \sqrt{\nu_t^i} dW_t^{\nu^i}, with\ d\left\langle W_t^{S^i}, W_t^{\nu^i}\right\rangle = \rho_{S^i,\nu^i} dt \\ dr_t = (\eta - \xi r_t)dt + \sqrt{\alpha r_t} dW_t^r, with\ d\left\langle W_t^{S^i}, W_t^r\right\rangle = \rho_{S^i,r} dt \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{dS_t^i}{S_t^i} = r_t dt + \sqrt{\nu_t^i} dW_t^{S^i}, with\ d\left\langle W_t^{S^i}, dW_t^{S^j}\right\rangle = \rho_{i,j} dt \\ d\nu_t^i = \kappa^i(\theta^i - \nu_t^i)dt + \gamma^i \sqrt{\nu_t^i}\left(\rho_{S^i,\nu^i} dW_t^{S^i} + \sqrt{1 - {\rho_{S^i,\nu^i}}^2} dZ_t^i\right) \\ dr_t = (\eta - \xi r_t)dt + \sqrt{\alpha r_t}\left(\rho_{S^i,r} dW_t^{S^i} + \sqrt{1 - {\rho_{S^i,r}}^2} dX_t\right) \end{cases}$$

with following notations:

- κ^i volatilities speed reversion to long-term mean.

- θ^i volatilities long-term mean.
- γ^i volatilities of volatility.
- ρ_{S^i,ν^i} the correlation between underlying S^i and its proper volatility ν^i.
- $\rho_{S^i,r}$ the volatility between underlying S^i and interest rate.
- ξ the interest rate speed reversion to long-term mean
- $\frac{\eta}{\xi}$ the interest rate long-term mean
- $\sqrt{\alpha}$ the volatility of interest rate, with positive rate constraint $\alpha < 2\eta$

The additional complexity lies in vector of underlying Brownian motions $(W^{S^1}, \ldots, W^{S^M})$, as, in general, we can't consider basket components are independent, then we need to introduce correlations between underlying through $d\,\langle W_t^{S^i}, W_t^{S^j}\rangle = \rho_{i,j}dt$, and we'll directly switch it to the sub-problem $\langle \varphi^{S^i}, \varphi^{S^j}\rangle = \rho_{i,j}$, as during discretization phase we'll focused on normal distributions.

We'll use the classical Cholesky decomposition in order to have a set of correlated Brownian motions, let's start by the correlation matrix and let's follow the algorithm path:

$$C = \begin{pmatrix} 1 & \cdots & \rho_{1,M} \\ \vdots & \ddots & \vdots \\ \rho_{M,1} & \cdots & 1 \end{pmatrix}$$

$$\Rightarrow \; Cholesky\ decomposition\ C \; = \; LU$$

$$\Rightarrow \; Generation\ of\ X \sim \mathcal{N}(0, I)$$

$$\Rightarrow \begin{pmatrix} \varphi^{S^1} \\ \vdots \\ \varphi^{S^M} \end{pmatrix} = XU$$

In addition, bank account dynamic is still:

$$B(t_0, t_k) = e^{\int_{t_0}^{t_k} r_s ds}$$

III – Algorithm

First, let's discretize stochastic processes:

$$\xrightarrow{Euler\ scheme} \begin{cases} S_{k+1}^i = \; S_k^i(r_k\delta t \; + \sqrt{\nu_k^i \delta t}\varphi^{S^i}) + S_k^i \\ \nu_{k+1}^i = \; \kappa^i(\theta^i - \nu_k^i)\delta t \; + \; \gamma^i\sqrt{\nu_k^i}\left(\rho_{S^i,\nu^i}\sqrt{\delta t}\varphi^{S^i} + \sqrt{1 - \rho_{S^i,\nu^i}{}^2}\sqrt{\delta t}\widetilde{\varphi^i}\right) + \nu_k^i \\ r_{k+1} = \; (\eta - \xi r_k)\delta t \; + \; \sqrt{\alpha r_k}\left(\rho_{S^i,r}\sqrt{\delta t}\varphi^{S^i} + \sqrt{1 - \rho_{S^i,r}{}^2}\sqrt{\delta t}\hat{\varphi}\right) + r_k \end{cases}$$

The algorithm implies many correlation and normal distributions quantities, before moving further and in order to avoid "lost in translation" effect, let's summarize all in following tab:

	Interest rate dynamic	Volatilities dynamics	Asset dynamics
Correlation factors	M $(\rho_{S^i,r})_{i=1\ldots M}$ correlation factors	M $(\rho_{S^i,v^i})_{i=1\ldots M}$ correlation factors	$\frac{M(M-1)}{2}$ $(\rho_{i,j})_{i,j}$ correlation factors
Normal distributions	1 iid $\hat{\varphi}$ normal distribution	M iid $(\widehat{\varphi^i})_{i=1\ldots M}$ normal distributions	M correlated $\begin{pmatrix}\varphi^{S^1}\\ \vdots \\ \varphi^{S^M}\end{pmatrix}_{i=1\ldots M}$ normal distributions

Now let's detail TRS market value components for components:

- $\sum_{i=1}^{M} \omega_i \cdot S^i(t_0, t_k)$ will be calculated with $\left(S_k^i\right)_k$ process, themselves based on both $\left(v_k^i\right)_k$ and $(r_k)_k$ processes.
- $B(t_0, t_k)$ will be calculated with $(r_k)_k$ process.
- s_{TRS} is considered as a constant input.
- $DCF(t_{k-1}, t_k) = t_k - t_{k-1}$, with $t_N = T$ the TRS maturity

And now let's detail the Monte-Carlo algorithm:

- A number n_{simul} of $\left(S_k^{i,j}\right)_{k,j}$,$\left(v_k^{i,j}\right)_{k,j}$ and $\left(r_k^j\right)_{k,j}$ processes will be simulated:
  - Under script means date.
  - Former superscript i indicates the i-th basket components.
  - Latter superscript j indicates the j-th simulated path.
- For each payment dates t_k, $k \in [1, N]$, following quantities will be calculated:
  - $\alpha_k \approx \frac{1}{n_{simul}} \sum_{j=1}^{n_{simul}} \frac{\sum_{i=1}^{M} \omega_i S^{i,j}(t_0,t_k) - \sum_{i=1}^{M} \omega_i S^{i,j}(t_0,t_{k-1})}{e^{\sum_{s=0}^{k} r_s^j}}$
  - $\beta_k \approx \frac{1}{n_{simul}} \sum_{j=1}^{n_{simul}} \frac{\sum_{i=1}^{M} \omega_i S^{i,j}(t_0,t_{k-1})}{e^{\sum_{s=0}^{k} r_s^j}}$
- Thanks to those two quantities, TRS market value can be calculated following:

$$TRS(t_0) = \sum_{k=1}^{N} \alpha_k - s_{TRS} \cdot \sum_{k=1}^{N} \beta_k \cdot (\, t_k - t_{k-1})$$

IV – Model discussions

- The implementation will be done with only three basket components and weights equal to 1; for simplification purpose.
- The model embeds several correlation quantities, which are notoriously difficult to catch and as instable (even more) than implied volatilities. The model then takes a major assumption that those all corelations are perfectly known and easily calibrated, which is clearly not acceptable for majority of cases. This model should then be seen as rather theoretical, and the author is quite dubious about its correctness in "real world".
- The model can be extended, once again following a very theoretical path, by using stochastic correlation dynamics. But this kind of model implies a calibration "switch" from correlation quantities themselves to correlation stochastic process's parameters, which is even more complex. As there are no major reason to proceed like this (no specific effect to "catch", only pricing methodology), the author is against this "step forward".