

## Total return swap – Fast Fourier Transformation – Bullet TRS with fixed rate

### I – Introduction

This paper aims to highlight Fast Fourier Transformation (FFT) in Total Return Swap (TRS) pricing context. The paper will start by a reminder about FFT methods, classically used for option pricing, this chapter will introduce general ideas which will be the “core” of all papers of FFT module. And, for this paper, a part will be dedicated to model itself for, in first approach, a very simple flavor of TRS.

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### II – Reminder of Fourier transformation and TRS generic formula

Let's assume  $f$  a probability density function of a variable  $X$ , which distribution has a characteristic function  $\phi$  (the gaussian is one of many examples of this kind of law).

The Fourier-cosine expansion of  $f$  on interval  $[a, b]$  is:

$$(\text{COS}) \begin{cases} f(x) = \sum_{j=0}^{\infty} F_j \cos\left(j\pi \frac{x-a}{b-a}\right)^* \\ F_j = \frac{2}{b-a} \mathcal{R}\left(\phi\left(\frac{j\pi}{b-a}; x\right) e^{-ij\pi \frac{a}{b-a}}\right) \end{cases}$$

The \* indicates that the first term of the sum should be weighted by 0.5.

The TRS risk-neutral valuation follows next equations, and at this step we won't make any assumption about use-case “reduction”, the TRS is assumed to follow one of most complex flavors, like several payments and floating financing rate:

$$\begin{aligned} TRS(S_0, r_0, t_0) &= \mathbb{E}^{\mathbb{Q}}\left(\sum_{k=1}^N \frac{V_k(S_k, r_k, t_k)}{B(t_0, t_k)} \middle| S_0, r_0\right) \\ \Rightarrow TRS(S_0, r_0, t_0) &= \sum_{k=1}^N \mathbb{E}^{\mathbb{Q}}\left(\frac{V_k(S_k, r_k, t_k)}{B(t_0, t_k)} \middle| S_0, r_0\right) \\ \Rightarrow TRS(S_0, r_0, t_0) &= \sum_{k=1}^N \iint_{\mathbb{R} \times \mathbb{R}} \frac{V_k(S_k, r_k, t_k)}{B(t_0, t_k)} f(S_k, r_k | S_0, r_0) dS dr \\ \Rightarrow TRS(x, y, t_0) &= \sum_{k=1}^N \iint_{\mathbb{R} \times \mathbb{R}} \frac{V_k(z, h, t_k)}{B(t_0, t_k)} f(z, h | x, y) dz dh \end{aligned}$$

With following notations:

- $V_k$  the TRS flows Pay-off.
- $S_k$  the TRS underlying asset, hidden through variable  $z$  in integration.
- $r_k$  the interest rate, hidden through variable  $h$  in integration.
- $B(t_0, t_k)$  the bank account.
- $S_0$  the spot underlying price, hidden through the conditional variable  $x$  in integration.
- $r_0$  the spot interest rate, hidden through the conditional variable  $y$  in integration.
- $f$  the joint probability function for both asset and interest rate.

The Pay-off for TRS flows and bank account are:

$$\begin{cases} \text{if Libor-like : } V_k(S, r, t_k) = S_{t_k} - K - r_{t_{k-1}} \cdot (t_k - t_{k-1}) \\ \text{if RFR : } V_k(S, r, t_k) = S_{t_k} - K - r_{t_k} \cdot (t_k - t_{k-1}) \\ B(t_0, t_k) = e^{\int_{t_0}^{t_k} r_s ds} \end{cases}$$

All those formulas are generic ones with the interesting two-variable density function, embedding asset and interest risk factors.

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### III – TRS pricing, model #1

For this first model, we took several assumptions:

- TRS has a bullet payment.
- Financing rate is fixed.
- Interest rate is considered as constant.
- Underlying asset follows a lognormal distribution (geometric Brownian motion)

With those previous notes, the previous TRS pricing scheme is heavily simplified and then follows:

$$\begin{aligned}
 TRS(S_0, t_0) &= e^{-r(T-t_0)} \mathbb{E}^{\mathbb{Q}}(V_T(S, T) | S_0) \\
 \Rightarrow TRS(x, t_0) &= e^{-r(T-t_0)} \int_{\mathbb{R}} V_T(z, T) f(z|x) dz \\
 \Rightarrow TRS(x, t_0) &\approx e^{-r(T-t_0)} \int_{[a,b]} V_T(z, T) f(z|x) dz \\
 \Rightarrow \begin{cases} TRS(x, t_0) \approx e^{-r(T-t_0)} \sum_{j=0}^{M-1} \mathcal{R}\left(\phi\left(\frac{j\pi}{b-a}; x\right) e^{-ij\pi \frac{a}{b-a}}\right) TRS_j^* \\ TRS_j \text{ specific components related to TRS payoff} \end{cases}
 \end{aligned}$$

For the previous equation, we used the fact that even if indeed the characteristic function of a lognormal distribution can't be directly calculated, we can use the following "trick", assuming  $Z \sim \ln \mathcal{N}(0,1)$ :

$$\begin{aligned}
 CDF_Z(z) &= \mathbb{P}(Z \leq z) = \mathbb{P}(e^X \leq z) = \mathbb{P}(X \leq \ln(z)) = CDF_X(\ln(z)) \\
 \Rightarrow f_Z(z) &= \frac{dCDF_Z(z)}{dz} = \frac{dCDF_X(\ln(z))}{d \ln(z)} \frac{d \ln(z)}{dz} = \frac{1}{z} f_X(\ln(z))
 \end{aligned}$$

And as  $X \sim \mathcal{N}(0,1)$  this time the characteristic function exists and hence we can use the previous (**COS**) result.

Taking following transformation into account:

$$\begin{aligned}
 V_T(S, T) &= S_T - K - s_{t_{TRS}} \\
 \Rightarrow V_T(S, T) &= \tilde{K} \left( \frac{S_T}{\tilde{K}} - 1 \right), \text{ with } \tilde{K} = K + s_{t_{TRS}} \\
 \Rightarrow V_T(z, T) &= \tilde{K} (e^z - 1), \text{ with } z = \ln \left( \frac{S}{\tilde{K}} \right)
 \end{aligned}$$

$TRS_j$  factors can be derived:

$$\begin{aligned}
 TRS_j &= \frac{2}{b-a} \int_a^b \tilde{K} (e^z - 1) \cos \left( j\pi \frac{z-a}{b-a} \right) dz \\
 \Rightarrow TRS_j &= \frac{2}{b-a} \tilde{K} (\chi_j(a, b) - \psi_j(a, b))
 \end{aligned}$$

with:

$$\begin{cases} \chi_j(a, b) = \int_a^b e^z \cos\left(j\pi \frac{z-a}{b-a}\right) dz \\ \psi_j(a, b) = \int_a^b \cos\left(j\pi \frac{z-a}{b-a}\right) dz \end{cases}$$

Taking direct  $[c, d] \rightarrow [a, b]$  variables (instead of highlighting integrals solving under  $[c, d] \subset [a, b]$  first, the reader can do it if he wants):

$$\Rightarrow \begin{cases} \chi_j(a, b) = \frac{1}{1 + \left(\frac{j\pi}{b-a}\right)^2} [\cos(j\pi) e^b - e^a] \\ \psi_j(a, b) = \begin{cases} j \neq 0 : 0 \\ j = 0 : b - a \end{cases} \end{cases}$$

The algorithm strictly follows previous equations and compare results with direct TRS pricing using risk-neutral, which is here quite trivial as the product is a pure delta one and the discounted asset is a martingale under  $\mathbb{Q}$  measure:

$$TRS = \mathbb{E}^{\mathbb{Q}} \left( \frac{S_T - K - S_{t_{TRS}}}{e^{r\tau}} \right) = S_0 - e^{-r\tau} (K + S_{t_{TRS}})$$


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#### **IV – Model discussion and next axis**

- Of course the current model is not very useful as the TRS is very vanilla and hence the direct pricing approach is far more useful. But it can be considered as a POC which highlights that FFT also works in TRS context and considered as a “core” for next models, which will be detailed in coming papers.
- Model #2 will relax the bullet payment assumption, allowing several payments.
- Model #3 will relax fixed financing rate assumption, allowing floating interest rate and hence an interest rate distribution, uncorrelated to asset distribution.
- Model #4 will do the same, but with joint-distribution asset/interest rate.