Total return swap – Fast Fourier Transformation – Bullet TRS with fixed rate

I – Introduction

This paper aims to highlight Fast Fourier Transformation (FFT) in Total Return Swap (TRS) pricing context. The paper will start by a reminder about FFT methods, classically used for option pricing, this chapter will introduce general ideas which will be the "core" of all papers of FFT module. And, for this paper, a part will be dedicated to model itself for, in first approach, a very simple flavor of TRS.

II – Reminder of Fourier transformation and TRS generic formula

Let's assume f a probability density function of a variable X, which distribution has a characteristic function ϕ (the gaussian is one of many examples of this kind of law).

The Fourier-cosine expansion of f on interval $[a, b]$ is:

$$(\boldsymbol{COS})\begin{cases} f(x) = {\sum_{j=0}^{\infty}}^{*} F_j cos\left(j\pi \frac{x-a}{b-a}\right) \\ F_j = \frac{2}{b-a}\mathcal{R}\left(\phi\left(\frac{j\pi}{b-a}; x\right) e^{-ij\pi\frac{a}{b-a}}\right) \end{cases}$$

The * indicates that the first term of the sum should be weighted by 0.5.

The TRS risk-neutral valuation follows next equations, and at this step we won't make any assumption about use-case "reduction", the TRS is assumed to follow one of most complex flavors, like several payments and floating financing rate:

$$\begin{aligned} TRS(S_0, r_0, t_0) \ &= \ \mathbb{E}^{\mathbb{Q}}\left(\textstyle\sum_{k=1}^{N} \frac{V_k(S_k, r_k, t_k)}{B(t_0, t_k)}\middle| S_0, r_0\right) \\ \Rightarrow TRS(S_0, r_0, t_0) \ &= \ \sum_{k=1}^{N} \mathbb{E}^{\mathbb{Q}}\left(\frac{V_k(S_k, r_k, t_k)}{B(t_0, t_k)}\middle| S_0, r_0\right) \\ \Rightarrow TRS(S_0, r_0, t_0) \ &= \ \sum_{k=1}^{N} \iint_{\mathbb{R}\times\mathbb{R}} \frac{V_k(S_k, r_k, t_k)}{B(t_0, t_k)} f(S_k, r_k | S_0, r_0)\, dSdr \\ \Rightarrow TRS(x, y, t_0) \ &= \ \sum_{k=1}^{N} \iint_{\mathbb{R}\times\mathbb{R}} \frac{V_k(z, h, t_k)}{B(t_0, t_k)} f(z, h | x, y)\, dzdh \end{aligned}$$

With following notations:

- V_k the TRS flows Pay-off.
- S_k the TRS underlying asset, hidden through variable z in integration.
- r_k the interest rate, hidden through variable h in integration.
- $B(t_0, t_k)$ the bank account.
- S_0 the spot underlying price, hidden through the conditional variable x in integration.
- r_0 the spot interest rate, hidden through the conditional variable y in integration.
- f the joint probability function for both asset and interest rate.

The Pay-off for TRS flows and bank account are:

$$\begin{cases} \text{if Libor} - \text{like}: \ V_k(S, r, t_k) \ = S_{t_k} - S_{t_{k-1}} - r_{t_{k-1}} \cdot (t_k \ - \ t_{k-1}) \\ \text{if RFR}: \ V_k(S, r, t_k) \ = S_{t_k} - S_{t_{k-1}} - r_{t_k} \cdot (t_k \ - \ t_{k-1}) \\ B(t_0, t_k) \ = \ e^{\int_{t_0}^{t_k} r_s ds} \end{cases}$$

All those formulas are generic ones with the interesting two-variable density function, embedding asset and interest risk factors.

III – TRS pricing, model #1

For this first model, we took several assumptions:

- TRS has a bullet payment.
- Financing rate is fixed.
- Interest rate is considered as constant.
- Underlying asset follows a lognormal distribution (geometric Brownian motion)

With those previous notes, the previous TRS pricing scheme is heavily simplified and then follows:

$$TRS(S_0, t_0) = e^{-r(T-t_0)}\,\mathbb{E}^{\mathbb{Q}}(V_T(S,T)|S_0)$$
$$\Rightarrow TRS(x, t_0) = e^{-r(T-t_0)} \int_{\mathbb{R}} V_T(z,T) f(z|x)\,dz$$
$$\Rightarrow TRS(x, t_0) \approx e^{-r(T-t_0)} \int_{[a,b]} V_T(z,T) f(z|x)\,dz$$

$$\Rightarrow \begin{cases} TRS(x, t_0) \approx e^{-r(T-t_0)} \sum_{j=0}^{M-1} \mathcal{R}\left(\phi\left(\frac{j\pi}{b-a}; x\right) e^{-ij\pi\frac{a}{b-a}}\right) TRS_j \\ TRS_j \text{ specific components related to TRS payoff} \end{cases}^{*}$$

For the previous equation, we used the fact that even if indeed the characteristic function of a lognormal distribution can't be directly calculated, we can use the following "trick", assuming $Z \sim ln\mathcal{N}(0,1)$:

$$CDF_Z(z) = \mathbb{P}(Z \le z) = \mathbb{P}(e^X \le z) = \mathbb{P}(X \le ln(z)) = CDF_X(ln(z))$$

$$\Rightarrow f_Z(z) = \frac{dCDF_Z(z)}{dz} = \frac{dCDF_X(ln(z))}{d\,ln(z)} \frac{d\,ln(z)}{dz} = \frac{1}{z} f_X(ln(z))$$

And as $X \sim \mathcal{N}(0,1)$ this time the characteristic function exists and hence we can use the previous ($\boldsymbol{COS}$) result.

Taking following transformation into account:

$$V_T(S,T) = S_T - K - s_{t_{TRS}}$$

$$\Longrightarrow V_T(S,T) = \widetilde{K}\left(\frac{S_T}{\widetilde{K}} - 1\right), with\ \widetilde{K} = K + s_{t_{TRS}}$$

$$\Longrightarrow V_T(z,T) = \widetilde{K}(e^z - 1), with\ z = ln\left(\frac{S}{\widetilde{K}}\right)$$

TRS_j factors can be derived:

$$TRS_j = \frac{2}{b-a}\int_a^b \widetilde{K}(e^z - 1) cos\left(j\pi \frac{z-a}{b-a}\right) dz$$

$$\Longrightarrow TRS_j = \frac{2}{b-a}\widetilde{K}\big(\chi_j(a,b) - \psi_j(a,b)\big)$$

with:

$$\begin{cases} \chi_j(a,b) = \int_a^b e^z cos\left(j\pi \frac{z-a}{b-a}\right) dz \\ \psi_j(a,b) = \int_a^b cos\left(j\pi \frac{z-a}{b-a}\right) dz \end{cases}$$

Taking direct $[c,d] \to [a,b]$ variables (instead of highlighting integrals solving under $[c,d] \subset [a,b]$ first, the reader can do it if he wants):

$$\Rightarrow \begin{cases} \chi_j(a,b) = \dfrac{1}{1+\left(\dfrac{j\pi}{b-a}\right)^2}[\cos(j\pi)e^b - e^a] \\ \psi_j(a,b) = \begin{cases} j \neq 0 : 0 \\ j = 0 : b-a \end{cases} \end{cases}$$

The algorithm strictly follows previous equations and compare results with direct TRS pricing using risk-neutral, which is here quite trivial as the product is a pure delta one and the discounted asset is a martingale under $\mathbb{Q}$ measure:

$$TRS = \mathbb{E}^{\mathbb{Q}}\left(\frac{S_T - K - s_{t_{TRS}}}{e^{r\tau}}\right) = S_0 - e^{-r\tau}(K + s_{t_{TRS}})$$

IV – Model discussion and next axis

- Of course the current model is not very useful as the TRS is very vanilla and hence the direct pricing approach is far more useful. But it can be considered as a POC which highlights that FFT also works in TRS context and considered as a “core” for next models, which will be detailed in coming papers.
- Model #2 will relax the bullet payment assumption, allowing several payments.
- Model #3 will relax fixed financing rate assumption, allowing floating interest rate and hence an interest rate distribution, uncorrelated to asset distribution.
- Model #4 will do the same, but with joint-distribution asset/interest rate.