

Total return swap – Fast Fourier Transformation – Bullet TRS with fixed rate

I – Introduction

This paper aims to highlight Fast Fourier Transformation (FFT) in Total Return Swap (TRS) pricing context. The paper will start by a reminder about FFT methods, classically used for option pricing, this chapter will introduce general ideas which will be the “core” of all papers of FFT module. And, for this paper, a part will be dedicated to model itself for, in first approach, a very simple flavor of TRS.

II – Reminder of Fourier transformation and TRS generic formula

Let's assume f a probability density function of a variable X , which distribution has a characteristic function $\phi(u) = \int_{\mathbb{R}} e^{i y u} f(y) dy$ (the gaussian is one of many examples of this kind of law).

The Fourier-cosine approximation of f on interval $[a, b]$ is:

$$(COS) \begin{cases} f(x) = \sum_{j=0}^{\infty} F_j \cos\left(j\pi \frac{x-a}{b-a}\right)^* \\ F_j = \frac{2}{b-a} \mathcal{R}\left(\phi\left(\frac{j\pi}{b-a}\right) e^{-ij\pi \frac{a}{b-a}}\right) \end{cases}$$

The * indicates that the first term of the sum should be weighted by 0.5 and \mathcal{R} indicates the real part of the complex component.

The TRS risk-neutral valuation follows next equations, and at this step we won't make any assumption about use-case “reduction”, the TRS is assumed to follow one of most complex flavors, like several payments and floating financing rate:

$$\begin{aligned} TRS(S_0, r_0, t_0) &= \mathbb{E}^{\mathbb{Q}}\left(\sum_{k=1}^N \frac{V_k(S_k, r_k, t_k)}{B(t_0, t_k)} \middle| S_0, r_0\right) \\ \Rightarrow TRS(S_0, r_0, t_0) &= \sum_{k=1}^N \mathbb{E}^{\mathbb{Q}}\left(\frac{V_k(S_k, r_k, t_k)}{B(t_0, t_k)} \middle| S_0, r_0\right) \\ \Rightarrow TRS(S_0, r_0, t_0) &= \sum_{k=1}^N \iint_{\mathbb{R} \times \mathbb{R}} \frac{V_k(S_k, r_k, t_k)}{B(t_0, t_k)} f(S_k, r_k | S_0, r_0) dS dr \\ \Rightarrow TRS(x, y, t_0) &= \sum_{k=1}^N \iint_{\mathbb{R} \times \mathbb{R}} \frac{V_k(z, h, t_k)}{B(t_0, t_k)} f(z, h | x, y) dz dh \end{aligned}$$

With following notations:

- V_k the TRS flows Pay-off.
- S_k the TRS underlying asset, hidden through variable z in integration.
- r_k the interest rate, hidden through variable h in integration.
- $B(t_0, t_k)$ the bank account.
- S_0 the spot underlying price, hidden through the conditional variable x in integration.
- r_0 the spot interest rate, hidden through the conditional variable y in integration.
- f the joint probability function for both asset and interest rate.

The Pay-off for TRS flows and bank account are:

$$\left\{ \begin{array}{l} \text{if Libor-like : } V_k(S, r, t_k) = S_{t_k} - S_{t_{k-1}} - r_{t_{k-1}} \cdot (t_k - t_{k-1}) \\ \text{if RFR : } V_k(S, r, t_k) = S_{t_k} - S_{t_{k-1}} - r_{t_k} \cdot (t_k - t_{k-1}) \end{array} \right.$$

$$B(t_0, t_k) = e^{\int_{t_0}^{t_k} r_s ds}$$

All those formulas are generic ones with the interesting two-variable density function, embedding asset and interest risk factors.

III – TRS pricing, model #1

For this first model, we took several assumptions:

- TRS has a bullet payment.
- Financing rate is fixed.
- Interest rate is considered as constant.
- Underlying asset follows a lognormal distribution (geometric Brownian motion)
- Formulas will highlight receive performance case, but the algorithm will of course handle both receiver/payer performance total return swaps.

With those previous notes, the previous TRS pricing scheme is heavily simplified and then follows:

$$\begin{aligned}
 TRS(S_0, t_0) &= e^{-r(T-t_0)} \mathbb{E}^{\mathbb{Q}}(V_T(S, T) | S_0) \\
 \Rightarrow TRS(x, t_0) &= e^{-r(T-t_0)} \int_{\mathbb{R}} V_T(z, T) f(z|x) dz \\
 \Rightarrow TRS(x, t_0) &\approx e^{-r(T-t_0)} \int_{[a,b]} V_T(z, T) f(z|x) dz \\
 \Rightarrow \begin{cases} TRS(x, t_0) \approx e^{-r(T-t_0)} \sum_{j=0}^{M-1} \mathcal{R} \left(\phi \left(\frac{j\pi}{b-a} \right) e^{-ij\pi \frac{a}{b-a}} \right) TRS_j^* \\ TRS_j \text{ specific components related to TRS payoff} \end{cases}
 \end{aligned}$$

With $\phi(u) = e^{\left(r - \frac{\sigma^2}{2}\right)iu(T-t_0) - \frac{1}{2}\sigma^2 u^2(T-t_0)}$ the characteristic function of the normal variable $\ln(S)$ under risk-neutral measure.

For the previous equation, we used the fact that even if indeed the characteristic function of a lognormal distribution can't be directly calculated, we can use the following "trick", assuming $Z \sim \ln \mathcal{N}(0,1)$:

$$\begin{aligned}
 CDF_Z(z) &= \mathbb{P}(Z \leq z) = \mathbb{P}(e^X \leq z) = \mathbb{P}(X \leq \ln(z)) = CDF_X(\ln(z)) \\
 \Rightarrow f_Z(z) &= \frac{dCDF_Z(z)}{dz} = \frac{dCDF_X(\ln(z))}{d \ln(z)} \frac{d \ln(z)}{dz} = \frac{1}{z} f_X(\ln(z))
 \end{aligned}$$

And as $X \sim \mathcal{N}(0,1)$ this time the characteristic function exists and hence we can use the previous (**COS**) result.

Taking following transformation into account:

$$\begin{aligned}
 V_T(S, T) &= S_T - K - s_{t_{TRS}} \\
 \Rightarrow V_T(S, T) &= \tilde{K} \left(\frac{S_T}{\tilde{K}} - 1 \right), \text{ with } \tilde{K} = K + s_{t_{TRS}} \\
 \Rightarrow V_T(z, T) &= \tilde{K} (e^z - 1), \text{ with } z = \ln \left(\frac{S}{\tilde{K}} \right)
 \end{aligned}$$

TRS_j factors can be derived:

$$TRS_j = \frac{2}{b-a} \int_a^b \tilde{K} (e^z - 1) \cos \left(j\pi \frac{z-a}{b-a} \right) dz$$

$$\Rightarrow TRS_j = \frac{2}{b-a} \tilde{K} (\chi_j(a, b) - \psi_j(a, b))$$

with:

$$\begin{cases} \chi_j(a, b) = \int_a^b e^z \cos\left(j\pi \frac{z-a}{b-a}\right) dz \\ \psi_j(a, b) = \int_a^b \cos\left(j\pi \frac{z-a}{b-a}\right) dz \end{cases}$$

Taking direct $[c, d] \rightarrow [a, b]$ variables (instead of highlighting integrals solving under $[c, d] \subset [a, b]$ first, the reader can do it if he wants):

$$\Rightarrow \begin{cases} \chi_j(a, b) = \frac{1}{1 + \left(\frac{j\pi}{b-a}\right)^2} [\cos(j\pi) e^b - e^a] \\ \psi_j(a, b) = \begin{cases} j \neq 0 : 0 \\ j = 0 : b - a \end{cases} \end{cases}$$

We can then rearrange the previous TRs pricing formula:

$$\begin{cases} TRS(x, t_0) \approx \tilde{K} e^{-r(T-t_0)} \mathcal{R} \left(\sum_{j=0}^{M-1} \phi\left(\frac{j\pi}{b-a}\right) e^{-ij\pi \frac{x-a}{b-a}} \cdot U_j \right)^*, \text{ with } x = \ln\left(\frac{S_0}{\tilde{K}}\right) \\ U_j = \frac{2}{b-a} (\chi_j(a, b) - \psi_j(a, b)) \end{cases}$$

The algorithm strictly follows previous equations and compare results with direct TRS pricing using risk-neutral, which is here quite trivial as the product is a pure delta one and the discounted asset is a martingale under \mathbb{Q} measure:

$$TRS = \mathbb{E}^{\mathbb{Q}} \left(\frac{S_T - K - S_{t_{TRS}}}{e^{r\tau}} \right) = S_0 - e^{-r\tau} (K + S_{t_{TRS}})$$

IV – Model discussion and next axis

- Of course, the current model is not very useful as the TRS is very vanilla and hence the direct pricing approach is far more useful. But it can be considered as a POC which highlights that FFT also works in TRS context and considered as a “core” for next models, which will be detailed in coming papers.
- Model #2 will relax fixed financing rate assumption, allowing floating interest rate and hence an interest rate distribution, with joint-distribution asset/interest rate.
- Model #3 will do the same, but with several TRS payments.