Counterparty credit risk under credit risk contagion using time-homogeneous phase-type distribution

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Declaration of Originality:
I hereby declare that this written work is original work which I alone have authored and written in my own words, with the exclusion of proposed corrections. Information derived from the published work of others has been acknowledged in the text and a list of references is given in the bibliography.

Abstract

With the current situation of credit spread contagion illustrated by the European sovereign bonds crisis and the chain reaction triggered by the derivatives books of Lehman Brothers, financial institutions have increasingly focused on pricing and risk management of counterparty credit risk. Recent credit contagion through financial contingent claims highlight the fact that contagion-links impact the value of products when investors are exposed to counterparty risk.

This thesis plan to build on reduced-form credit risk models to assess the credit risk contagion that is inherent in a obligor multivariate framework. The aim is to evaluate the requirements that are necessary in generating a mathematical framework consistent with the valuation of financial claims, credit and non-credit related, where the parties of those claims exhibit credit risk contagion.

By applying a multivariate framework of credit contagion to counterparty credit risk based on a queueing theory, called phase-type distribution, we hope to highlight the benefit of bottom-up models versus top-down ones in terms of extracting information relative to dependence within an identifiable obligor set. We will review the mathematical literature in addressing credit dependence modelling in dynamic and static format. This will be the opportunity to value a set of claims under our model to show that claims that contain "credit leverage" are particularly sensible to credit risk contagion and could benefit from our developed framework to gain adequate counterparty credit risk pricing.

Contents

0.1 Introduction and overview of content.

The title of this document is "Counterparty Credit risk under credit risk contagion using the multivariate phase-type distribution".

In relation with the title, the content of this thesis is structured a follow

- First, an introduction in chapter 1 on the various aspect of credit risk, describing the characteristics of credit risk and their diversity in terms of definition. Then, a presentation of the type of financial products that are based around credit. Those are categorised as "defaultable"products meaning that a credit related event is the trigger of the valuation of such products such as the single-name credit default swaps and the index credit default swaps. The product coverage will span towards "non-defaultable"products such as the interest-rate swaps which will be credit-linked indirectly due to the credit sensitivity of the parties involved in this bilateral claim. This will introduce the need to cover counterparty credit risk.

- Second, a general presentation on one specific area of credit risk, called Counterparty Credit Risk, in chapter 2, that will be the core of the focus of the risk we want to value in this document. This part will mainly present the reasons of interest in this metric from the industry and its coverage according to regulatory requirements.

- In chapter 3, we then introduce the mathematical framework necessary to value the contingent claims. The claims space described in section 3.1.1 will be separated into defaultable and non-defaultable claims. The separation line is mainly the distinction between credit and non-credit linked value and the different approach to establish price under risk-neutral expectation using the pricing measure $\mathbb{Q}$. Then, in section 3.1.2, we adjust those formulas under $\mathbb{Q}$ to take into account the riskiness introduced by the credit risk of both parties introducing the well-known Credit Valuation Adjustment.

- In chapter 4,in section4.1, we enlarge the probability space to a multivariate settings with a set of m obligors where the filtration is also enlarged to $\mathbb{G}$ to encompass multi-name default observations in $\mathbb{H}$ and state variables in the filtration $\mathbb{F}$. The previous section 3.1.2 is adjusted to redefine Counterparty Credit Risk in the valuation of claims under the new multivariate framework. Addtionnal technical assumptions like the $(\mathcal{H})$-hypothesis, no-simultaneous default, and the separability of processes in section 4.1 are specified to ensure valuation across filtrations for a finite sequence of stopping times.

- The next section introduces credit risk models to move towards numerical valuations. Section 5.1 will review the litterature on credit models, structural models and reduced-form models, then focus on dependence credit modelling with a distinction between "bottom-up"and "top-down "models.

- After reviewing the credit dependency models, we focus in section 6.1 on a Markov jump process, called phase-type distributions, that will be able to modelise dependency for m obligors with closed-form formula. We adjust the formula in section 6.1.2 in the Credit Valuation formula to account for credit dependency or credit contagion under the multivariate framework in section 4. We realise valuation under those credit contagion scenario for defaultable and non-defaultable claims to realise the potential impact of stressed event. We believe that the leveraged credit default swap will exhibit high sensitivity to those dependency events thus displaying the impact of the "gap risk "concept.

- Finally, due to the optionality aspect of the Credit Counterparty Risk and its sensitivity to credit spread volatility, we adjust the phase-type distribution model, in section 6.3, to encompass regime shifts in the markov jump parameters. We try to assess the conditions under which such a framework could exist.

Chapter 1

Credit risk

In this section, we give a short overview of some aspects of credit risk. A much more extensive discussions can be found in [Bielecki01], [Duffie03], [Lando04] and [Schőnbucher03]. There are several ways of defining the credit risk such as in [Duffie03]

- as "*the risk of changes in value associated with unexpected changes in credit quality.*"

or as in [Schőnbucher03]

- as "*the risk that an obligor does not honour its payments.*"

The concept of credit risk has to be put in the framework where it will be used:

- single obligor level, or on a portfolio level,
- in a static or dynamic environment,
- from a risk management or a pricing point of view,
- from a counterparty-risk exposure or standalone point of view

Additionally, as presented in [Schőnbucher03], page 2, many of the different framework of credit risk have to address the characteristics of the associated credit event with

- **arrival risk**: i.e. the occurrence of credit event over a given time period,
- **timing risk**: i.e. the uncertainty of the time of arrival risk,
- **recovery risk**: i.e. the size of the actual loss when the arrival risk is observed,
- **market risk or credit spread risk**: i.e. the deterioration or improvement of credit quality prior to the arrival risk,
- **interaction between market risk and arrival risk**: i.e. the joint dependence structure of credit event with parameters such as interest rates, equity prices, liquidity, etc.
- **cross-arrival risk**: i.e. the risk of several obligors jointly defaulting during some specific time period. This is of particular interest in a context of credit portfolio.

It is of course interesting to account as many of these topics as possible in any model. This will however lead to complex models that might be hard to calibrate from observable market data. Thus, given a specific credit risk problem one has to make a decision which components that should be taken into account.

In the case of a big wholesale credit portfolio at a commercial bank, such as credit card loans or Small and Medium Enterprise loans (SME), it is important to study the macro-economic factors that affects the households and the businesses. Those factors will be typically the level of interest rates or the credit scoring by opposition to modelling each of the thousand loans under consideration. The timing of default of each loans will be of reduced interest.

In the case of a portfolio of commercial loans, it will be of interest to model the loans but also the dependency arising from business interactions per industries. This will lead to some dimension reduction to make the problem tractable.

In the case of a portfolio up to a dozen of obligors, it is possible to model the obligor dependency behaviour in a portfolio context. This case is the situation that we will investigate by modelling the dependency links between those obligors and its impact in terms of valuation for bilateral agreement.

A *default risk* is a possibility that a counterparty to a financial contract will not fulfil a contractual commitment to meet her obligations stated in the contract. If this actually happens, we say that the party defaults, or that the default event occurs. More generally, by a *credit risk* we mean the risk associated with any kind of credit-linked events, such a: changes in the credit quality (including downgrades or upgrades in credit ratings), variations of credit spreads, and the default event. The *spread risk* is thus another component of the credit risk. To facilitate the analysis of complex agreements, it is important to make a clear distinction between the *reference (credit) risk* and the *counterparty (credit) risk* . The first generic term refers to the situation when both parties of a contract are assumed to be default-free, but due to specific features of the contract the credit risk of some reference entity appears to play an essential role in the contract's settlement. *Credit derivatives* are financial instruments that allow market participants to isolate and trade the reference credit risk. The main goal of a credit derivative is to transfer the reference risk either completely or partially, between the counterparties. In most cases, one of the parties can be seen as a buyer of insurance against the reference risk; consequently, the party that bears the reference risk is referred to as its buyer.

Let us now focus on the counterparty credit risk. An important feature of all the over-the-counter (OTC) derivatives is that, unlike exchanged-traded contracts, they are not backed by the guarantee of a clearinghouse or an exchange, so that each counterparty is exposed to the default risk of the other party. In practice, parties are required to post collateral or mark to market periodically. The counterparty risk emerges in a clear way in such contracts as *vulnerable claims* and *defaultable swaps*. In both cases, one needs to quantify the default risk of both parties in order to correctly assess the contract's value. Depending on whether the default risk of one or both parties is taken into account, we say that a contract involves the *unilateral* (one sided) or the bilateral *two-sided* default risk. We review now some of the products associated with credit exposure.

1.1 Products

1.1.1 Corporate Bonds

Corporate bonds are debt instruments issued by corporations. They are part of the capital structure of the firm (along with the equity). By issuing bonds, a corporation commits itself to make specific payments to the bondholders at some future dates and the corporation charges a fee for this commitment. However, the firm may default on its commitment in which case the bondholders will not receive the promised payments in full and, thus, will suffer financial loss. Of course, the occurrence of default, possibly caused by the firm's bankruptcy, is meaningful only during the life time of a particular bond. A corporate bond is an example of a *defaultable claim.* If we fix the maturity date of the bond as T, the price at time t of a T-maturity defaultable bond will be denoted by

$$D(t,T)$$

By contrast the notation $B(t,T)$ is used to denote the price at time t of a T-maturity default-free bond with the face value 1.

The *defaultable term structure* is the term structure implied by the yields on the default prone corporate bonds or sovereign bonds. A large portion of the credit risk literature is devoted on the modelling of a defaultable term structure as well as to pricing related credit derivatives.

Recovery rules

The recovery payment is frequently specified by the recovery rate δ, i.e. the fraction of the bond's face value amount paid to the bondholders in case of default with time of default τ. The timing of the recovery payoff is of course another essential parameter. The most common models are (with bond face value $L = 1$) with their associated bond payoff at T

- *fractional recovery of par value,*

$$D(T,T) = 1_{\{\tau>T\}} + \delta B^{-1}(\tau,T)1_{\{\tau\leq T\}}$$

- *fractional recovery of Treasury value*

$$D(T,T) = 1_{\{\tau>T\}} + \delta 1_{\{\tau\leq T\}}$$

- *fractional recovery of market value*

$$D(T,T) = 1_{\{\tau>T\}} + \delta D(\tau-,T)B^{-1}(\tau,T)1_{\{\tau\leq T\}}$$

  where $D(\tau-,T)$ is the value of the bond just before default time.

It is more common to use the generic term *loss given default* (LGD) to describe the likely loss of value in case of default with LGD being equal to 1 minus recovery rate.

In a more abstract formulation to the recovery rule it is assumed that if a bond defaults during its lifetime then the recovery payment is made either at the default time τ or at the maturity T of the bond. In the first case the payment is denoted by the value Z_τ at default time of the *recovery process* Z. In the second case, the recovery payment is determined by the realisation of the *recovery claim* $\tilde{X}$.

Credit spreads

A *credit spread* measures the excess return on a corporate bond over the return of an equivalent Treasury bond. A credit spread may be expressed as the difference between respective yields to maturity or as the difference between respective forward rates. The *term structure of credit spreads* will refer to the term structure of those differences. The dertermination of the credit spread is important in credit risk modeling as the information encapsulated in this spread with respect to the default free case information gives indication of the likelihood of default of the credit entity under consideration. Typically *Distressed securities* are defined by the high level of credit spreads yielded by some corporate securities.

Credit ratings

A firm's *credit rating* is a measure of the firm's propensity to default. Credit ratings are typically identified with elements of a finite set, called the *credit grades*. The credit ratings are attributed by commercial rating agencies notably Moody's Investors Service an Standard and Poor's Corporation. It is worth noting that the ratings primarily reflect the likelihood of default and thus provide not the most adequate assessment of the debt credit quality.

Typical products

Typical products include

- Corporate Coupon Bonds: Consider a corporate coupon with face value L which matures at time $T = T_n$ and promises to pay (fixed or variable) coupons c_i at times $T_1 < T_2 < \cdots < T_n = T$. Assume that the recovery payment is proportional to the face value and this it is made at maturity T in case the default event occurs before or at the maturity date. The bond cash flows are

$$\sum_{i=1}^{n} c_i 1_{\{\tau > T_i\}} 1_{T_i}(t) + (L 1_{\{\tau > T\}} + \delta L 1_{\{\tau \leq T\}}) 1_T(t)$$

- Fixed and Floating Rate Notes: Consider two fixed coupon bonds, a defaultable bond paying c_1 and a risk-free bond paying c_2. Thus to compensate for default risk, the coupon rate of the corporate bond would be greater than of the risk-free bond and be referenced as the fixed-rate credit spread $S := c_1 - c_2$.

- Floating Rate Note: Each of the coupon payments is made according to the floating interest rate value on the coupon's date. As previously if $L(T_i)$ is the floating interest rate for the risk-free borrowing at T_i and lending over the accrual period $[T_i T_{i+1}]$, the payment on coupon date T_i is the risk adjusted floating rate $\hat{L}(T_i) = L(T_i) + s$. The non-negative constant s represents the bond-specific floating-rate credit spread. The higher level of credit spread s usually corresponds to the lower credit quality of the issuer.

1.1.2 Credit Derivatives

A credit derivative is a financial instrument that allows banks, insurance and other participants to isolate, manage and trade their credit sensitive investments. Initially credit derivatives were designed as tools to partially or completely remove credit risks. However credit derivatives is a very broad class of derivatives and it is hard to give an exact mathematical definition that covers all the different versions. [Schőnbucher03] provides a general feature definition

- "*A credit derivative is a derivative security that is primarily used to transfer, manage or hedge credit risk.*"
- "*A credit derivative is a derivative security whose payoff is materially affected by credit risk.*"

The credit derivatives space can be categorised according to [Bielecki01] as either

- *default products*: Those are credit derivatives that are linked exclusively to the default event. It includes contracts with the payoff determined by the default event as opposed to changes in the credit quality of the underlying instruments. The most notable products are *Default swaps, first-to-default swaps*, *k^{th}-to-default swaps* and *basket default swaps.*
- *spread products*: Those are credit derivatives whose payoff is determined by the change in credit quality of the reference obligor. The most common products are *credit spread swaps*, *credit spread options* and *credit linked notes.*
- *credit-transferred products*: Those products allow for the transfer of total risk of assets. The most common products are *Total return swap* and *collateralised debt obligation (CDO).*

A full coverage of credit derivatives products can be found in [Das98a], in [Gallo06] for a good coverage of credit index derivatives, and in [Batchvarov01] for credit-transferred product (or more commonly called securitised products).

Credit default swaps

Credit default swaps are the most traded credit derivatives instruments and are used as building blocks for more exotic credit derivatives such as basket default swaps and CDO's. The single-name default swaps will then be used as a hedging tool with other instruments such as bonds or asset swaps, etc. They are also used to hedge counterparty-risk that arise in a more traditional interest-rate derivatives such as swaps or caps and floors and any other OTC products where there is a risk of a counterparty not fulfilling its obligations as in the case of vulnerable claims.

[Schőnbucher03] gives a definition of credit derivatives of the default type along with notations that will be useful in the reminder of this document

Definition 1.1.1. *Consider two counterparties entity A and B. The credit derivatives of default product type is a bilateral agreement entered between A and B that has a payoff tied to a default event. The default event is tied to a reference composed either of an obligor C, a collection of obligors* $(C_1, \ldots, C_n)$ *or reference credit assets issued by those references. If there is a default event over the course of the financial agreement, the entity A will make a default payment to B according to the rules set at the inception of this agreement.*

We will generally by referring to a credit derivatives imply that this derivatives is of the default type unless stated otherwise.

This is equivalent to an "insurance"contract with the entity A referred as the "protection buyer", the entity B as "protection seller"while C or $(C_1, \ldots, C_n)$ are the reference obligor(s).

The default event is characterised through a set of rules established by the International Swaps and Derivatives Association (ISDA -www.isda.org) with the following events listed:

- bankruptcy
- failure to pay
- obligation default
- obligation acceleration
- repudiation/moratorium
- restructuring
- ratings downgrade below given threshold (for ratings-triggered credit derivatives)
- changes in the credit spread (for credit spread-triggered credit derivatives)

However for modelling reasons, the main default event under consideration will the bankruptcy of the obligors in the reference set $(C_1, \ldots, C_n)$. The time of the default event connected to the reference obligor C will be a crucial variable in pricing, hedging and managing dynamic credit risk. This positive random variable will be denoted τ and mainly modelled as a hitting time.

Single-name credit default swaps

A single-name credit default swaps is characterised as in definition 1.1.1, page 11, with a unique reference obligor C. Those products are the most actively traded credit derivatives and are standardly characterised by the following elements:

Consider a single-name credit derivatives that starts at time t with a maturity set at T the fee payment that A makes to B to acquire protection w.r.t obligor C is paid in two streams

- A pays an upfront-fee at t
- A pays a recurrent-fee to B until $T \wedge \tau$

Both parties enter the agreement having agreed on

- the reference obligor C and its reference credit asset
- the notional amount N of the contract
- A detailed definition of the default event τ in accordance with ISDA rules
- the time period validity of the contract, i.e. $[t, T]$
- the fee amount (also called *default swap premium or default swap rate* expressed as basis point (bps) per annum on the notional
- the payment frequency, typically annually, semi-annually, quarterly
- the timing and size of default payments from B to A propotional to N. Those payments can be specified either *cash-settled* or *physically-settled*,

After inception, for most common reference obligor C, the default swap rate is traded on secondary markets and the value of this rate represents perceptions by the market of the changes in the credit quality of the underlying, other valuation factors such as the level of interest rates and technical factors such as liquidity in the market or expectations. This will be a source of valuable information w.r.t to the default time τ of the obligor C.

Basket default swaps

Basket default swaps are identical in principle to single-name default swaps with the most notable difference being the fact that the reference is constituted of a set of obligors $(C_1, \ldots, C_n)$ or a collection of assets issued by this set. The main difference is how the default event is characterised out of the set of obligors and how the default payment is defined.

Consider m obligors in the set. Let τ_i denote the time of the default time associated with the i^{th} instrument and $\tau^{(i)}$ the i^{th} default time in the obligor set. The two most common denomination for this family of products are categorised as

- *first-to-default swap*: We denote τ as the time when the first of these credit events occurs, i.e. $\tau = \min(\tau_1, \ldots, \tau_m)$ where it is commonly assumed that $\tau_i \neq \tau_j$,. It the first credit event happens no later than at time T a contingent payment linked to the defaulted reference is made.
- k^{th}*-to-default swap*: the credit event is redefined as $\tau = \tau^{(k)}$, $1 < k \leq m$ and the contingent payment is the reference name to be the k^{th} default name out of the obligor set.

Collateralised debt obligations

A collateralised debt obligation is structured through securitisation of a pool of defaultable assets being a pool of loans, bonds, credit default swaps, etc. This pool is sold to a entity called a Special Purpose Vehicle which will issue obligations whose payments are based by the income generating pool. The service of the obligations is structured through a payment priority structure called tranches to reflect seniority.

Let's consider an underlying pools constituted of defaultable assets of m obligors $(C_1, \ldots, C_m)$ with the corresponding notional amounts $N_1, \ldots, N_m$. The total notional of the CDO is thus $N = \sum_{i=1}^{m} N_i$. Let's consider γ tranches dividing the notional N with $N = \sum_{k=1}^{\gamma} T_i$ and then define each tranche T_k by its attachment point k_a and detachment point k_d. The upper tranche is called the *super senior tranche* while the lower tranche is often called the *equity tranche*.

The income generated by the loans will be paid at coupon dates $t_1, \ldots, t_n = T$ to the tranche holders according to the schedule of payments that is usually a function of the tranche seniority. Let introduce L_t the accumulated loss at time t due to the defaults from the underlying pool accrued over the time period from inception until time t.

$$L_t = \sum_{i=1}^{m} N_i(1 - REC_i)1_{\{\tau_i \leq t\}}$$

with REC_i the recovery rate of obligor i.

The financial instrument tranche γ with maturity T is a bilateral contract where the protection seller B agrees to pay the protection buyer A, all losses that occur in the interval $[k_{\gamma-1}, k_\gamma]$ derived from L_t up to time T. The payments are made at default times it they are prior to T. The expected value of this payment is called *the protection leg*. As a compensation for this A pays B a periodic fee proportional to the current outstanding (reduced due to losses) value of tranche γ up to time T. The value of this payment constitues the *premium leg*. The tranche spread is the premium fee that equates the value of both legs.

The loss L_t will be impacted to the notional of the notes in a seniority inverted logic, thus first impacting the equity tranche, then the second tranche when the losses L_t is greater than the notional of tranche T_1, etc. The accumulated Loss $L^{(k)}$ of the tranche $k \in \{1, \ldots, \gamma\}$ at time t is defined as

$$L_t^{(k)} = (L_t - T_{\gamma-1})1_{\{L_t \in [T_{k-1}, T_k]\}} + (T_k - T_{k-1})1_{\{L_t > T_k\}}$$

and correspondingly will reduce the payments received by the notes holder. Note that in this formula the loss L_t is adjusted with the recovery of defaultable securities which is not taken into account in the case of index default swaps. (defined next)

The structure is of course a way to securitise loans. If the underlying credit portfolio contains traditional loans which can not be readily traded on the market then the CDO allows market participants to invest in notes connected to these loans, thus making it possible to put a market price on loans. Those notes are characterised as *cash CDO*.

As a long position in a corporate bond has a similar risk to a short position in a CDS when the reference entity in the CDS is the company issuing the bond it is possible to issue notes backed by a pool of CDS. Those notes are characterised as *synthetic CDO*.

We can point at this time that CDO's value are dependent of joint default behaviour of its underlying pool and are also impacted by the structural features of the Notes. Although CDO will not be valued in this document, they are a key structure used to model dependency structure of credit risk with related papers.

Index default swaps

The introduction of CDS indices such as the iTraxx in Europe and Asia, and CDX in North America have revolutionized the trading of credit risk due to their liquidity, flexibility and standardisation. Index default swaps are similar to synthetic collateralised debt obligations (CDO). An index CDS with maturity T, has almost the same structure as a corresponding CDO but with two main differences. The protection is on all credit losses that occurs in the underlying portfolio up to time T. Secondly, in the premium leg the spread is paid on a notional proportional to the number of obligors left in the portfolio at each payment date. Thus there is no account of the recovery rate of defaulted entities. So with $N_t = \sum_{i=1}^{m} 1_{\{\tau_i \leq t\}}$ the number of default obligors among the underlying pool at time t then the fee is paid on the notional $(1 - \frac{N_t}{m})$. A list of all the specificities can be found in [Gallo06], page 73, in [Ginty04a] and in [Ginty04b]. So, similarly to CDO, index CDS are by nature products whose prices are determined by the correlation of credit risk in the underlying portfolio.

Leverage Super Senior

Leverage Super Senior (LSS) products are categorised as spread products build on top of super senior tranche of CDO. LSS notes apply leverage outside the CDO deal to allow investors to achieve a higher yield. The LSS notes structure means to attract a broader investor base by offereing a portion of the super-senoir tranche in the form of a cash note. The notional of the LSS note is a fraction of the notional of the super senior tranche. From that perspective, the LSS tranche receives the benefit of the cash flows allocated to the full super senior tranche thus allowing it to pay a considerably higher spread. However at the same time its exposure to loss is capped at its size. In other words, the LSS note investment has limited recourse to the funded amount. Should losses accumulate to exceed the LSS notional, this discrepancy, unless corrected with triggers would be absorbed by the protection buyer. The triggers are based on either the market value or collateral losses exceeding a threshold usually consistent with a downgrade below AAA. Triggers are designed to mitigate the protection buyer's "gap risk". *The protection buyer is exposed to the risk that gapping spreads will result in a tranche mark-to-market exceeding the LSS note investor's tranche notional.* To protect against such a scenario, the protection buyer desires the structure to delevarage (increase the funded amount) or unwind, should the probability of such a scenario rise significantly. To gauge the probability and protect against such risk, one of the following triggers are used.

- tranche mark-to-market,
- weighted average spread (WAS) on the portfolio,
- Losses (of subordination/collateral).

LSS are very interesting products since the triggers are very sensitive of the correlation of credit risk and not only credit default implying a necessity to model credit risk dependence and not only credit default dependence. A detailed explanation can be found in [Gallo06], page 148.

We now introduce the vulnerable claims category which can be defined as claims whose value driver is non credit risk related as previously but might be vulnerable to counterparty credit risk.

1.1.3 Vulnerable claims

Vulnerable claims are contingent agreement that are traded over-the-counter between default-prone parties: each side of the contract is thus exposed to the counterparty risk of the other party. The default risk of a counterparty (or of both parties) is thus an important component of financial risk embedded in a vulnerable claim.; it should necessarily be taken into account in valuation and hedging procedures for vulnerable claims. On the other hand, the underlying (reference) assets are assumed to be insensitive to credit risk. This assumption however might be relaxed and *cross-effect* might be evaluated since for example higher credit risk might also correspond empirically in high interest rate environment. *Credit Derivatives* are financial instruments that allow trading in the reference credit might also be an example though unless the counterparty risk is negligible.

The classical example of a vulnerable contingent claim with unilateral default risk is an option contract in which the option writer may default on its obligations. Thus, the payoff at T depends on whether a default event associated with the option's writer has occurred before or on the maturity date, or not. The default risk of the holder of the option is not relevant here due to the asymmetricallity of the payoff. Thus if C_T is the payoff of the option, the vulnerable claim adjustment from the option holder is

$$C_T^d = C_T 1_{\{\tau > T\}} + \delta C_T 1_{\{\tau \leq T\}}$$

We will focus on a list of derivatives, thus exposed to bilateral counterparty risk, for the main class of assets : interest-rate, currencies and equities. Those asset classes are considered to be risk-free although in the case of equities that might not be an acceptable assumption. [Hull11] provides a good account of all derivatives products per asset classes.

Interest rate derivatives

In an interest rate swap (IRS) is an agreement where one party A agrees to pay a fixed coupon to a party B in exchange for a floating coupon over some sequence of date, generally at a semi-annual frequency. The interest rate swap is thus characterised by

- a Notional amount N
- a floating-rate reference L,
- a fixed-rate coupon c
- a payment sequence $t_1, \ldots, t_n$

An interest rate swap can be understood as an agreement between two parties: one long a coupon bond and short a floating-rate note (FRN) with the same maturity and notional, and the other party taking the offsetting position. The two parties are called the (fixed-rate) payer and the (fixed-rate) receiver. Of note, the payer and the receiver do not exchange the notional amount at inception and maturity and only exchange the net difference between the fixed and floating cash flows occurring at the payment dates, a risk management practice calling netting. However. from a valuation perspective its interesting to keep the notional exchange to link with bonds and FRN. The fixed-rate coupon, called the swap-rate is determined so that the swap has value 0 at inception time t. The condition for c to be the swap rate is

$$FRN = 1 = c \sum_{i=0}^{n-1} \theta_i D(t, t_{i+1}) + D(t, t_n)$$

with θ_i the basis , or

$$c = \frac{1 - D(t, t_n)}{\sum_{i=0}^{n-1} \theta_i D(t, t_{i+1})}$$

Swap rates are quoted and as the CDS rate should be viewed as market data. A more detailed list of the interest rate derivatives products can be found in [Fabozzi12].

Currencies (or FX) derivatives

Another popular type of swap is known as *cross-currency swap* (CCS). This involves exchanging principal and interest payments in one currency for principal and interest payments in another currency. A currency swap agreement requires the principal to be specified in each of the two currencies. The principal amount are exchanged at the beginning and at the end of the life of the swap. Usually the principal amounts are chosen to be approximately equivalent using the exchange rate at the swap's inception. When they are exchanged at the end of the life of the swap their values may be quite different. The type of the interest payments defines the nature of the swap

- *fixed-for-fixed currency swaps*: where a fixed interest rate in one currency is exchanged for a fixed interest rate in another currency.
- *fixed-for-floating currency swaps* where a fixed interest rate in one currency is exchanged for a floating interest rate in another currency.
- *floating-for-floating currency swaps* where a floating interest rate in one currency is exchanged for a floating interest rate in another currency.

Equity derivatives

In a equity swap, one party promises to pay the return of an equity index on a notional principal while the other promises to pay a fixed or floating return on a notional principal. Equity swaps enable fund managers to increase or reduce their exposure to an index without buying or selling stock. An equity swap is a convenient way of packaging a series of forward contracts on a index.

Another product is an *Equity default swap* which is equivalent to a credit default swap but where the reference asset is some company's stock and protection is provided against a dramatic decline in the price of that stock. (For example the trigger event could be 70% drop in the price of the stock w.r.t some level at the contract inception date). The fact that the trigger is so low will also be associated with a sharp deterioration of the credit quality of the reference. However, we do not consider in this document the case of Hybrid products like Equity Default swaps since they mix credit and equity risk.

In conclusion, from this list of products, most of those contingent claims are, even if their underlying risk is not of credit type, exposed to the credit risk of their counterpart. Thus, the risk of the counterparty defaulting during a transaction and also the impact of this default on the ongoing transaction need to be valued at inception in order to reflect the riskiness of the counterpart. Thus, counterparty credit risk has long being a risk that has captured regulators attention and is reported in details.

Chapter 2

Context Introduction

This part is mainly a presentation of the historical treatment of the counterparty credit risk and its rationale in terms of integration in valuing contingency claims. Readers interested in the modelling aspects could skip this part.

2.1 Introduction of counterparty credit risk

Following the events of the financial crisis and the major changes that it implied in terms of the accuracy of pricing and risk management of correlation products, the widely used model of the Gaussian copula has demonstrated that it failed to fully capture tail risk, especially through its non-dynamic aspect of credit portfolio dependence. As a consequence of the growing popularity of credit portfolio financial products in recent years, the understanding and the modelling of default dependency, as well as of counterparty risk, has attracted much attention like in [Brig11a]. The book [Brig11c] accounts very desriptively of all the modeling issues of credit risk dependency that arose with the financial crises. We will try in this thesis to explore the valuation of counterparty default risk regarding financial derivatives products while the market is currently undergoing heavy financial stress. The intra-obligor "contagion"situation has recently attracted press coverage interest.

- In [Xydias11], *Synthetic Exchange Traded Funds represent risks of enhanced contagion channels across previously non correlated markets.*
- In [Carver11], *the Regulatory negative feedback loop between Credit Default Swaps products value and their use in hedging Counteparty Valuation Adjustment risk might limit the effectiveness or availability of the hedged position.*
- In [Braithwaite12], *leveraged credit positions such as in* k^{th}*-to-default swaps (KTD) or Leveraged super senior claims (LSS) exposes the challenges for valuation of claims that exhibit "Gap Risk"which can be defined as negative change of value that are not accounted by models.*

Historically, the valuation and the management of counterpart risk has always been centre to the preoccupation of financial institutions. As mentioned in [Canabarro03], this risk has evolved from lending risk provisions to dynamically-hedged counterparty credit risk. This evolution has naturally developed in a rich array of quantitative metrics for banks to represent their ongoing risk for regulation purposes. Regulatory capital documents from the Bank of International Settlements, [BIS10a], [BIS01a] and [BIS98], cover the general description of risk from Economic Capital concept to the more dynamic Counterparty Valuation Adjustment calculation. More recent papers, [BIS05] and [BIS10b], focus on the counterparty risk of centralised exchanges or the risk of hedged exposure through double-default situations highlighting the constant evolution of this type of risk due to new instruments or new types of counterparties. The current models implemented across the industry place also emphasis on different levels of complexity in terms of the representation of risk permitted by the regulator.

But let's first see, in the next section, how we define contagion?

2.1.1 Correlation/Contagion - the usual suspects of dependence

From the Oxford Dictionary, *contagion* is defined as

- *noun- the communication of disease from one person to another by close contact - in a figurative way - the spreading of a harmful idea or practice.* It is stated that the base of Latin refers from *con - together with* and the base *tangere - to touch.*

Additionally, correlation is defined as

- *a mutual relationship or connection between two or more things* from the Latin *cor - together* and *relatio - relation.*

These two definitions although encapsulating a similar behaviour, shed light to the causality aspect of the contagion and the symmetrically of correlation. It is the modelling of this causality behaviour in relation to counterparty risk that we are trying to investigate in this thesis.

Interestingly, the causality of the contagion effect has been already extensively investigated in other areas of finance such as macroeconomy or asset management.

For example, in a microeconomic context, companies rely on a chain of suppliers and service providers to deliver the goods and services they need to meet customer demand. That journey from source to destination has always had its risks and uncertainties and changes on nearly every front have dramatically altered the dynamics of the supply chain for good and for bad. Typically, shifts in global sourcing and production, information and communications technology, consumer expectations, pricing volatility, product availability, financial conditions, regulation and compliance rules ... combine to make managing the supply chain a new game now. These new vulnerabilities all define a set of channels that in a stressed environment can generate rise to a contagion effect.

Additionally, the dependency of factors and risks is extensively studied in Management and Operational Corporate Research like in [Chopra04] with financial risks such as supplier-bankruptcy, exchange-rate risk or creditworthiness of customers. This web of interconnections that channels contagion is also covered in the Asset Management Research space like in [Cohen06] where return predictability through economically-linked firms can generate a monthly alphas of over 150 bps or over 18% per year. The approach described in [Cohen06] is to associate companies which have a 20% supplier-customer link based on their financial accounts and enter trading pairs when earnings announcements are not impacted on the paired company in order to play an Efficient Market normalisation effect.

Finally, in the Asset Management world, another common application of contagion is called pairs trading where [Zee06] gives an intuitive illustration of statistically cointegrated pairs like General Motors and Ford that can be traded for profit using their mean-reverting feature. A good introduction of those strategies can be found in [Vidyamurthy04] with [Vidyamurthy04] capturing correlation while [Cohen06] captures contagion.

Moving back to credit risk, recent press coverage, like in [Economist12], has focused on assets correlations spiking to one and on the contagion of euro-zone sovereign spreads in 2011. Interestingly, [Economist12] extended financial contagion between major countries that are trading partners, like Mexico and the United-States, to less related ones involving Thailand, Indonesia and the United-States. Those correlation to "pure contagion" aspects are also addressed in [Hartmann04] which focuses on statistical methods for disentangling contagion from correlation by looking at the asset returns linkage through extremal dependence measure instead of conditional correlation analysis.

All those examples highlight the sound foundation of business relations that give rise to correlation-contagion links in the financial world. Then, it is natural that all those links or relationship would translate, in a credit risk setting, into credit contagion scenarios in the context of counterparty credit risk exposure.

Historically, risks exposed to counterparty risk have been at the core of the banking industry through its lending practice and was accounted through the metric defined as **Economic Capital**. While there is no contagion feature embedded in it, we view it as an interesting starting point to cover the set of metrics and their modelling that capture counterparty risk.

2.1.2 Economic Capital: early coverage of counterparty credit risk

The evolution from lending risk towards Counterparty Credit Risk through derivatives is encapsulated in the bilateral aspect of the risk depending on the future state of the market. Typically, Economic Capital defines for a loan portfolio a measure of risk representing the potential Unexpected Loss (UL) of the Economic Value of a portfolio or business over some long time horizon. Thus, as represented in figure 2.1 on page 22, the exposure evaluation will require a double level of simulation:

- simulation in market rates (covered by desks specific risk management), and,
- simulation of the types of credit event (covered by Counterparty risk desks).

Additionally, [Picoult05] introduces also the concept of **Current Counterparty Exposure** which is defined as the *immediate exposure to a counterparty representing the current replacement cost of the contracts under an immediate default.* Extending to the case of multiple transactions, the two most common ways of measuring this "Potential Exposure" of a counterparty are either a

- **Simple Transaction** methodology, or,
- a more precise and sophisticated portfolio **Simulation Methodology**.

Thus, to analyse the wide array of different transactions and their different precision needs, a set of different metrics for Economic Capital has been developed to cover the cases ranging from

- a static position as in bank lending with a known profile to
- a dynamic position depending on market variables and requiring a full simulation.

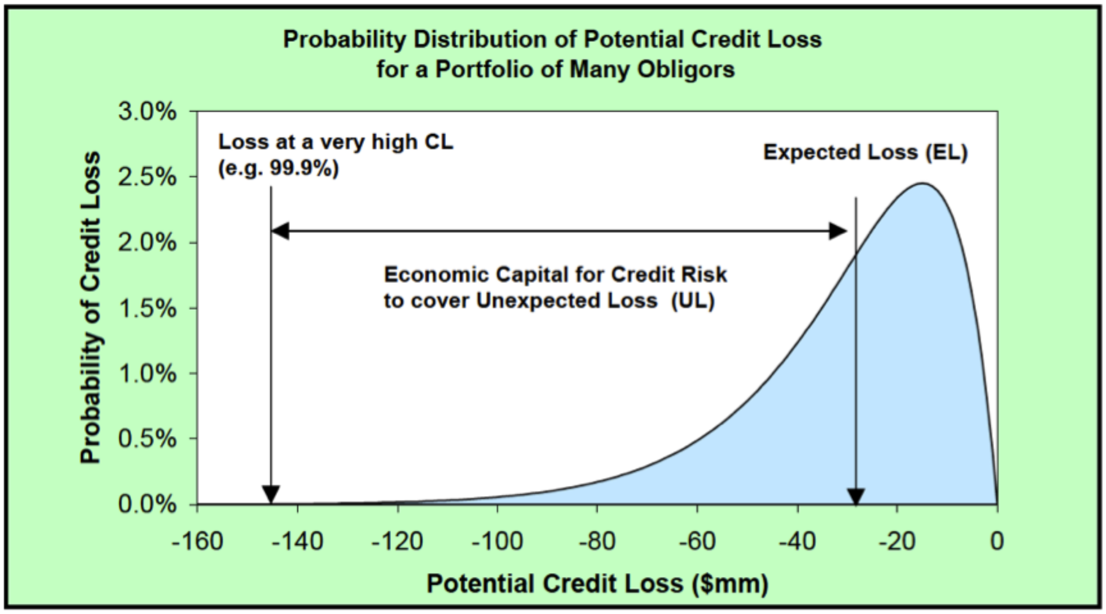


Figure 2.1: Definition of Economic Capital - Source: [Picoult05]

This last approach is close conceptually to VaR (*Value at Risk*) measurement to quantify the potential exposure even if market-factor sensitivities are not normally distributed. Since the counterparty loss exposure is mostly expressed from the investor point of view when a position is in the money, and with the exposure profile calculated as of today (time t), the basis of the VaR-equivalent approach is defined as the **average positive exposure**. A typical simulation result for an exposure profile will have the shape presented in figure 2.2 in the case of an Interest Rate Swap (IRS). The exposure profile over the forecasting time horizon will be used to produce Counterparty Credit Risk (CCR) metrics (which will be detailed later) such as:

- **EPE**: Expected Positive Exposure, and,
- **PFE**: Potential Future Exposure.

In the early 90s, some investment banks began to make a price adjustment called **Counterparty Value Adjustement** (**CVA**) (This was first mentioned in the paper [Sorensen94]) to take into account the counterparty's credit risk with a valuation adjustment reported in transactions as:

$$\text{Market Value}_{Counterparty_k} = \sum \text{Present Value}_{risk-free} - \text{CVA}_k$$

Thus, as an analogy with bond price being the risk-free bond minus the risky adjustment, CVA can be viewed as the *difference between the risk-free value of a derivative portfolio and the value after taking the counterparty risk into account.* And, as in valuing any derivatives product, the challenge of defining the Credit Charge (or **CVA**) is to take into account the effect of market spreads on the market value of a derivative portfolio.

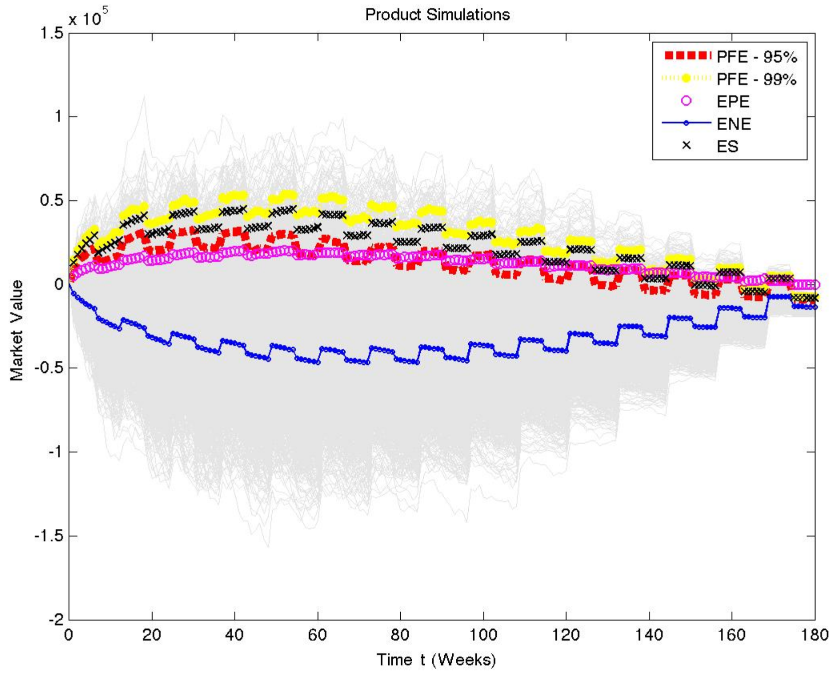


Figure 2.2: Exposure Profile for an Interest Rate Swap

So, in conclusion, while Economic Capital for counterparty credit risk has evolved from static to dynamic measurement to account for pricing requirements, the origin of valuing counterparty exposure comes from regulatory reporting to encapsulate the extent of counterparty risk taken by an institution. Therefore, for historic reasons, Regulatory Capital requirement provided the base of definitions to account for this risk. We will present those requirement as they will also form the basis of reporting the results in this thesis.

2.2 Counterparty Credit Risk Concepts

2.2.1 Is CVA valuation important?

The press releases

- [Citigroup10] of Citigroup Full year 2010 results, *Fourth Quarter Included Negative CVA of $1.1 Billion Pre-Tax Due to Citi Spreads Tightening*, and,

- [Citigroup09] of Citigroup Q1 2009 Results, *A net $2.5 billion positive CVA on derivative positions, excluding monolines, mainly due to the widening of Citis CDS spreads.*

give an immediate indication of the importance of the CVA. Especially, as it is mentioned in page 9 - Appendix A of [Citigroup10] and reproduced in table 2.1, total CVA has evolved from $+2,659 millions in Q1-2009 to $-1,897 millions in Q4-2009 to finish at $+288 millions in Q1-2010. The staggering amounts and volatility show how it is important to have a precise and consistent model throughout the accounting book and different over-the-counter (OTC) products for financial institutions in order to reflect correctly the risks that are taken during the business cycles.

Table 2.1: CITICORP - Securities and Banking CVA - Source: [Citigroup10], [Citigroup09]

(in millions dollars)	**1Q-2010**	**4Q-2009**	**1Q-2009**
CVA on Citi Liabilities at Fair Value Option	(2)	(1,764)	197
Derivatives CVA	290	(133)	2,462
Total CVA	288	(1,897)	2,659

The magnitude of valuation changes, especially in a period of financial stress, indicates the importance of a precise valuation of CVA and a consistent model to capture the diversity of risk dynamics to which a financial institution is exposed. More recently, [Carver11] has focused and demonstrated how a regulatory built feedback loop between sovereign CDS and CVA valuation has increased the systemic risk encountered through financial institutions. [Carver11] especially focused on the Basel III charges for Credit Value Adjustment (CVA) with the new regulation using credit default swap spreads to calculate counterparty exposure in derivatives trades, and requiring banks to hold capital against that number. However, it also allows banks to mitigate the capital requirement by buying CDS protection. The result, dealers say, is pro-cyclical: if the CVA charge increases, they are incentivised to buy protection; if they buy protection, spreads rise and the charge increases further. That dynamic exists today – banks already calculate and hedge CVA exposure using CDSs – but dealers argue the introduction of a specific capital requirement in 2013 will increase demand for protection in a market that is not liquid enough to support it.

Additionally, building on the increased volatility in CVA from [Carver11], the paper [Singh12] details how multiple re-pledging of collateral between institutions unlocked vast stores of "new" cash and how subsequent interbank mistrust and a shrinkage of what's acceptable as collateral drained the pool. The process of "re-hypothecation", describes how a hedge-fund or mutual-fund borrows cash from a bank by posting a bond as collateral and then how that bank subsequently re-pledges that same bond as collateral for its own purposes. According to [Singh12] at 2007 peaks, the re-use rate of primary collateral from funds and custodians by the largest banks was about 3 times – creating a total web of intricate collateral chains in excess of $10 trillion. That compares with a U.S. M2 money supply back then of some $7 trillion. But by 2011, this re-use rate was already down to 2.4, involving almost a halving of the total collateral pool. The loss in collateral flow is estimated at $4-5 trillion, stemming from both shorter collateral chains and increased "idle"collateral due to institutional ring-fencing. The knock-on impact is higher credit costs for the economy as well as a higher contagion or correlation of the values of financial claims.

Those examples highlights the fact that a starting point for building a state-of-the-art model is to understand the regulatory requirements to **Counterparty Credit Risk** and the regulatory validation process of in-house Counterparty Credit Risk structures.

2.2.2 Counterparty Credit Risk regulatory approach

In the regulatory framework detailed in [BIS05], the **Bank of International Settlements** (BIS) encourages banks to identify the risks they may face, today and in the future, and to develop or improve their abilities to manage those risks. The implementation must cover:

- The treatment of the **Counterparty Credit Risk** (CCR) for OTC derivatives, repo-style securities financing transactions and the treatment of cross-product netting agreements;
- The treatment of **Double-Default** effects for covered transactions (i.e. regulatory capital treatment of CVA-hedged transactions);

To realise this, the quantification of exposure requires the estimate of **Exposure At Default** (EAD) for transactions. This can be viewed as the equivalent to the probability distribution of a derivative as introduced earlier in section 2.1.2. The CCR generally refers to the bilateral risk of transactions with uncertain exposures that can vary over time with the movement of underlying factors. In summary, three methods are advanced for calculating the EAD for transactions involving CCR in the banking or the trading book under the "Revised Framework"introduced in [BIS05]:

1. the **Current Exposure Method** (CEM) as described in appendix A.2.
2. the **Standardised Method** (SM) as described in appendix A.2,
3. the **Internal Model Method** (IMM) as described in appendix A.2. That method develops the concept of the **Expected Positive Exposure** (EPE) that we have already mentioned in section 2.1.2, page 21, and is defined in section 2.2.4, page 29.

The three methods are intended to represent different points along a continuum of sophistication in risk management practices and are structured to provide incentives for banks to improve their management of Counterparty Credit Risk. The common aspects of those methods include collateral to mitigate risk, legal netting or rights of offset contracts and re-margining agreements. They

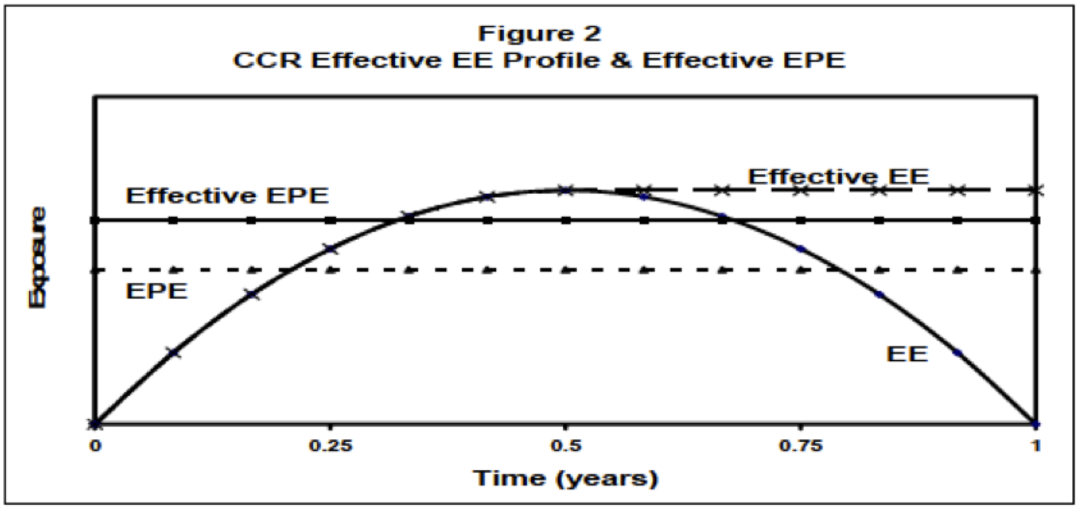


Figure 2.3: Stylised CCR Effective Expected Exposure and Effective Expected Positive Exposure illustration - Source: Bank of International Settlements

rely on the knowledge of the credit exposure in the future for a given derivatives like the exposure conceptually illustrated in figure 2.3, page 26. Typically, for regulatory purposes, the horizon knowledge of the credit exposure is set to one year notably for liquidity of hedging and the precision of the forecasting horizon. With this set of exposures, the CEM is the most simple approach with add-on factor α to account for the volatility of underlying parameters. The SM incorporates in this valuation approach the benefit of bucketing of risk and their subsequent hedging. The volatility of the underlying parameters used to generate the credit exposure is then incorporated in a factor β. Interestingly, both α and β are set at 1.4, but supervisors have the flexibility to raise either parameter in appropriate situations. This highlights to a certain extent the non-dynamical feature of those risk measures. We will, in this thesis, try to address the applicability of static volatility coefficients in different cases of credit risk regimes.

The last method, IMM, revolves around a full simulation of the underlying driver parameters through a classical Monte-Carlo simulation as detailed in [Glasserman04]. This has the benefit of generating a full distribution of each product enabling netting effect. Additionally, IMM with the full distribution knowledge generates confidence interval based metrics, detailed a bit later in section 2.2.4, page 29, such as:

- **Expected Positive Exposure** (EPE),
- **Expected Exposure** (EE), and
- **Potential Future Exposure** (PFE).

Banks typically compute and report the metrics named **EPE**, **EE**, and **PFE** using a common stochastic model. *The EPE is generally viewed as the appropriate Exposure At Default (EAD) measure to determine capital for CCR.* Additionally, the EAD for instruments with CCR must also be determined conservatively and conditionally on a “bad state”of the economy with the CEM and SM scaling EPE using multipliers, termed “alpha”-α and “beta”-β, respectively. However, in [BIS10a], following the financial crisis and to take into better account the volatility of exposure, recent changes have focused on the risk reporting of hedged transaction or the use of central counterparties like exchanges. Thus, to conclude, the regulatory metrics for the upside risk are viewed as a VaR on the downside risk, where counterparty credit risk makes uses of the full distribution knowledge of the product distribution and requires the knowledge of all type of risk and risk mitigation requiring the implementation of institution-wide pricing engine. [Cesari09] and [Gregory10] provide a very detailed coverage of asset-class wide implementation of such pricing engine among financial institutions.

2.2.3 The need to capture everything?

As mentioned in [Goldman10] on Goldman Sachs' website and titled *Overview of Goldman Sachs Interaction with AIG and Goldman Sachs Approach to Risk Management*, even if the counterparty of a credit default swap trade is rated AAA among rating agencies and under Basel-II rules doesn't require to post collateral, events of the 2008 financial crisis highlights the benefits of dynamically managing counterparty risk through the introduction of so-called *Master Margining Agreements*. The speed of default of the counterpart might not give enough time to enter new margining agreement and hedging might soon become too expensive. This was typically the case for the collateral trades with the insurance company AIG as the figure 2.4 page 28 illustrates with credit protection reaching levels where protection buying was too expensive (i.e $> 500bps$).

This systematic approach to counterparties is highlighted by Goldman's practice in [Goldman10]: “*AIG was a AAA-rated company, one of the largest and considered one of the most sophisticated trading counterparts in the world. Goldman Sachs established credit terms with them commensurate with those extended to other major counterparts including a willingness to do substantial trading volumes, but subject to collateral arrangements that were tightly managed. These arrangements included the requirement that AIG give Goldman Sachs collateral to protect us against possible future loss on the securities protected.*”

Recently, counterparty risks have mushroomed in the financial markets due to

- the usual practice of offsetting rather than unwinding derivative positions, and,
- the array of inter-dealer trades required to connect final risk-takers.

This has led, in the current environment of financial stress, to derivatives and counterparties risks being the focal points for market participants, policymakers, regulators, accountants, tax authorities and many others. Over-The-Counter (OTC) derivatives such as credit default swaps have been a focal point of attention in risk concentration due to the fact of being non centralised instruments. [Canabarro10] and [Canabarro03],page 124, depict the challenges in connection with the resources to build sophisticated PFE measurement systems with

- databases (incorporating trades, agreements, legal entities, legal opinions, collateral holdings, risk limits, ...),

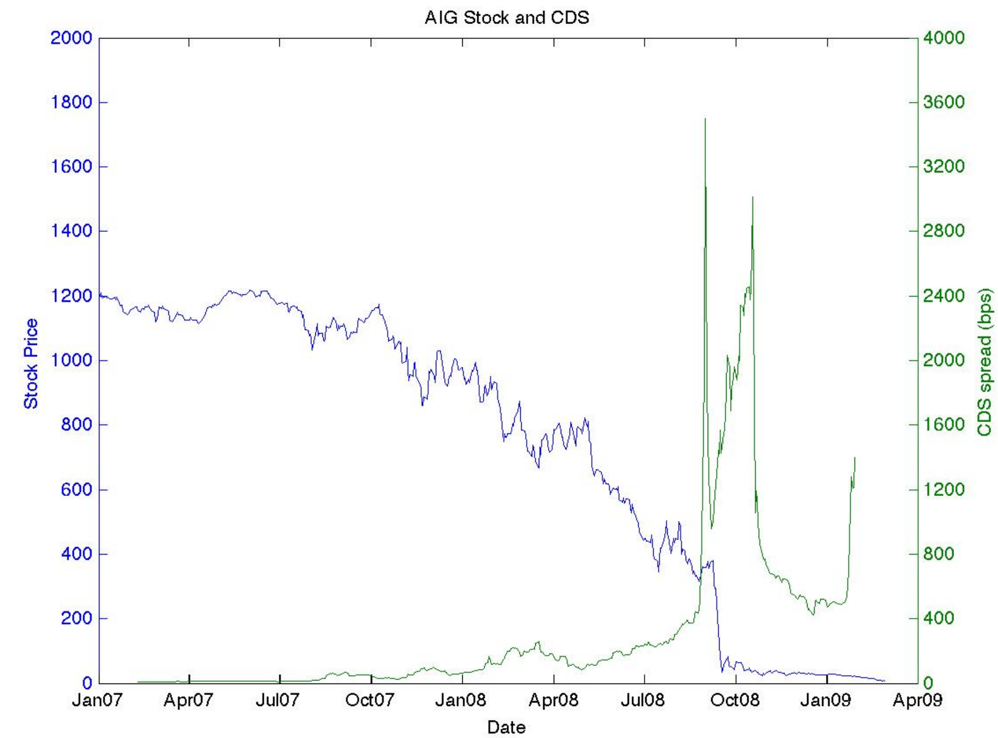


Figure 2.4: AIG stock value and protection cost around the September 2008 bailout - *LHS: AIG Stock Price, RHS: CDS spread (bps)* - Source: Datastream

- Monte-Carlo simulation engines,
- trade pricing calculators,
- exposure calculators, and,
- reporting tools.

It is, thus, necessary to capture all those elements to build an integrated platform where different market instruments require the specification of different stochastic processes to characterise their evolutions through time. Classically, as listed in [Cesari09]

- Interest rates in developed economies are modelled as normal or lognormal diffusion processes. When interest rates are low, the normal diffusion may be appropriate. When interest rates are high the lognormal diffusion may be appropriate.
- Major foreign exchange rates are usually modelled as lognormal diffusions. In contrast, emerging foreign exchange rates incorporate jumps.
- Commodity and equity prices are modelled usually as lognormal diffusions with jumps for some less liquid commodities and equities.

In order to capture such a diversity of risks and exposures both for regulatory and internal needs, a set of generic metric has to be computed as presented in section 2.2.4.

2.2.4 Counterparty Credit Risk metrics definitions

As defined in [Cesari09] and already mentioned, the **Counterparty Credit Risk** is the amount a company could potentially loose in the event of one of its counterparties defaulting. Let us generically consider a distribution of portfolio prices V_t that can incorporate collateral and priced used either with neutral probability measure $\mathbb{Q}$ or real probability measure $\mathbb{P}$.

Risk Measures

We define the following metrics:

- the **Potential Future Exposure**

$$\mathrm{PFE}_{\alpha,t} = q_{\alpha,t} = \inf\{x : \mathbb{P}(V_t \leq x) \geq \alpha)\} \tag{2.1}$$

  is the exposure of V_t in case of counterparty default with a $\alpha\%$ confidence level under a given probability measure $\mathbb{P}$. Typically a high level quantile such as 97.5% or 99%.

- the **Expected Shortfall**

$$\mathrm{ES}_{\alpha,t} = \mathbb{E}(V_t | V_t \geq q_{\alpha,t}) \tag{2.2}$$

  gives in the $\mathbb{P}$ measure a sense of some tail risk by expressing the average loss outside of the confidence interval.

- the **Expected Positive Exposure**

$$\text{EPE}_t = \mathbb{E}(V_t^+) \tag{2.3}$$

  is the mean of the positive part of the price distribution.

  This measure is used in practice for hedging counterpary exposure and for computing risk weighted assets and capital with the use of CDS to hedge the counterparty by buying a CDS with a notional related to the EPE profile.

In order to qualify as a proper risk metrics, a set of properties will be required as mentioned in the next section.

Risk Measures Categorisation:

As subsequently developed in [Zagst02] and [Elliott04], [Artzner99] presented a set of desirable properties which a risk measure should satisfy and called all risk measures holding their conditions as ***coherent***. The risk of a position X, defined as $\rho(X)$, is the additional amount of capital which has to be allocated by a banking institution to cover the risk position X. Assuming that the interest on allocated capital is zero, the conditions are:

- **Monotonicity**: $\forall X, Y \in \mathcal{X}$ with $X \leq Y$,

$$\rho(X) \geq \rho(Y) \tag{2.4}$$

  A risk position with a lower value should be assigned a higher risk in the case of downside risk and, a risk position with a higher value should be assigned a higher risk in the case of counterparty risk.

- **Translation - Invariance**: $\forall X, Y \in \mathcal{X}$ and $\forall c \in \mathbb{R}$,

$$\rho(X + c) = \rho(X) - c \tag{2.5}$$

  An additional position of risk-less capital should reduce the risk by exactly this amount of money.

- **Positive Homogeneity**: $\forall X, Y \in \mathcal{X}$ and $\forall\ \lambda\ \geq\ 0$,

$$\rho(\lambda.X) = \lambda.\rho(X) \tag{2.6}$$

  The multiplication of an actual risk position should simply result in a corresponding multiplication of risk.

- **Subadditivity**: $\forall X, Y \in \mathcal{X}$,

$$\rho(X + Y) \leq \rho(X) + \rho(Y) \tag{2.7}$$

  Diversification effect with the addition of two risk positions should lead to correlation effects and reduce risk.

Typically like a reversed VaR , PFE_α is not a coherent risk measure. On the other hand, ES_α or TCE_α define coherent risk measures. PFE_α has been historically the measure of choice, however the ES_α has gained more use thanks to it's ability to indicate potential losses in tail events.

In conclusion of this chapter, in order to focus on the effect of credit risk contagion in a counterparty credit risk context, we need to use only the classical risk metrics defined here and forget about the risk management structure that is required for regulatory reasons. Thus, we will focus on typical financial claims to illustrate the implementation of a Counterparty Credit Risk model and to generate the distribution of products in the future leading to the relevant risk metrics. In order to generate those metrics we will need the following mathematical setting:

Chapter 3

Mathematical setting

We present under this chapter the main assumptions and previously established results that will be of interest in the context of valuing the counterparty credit risk for contingent claims when reference obligors undergo credit risk contagion. Then we will, successively, introduce:

- the general mathematical background to value financial claims,
- the different approaches for multivariate obligor default-time modelling,
- the Credit Valuation Adjustment (CVA), and,
- the way of modelling dependence notably through Markov chains.

3.1 Problem introduction

The mathematical setting and notations used thereafter are mainly introduced in [Bielecki01] and [Musiela97]. [Bingham98] details risk-neutral valuation.

3.1.1 Mathematical settings

Consider a probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$ with $\omega \in \Omega$ a realisation of Ω representing all the possible outcomes of random experiment. The $\sigma-$field $\mathcal{G}$ filtration represents the sets of events $A \subset \Omega$ and where the $\sigma-$ field $\mathcal{G}_t$ represents the information up to time t.

$$\forall u \leq t, \mathcal{G}_u \subseteq \mathcal{G}_t \subseteq \mathcal{G}$$

The probability measure $\mathbb{Q}$ is the risk-neutral measure or the pricing measure and we define the locally risk-free bank account numeraire β_t evolving as

$$d\beta_t = r_t \beta_t dt$$

where r is the risk-free rate along with

$$S_t^0 = \beta_t, \dots, S_t^n$$

be $n+1$ $\mathcal{G}_t$-adapted *càdlàg* semi-martingales that all are strictly positive and represents the prices of $n+1$ traded assets at time t. Under this measure all prices of tradable assets divided by β_t are $\mathcal{G}_t$-adapted *càdlàg* martingales (Lemma 2.1.1, p. 37, in [Bielecki01]).

Let's consider a set of m obligors with $E = \{1, \ldots, m\}$ a finite set of obligors. The default times $\tau = \tau_i$ for any obligor i is an arbitrary non-negative random variable defined on the probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$. Each default is associated with a jump process H, the default process, defined as $H_t = 1_{\{\tau \leq t\}}, \forall t \in \mathbb{R}_+$. H_t is thus a right-continuous process with an associated filtration $\mathbb{H}$.

The probability space is endowed with a right-continuous and complete sub-filtration $\mathcal{F}_t$ representing all the observable market quantities but the default event (hence $\mathcal{F}_t \subseteq \mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$ where $\mathcal{H}_t = \sigma(\{\tau \leq u\} : u \leq t)$ is the right-continuous filtration generated by the default event). Thus, $\mathcal{G}_t$ is the filtration modelling the market information up to time t including explicit default monitoring up to time t whereas $\mathcal{F}_t$ is the default-free market market information up to time t (Currencies, interest rates, etc) without default monitoring.

We distinguish claims in two universe:

- **Non-defaultable claims**: financial claims where the value is not linked to a credit event. The stochastic drivers of value will be of type: stocks, currencies, interest rates, commodities, etc. Typically the information will be encapsulated in the sub-filtration $\mathcal{F}_t$.
- **Defaultable claims**: financial claims where the value is linked to a credit event. Naturally such claims are named credit claims with credit as an underlying such as Credit Default swaps, Collateral Debt Obligations, etc. Here the information will be encapsulated in the sub-filtration $\mathcal{H}_t$. It could be mentioned here that some hybrid claims such as Equity Default Swaps could be categorised across the divide but we will only focus in this thesis on general claims and not exotic ones.

Remark: The defaultable non defaultable vs defaultable name categorisation is mainly centred on the fact that the refrence of the derivatives claims is credit risk based or not. We will keep this notation in the reminder of this document.

Let's now present the valuation of those claims starting with the non-defaultable case.

Non-defaultable case:

Let's now consider a contingent claim X, an $\mathcal{F}_T$-measurable random variable, that represents from a financial point of view a contract with a stochastic payoff, i.e. stochastic cash-flow that depends on the information available at time T. From [Bingham98], we say that a contingent claim X is $\mathbb{Q}$-attainable if there exists a $\mathbb{Q}$-admissible trading strategy ϕ (called replicating strategy) such that

$$V_T(\phi) = X$$

This is established with the two theorems, with [Harrison81] establishing that

Theorem 3.1.1 (Default-free attainable claim)**.** *If X is an attainable contingent claim such that $(\beta X) \in L^1(\mathbb{Q}, \mathcal{F}_T)$, then the arbitrage-free price V_t^X of X at time t is given by*

$$V_t^X = S_t^0 \mathbb{E}^{\mathbb{Q}} \left[\frac{X}{S_T^0} \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}} \left[\int_t^T \beta_u dX_u \middle| \mathcal{F}_t \right] \tag{3.1}$$

since X is an $\mathcal{F}_T$-measurable random variable and $\beta_t = 1$. Note that in the reminder of the document we will use the notation $X(t)$ to define the claim instead of the notation V_t^X to refer more to the claim rather than its replicating portfolio. The replicability of the claim is obtained in a complete financial market $\mathcal{M}$ with the following lemma

Lemma 3.1.1. *Let $\mathbb{Q}$ be a martingale measure. Let X be a contingent claim such that $(\beta_t X_t) \in L^1(\mathbb{Q}, \mathcal{F}_T)$. Define the martingale M_t as*

$$M_t = \mathbb{E}^{\mathbb{Q}} \left[\frac{X}{\beta_T} \mid \mathcal{F}_t \right]$$

If M_t could be represented as

$$M_t = p + \sum_{k=0}^{n} \int_0^t \phi_u^k d\tilde{S}_u^k$$

for some $\phi_t = (\phi_t^1, \dots, \phi_t^n)$ where ϕ_t^k is $\mathcal{F}$-predictable and locally bounded for $k = 1, \dots, n$ and x is a constant, then X is attainable.

Let's do now the case of a Defaultable claim:

Defaultable case:

Notation: Defaultable claim define in the current document claims that are credit-linked such as credit default swaps, k^{th}-to-default where the underlying driver of valuation is credit risk derived. In this context, an equity default swap can be defined as being a defaultable claim even if the underlying is the equity price of the reference name.

We fix the finite horizon date T. We suppose that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ endowed with some filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ is sufficiently rich to support the processes into which the Defaultable claim will be decomposed.

We consider a default time τ connected to some reference obligor $i \in E$ with any defaultable contingent claim of maturity T as the set

$$\Pi(t, T) = (X, A, \tilde{X}, Z, \tau, T) \tag{3.2}$$

where

- X represents a payoff that the holder of the claim receives at T if $\tau > T$,

- $A = (A_t)_{0 \leq t \leq T}$ is a process representing the accumulated promised cash flows received by the owner of the claim up to time t, given that $\tau > t$.
- $Z = (Z_t)_{0 \leq t \leq T}$ is a process representing the recovery payoff received by the holder of the claim at τ if $\tau \leq T$, i.e. Z is what is recovered at the default time τ, if $\tau \leq T$,
- $\tilde{X}$ represent a payoff that the holder of the claim receives at T if $\tau \leq T$.

The processes X and $\tilde{X}$ are $\mathcal{F}_t$-measurable, and the processes A_t and Z_t are $\mathcal{F}_t$-predictable, with A_t of finite variation and $A_0 = 0$. Usually, by convention in claims either $\tilde{X} = 0$ or $Z = 0$. In the remainder of this document, we will consider $\tilde{X} = 0$ as if the recovery payoff Z is received at time of default τ.

With the default process H_t of obligor i as $H_t^i = 1_{\{\tau = \tau_i \leq t\}}$ and its filtration up to default time as $\mathcal{H}_\tau^i = \sigma(\tau \leq s, s \leq t)$, from Proposition 8.1.1, p. 224 in [Bielecki01], the defaultable contingent claim can be decomposed at time T as

$$X_T^d = X 1_{\{\tau > T\}} + \tilde{X} 1_{\{\tau \leq T\}}$$

and viewed as a continuous dividend process D_t of $\Pi(t,t) = (X, A, \tilde{X}, Z, \tau, T)$ of finite variation on compacts in $[0, T]$ with

$$D_t = X_t^d 1_{\{t \geq T\}} + \int_0^t (1 - H_s) dA_s + \int_0^t Z_s dH_s$$

where, since $1 - H_s = 1_{\{\tau > s\}}$

$$\int_0^t (1 - H_s) dA_s = \int_0^t 1_{\{\tau > s\}} dA_s = A_{\tau -} 1_{\{\tau \leq t\}} + A_t 1_{\{\tau > t\}}$$

and

$$\int_0^t Z_s dH_s = Z_\tau 1_{\{\tau \leq t\}}$$

Similary, as for formula (3.1), to find the arbitrage-free price V_t^{DCT} of the claim $\Pi(t,T)$ at time t, let assume there exists $S_t^0, \ldots, S_t^n$ $n+1$ $\mathcal{F}$-adapted semi-martingales under a risk-neutral measure $\mathbb{Q}$ on the market $\mathcal{M}$. Additionally, for certain claims like CDS, let us assume that the defaultable claim $\Pi(t,T)$ is also traded on this market, and S_t^{n+1} be the strictly positive price of the claim $\Pi(t,T)$ at time t. As in the default-free case, we define a trading strategy ϕ_t (w.r.t S_t) as a locally bounded, $\mathcal{F}$-predictable $n+2$ dimensional process

$$\phi_t = (\phi_t^0, \ldots, \phi_t^n, \phi_t^{n+1})$$

where the intuitive meaning of ϕ_t is as in the default-free case and where ϕ_t^{n+1} is the amount of the defaultable claim held at t result in the portfolio value V_t^{DCT}:

$$V_t^{DCT}(\phi_t) = \phi_t S_t = \sum_{k=0}^{n+1} \phi_t^k S_t^k$$

[Bielecki01] argues that this implies that the arbitrage-free price V_t^{DCT} of the $\Pi(t,T)$ at time t is given by $V_t^{DCT} = S_t^{n+1}$ where

$$S_t^{n+1} = B_t \mathbb{E}^{\mathbb{Q}} \left[\int_t^T \beta_s dD_s \middle| \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t \right]$$

where the expectation is taken under $\mathbb{Q}$. Hence, if ϕ is self-financing and $\tilde{V}_t^{DCT}$ is a local-$\mathbb{Q}$-local martingale we could still use theorem 3.1.1 page 34 (but now with the enlarged filtration) to find the arbitrage-free price of a defaultable claim $\Pi(t,T)$. Thus, unless explicit stated, arbitrage-free prices of claims (default free or defaultable) will be computed simply by discounting their future promised cash flows under the risk-neutral measure $\mathbb{Q}$.

Theorem 3.1.2 (Defaultable attainable claim)**.** *If $\Pi(t,T)$ is an attainable defaultable contingent claim, then the arbitrage-free price V_t^{DCT} of $\Pi(t,T)$ at time t is given by*

$$S_t^{n+1} = B_t \mathbb{E}^{\mathbb{Q}} \left[\int_t^T \beta_s (1 - H_s) dA_s + \int_t^T \beta_s Z_s dH_s + \beta_T X_t^d \middle| \mathcal{F}_t \vee \mathcal{H}_t \right] \tag{3.3}$$

3.1.2 Counterparty risk impact on valuation

Moving from the previous section, the valuation equations (3.1) and (3.3) of a contingent claim of final maturity T highlight the influence of the default of reference obligor i. Additionally, financial claims, especially when designed as bilateral agreements, are exposed to default of the counterpart entity adding an extra layer of defaultability even in equation (3.1), making it now default sensitive.

Conceptually, investing in default risky assets requires a *risk premium* as a reward for assuming the default risk with the difference of the yield to the equivalent treasury bond being called the *credit spread*. Thus, logically, the value of a generic claim traded with a counterparty subject to default risk is always smaller than the value of the same claim traded with a counterparty having a null default probability.

Motivated by the events of Asian crisis in 1998 which highlighted credit contagion default, [Jarrow01] presented an interesting attempt to capture the influence of the counterparty riskiness in the valuation of the contigent claims through a set "secondary firms"influencing the default probability of "primary firms "and thus the valuation of claims linked to those firms.

While [Jarrow01] focused on claims that can be categorised as non-defaultable, [Leung05] focused on the defaultable ones by assessing credit default swap valuation with counterparty credit risk. [Leung05] highlighted the tendency of those claims to exhibit "wrong-way"risk, namely, the fact that under credit risk dependency the higher risk of the counterpart not fulfilling its obligation is also the moment the value of this obligation is the most valuable. In effect, that aspect should also being considered in the original valuation of the claim.

[Brig06] established a link between the default-free valuation and the defaultable one called **Credit Value Adjustment** (CVA). This adjustment will be central in valuing the effect of credit contagion among a set of obligors especially for credit-basket sensitive claims like k^{th}-to-default (KTD) credit default swaps or Leveraged Super Senior (LSS) claims introduced earlier.

[Brig06] illustrated how the inclusion of counterparty risk in the valuation can make a payoff model dependent by adding one level of optionality, use the risk neutral default probability for the counterparty by extracting it from CDS data and account for the correlation between the underlying of the contract and the default of the counterparty.

[Brig06] shows that "**the derivative price in presence of counterparty risk is just the default free price minus a discounted option term in scenarios of early default**". **The option underlying is the residual present value of the derivative at time of default**. Thus, even payoffs whose valuation is model independent become model dependent due to counterparty risk in a way that does not destroy the default-free valuation models.

Moving back to our setting:

In the set of obligor $E = \{1, \ldots, m\}$, consider the two parties in a deal as the investor I and the counterpart C with claims potentially encompassing reference obligors i with their associated default process H_t^I, H_t^C and $H_t^i, \forall i \in E \backslash \{I, C\}$. Thus, the filtration $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$ can be decomposed as:

$$\mathcal{H}_t = \mathcal{H}_t^I \vee \mathcal{H}_t^C \vee \mathcal{H}_t^i, \ \forall t \in [0, T]$$

with the respective filtrations up to default time as $\mathcal{H}^i = \sigma(\tau_i \leq s, s \leq t), \ \forall i \in E$.

Comment: As it is common usage, we present the calculations form the view point of the investor I.

We make also the following assumption regarding the default of obligors in E:

Assumption 3.1.1 (No simultaneous defaults)**.** *In the current setting in the set of obligors E we rule out the case of simultaneous defaults, i.e.*

$$\forall i, f \in E, \ i \neq j, \ \{\tau_i = \tau_j\} = \emptyset$$

Two cases are possible for the valuation of counterpart credit risk:

- **Unilateral case**: only the counterpart C default time is taken into account in its impact on claim valuation. This was mainly the case pre-2008 and the financial crisis where bank where considered risk-free especially w.r.t their corporate clients.
- **Bilateral case**: the counterpart C and the investor I, both, can affect the valuation of claims.

We now express the counterparty risk metrics defined in section 2.2.4, page 29, in our current setting with first in the unilateral case and then in the bilateral case.

Counterparty risk metrics - Unilateral case:

Consider the filtration $\mathcal{H}_t^C = (\{\tau_C \leq t\}, t \in \mathbb{R})$ generated by default time τ_C of the counterparty C. The filtration is thus, $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^C$. If we define $X(\tau_C)$ as the termination value based on fair value (i.e entering a new contract), the metrics are

- ED: Exposure at default is a $\mathcal{G}_\tau$-measurable random variable with

$$ED(\tau) \ := \ X(\tau_C) - (R_C X(\tau_C)^+ - X(\tau_C)^-) \tag{3.4}$$

- CVA: Credit Valuation Adjustment. CVA process is killed at $\tau_C \wedge t$ defined as $t \in [0, T]$

$$\beta_t CVA_t \ := \ \mathbb{E}_t(1_{\{t < \tau_C\}} \beta_{\tau_C} ED(\tau_C)) \tag{3.5}$$

- EPE: Expected Positive Exposure. EPE is the function of time defined as

$$EPE(t) \ := \ \mathbb{E}_t(ED(\tau_C) \mid \tau_C = t) \tag{3.6}$$

Typically, the equations (3.4) and (3.6) are used in a context of risk management under the $\mathbb{P}$-measure, while equation (3.5) is used in a context of pricing under the $\mathbb{Q}$ -measure.

Comment: Note that the claim $X(\tau_C)$ above corresponds to both the non-defaultable claim X_t in equation (3.1) and to the defaultable claim in equation (3.3). We will in the reminder of the document use the notation X as a general claim when there is no need to differentiate between the two cases.

Counterparty risk metrics - Bilateral case:

Consider the filtration $\mathcal{H}_t^C = (\{\tau_C \leq t\}, t \in \mathbb{R})$ generated by default time τ_C of the counterparty C and the filtration $\mathcal{H}_t^I = (\{\tau_I \leq t\}, t \in \mathbb{R})$ generated by default time τ_I of the investor I . The filtration is thus, $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ with $\mathcal{H}_t = \mathcal{H}_t^I \vee \mathcal{H}_t^C$. We excluded simultaneous defaults of the Investor and the counterpart i.e. $\mathbb{Q}(1_{\tau_I=\tau_C}) = 0$.

From the unilateral case, the equations (3.4) (3.5) and (3.6) and are rewritten

- ED: Exposure at default is a $\mathcal{G}_\tau$-measurable random variable where $\tau = \min(\tau_I, \tau_C)$with

$$\begin{aligned} ED(\tau) \;&:=\; 1_{\{\tau=\tau_C\}}(X(\tau_C) - (R_C X(\tau_C)^+ - X(\tau_C)^-)) \\ &+\; 1_{\{\tau=\tau_I\}}(X(\tau_I) - (X(\tau_I)^+ - R_I X(\tau_I)^-)) \end{aligned} \tag{3.7}$$

- CVA: Credit Valuation Adjustment. CVA process is killed at $\tau_C \wedge t$ defined as $t \in [0, T]$

$$\beta_t CVA_t := \mathbb{E}_t(1_{\{t<\tau_C\}} \beta_{\tau_C} ED(\tau_C)) \tag{3.8}$$

- DVA: Debt Valuation Adjustment. DVA process is killed at $\tau_I \wedge t$ defined as $t \in [0, T]$

$$\beta_t DVA_t := \mathbb{E}_t(1_{\{t<\tau_I\}} \beta_{\tau_I} ED(\tau_I)) \tag{3.9}$$

- EPE: Expected Positive Exposure. EPE is the function of time defined as

$$EPE(t) := \mathbb{E}_t(ED(\tau_C) \mid \tau_C = t) \tag{3.10}$$

- ENE: Expected Negative Exposure. ENE is the function of time defined as

$$ENE(t) := \mathbb{E}_t(ED(\tau_I) \mid \tau_I = t) \tag{3.11}$$

We those definitions we now express the defaultable and non defaultable claims adjusted for their counterparty exposure by linking the Credit Valuation Adjustment with the counterpart-free value

Non defaultable claim - Unilateral case:

Consider a non-defaultable claim X under no counterparty risk but with τ a $\mathcal{F}_t$-adapted stopping time with $\tau \in]t, T]$. In this case the filtrations are $\mathcal{G}_t = \mathcal{F}_t$ with $\mathcal{H}_t = \emptyset$

We express the non defaultable claim in equation (3.1) under counterparty credit risk as

$$\begin{aligned}
\beta_t X(t,T)^d &= \int_t^{T\wedge\tau_C} \beta_u dX_u + 1_{\{t<\tau_C\leq T\}}\beta_{\tau_C}(R_C X(\tau_C)^+ - X(\tau_C)^-) \\
&= \int_t^{T} 1_{\{\tau_C>u\}}\beta_u dX_u + 1_{\{t<\tau_C\leq T\}}\beta_{\tau_C}(R_C X(\tau_C)^+ - X(\tau_C)^-) \\
&= \int_t^{T} \beta_u dX_u + \int_t^{T} 1_{\{\tau_C\leq u\}}\beta_u \phi_u dS_u + 1_{\{t<\tau_C\leq T\}}\beta_{\tau_C}(R_C X(\tau_C)^+ - X(\tau_C)^-) \quad (3.12)
\end{aligned}$$

We have adjusted the claim where

- the first term is the risk-neutral valuation of the payoff if no early default or the payments received before default occurs,

and in case of early default,

- the second term is the residual net present value if the expected exposure is positive and adjusted for the recovery of the counterpart.

Note that $X(\tau_C)$ refers to the valuation of the claims as of equation (3.1) i.e. equivalent to entering a new contract as of τ_C. The notation will be used across all the counterparty valuation formula under all cases.

Proof. The proof can be find in [Brig06] where by taking the expectation under $\mathcal{G}_t$

$$\begin{aligned}
&\mathbb{E}_t(\beta_t X(t,T)^d) \\
= \; &\mathbb{E}\left(\int_t^T \beta_u dX_u + \int_t^T 1_{\{\tau_C\leq u\}}\beta_u dX_u + 1_{\{t<\tau_C\leq T\}}\beta_{\tau_C}(R_C X(\tau_C)^+ - X(\tau_C)^-)\middle|\mathcal{G}_t\right)
\end{aligned}$$

using $f = f^+ - f^-$ and the law of iterated expectations we get

$$= \mathbb{E}\left((1_{\{\tau_C>T\}} + 1_{\{t<\tau_C\leq T\}})\int_t^T \beta_u dx_u\middle|\mathcal{G}_t\right) - \mathbb{E}\left(1_{\{t<\tau_C\leq T\}}\beta_{\tau_C} LGD_C X(\tau_C)^+\middle|\mathcal{G}_t\right)$$

□

The equation (3.1) is thus valued under unilateral counterparty risk as

$$\begin{aligned}
\mathbb{E}_t(\beta_t X(t,T)^d) &= \mathbb{E}\left((1_{\{\tau_C>T\}} + 1_{\{t<\tau_C\leq T\}})\int_t^T \beta_u dX_u\middle|\mathcal{G}_t\right) \\
&- \mathbb{E}\left(1_{\{t<\tau_C\leq T\}}\beta_{\tau_C} LGD_C X(\tau_C)^+\middle|\mathcal{G}_t\right) \quad (3.13)
\end{aligned}$$

The value of a **defaultable claim** is then decomposed in a unilateral case as the value of the corresponding **default-free claim** under the filtration $\mathcal{G}_t$

- plus a **long** position in a **put** position with a zero strike on the Expected Exposure if the investor is the earliest to default before maturity,

As in [Brig06], we quote the non defaultable claim X defined in (3.1) under unilateral counterparty credit risk as

Proposition 3.1.1 (Unilateral CVA for non-defaultable claim)**.** *Consider a $\mathcal{F}_T$-measurable defaultable claim X defined in (3.1) under counterparty credit risk as*

$$\begin{aligned} X(t,T)^d = \mathbb{E}_t(\beta_t X(t,T)^d) &= \mathbb{E}\left(\int_t^T \beta_u dX_u \middle| \mathcal{G}_t\right) \\ &- \mathbb{E}\left(1_{\{t<\tau_C\leq T\}}\beta_{\tau_C} LGD_C X(\tau_C)^+ \middle| \mathcal{G}_t\right) \\ &= X(t,T) - CVA_t \end{aligned} \tag{3.14}$$

Thus,

$$CVA_t := X(t,T) - X^d(t,T)$$

with

$$CVA_t := \mathbb{E}(1_{\{t<\tau_C\leq T\}}\beta_{\tau_C} ED(\tau_C) \mid \mathcal{G}_t) \tag{3.15}$$

Note that the claim Note that $X(t,T)$ refers to the valuation of the claims as of equation (3.1). However the valuation of the claim in the current case is under the filtration $\mathbb{G}$ while in (3.1) it is under the filtration $\mathbb{F}$. We will address this issue notably with $(\mathcal{H})$-hypothesis stated in proposition 4.1.5, page 58. We keep the notation $X(t,T)$ at the moment to highlight the non identity of the formulae.

Proof. Using results form the previous proof and

$$\begin{aligned} CVA_t &= \mathbb{E}(1_{\{t<\tau_C\leq T\}}\beta_{\tau_C} LGD_C X(\tau_C)^+ \mid \mathcal{G}_t) \\ &= \mathbb{E}(1_{\{t<\tau_C\leq T\}}\beta_{\tau_C}(1-R_C)X(\tau_C)^+ \mid \mathcal{G}_t) \\ &= \mathbb{E}(1_{\{t<\tau_C\leq T\}}\beta_{\tau_C}(X(\tau_C)^+ - R_C X(\tau_C)^+) \mid \mathcal{G}_t) \\ &= \mathbb{E}(1_{\{t<\tau_C\leq T\}}\beta_{\tau_C}(X(\tau_C) + X(\tau_C)^- - R_C X(\tau_C)^+) \mid \mathcal{G}_t) \\ &= \mathbb{E}(1_{\{t<\tau_C\leq T\}}\beta_{\tau_C} X(\tau_C) - 1_{\{t<\tau_C\leq T\}}\beta_{\tau_C}(R_C X(\tau_C)^+ - X(\tau_C)^-) \mid \mathcal{G}_t) \\ &= \mathbb{E}(1_{\{t<\tau_C\leq T\}}\beta_{\tau_C} ED(\tau_C) \mid \mathcal{G}_t) \end{aligned}$$

□

Non defaultable claim - bilateral case

Similarly in the bilateral case, we introduce the disjoints following events and excluding simultaneous defaults where $\mathbb{Q}(1_{\tau_C=\tau_I}) = 0$

$$\mathcal{I}_1 = \{\tau_I < \tau_C < T\},\ \mathcal{I}_2 = \{\tau_I < T \leq \tau_C\},\ \mathcal{I}_3 = \{\tau_C < \tau_I < T\} \tag{3.16}$$

$$\mathcal{I}_4 = \{\tau_C < T \leq \tau_I\},\ \mathcal{I}_5 = \{T \leq \tau_I < \tau_C\},\ \mathcal{I}_6 = \{T \leq \tau_C < \tau_I\} \tag{3.17}$$

with

$$\mathcal{I}_I = \mathcal{I}_1 \cup \mathcal{I}_2,\ \mathcal{I}_C = \mathcal{I}_3 \cup \mathcal{I}_4,\ \mathcal{I}_O = \mathcal{I}_5 \cup \mathcal{I}_6 \tag{3.18}$$

and

$$\tau = \min\{\tau_I, \tau_C\} \tag{3.19}$$

We define the non defaultable claim under bilateral counterpart credit risk adjusted for the default of investor I as

$$\beta_t X(t,T)^d = \int_t^{T\wedge\tau_C\wedge\tau_I} \beta_u dX_u + 1_{\mathcal{I}_C}\beta_{\tau_C}(R_C X(\tau_C)^+ - X(\tau_C)^-) + 1_{\mathcal{I}_I}\beta_{\tau_I}(X(\tau_I)^+ - R_I X(\tau_I)^-)$$

The proof is in [Brig06] and we can express the non defaultable claim X defined in equation (3.1) under bilateral counterparty credit risk as

Proposition 3.1.2 (Bilateral CVA for non-defaultble claim)**.** *Consider an $\mathcal{F}_T$-measurable non-defaultable claim X defined in (3.1) under bilateral counterpart credit risk as*

$$\begin{aligned} X(t,T)^d = \mathbb{E}_t(\beta_t X(t,T)^d) &= \mathbb{E}\left(\int_t^T \beta_u dX_u \middle| \mathcal{G}_t\right) \\ &- \mathbb{E}\left(1_{\mathcal{I}_C}(\beta_{\tau_C} LGD_C X(\tau_C)^+) \mid \mathcal{G}_t\right) \\ &+ \mathbb{E}\left(1_{\mathcal{I}_I}(\beta_{\tau_I} LGD_I X(\tau_I)^-) \mid \mathcal{G}_t\right) \\ &= X(t,T) - BCVA_t \end{aligned} \tag{3.20}$$

with

$$\begin{aligned} BCVA_t &:= CVA_t - DVA_t = X(t,T) - X^d(t,T) \\ &:= \mathbb{E}\left(1_{\mathcal{I}_C}(\beta_{\tau_C} LGD_C X(\tau_C)^+) \mid \mathcal{G}_t\right) - \mathbb{E}\left(1_{\mathcal{I}_I}(\beta_{\tau_I} LGD_I X(\tau_I)^-) \mid \mathcal{G}_t\right) \\ &:= \mathbb{E}\left(1_{\{t<\tau\le T\}}\beta_\tau[ED(\tau)] \mid \mathcal{G}_t\right) \end{aligned} \tag{3.21}$$

and

$$CVA_t := \mathbb{E}\left(1_{\mathcal{I}_C}(\beta_{\tau_C} LGD_C X(\tau_C)^+) \mid \mathcal{G}_t\right) \tag{3.22}$$

and

$$DVA_t := \mathbb{E}\left(1_{\mathcal{I}_I}(\beta_{\tau_I} LGD_I X(\tau_I)^-) \mid \mathcal{G}_t\right) \tag{3.23}$$

So clearly, as mentioned earlier, the valuation of the counterparty impact in equation (3.22) and of the investor impact in equation (3.23), require the joint knowledge of the future distribution of the claim X and the ordered events detailed in equations (3.16).

Defaultable claim - unilateral case

As defined previously in equation (3.2), consider a defaultable claim $\Pi(t,T) = (X, A, \tilde{X}, Z, \tau_R, T)$ where τ_R defines the default time of the claim reference entity $R = Ref$ (Those reference could be multiple but at the moment we consider a unique reference). The filtration is enlarged such that,

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t \tag{3.24}$$

with

$$\mathcal{H}_t = \mathcal{H}_t^C \vee \mathcal{H}_t^R \tag{3.25}$$

As previously, the claim under no counterpart risk is given by theorem 3.1.2 and equation (3.3)

$$S_t^{n+1} = B_t \mathbb{E}^{\mathbb{Q}} \left[\int_t^{T\wedge\tau_R} \beta_s dA_s + \beta_{\tau_R} Z_{\tau_R} 1_{\{t<\tau_R\leq T\}} + \beta_T X_t^d \middle| \mathcal{F}_t \vee \mathcal{H}_t^R \right] \tag{3.26}$$

In the filtration $\mathcal{G}_t$, we introduce the disjoints following events of ordered defaults excluding simultaneous defaults according to assumption 3.1.1 especially with $\mathbb{Q}(1_{\tau_C=\tau_R}) = 0$

$$\begin{aligned} &\mathcal{I}_1 = \{\tau_R < \tau_C < T\},\ \mathcal{I}_2 = \{\tau_R < T \leq \tau_C\},\ \mathcal{I}_3 = \{\tau_C < \tau_R < T\} \\ &\mathcal{I}_4 = \{\tau_C < T \leq \tau_R\},\ \mathcal{I}_5 = \{T \leq \tau_R < \tau_C\},\ \mathcal{I}_6 = \{T \leq \tau_C < \tau_R\} \end{aligned} \tag{3.27}$$

with the regroupment of events characterising first-to-default events

$$\mathcal{I}_R = \mathcal{I}_1 \cup \mathcal{I}_2,\ \mathcal{I}_C = \mathcal{I}_3 \cup \mathcal{I}_4,\ \mathcal{I}_O = \mathcal{I}_5 \cup \mathcal{I}_6 \tag{3.28}$$

Using the unilateral equation (3.5), the $\mathcal{G}$-measurable defaultable claim $S_t^{n+1,d}$ under unilateral counterpart risk is

$$\beta_t S_t^{n+1,d} = \int_t^{T\wedge\tau_R\wedge\tau_C} \beta_s dA_s + \beta_{\tau_R} Z_{\tau_R} 1_{\mathcal{I}_R} + \beta_T X_t^d 1_{\mathcal{I}_O} + 1_{\mathcal{I}_C} \beta_{\tau_C} (R_C (S_{\tau_C}^{n+1,d})^+ - (S_{\tau_C}^{n+1,d})^-) \tag{3.29}$$

Proof. The proof is identical as in [Brig06] in terms of the flow and arguments but with a slight difference due to the introduction of the reference early termination of the claim so we just outline the process. Thus, as previously, by taking the expectation under $\mathcal{G}_t$

$$
\begin{aligned}
& \mathbb{E}_t(\beta_t S_t^{n+1,d}) \\
= \ & \mathbb{E}\left(\int_t^{T\wedge\tau_R\wedge\tau_C} \beta_s dA_s + \beta_{\tau_R} Z_{\tau_R} 1_{\mathcal{I}_R} + \beta_T X_t^d 1_{\mathcal{I}_O} + 1_{\mathcal{I}_C}\beta_{\tau_C}(R_C (S_{\tau_C}^{n+1,d})^+ - (S_{\tau_C}^{n+1,d})^-)\Big|\mathcal{G}_t\right) \\
= \ & \mathbb{E}\left(\int_t^{T\wedge\tau_R\wedge\tau_C} \beta_s dA_s + \beta_{\tau_R} Z_{\tau_R} 1_{\mathcal{I}_R} + \beta_T X_t^d 1_{\mathcal{I}_O} + 1_{\mathcal{I}_C}(\beta_{\tau_C}(S_{\tau_C}^{n+1,d}) + (R_C - 1)\beta_{\tau_C}(S_{\tau_C}^{n+1,d})^+)\Big|\mathcal{G}_t\right) \\
= \ & \mathbb{E}\left(\int_t^{T\wedge\tau_R} (1_{\mathcal{I}_O} + 1_{\mathcal{I}_C})\beta_s dA_s + \beta_{\tau_R} Z_{\tau_R} 1_{\mathcal{I}_R} + \beta_T X_t^d 1_{\mathcal{I}_O} + 1_{\mathcal{I}_C}(\beta_{\tau_C}(S_{\tau_C}^{n+1,d}) + (R_C - 1)\beta_{\tau_C}(S_{\tau_C}^{n+1,d})^+)\Big|\mathcal{G}_t\right) \\
= \ & \mathbb{E}\Bigg(1_{\mathcal{I}_O}\int_t^{T\wedge\tau_R} \beta_s dA_s + \beta_{\tau_R} Z_{\tau_R} 1_{\mathcal{I}_R} + \beta_T X_t^d 1_{\mathcal{I}_O} \\
+ \ & 1_{\mathcal{I}_C}\left(\int_t^{\tau_C} \beta_s dA_s + \beta_{\tau_C}(S_{\tau_C}^{n+1,d}) + (R_C - 1)\beta_{\tau_C}(S_{\tau_C}^{n+1,d})^+\right)\Big|\mathcal{G}_t\Bigg) \\
= \ & \mathbb{E}\Bigg(1_{\mathcal{I}_O}\int_t^{T\wedge\tau_R} \beta_s dA_s + \beta_{\tau_R} Z_{\tau_R} 1_{\mathcal{I}_R} + \beta_T X_t^d 1_{\mathcal{I}_O} \\
+ \ & 1_{\mathcal{I}_C}\left(\int_t^{\tau_C} \beta_s dA_s + \mathbb{E}\left(\int_{\tau_C}^{T\wedge\tau_R} \beta_s dA_s + \beta_{\tau_R} Z_{\tau_R} 1_{\mathcal{I}_3} + \beta_T X_t^d 1_{\mathcal{I}_4} \mid \mathcal{G}_{\tau_C}\right) + (R_C - 1)\beta_{\tau_C}(S_{\tau_C}^{n+1,d})^+\right)\Big|\mathcal{G}_t\Bigg) \\
= \ & \mathbb{E}\Bigg(1_{\mathcal{I}_O}\int_t^{T\wedge\tau_R} \beta_s dA_s + \beta_{\tau_R} Z_{\tau_R} 1_{\mathcal{I}_R} + \beta_T X_t^d 1_{\mathcal{I}_O} \\
+ \ & 1_{\mathcal{I}_C}\left(\int_t^{\tau_C} \beta_s dA_s + \int_{\tau_C}^{T\wedge\tau_R} \beta_s dA_s + \beta_{\tau_R} Z_{\tau_R} 1_{\mathcal{I}_3} + \beta_T X_t^d 1_{\mathcal{I}_4} + (R_C - 1)\beta_{\tau_C}(S_{\tau_C}^{n+1,d})^+\right)\Big|\mathcal{G}_t\Bigg) \\
= \ & \mathbb{E}\Bigg(1_{\mathcal{I}_O}\int_t^{T\wedge\tau_R} \beta_s dA_s + \beta_{\tau_R} Z_{\tau_R} 1_{\mathcal{I}_R} + \beta_T X_t^d 1_{\mathcal{I}_O} \\
+ \ & 1_{\mathcal{I}_C}\int_t^{T\wedge\tau_R} \beta_s dA_s + \beta_{\tau_R} Z_{\tau_R} 1_{\mathcal{I}_C} 1_{\mathcal{I}_3} + \beta_T X_t^d 1_{\mathcal{I}_C} 1_{\mathcal{I}_4} + 1_{\mathcal{I}_C}(R_C - 1)\beta_{\tau_C}(S_{\tau_C}^{n+1,d})^+ \mid \mathcal{G}_t\Bigg) \\
= \ & \mathbb{E}\left((1_{\mathcal{I}_O} + 1_{\mathcal{I}_C})\int_t^{T\wedge\tau_R} \beta_s dA_s + \beta_{\tau_R} Z_{\tau_R}(1_{\mathcal{I}_R} + 1_{\mathcal{I}_3}) + \beta_T X_t^d (1_{\mathcal{I}_O} + 1_{\mathcal{I}_4}) \mid \mathcal{G}_t\right) \\
- \ & \mathbb{E}\left(1_{\mathcal{I}_C} LGD_C \beta_{\tau_C}(S_{\tau_C}^{n+1,d})^+ \mid \mathcal{G}_t\right)
\end{aligned}
$$

using $f = f^+ - f^-$ and the law of iterated expectations. □

Remark: It is interesting to notice that the valuation of Defaultable claim will require the identification of scenarios like the event $\mathcal{I}_3 = \{\tau_C < \tau_R < T\}$. This will need to identify an ordered sequence of defaults in E and realise thus a multivariate simulation of obligors default and not just the time of first default τ of the counterpart C or the investor I. Those sequence of defaults will be the basis of financial stress scenario like in the Financial crisis in 2008 where credit default contagion was at its peak.

We can express the defaultable claim $S_t^{n+1,d}$ defined in (3.3) under unilateral counterparty credit risk as

Proposition 3.1.3 (Unilateral CVA for defaultable claim)**.** *Consider a $\mathcal{G}$-measurable defaultable claim $S_t^{n+1,d}$ defined in (3.3) under counterparty credit risk as*

$$\begin{aligned} S_t^{n+1,d} = \mathbb{E}_t(\beta_t S_t^{n+1,d}) &= \mathbb{E}\left((1_{\mathcal{I}_O} + 1_{\mathcal{I}_C}) \int_t^{T \wedge \tau_R} \beta_s dA_s + \beta_{\tau_R} Z_{\tau_R} (1_{\mathcal{I}_R} + 1_{\mathcal{I}_3}) + \beta_T X_t^d (1_{\mathcal{I}_O} + 1_{\mathcal{I}_4}) \mid \mathcal{G}_t \right) \\ &- \mathbb{E}\left(1_{\mathcal{I}_C} LGD_C \beta_{\tau_C} (S_{\tau_C}^{n+1,d})^+ \mid \mathcal{G}_t \right) \\ &= S_t^{n+1} - CVA_t \end{aligned} \tag{3.30}$$

Thus,

$$CVA_t := S_t^{n+1} - S_t^{n+1,d}$$

with

$$CVA_t := \mathbb{E}\left(1_{\mathcal{I}_C} LGD_C \beta_{\tau_C} ED(\tau_C) \mid \mathcal{G}_t \right) \tag{3.31}$$

Defaultable claim - Bilateral case

Since the Bilateral case is identical with longer proof and equations, we just state the results in this case with an extended set of ordered events. The filtrations are enlarged such that

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t \tag{3.32}$$

with

$$\mathcal{H}_t = \mathcal{H}_t^C \vee \mathcal{H}_t^R \vee \mathcal{H}_t^I \tag{3.33}$$

For the filtration $\mathcal{G}_t$, we introduce the disjoints following events of ordered defaults without simultaneous defaults $\mathbb{Q}(1_{\tau_i = \tau_j}) = 0, \forall i, j \in \{C, I, R\}, i \neq j$

$$\begin{aligned} \mathcal{I}_1 = \{\tau_R < \tau_C < \tau_I < T\},\ \mathcal{I}_2 = \{\tau_R < \tau_C < T < \tau_I\},\ \mathcal{I}_3 = \{\tau_R < \tau_I < \tau_C < T\}, \\ \mathcal{I}_4 = \{\tau_R < \tau_I < T < \tau_C\},\ \mathcal{I}_5 = \{\tau_R < T < \tau_I < \tau_C\},\ \mathcal{I}_6 = \{\tau_R < T < \tau_C < \tau_I\} \\ \mathcal{I}_7 = \{\tau_C < \tau_R < \tau_I < T\},\ \mathcal{I}_8 = \{\tau_C < \tau_R < T < \tau_I\},\ \mathcal{I}_9 = \{\tau_C < \tau_I < \tau_R < T\}, \\ \mathcal{I}_{10} = \{\tau_C < \tau_I < T < \tau_R\},\ \mathcal{I}_{11} = \{\tau_C < T < \tau_I < \tau_R\},\ \mathcal{I}_{12} = \{\tau_C < T < \tau_R < \tau_I\} \\ \mathcal{I}_{13} = \{\tau_I < \tau_R < \tau_C < T\},\ \mathcal{I}_{14} = \{\tau_I < \tau_R < T < \tau_C\},\ \mathcal{I}_{15} = \{\tau_I < \tau_C < \tau_R < T\}, \\ \mathcal{I}_{16} = \{\tau_I < \tau_C < T < \tau_R\},\ \mathcal{I}_{17} = \{\tau_I < T < \tau_C < \tau_R\},\ \mathcal{I}_{18} = \{\tau_I < T < \tau_R < \tau_C\} \\ \mathcal{I}_{19} = \{T \leq \tau_R < \tau_C < \tau_I\},\ \mathcal{I}_{20} = \{T \leq \tau_R < \tau_I < \tau_C\},\ \mathcal{I}_{21} = \{T \leq \tau_I < \tau_C < \tau_R\}, \\ \mathcal{I}_{22} = \{T \leq \tau_I < \tau_R < \tau_C\},\ \mathcal{I}_{23} = \{T \leq \tau_C < \tau_I < \tau_R\},\ \mathcal{I}_{24} = \{T \leq \tau_C < \tau_R < \tau_I\} \end{aligned} \tag{3.34}$$

with the regroupment of events characterising first-to-default events

$$\begin{aligned} \mathcal{I}_R = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3 \cup \mathcal{I}_4 \cup \mathcal{I}_5 \cup \mathcal{I}_6,\ \mathcal{I}_C = \mathcal{I}_7 \cup \mathcal{I}_8 \cup \mathcal{I}_9 \cup \mathcal{I}_{10} \cup \mathcal{I}_{11} \cup \mathcal{I}_{12} \\ \mathcal{I}_I = \mathcal{I}_{13} \cup \mathcal{I}_{14} \cup \mathcal{I}_{15} \cup \mathcal{I}_{16} \cup \mathcal{I}_{17} \cup \mathcal{I}_{18},\ \mathcal{I}_O = \mathcal{I}_{19} \cup \mathcal{I}_{20} \cup \mathcal{I}_{21} \cup \mathcal{I}_{22} \cup \mathcal{I}_{23} \cup \mathcal{I}_{24} \end{aligned} \tag{3.35}$$

We can express the defaultable claim $S_t^{n+1,d}$ defined in (3.3) under bilateral counterparty credit risk as

Proposition 3.1.4 (Bilateral CVA for defaultable claim). *Consider a $\mathcal{G}_t$-measurable defaultable claim $S_t^{n+1,d}$ defined in (3.3) under bilateral counterparty credit risk as*

$$
\begin{aligned}
S_t^{n+1,d} = \mathbb{E}_t(\beta_t S_t^{n+1,d}) \;&=\; \mathbb{E}\Bigg((1_{\mathcal{I}_O} + 1_{\mathcal{I}_C} + 1_{\mathcal{I}_I}) \int_t^{T\wedge\tau_R} \beta_s dA_s + \beta_{\tau_R} Z_{\tau_R}(1_{\mathcal{I}_R} \\
&+\; 1_{\mathcal{I}_7} + 1_{\mathcal{I}_8} + 1_{\mathcal{I}_{13}} + 1_{\mathcal{I}_{14}}) + \beta_T X_t^d (1_{\mathcal{I}_O} + 1_{\mathcal{I}_{11}} + 1_{\mathcal{I}_{12}} + 1_{\mathcal{I}_{17}} + 1_{\mathcal{I}_{18}}) \Bigg| \mathcal{G}_t \Bigg) \\
&+\; \mathbb{E}\left(1_{\mathcal{I}_C} \beta_{\tau_C}(R_C - 1)(S_{\tau_C}^{n+1,d})^+ \mid \mathcal{G}_t\right) \\
&+\; \mathbb{E}\left(1_{\mathcal{I}_I} \beta_{\tau_I}(1 - R_I)(S_{\tau_I}^{n+1,d})^- \mid \mathcal{G}_t\right) \\
&=\; S_t^{n+1} - BCVA_t
\end{aligned}
\tag{3.36}
$$

Thus,

$$
\begin{aligned}
BCVA_t \;&:=\; CVA_t - DVA_t := S_t^{n+1} - S_t^{n+1,d} \\
&:=\; \mathbb{E}\left(1_{\mathcal{I}_C} \beta_{\tau_C}(R_C - 1)(S_{\tau_C}^{n+1,d})^+ \mid \mathcal{G}_t\right) - \mathbb{E}\left(1_{\mathcal{I}_I} \beta_{\tau_I}(1 - R_I)(S_{\tau_I}^{n+1,d})^- \mid \mathcal{G}_t\right) \\
&:=\; \mathbb{E}\left(1_{\{t<\tau\leq T\}} \beta_\tau ED(\tau) \mid \mathcal{G}_t\right)
\end{aligned}
\tag{3.37}
$$

and

$$
CVA_t := \mathbb{E}\left(1_{\mathcal{I}_C} \beta_{\tau_C}(R_C - 1)(S_{\tau_C}^{n+1,d})^+ \mid \mathcal{G}_t\right) \tag{3.38}
$$

and

$$
DVA_t := \mathbb{E}\left(1_{\mathcal{I}_I} \beta_{\tau_I}(1 - R_I)(S_{\tau_I}^{n+1,d})^- \mid \mathcal{G}_t\right) \tag{3.39}
$$

The propositions 3.1.1, 3.1.2, 3.1.3 and 3.1.4 are central in calculating the Credit Valuation Adjustment in the case of credit contagion between a set of obligors Ref, C, I (Note that Ref or R are used interchangeably but with R identifying a reference obligor while Ref identifies references that could also be multiple).

In terms of numerical analysis, the book [Brig10a] contains several papers where the impact of credit dependence between the credit risk obligors and the state variables (notably interest rates). A special attention is also put on so-called "wrong-way risk", in [Brig08a]. However, most of those contagion aspect are presented in a direct way with the drivers of credit risk of the parties in the claim and/or the state variables of the claim. We aim to value contingent claims in a context where the credit risk is impacted by other obligors in a more natural way with for example obligor non related to the claim that default and still impact the counterparts of such claim. We will need to express the Credit Valuation Adjustment in a multi-variate credit risk context.

Chapter 4

Counterparty credit risk in multivariate setting

4.1 Multivariate setting

Consider now the previous set of obligors $E = \{1, \ldots, m\}$ and define the ordering of defaults such as

$$\tau_{(1)} < \tau_{(2)} < \ldots < \tau_{(m)}$$

with $\tau_{(i)}$: the non-negative random variable of the i^{th} default. The set of n permutations of the ordering of defaults is defined as Π which identifies the obligors within the ordering for a finite number of obligor E with the finite cardinalities $\mid \Pi \mid = n \in \mathbb{N}$ and $\forall \pi \in \Pi, \mid \pi \mid = m$.

Definition 4.1.1. *Let us define the set of event triggers in the case of a counterparty valuation as the set containing the obligors among E such as the counterpart C, the investor I and the claim reference Ref. In the case of non defaultable claims like equity, currencies or commodities claims, the set $Ref = \emptyset$, while in the case of single-name credit default swap it is a single obligor and for the index credit claims it is 125 names.*

Thus, be $\mathcal{T}$ the set of event triggers, called the "trigger set"

$$\mathcal{T} = \{C, I, Ref\} \tag{4.1}$$

where the "trigger event"is the random variable

$$\tau = \min_{i \in \mathcal{T}} \tau_i \tag{4.2}$$

and all the permutations $\pi \in \Pi$ can be decomposed as

$$\sum_{\pi \in \Pi} 1_{\{\tau(\pi-) < \tau < \tau(\pi+)\}} \tag{4.3}$$

with the sub-sets $\tau(\pi-)$ being the ordering of all the obligors defaulting before the trigger $\tau = \tau_{(j)}, j \in \mathcal{T}$ and $\tau(\pi+)$ the sub-sets of all the obligors defaulting after the default time of the trigger obligor.

We have the properties

$$\tau(\pi-) \cap \tau(\pi+) = \emptyset,\ \tau(\pi-) \cup j \cup \tau(\pi+) = E.$$

- In the non-defaultable case,

$$\mid \tau(\pi-) \mid = \{0, \ldots, m-2\},\ \mid \mathcal{T} \mid = 2 \text{ and } \mid \tau(\pi+) \mid = \{1, \ldots, m-1\}$$

- In the defaultable case,

$$\mid \tau(\pi-) \mid = \{0, \ldots, m-3\},\ \mid \mathcal{T} \mid = 3 \text{ and } \mid \tau(\pi+) \mid = \{2, \ldots, m-1\}$$

Remark: We consider in the reminder of the document for notational purposes that the cardinality of the trigger to be one and thus the remaining obligors in the trigger set $\mathcal{T}$ to be part of the ordering partition $\tau(\pi+)$ since the trigger event is hit by the first of the obligor in the set $\mathcal{T}$ to default. This also includes multi-name reference sets Ref like k^{th}-to-default swaps since the trigger entity is always a unique obligor. However, index derivatives are claims that are characterised by the fact that they are not ended when a default event occurred and thus might not fit the case of a cardinality of one for the trigger event τ.

We extend the definition per trigger event again, assuming no simultaneous defaults in E as per assumption 3.1.1, with

- the set of permutation Π^C where the trigger is the counterpart C

$$\Pi^C = \Pi \cap \{\pi \in \Pi, \tau = \tau_C\} \tag{4.4}$$
$$\tag{4.5}$$

- the set of permutation Π^I where the trigger is the investor I

$$\Pi^I = \Pi \cap \{\pi \in \Pi, \tau = \tau_I\} \tag{4.6}$$
$$\tag{4.7}$$

- the set of permutation Π^{Ref} where the trigger is the obligor reference $Ref \in E/\{I, C\}$ of the valued claim under consideration

$$\Pi^{Ref} = \Pi \cap \{\pi \in \Pi, \tau = \tau_R\} \tag{4.8}$$

the decomposition property

$$\Pi = \Pi^{Ref} \cup \Pi^I \cup \Pi^C, \tag{4.9}$$

or

$$\sum_{\pi \in \Pi} 1_{\{\tau(\pi-) < \tau < \tau(\pi+)\}} = \sum_{\pi \in \Pi^C} 1_{\{\tau(\pi-) < \tau_C < \tau(\pi+)\}} + \sum_{\pi \in \Pi^I} 1_{\{\tau(\pi-) < \tau_I < \tau(\pi+)\}} + \sum_{\pi \in \Pi^{Ref}} 1_{\{\tau(\pi-) < \tau_R < \tau(\pi+)\}}$$

and pairwise independence

$$\Pi^{Ref} \cap \Pi^I = \emptyset,\ \Pi^{Ref} \cap \Pi^C = \emptyset,\ \Pi^C \cap \Pi^I = \emptyset,$$

Decomposition of a measurable process per default time events: In order to value claims represented by $\mathcal{F}$-measurable random variable H along the time interval $(t, T]$, we have to make sure that these claims are measurable across the filtration $\mathcal{G}$ stopped at each stopping time $\tau_{\pi(i)}, \forall i \in \pi$ with $\pi \in \Pi$. From the viewpoint of the filtration $(\mathcal{F}_t)_{t\geq 0}$, the valuation is similar to a case of progressive enlargement with τ a finite random times and define the extended filtration as

$$\mathcal{F}_t^\tau = \bigcap_{\epsilon>0} \{\mathcal{F}_{t+\epsilon} \vee \sigma(\tau \wedge (t+\epsilon))\}$$

In multivariate case, we quote the following proposition:

Proposition 4.1.1 (Separability of measurable functions according to a sequence of stopping times)**.** *Consider a $\mathbb{G}$-predictable process H and $\forall\pi \in \Pi, \tau_{\pi(i)}$ permutations with $i \in \{1, \ldots, m\}$ an increasing sequence of honest times, $\exists m+1$ $\mathbb{F}$-predictable process h^i such that*

$$H_t = \mathbb{E}(H \mid \mathcal{G}_t) = h_t^1 1_{\{t<\tau_{\pi(1)}\}} + \sum_{i=2}^{m} h_t^i 1_{\{\tau_{\pi(i-1)} \leq t < \tau_{\pi(i)}\}} + h_t^{m+1} 1_{\{t \geq \tau_{\pi(m)}\}}$$

with the process h^i being ratios of $\mathbb{F}$-predictable projection as

$$h_t^1 = \frac{\mathbb{E}(H_t 1_{\{t<\tau_{\pi(1)}\}} \mid \mathcal{F}_t)}{\mathbb{P}(t < \tau_{\pi(1)} \mid \mathcal{F}_t)},\ h_t^i = \frac{\mathbb{E}(H_t 1_{\{\tau_{\pi(i-1)} \leq t < \tau_{\pi(i)}\}} \mid \mathcal{F}_t)}{\mathbb{P}(\tau_{\pi(i-1)} \leq t < \tau_{\pi(i)} \mid \mathcal{F}_t)},\ h_t^{m+1} = \frac{\mathbb{E}(H_t 1_{\{t \geq \tau_{\pi(m)}\}} \mid \mathcal{F}_t)}{\mathbb{P}(t \geq \tau_{\pi(m)} \mid \mathcal{F}_t)}$$

Proof. First quote the definition of an honest time: A random time is honest if it is the end of a predictable set i.e. $\tau(\omega) = \sup\{t : (t, \omega) \in \Omega\}$ where Ω is a $\mathbb{F}$-predictable set. The honest time is $\mathcal{F}_\infty$-measurable.

The proof is an aggregation of the results from [Dellacherie92] about the decomposition across honest times and [Jeulin80] about infinite sequence of honest times. From [Dellacherie92]

- Given L an honest variable, i.e the end of a predictable set, and since any stopping time is honest [Protter90]. The sets of the form

  $$C = (A \cap \{t < L\}) \cup (C = (B \cap \{L \leq t\}),\ A, B \in \mathcal{F}_t$$

  are a tribe on $\mathcal{G}_t$ and the family $(\mathcal{G}_t)$ is a filtration satisfying the usual conditions with L being a stopping time.

- A process H is $\mathbb{G}$-predictable if and only if it admits a representation such as

  $$H = J 1_{]0,L]} + K 1_{]L,\infty[}$$

  where the processes J and K are $\mathbb{F}$-predictable.

Additionally, in [Dellacherie92] p186, the process H is further decomposed as

$$H_t = \mathbb{E}(H \mid \mathcal{G}_t) = \frac{\mathbb{E}(H 1_{\{t<L\}} \mid \mathcal{F}_t)}{\mathbb{P}(t < L \mid \mathcal{F}_t)} 1_{\{t<L\}} + \frac{\mathbb{E}(H 1_{\{t\geq L\}} \mid \mathcal{F}_t)}{\mathbb{P}(t \geq L) \mid \mathcal{F}_t)} 1_{\{t\geq L\}}$$

To expand to the multivariate case with a finite number of stopping times, [Jeulin80] page 88, establishes infinite successive enlargement with Corollary 5.22 that states that given $(L_n)_{n\in\mathbb{N}}$ an increasing sequence of $\mathbb{F}$-honest variables such that $L_0 = 0$ and $\sup_n L_n = +\infty$, then $\mathcal{F}^{(L_n)}$ is the right continuous smallest filtration where $\mathcal{F} \subset \mathcal{F}^{(L_n)}$ and (L_n) are stopping times. $\square$

Additionally, in our multivariate setting of order defaults, we can define successive filtrations:

$$\mathcal{F}_{\tau(1)} \subsetneq \mathcal{F}_{\tau(2)} \subsetneq \ldots \subsetneq \mathcal{F}_{\tau(m)} \subsetneq \mathcal{F}_{\infty}$$

With the separability of random variables, we can now value claims along the default permutations $\pi \in \Pi$ and look for the impact of the Credit Valuation Adjustment.

4.1.1 Counterparty Valuation Adjustement

Non defaultable claim - Multivariate Bilateral case

In the multivariate setting, with the condition of proposition 4.1.1 valid, enabling the existence of a measurable process along any permutation π, we define a non defaultable claim as

$$\begin{aligned}
\beta_t X(t,T)^d &= \int_t^{T\wedge\tau_C\wedge\tau_I} \sum_{\pi\in\Pi} 1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}}\beta_u dX_u \\
&+ \sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}} 1_{\{\tau_C<T\}}\beta_{\tau_C}(R_C X(\tau_C)^+ - X(\tau_C)^-) \\
&+ \sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}} 1_{\{\tau_I<T\}}\beta_{\tau_I}(X(\tau_I)^+ - R_I X(\tau_I)^-) \qquad (4.10)
\end{aligned}$$

Thus, from the viewpoint of the company facing counterparty risk, i.e. the investor I, the value can be decomposed as:

- the first term is the risk-neutral valuation of the payoff under the scenario of no early default or the payments received before default occurs, default being either the counterpart C or the investor I. This equivalent to all set of permutations Π.
- the second term is the residual net present value of the expected exposure in case of $\pi \in \Pi^C$, and,
- the third term is the residual net present value of the expected exposure in case of $\pi \in \Pi^I$.

Proof. Thus, by taking the expectation under $\mathcal{G}_t$, using the same technique as in [Brig06] with the separability of the $\mathcal{F}_T$-measurable process X along the stopping times $\tau_i, i \in E$ of any permutation $\pi \in \Pi$, we have

$$
\begin{aligned}
& \mathbb{E}_t(\beta_t X(t,T)^d) \\
= \; & \mathbb{E}\left(\int_t^{T\wedge\tau_C\wedge\tau_I} \sum_{\pi\in\Pi} 1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}}(1_{\{\tau<T\}}+1_{\{\tau\geq T\}})\beta_u dX_u\right. \\
+ \; & \sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}}1_{\{\tau_C<T\}}\beta_{\tau_C}(R_C X(\tau_C)^+ - X(\tau_C)^-) \\
+ \; & \left.\sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}}1_{\{\tau_I<T\}}\beta_{\tau_I}(X(\tau_I)^+ - R_I X(\tau_I)^-)\middle|\mathcal{G}_t\right) \\
= \; & \mathbb{E}\left(\int_t^{T} \sum_{\pi\in\Pi} 1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}}1_{\{\tau\geq T\}}\beta_u dX_u\right. \\
+ \; & 1_{\{\tau_C<T\}}\left(\int_t^{\tau_C} \sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}}\beta_u dX_u + \sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}}\beta_{\tau_C}(R_C X(\tau_C)^+ - X(\tau_C)^-)\right) \\
+ \; & \left.1_{\{\tau_I<T\}}\left(\int_t^{\tau_I} \sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}}\beta_u dX_u + \sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}}\beta_{\tau_I}(X(\tau_I)^+ - R_I X(\tau_I)^-)\right)\middle|\mathcal{G}_t\right) \\
= \; & \mathbb{E}\left(\int_t^{T} \sum_{\pi\in\Pi} 1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}}1_{\{\tau\geq T\}}\beta_u dX_u\right. \\
+ \; & 1_{\{\tau_C<T\}}\left(\int_t^{\tau_C} \sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}}\beta_u dX_u + \sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}}\beta_{\tau_C}(R_C - 1) X(\tau_C)^+\right. \\
+ \; & \left.\sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}}\mathbb{E}\left(\int_{\tau_C}^T \beta_u dX_u \mid \mathcal{G}_{\tau_C}\right)\right) \\
+ \; & 1_{\{\tau_I<T\}}\left(\int_t^{\tau_I} \sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}}\beta_u dX_u + \sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}}\beta_{\tau_I}(1 - R_I) X(\tau_I)^-\right. \\
+ \; & \left.\left.\sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}}\mathbb{E}\left(\int_{\tau_I}^T \beta_u dX_u \mid \mathcal{G}_{\tau_I}\right)\right)\middle|\mathcal{G}_t\right) \\
= \; & \mathbb{E}\left(\int_t^{T} \sum_{\pi\in\Pi} 1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}}\beta_u dX_u\middle|\mathcal{G}_t\right) \\
- \; & \mathbb{E}\left(\sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}}1_{\{\tau_C<T\}}\beta_{\tau_C} LGD_C X(\tau_C)^+\middle|\mathcal{G}_t\right) \\
+ \; & \mathbb{E}\left(\sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}}1_{\{\tau_I<T\}}\beta_{\tau_I} LGD_I X(\tau_I)^-\middle|\mathcal{G}_t\right)
\end{aligned}
$$

using $f = f^+ - f^-$, the law of iterated expectations and proposition 4.1.1. $\square$

Thus we can restate the non-defaultable claim X defined in 3.1 under bilateral counterparty credit risk as

Proposition 4.1.2 (Bilateral CVA for non-defaultble claim in multivariate setting)**.** *Consider an $\mathcal{F}_T$-measurable non-defaultable claim X defined in (3.1) under bilateral counterpart credit risk as*

$$\begin{aligned}
\mathbb{E}_t(\beta_t X(t,T)^d) &= \mathbb{E}\left(\int_t^T \beta_u dX_u \middle| \mathcal{G}_t\right) \\
&- \mathbb{E}\left(\sum_{\pi \in \Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}} 1_{\{\tau_C<T\}} \beta_{\tau_C} LGD_C X(\tau_C)^+ \middle| \mathcal{G}_t\right) \\
&+ \mathbb{E}\left(\sum_{\pi \in \Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}} 1_{\{\tau_I<T\}} \beta_{\tau_I} LGD_I X(\tau_I)^- \middle| \mathcal{G}_t\right) \\
&= X(t,T) - BCVA_t \qquad (4.11)
\end{aligned}$$

with

$$\begin{aligned}
BCVA_t &:= CVA_t - DVA_t = X(t,T) - X^d(t,T) \\
&:= \mathbb{E}\left(\sum_{\pi \in \Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}} 1_{\{\tau_C<T\}} \beta_{\tau_C} LGD_C X(\tau_C)^+ \middle| \mathcal{G}_t\right) \\
&- \mathbb{E}\left(\sum_{\pi \in \Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}} 1_{\{\tau_I<T\}} \beta_{\tau_I} LGD_I X(\tau_I)^- \middle| \mathcal{G}_t\right) \\
&:= \mathbb{E}\left(1_{\{t<\tau\leq T\}} \sum_{\pi \in \Pi} 1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}} \beta_\tau [ED(\tau)] \mid \mathcal{G}_t\right) \qquad (4.12)
\end{aligned}$$

We similarly now process the case of the defaultable claim.

Defaultable claim - Multivariate Bilateral case

Similarly from equation (3.3) and in the multivariate setting, a defaultable claim is defined as

$$\begin{aligned}
\beta_t S_t^{n+1,d} &= \int_t^{T\wedge\tau_R\wedge\tau_C\wedge\tau_I} \sum_{\pi\in\Pi} 1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}}\beta_s dA_s \\
&+ \sum_{\pi\in\Pi^R} 1_{\{\tau(\pi-)<\tau_R<\tau(\pi+)\}} 1_{\tau_R<T}\beta_{\tau_R} Z_{\tau_R} \\
&+ \sum_{\pi\in\Pi} 1_{\{\tau(\pi-)<\tau_R<\tau(\pi+)\}} 1_{\tau\geq T}\beta_T X_t^d \\
&+ \sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_R<\tau(\pi+)\}} 1_{\tau_C<T}\beta_{\tau_C}(R_C X_{\tau_I}^+ - X_{\tau_C}^-) \\
&+ \sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_R<\tau(\pi+)\}} 1_{\tau_I<T}\beta_{\tau_I}(X_{\tau_I}^+ - R_I X_{\tau_I}^-) \qquad (4.13)
\end{aligned}$$

Thus, from the viewpoint of the company facing counterparty risk, i.e. the investor I, the value can be decomposed as:

- the first term is the risk-neutral valuation of the acrued payments under the scenario of no early default or the payments received before default occurs, default being either the counterpart C, the investor I or the reference Ref. This equivalent to all set of permutations Π.
- the second term is the residual net present value of the recovered payment in case of $\pi \in \Pi^{Ref}$, and,
- the third term is the residual net present value of the payment at maturity in case of no early defaults, and,
- the fourth term is the residual net present value of the expected exposure in case of $\pi \in \Pi^C$, and,
- the fifth term is the residual net present value of the expected exposure in case of $\pi \in \Pi^I$.

Proof. Thus, by taking the expectation under $\mathcal{G}_t$ again with the separability of the $\mathcal{F}_T$-measurable processes that compose the defaultable claim $\Pi = (A, Z, X, \tau, T)$ we have,

$$\begin{aligned}
&\int_t^{T\wedge\tau_R\wedge\tau_C\wedge\tau_I} \sum_{\pi\in\Pi} 1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}}\beta_s dA_s \\
= &\int_t^T \left(\sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}}1_{\tau_C\geq T} + \sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}}1_{\tau_I\geq T} + \sum_{\pi\in\Pi^R} 1_{\{\tau(\pi-)<\tau_R<\tau(\pi+)\}}1_{\tau_R\geq T} \right)\beta_s dA_s \\
+ &\int_t^{\tau_I} \sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}}1_{\tau_I<T}\beta_s dA_s \\
+ &\int_t^{\tau_R} \sum_{\pi\in\Pi^R} 1_{\{\tau(\pi-)<\tau_R<\tau(\pi+)\}}1_{\tau_R<T}\beta_s dA_s \\
+ &\int_t^{\tau_C} \sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}}1_{\tau_C<T}\beta_s dA_s
\end{aligned}$$

We can extract from the integral all the permutations and identifiant functions that do not influence the integration result

$$\begin{aligned}
&\int_t^{T\wedge\tau_R\wedge\tau_C\wedge\tau_I} \sum_{\pi\in\Pi} 1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}}\beta_s dA_s \\
= &\int_t^T \left(\sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}}1_{\tau_C\geq T} + \sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}}1_{\tau_I\geq T} + \sum_{\pi\in\Pi^R} 1_{\{\tau(\pi-)<\tau_R<\tau(\pi+)\}}1_{\tau_R\geq T} \right)\beta_s dA_s \\
+ &\sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}}1_{\tau_I<T}\int_t^{\tau_I} \beta_s dA_s \\
+ &\sum_{\pi\in\Pi^R} 1_{\{\tau(\pi-)<\tau_R<\tau(\pi+)\}}1_{\tau_R<T}\int_t^{\tau_R} \beta_s dA_s \\
+ &\sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}}1_{\tau_C<T}\int_t^{\tau_C} \beta_s dA_s
\end{aligned}$$

Those partial integral with respect to the interval of the claim $[t, T]$ will be completed with the corresponding defaulted event for each trigger identifier. In the case of the counterpart C under $\mathbb{E}(\cdot \mid \mathcal{G}_t)$

$$\begin{aligned}
&\sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}}1_{\tau_C<T}\beta_{\tau_C}(R_C X^+_{\tau_I} - X^-_{\tau_C}) \\
= &\sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_R<\tau(\pi+)\}}1_{\tau_C<T}\beta_{\tau_C}(R_C - 1)X^+_{\tau_I} + \sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_R<\tau(\pi+)\}}1_{\tau_C<T}\beta_{\tau_C}X_{\tau_C}
\end{aligned}$$

We decompose X_{τ_C} as

$$\mathbb{E}\left(\int_{\tau_C}^{T\wedge\tau_R} \beta_s dA_s + \sum_{\pi\in\Pi^R} 1_{\{\tau(\pi-)<\tau_R<\tau(\pi+)\}}1_{\tau_R\geq T}\beta_{\tau_R}Z_{\tau_R} + \sum_{\pi\in\Pi} 1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}}1_{\tau\geq T}\beta_T X^d_t \mid \mathcal{G}_{\tau_C} \right)$$

and with the law of the iterated expectation, we can aggregate in the $\mathbb{E}(\cdot \mid \mathcal{G}_t)$ general term by dropping the two last terms since $\Pi^C \cap \Pi^R = \emptyset$ and $\{\Pi^C \cup \{\tau_C < T\}\} \cap \{\Pi^C \cup \{\tau_C \geq T\}\} = \emptyset$ leaving the term

$$\sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}} 1_{\tau_C<T} \int_{\tau_C}^{T\wedge\tau_R} \beta_s dA_s$$

to add to reconstruct the counterpart-free default claim.

We repeat the same approach with investor I and Reference R and we can reconstruct the counterparty-free valuation formula with

$$\begin{aligned}
& \int_t^{T\wedge\tau_R} \beta_s dA_s \\
= & \int_t^T \left(\sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}} 1_{\tau_C\geq T} + \sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}} 1_{\tau_I\geq T} + \sum_{\pi\in\Pi^R} 1_{\{\tau(\pi-)<\tau_R<\tau(\pi+)\}} 1_{\tau_R\geq T} \right) \beta_s dA_s \\
+ & \sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}} 1_{\tau_C<T} \int_{\tau_C}^{T\wedge\tau_R} \beta_s dA_s \\
+ & \sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}} 1_{\tau_I<T} \int_{\tau_I}^{T\wedge\tau_R} \beta_s dA_s \\
+ & \sum_{\pi\in\Pi^R} 1_{\{\tau(\pi-)<\tau_R<\tau(\pi+)\}} 1_{\tau_R<T} \int_t^{\tau_R} \beta_s dA_s
\end{aligned}$$

□

Thus we can express the defaultable claim S_t^{n+1} defined in 3.3 under bilateral counterparty credit risk as

Proposition 4.1.3 (Bilateral CVA for defaultble claim in multivariate setting)**.** *Consider an $\mathcal{F}_T$-measurable defaultable claim S_t^{n+1} defined in (3.3) under bilateral counterpart credit risk as*

$$\begin{aligned}
\mathbb{E}_t(\beta_t S_t^{n+1,d}) &= \mathbb{E}\left(\int_t^{T\wedge\tau_R} \beta_s dA_s \right. \\
&+ \sum_{\pi\in\Pi^R} 1_{\{\tau(\pi-)<\tau_R<\tau(\pi+)\}} 1_{\tau_R<T} \beta_{\tau_R} Z_{\tau_R} \\
&+ \left. \sum_{\pi\in\Pi} 1_{\{\tau(\pi-)<\tau_R<\tau(\pi+)\}} 1_{\tau\geq T} \beta_T X_t^d \middle| \mathcal{G}_t \right) \\
&- \mathbb{E}\left(\sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}} 1_{\{\tau_C<T\}} \beta_{\tau_C} LGD_C X(\tau_C)^+ \middle| \mathcal{G}_t \right) \\
&+ \mathbb{E}\left(\sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}} 1_{\{\tau_I<T\}} \beta_{\tau_I} LGD_I X(\tau_I)^- \middle| \mathcal{G}_t \right) \\
&= S_t^{n+1} - BCVA_t
\end{aligned} \tag{4.14}$$

with

$$\begin{aligned}
BCVA_t &:= CVA_t - DVA_t \\
&:= \mathbb{E}\left(\sum_{\pi \in \Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}} 1_{\{\tau_C<T\}} \beta_{\tau_C} LGD_C X(\tau_C)^+ \middle| \mathcal{G}_t\right) \\
&- \mathbb{E}\left(\sum_{\pi \in \Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}} 1_{\{\tau_I<T\}} \beta_{\tau_I} LGD_I X(\tau_I)^- \middle| \mathcal{G}_t\right) \\
&:= \mathbb{E}\left(1_{\{t<\tau\le T\}} \sum_{\pi \in \Pi} 1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}} \beta_\tau [ED(\tau)] \mid \mathcal{G}_t\right) \qquad (4.15)
\end{aligned}$$

As pointed in [Jarrow01] and [Brig10a], the defaultable claim is the sum of the value of the corresponding of the counterpart-free and investor-free claim adjusted by

- a call option with zero strike on Expected Exposure when $\{\tau_C < T\}, \forall \pi \in \Pi^C$, i.e if the counterparty is the earliest to default before maturity.
- a put option with zero strike on Expected Exposure when $\{\tau_I < T\}, \forall \pi \in \Pi^I$, i.e. if the investor is the earliest to default before maturity.

We can make the same remark as mentioned in the numerical part of [Brig10a] where any claim priced in a model-free fashion will be now model dependent due to the embedded optionality in counterpart or investor term.

The propositions 4.1.2, and 4.1.3 will be central in expressing the counterparty valuation adjustment in the case of credit contagion between a set of obligors. It will be necessary in an expression like

$$\sum_{\pi \in \Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}} 1_{\{\tau_C<T\}}$$

to built marginal distribution knowledge of the obligors in $\mathcal{T}$ out of the multivariate distributions represented by the permutations. We need, thus, to make sure that the probability space $(\Omega, \mathbb{G}, \mathbb{Q})$ we work-in is embedded with such distributions.

4.1.2 Additional Hypothesis

In order to realise computation of the claim under successive default scenarios where obligor undergo default contagion, it is necessary to specify some other properties of the probability space.

As stated before, the filtration $\mathbb{F}$ does not contain default-related information. The extra information due to the enlargement modify the predictions of future events implying that $\mathbb{F}$-martingales might not stay $\mathbb{G}$-martingales. The main assumption from [Bremaud78], ensures that $\mathbb{F}$-martingales stay $\mathbb{G}$-martingales

Proposition 4.1.4 (($\mathcal{H}$)-hypothesis)**.** *Every square integrable $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale*

Alternatively, the filtration $\mathbb{F}$ is said to be **immersed** in $\mathbb{G}$, if any square integrable $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale and is also equivalent to the following propositions with proof in [Jeanblanc10] p 316 or conditions $(G1), (M1)$ and $(M2)$ under the Martingale Invariance Property in [Bielecki01] p141 and p 166.

Proposition 4.1.5 (($\mathcal{H}$)-hypothesis equivalence)**.** *The ($\mathcal{H}$)-hypothesis is equivalent to any of the following properties.*

- $(\mathcal{H}1)$ $\forall t \geq 0$, *the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent of $\mathcal{F}_t$*
- $(\mathcal{H}2)$ $\forall t \geq 0,\ \forall G_t \in L^1(\mathcal{G}_t)\ \mathbb{E}(G_t \mid \mathcal{F}_\infty) = \mathbb{E}(G_t \mid \mathcal{F}_t)$
- $(\mathcal{H}3)$ $\forall t \geq 0,\ \forall F \in L^1(\mathcal{F}_\infty)\ \mathbb{E}(F \mid \mathcal{G}_t) = \mathbb{E}(F \mid \mathcal{F}_t)$

The ($\mathcal{H}$)-hypothesis implies that in the case of the non-defaultable claim the expectation under $\mathcal{G}_t$ is the same as the exprectation under $\mathcal{F}_t$. This implies in reality that the valuation of ex-credit desks are still valid to be integrated into a CVA desks where the default simulations are done. Although, this requires that the time-grid discretised simulations are consistent across desks as covered in [Cesari09].

With the martingale property is conserved in the enlarged filtration, it is also necessary to verify that the ($\mathcal{H}$)-hypothesis holds for a change of probability measure. Since some exposure metrics are computed under the risk-neutral measure $\mathbb{Q}$ like CVA and the real world measure $\mathbb{P}$ under regulatory requirements, it would be of interest to have conservation of the immersion property under such change. The following proposition in [Jeanblanc10] demonstrates the conservation

Proposition 4.1.6 (Immersion property and change of probability measure)**.** *Assume that the filtration $\mathbb{F}$ is immersed in filtration $\mathbb{G}$ under the probability measure $\mathbb{P}$ and define $\mathbb{Q}\mid_{\mathcal{G}_t} = L_t \mathbb{P}\mid_{\mathcal{G}_t}$ where the Radon-Nikodym density L is assumed to be $\mathbb{F}$-adapted. Then $\mathbb{F}$ is immersed in $\mathbb{G}$ under the probability measure $\mathbb{Q}$*

Proof. Let N_t be a $(\mathbb{F}, \mathbb{Q})$-martingale, then $(N_t L_t^{-1})$ is a $(\mathbb{F}, \mathbb{P})$-martingale and since $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{P}$, $(N_t L_t^{-1})$ is a $(\mathbb{G}, \mathbb{P})$ martingale which implies that N_t is a $(\mathbb{G}, \mathbb{Q})$-martingale. $\square$

We have earlier mentioned that the fact of traded defaultable securities completed the market by ensuring the uniqueness of the attained price in $(\mathbb{G}, \mathbb{Q})$. However, in the case of non-defaultable claims the completeness of Market $\mathcal{M}$ has been established under $(\mathbb{F}, \mathbb{Q})$ and should be conserved with the ($\mathcal{H}$)-hypothesis under $(\mathbb{G}, \mathbb{Q})$.

Proposition 4.1.7 (($\mathcal{H}$)-hypothesis and Arbitrage)**.** *Assume that there exist a probability $\hat{\mathbb{Q}}$ equivalent to $\mathbb{P}$ on $\mathbb{G}_T = \mathbb{F}^S \vee \mathcal{H}_T$ such that $(S_t\beta_t, 0 \leq t \leq T)$ is a $\mathbb{G}$-martingale under probability $\hat{\mathbb{Q}}$. Then $(\mathcal{H})$ hold under $\hat{\mathbb{Q}}$ and the restriction $\hat{\mathbb{Q}}$ to $\mathcal{F}_T^S$ is equal $\mathbb{Q}$.*

Proof. Let the random variable $X \in L^2(\mathbb{Q}, \mathcal{F}_T)$ define a contingent claim with $\hat{X} = \beta_T X$ its discounted value. Since the same claim exists in the larger market, which is assumed to be arbitrage-free, the discounted value of that claim is a $(\mathbb{G}, \hat{\mathbb{Q}})$-martingale. From the uniqueness of the price for a hedgeable claim, for any contingent claim $X \in \mathcal{F}_T^S$ and any $\mathbb{G}$-equivalent martingale measure $\hat{\mathbb{Q}}$

$$\mathbb{E}^{\mathbb{Q}}(X\beta_T \mid \mathcal{F}_T^S) = \mathbb{E}^{\hat{\mathbb{Q}}}(X\beta_T \mid \mathcal{G}_t)$$

which yields that $\hat{\mathbb{Q}} = \mathbb{Q}$ on $\mathcal{F}_T$.
Morever since every square integrable $(\mathcal{F}^S, \mathbb{Q})$ can be written as

$$\mathbb{E}^{\hat{\mathbb{Q}}}[\beta_T X \mid \mathcal{F}_t] = \mathbb{E}^{\hat{\mathbb{Q}}}[\beta_T X \mid \mathcal{G}_t]$$

which is equivalent to $(\mathcal{H})$. Thus in this setting $(\mathcal{H})$ holds under $\hat{\mathbb{Q}}$ and the restriction of $\hat{\mathbb{Q}}$ to $\mathcal{F}_T^S$ is $\mathbb{Q}$.

$\square$

With this set of properties, we can now turn to value credit risk modelling in a multivariate setting. After reviewing the different univariate and multivariate credit models, we will focus on multivariate models that can integrate credit risk contagion and at the same time provide marginal distributions notably for obligors in the event trigger set $\mathcal{T}$.

Chapter 5

Credit risk modelling

5.1 Review of Credit risk models

Credit modelling is categorised in two families: structured-form and reduced-form.

- Structural papers are based on the work of [Merton74], in which a firm life is linked to its ability to pay back its debt. The model states that a firm issues a bond to finance its activities and also that this bond has a maturity T. At final time T, if the firm is not able to reimburse all the bondholders there has been a default event. In this context, default may occur only at final time T and is triggered by the value of the firm being below the debt level.

  Formally, suppose there is a debt maturity T, face value L and the company defaults at final maturity (and only then) if the value of the firm V_T is below the debt L to be paid. The debt value at time $t < T$ is

$$\begin{aligned} D_t &= \mathbb{E}_t[D(t,T)\min(V_T, L)] \\ &= \mathbb{E}_t[D(t,T)[V_T - (V_T - L)^+]] \\ &= P(t,T)L - \mathrm{Put}(t,T;V_t,L) \end{aligned}$$

  with $D(t,T) = B(t)/B(T)$ where $B(t) = \exp(-\int_0^t r_u du)$ denotes the bank account numéraire and r the instantaneous short interest rate. The equity value can be derived as the difference between the value of the firm and the debt:

$$\begin{aligned} S_t = V_t - D_t &= V_t - P(t,T)L + \mathrm{Put}(t,T;V_t,L) \\ &= \mathrm{Call}(t,T;V_t,L) \end{aligned}$$

  So the equity can be interpreted as a call option on the value of the firm and debt investor write a put on the assets.

  An immediate limitation of the Merton model is the fact that the driver of the value is the firm value process V_t which is not observable and also that default can only happen at maturity.

[Black76] introduces a more realistic and sophisticated model called the *first-passage time models*. In the Black and Cox model, default can also happen before the maturity with safety covenants in place force the firm to reimburse as soon as its asset value V_t hits a low enough "safety level" $\hat{H}(t)$. The choice of that safety level is not easy as it could be the present value of the final debt value $P(t,T)L$ or incorporate also some slack to the counterparty such as "grace period". If τ defines the first time where the process hits the barrier from above, the price of this down-and-out digital option/bond is given by:

$$DOB(0,T) = \mathbb{E}\{D(0,T)\mathbb{1}_{\{\tau>T\}}\}$$

and with the assumption of deterministic rates:

$$DOB(0,T) = P(0,T)\mathbb{E}\{\mathbb{1}_{\{\tau>T\}}\} = P(0,T)\mathbb{Q}\{\tau > T\}$$

with $\mathbb{Q}\{\tau > T\}$ being the risk-neutral probability of never touching the barrier before T or *survival probability*. [Bielecki01] gives an the explicit formula for barrier option with asset process following a geometric Brownian Motion:

$$\mathbb{Q}\{\tau > T\} = \left[\Phi(d_1) - \left(\frac{V_0}{\hat{H}}\right)^{1-2(r-q)/\sigma^2} \Phi(d_2)\right] \tag{5.1}$$

with $d_{1,2} = \left(\pm \log \frac{V_0}{\hat{H}} + (r - q - \frac{\sigma^2}{2})\right) / (\sigma\sqrt{T})$, V_0 the asset value at time 0, q the continuous dividend yield, r the instantaneous interest rate, σ the volatility of the asset value process V_t and Φ the cumulative normal distribution.

In [Bielecki01] similar formulas can also be obtained for a barrier that is not necessarily constant in time but with a particular exponential shape. When relaxing the assumption of constant parameters in the V dynamics, the situation becomes much more complicated and it is not possible to find closed-form formulas even if [Brig04] shows that survival distribution closed-form formulas are achievable in the case of time dependent parameters with a particular curved shape .

In first-passage time models, the default time is thus the first instant where the firm value hits from above either deterministic (possibly time varying) or stochastic barrier. This firm value follows a random process that is similar to the one used to describe generic stock in equity markets. In this case, the filtrations are $\mathcal{F}_t = \mathcal{G}_t$ and the default of the firm can be seen as a predictable process. However standard barriers have few parameters and cannot be calibrated exactly to structured data such as CDS quotes. Interestingly however [Brig04] introduces time-varying exponentially-shaped barriers with calibration identical to reduced-form models.

- Reduced-form models (also called intensity models when a suitable context is present) describe default through an exogenous jump process: more precisely, the default time τ is the first jump time of a Poisson process with deterministic or stochastic (Cox processes) intensity. Thus, default is not triggered by basic market observables, $\mathcal{F}_t$ adapted processes, but by an exogenous component independent of all the default free market information, $\mathcal{H}_t$-adapted process. Monitoring the default free market does not give complete information on the default process and there is no economic rationale behind the default by opposition to the structural models.

Using the description in [Bremaud81] and [Daley02], the classical intensity model is designed as a non-explosive univariate point process $(T_n)_{n=0}^{\infty}$ on the positive real line $[0, \infty[$ describing a strictly increasing sequence of positive random variables such that.

$$\begin{aligned} T_0 &= 0 \; a.s \\ \lim_{n\to\infty} T_n &= \infty \end{aligned}$$

with the corresponding *counting process* N_t related to the point process T_n identified as

$$N_t = \sum_{n=1}^{\infty} 1_{\{T_n \leq t\}}$$

In [Bielecki01] and Theorem 4 p25 in [Bremaud81], The **Wanatabe's Characterisation** provides the definition of the intensity process associated with the counting process as:

Definition 5.1.1 (Intensity Definition)**.** *Let N_t be a point process adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and let λ_t be a non-negative $\mathcal{F}_t$-progressive process such that for all $t \geq 0$*

$$\int_0^s \lambda_s ds < \infty \; \mathbb{Q} - a.s$$

Then if for all non-negative $\mathcal{F}_t$-predictable process C_t

$$\mathbb{E}\left[\int_0^{\infty} C_s dN_s\right] = \mathbb{E}\left[\int_0^{\infty} C_s \lambda_s ds\right]$$

we say that N_t admits the $(\mathbb{Q}, \mathcal{F}_t)$ intensity λ_t, or for short N_t admits the $\mathcal{F}_t$-intensity λ_t

In [Bremaud81], the intensity λ_s of point processes is introduced as a $\mathcal{F}_t$-progressive and $\mathcal{F}_t$-predictable process as

Theorem 5.1.1. *Let N_t admits the $\mathcal{F}_t$-intensity λ_t which satisfies*

$$\int_0^s \lambda_s ds < \infty \; \mathbb{Q} - a.s$$

Then the process M_t given by

$$M_t = N_t - \int_0^t \lambda_s ds$$

is a $\mathcal{F}_t$-martingale.

where the process $A_t = \int_0^t \tilde{\lambda}_s ds$ is called the compensator of N_t.

In [Bremaud81] and [Bielecki01], the link between the intensity λ_s and the survival probability $\mathbb{Q}[\tau > t]$ is presented in property as

Proposition 5.1.1. *Let τ be a positive random variable linked to the first jump T_1 and $(\mathcal{H}_t)_{t\geq 0}$ a default filtration to information until time t. Define N_t as*

$$N_t = 1_{\{\tau \leq t\}}$$

and let N_t admit the $\mathcal{H}_t$ intensity λ_t where λ_t is càdlàg. Then

$$-\left(\frac{\partial}{\partial T}\mathbb{Q}[\tau > T|\mathcal{H}_t]\right)|_{T=t} = \lambda_t$$

where $\frac{\partial^+}{\partial T}$ denotes differentiation from the right.

Thus, with λ_t a non-negative $\mathcal{H}_t$-adapted process, τ is defined as

$$\tau = \inf\left\{t : \int_0^t \lambda_s ds \geq \mathrm{e}\right\}$$

where e= exp(1), and thus the survival distribution is defined as:

$$\mathbb{Q}[\tau > t] = \mathbb{E}\left[\exp\left(-\int_0^t \lambda_s ds\right)\right]$$

Thus, the distribution of N_t or equivalently, the distribution of τ is completely determined by the intensity λ_t for point processes.

5.1.1 Valuation in a reduced-form context

In order to value non-defaultable contingent claim (3.1) and defaultable contingent claim (3.3), reduced-form models of the jump process H_t are either incorporated via the Hazard process or via the martingale characterisation of the intensity λ_t in counterpart adjusted formulas (3.13), (3.20), (3.30) and (3.36) or in multivariate equations (4.11) and (4.14).

Valuation via the Hazard process

The hazard process of the default time given the flow of information represented by the filtration $\mathbb{F}$ is defined as

Definition 5.1.2. *The $\mathbb{F}$-hazard process of τ under $\mathbb{Q}$ denoted by Γ is defined through the formula $1 - F_t = e^{-\Gamma_t}$ or $\Gamma_t := -\ln G_t = -\ln(1 - F_t), \forall t \in \mathbb{R}_+$*

with default process $F_t = \mathbb{Q}\{\tau \le t \mid \mathcal{F}_t\}$ and the survival process $G_t = 1 - F_t = \mathbb{Q}\{\tau > t \mid \mathcal{F}_t\}$ of the random time τ with respect to the reference filtration $\mathbb{F}$. Since $\{\tau \le t\} \subseteq \{\tau \le s\}$, $\forall 0 \le t \le s$ and

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(F_s \mid \mathcal{F}_t) &= \mathbb{E}_{\mathbb{Q}}(\mathbb{Q}\{\tau \le s \mid \mathcal{F}_s\} \mid \mathcal{F}_t) \\ &= \mathbb{Q}\{\tau \le s \mid \mathcal{F}_t\} \ge \mathbb{Q}\{\tau \le t \mid \mathcal{F}_s\} = F_t\end{aligned}$$

the process F (the survival process G, resp.) follows a bounded, non-negative $\mathbb{F}$-submartingale ($\mathbb{F}$-supermartingale, resp.) under $\mathbb{Q}$.

Multivariate context: In the multivariate setting, the occurrence of default time τ_i is $\mathbb{H}^i = (H_t^i)_{t>0}$ and generated by the default indicator process with $H_t^i = 1_{\{\tau_i \le t\}}$.

We also define the occurrence in the case of permutation of the ordered default time as $\forall \pi \in \Pi, \tau_{(i)} = \tau_{\pi(i)}$ with the filtration

$$\mathbb{H}^{(i)} = (H_t^{(i)})_{t>0}$$

with $H_t^{(i)} = 1_{\{\tau_{(i)} \le t\}}$.

The processes $H^i, H^{(i)}$ are $\mathbb{G}$-adapted but not $\mathbb{F}$-adapted (τ_i and $\tau_{(i)}$ are $\mathbb{G}$ stopping times but not $\mathbb{F}$ stopping times).

Define the associated $\mathbb{F}$-survival processes

$$G_t^i := \mathbb{Q}(\tau_i > t \mid \mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}[1_{\{\tau_i > t\}} \mid \mathcal{F}_t]$$
$$G_t^{(i)} := \mathbb{Q}(\tau_{(i)} > t \mid \mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}[1_{\{\tau_{(i)} > t\}} \mid \mathcal{F}_t]$$

and $\mathbb{F}$-default processes

$$F_t^i := 1 - G_t^i = \mathbb{Q}(\tau_i \le t \mid \mathcal{F}_t)$$
$$F_t^{(i)} := 1 - G_t^{(i)} = \mathbb{Q}(\tau_{(i)} \le t \mid \mathcal{F}_t)$$

The processes $G^i, G^{(i)}$ are non-negative $\mathbb{F}$-super martingale. ($F^i, F^{(i)}$ submartingles)

Proof. $\forall s < t$

$$\mathbb{E}^{\mathbb{Q}}[G_t^{(i)} \mid \mathcal{F}_s] = \mathbb{E}^{\mathbb{Q}}[1_{\{\tau_{(i)} > t\}} \mid \mathcal{F}_s] \le \mathbb{E}^{\mathbb{Q}}[1_{\{\tau_{(i)} > s\}} \mid \mathcal{F}_s] = G_s^{(i)}$$

With identical proof for G^i. □

We assume that $\forall t \in \mathbb{R}^+, F_t^i < 1$ and $F_t^{(i)} < 1$ and are continuous decreasing process thus enabling the existence of $\mathbb{F}$-progressively measurable processes $F_t^i = \int_0^t f_s^i ds$, marginal distribution of obligor i and $F_t^{(i)} = \int_0^t f_s^{(i)} ds$ marginal distribution of the i^{th} obligor to default in set E

Similarly, the definition of the $\mathbb{F}$-hazard process of $\tau_i, \tau_{(i)}$ under $\mathbb{Q}$ denoted $\Gamma^i, \Gamma^{(i)}$ is given by

$$G_t^i := e^{-\Gamma_t^i} \text{ or } \Gamma_t^i = -\ln G_t^i$$
$$G_t^{(i)} := e^{-\Gamma_t^{(i)}} \text{ or } \Gamma_t^{(i)} = -\ln G_t^{(i)}$$

Introducing permutations: Considering the permutation $\pi \in \Pi$ with the associated event indicator function of ordered defaults $1_{\{\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}}$, we define the multivariate $\mathbb{F}$-survival process and $\mathbb{F}$-default process

$$\begin{aligned} G_t &:= \mathbb{Q}(\tau_1 > t, \ldots, \tau_m > t \mid \mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}[1_{\{\tau_1>t,\ldots,\tau_m>t\}} \mid \mathcal{F}_t] \\ F_t &:= \mathbb{Q}(\tau_1 \leq t, \ldots, \tau_m \leq t \mid \mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}[1_{\{\tau_1\leq t,\ldots,\tau_m\leq t\}} \mid \mathcal{F}_t] \end{aligned} \tag{5.2}$$

However, we assumed that the set of obligor E can be ordered according to their default times, under no-simultaneous default assumption, into a set of permutations with the corresponding processes per permutation π or for all permutations Π

$$G_t^{\pi} := \mathbb{Q}(t < \tau_{\pi(-)} < \tau < \tau_{\pi(+)} \mid \mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}[1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} \mid \mathcal{F}_t] \tag{5.3}$$

$$G_t^{\Pi} := \mathbb{Q}(\sum_{\pi\in\Pi} t < \tau_{\pi(-)} < \tau < \tau_{\pi(+)} \mid \mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}[\sum_{\pi\in\Pi} 1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} \mid \mathcal{F}_t] \tag{5.4}$$

and their corresponding $\mathbb{F}$-default processes F_t^{π} and F_t^{Π}. In this case we have the equality

$$G_t^{\Pi} = G_t$$

and due to the separability of the events of each permutations π

$$G_t^{\Pi} = \sum_{\pi\in\Pi} G_t^{\pi} \tag{5.5}$$

We now need to express the density of the default time τ associated with the trigger event in the context of the permutations π. Similarly, we define the $\mathbb{F}$-survival processes

$$\begin{aligned} G_t^{\pi,\tau} &:= \mathbb{Q}(\{\tau_{\pi(-)} < \tau < \tau_{\pi(+)}\} \cap \{t < \tau\} \mid \mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}[1_{\{\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}\}\cap\{t<\tau\}} \mid \mathcal{F}_t] \\ G_t^{\tau} &:= \mathbb{Q}(\sum_{\pi\in\Pi}\{\tau_{\pi(-)} < \tau < \tau_{\pi(+)}\} \cap \{t < \tau\} \mid \mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}[\sum_{\pi\in\Pi} 1_{\{\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}\}\cap\{t<\tau\}} \mid \mathcal{F}_t] \end{aligned}$$

with the associated $\mathbb{F}$-default processes $F_t^{\pi,\tau}, F_t^{\tau}$ and again due to the separability of the events of each permutations π, we have the relation:

$$G_t^{\tau} = \sum_{\pi\in\Pi} G_t^{\pi,\tau} \tag{5.6}$$

Although the processes $G_t^{\pi,\tau}$ and G_t^{τ} seem to be multivariate processes, they are univariate processes. We use the default time τ of the trigger in the notation $G_t^{\pi,\tau}, G_t^{\tau}$ as the default event of the trigger that is modelled and thus, is univariate although the defaults $\tau_{\pi(-)}$ will be reflected in the default event τ through credit dependence. Its associated density need to express the survival probability of τ associated with the trigger event. This will in the following sections being expressed in closed-form through a Markov chain called a multivariate phase-type distribution.

Since, in reduced-form models of credit risk, the hazard process $\Gamma^{\bullet}$ is postulated to have absolutely continuous sample paths with respect to the Lebesgue measure, it admits the following integral representation

$$\Gamma_t^{\bullet} := \int_0^t \gamma_u^{\bullet} du, \ \forall t \in \mathbb{R}_+$$

for some non negative $\mathbb{F}$-progressivey measurable stochastic process $\gamma^\bullet$ with integrable sample paths. In terms of notation in the literature, the process $\gamma^\bullet$ is called the $\mathbb{F}$-hazard rate or $\mathbb{F}$-intensity of $\tau_\bullet$ or stochastic intensity. It is called intensity function of $\tau_\bullet$ when the intensity of a default time is non-random and written $\gamma^\bullet(t)$ implying the trivial filtration as the reference filtration $\mathbb{F}$ so that $\mathbb{G} = \mathbb{H}$.

The continuity of processes $F_t^i, F_t^{(i)}, F_t^{\pi,\tau}, F_t^\tau$ introduces the corresponding hazard rate of $\tau^{(i)}, \tau^i, \tau^\pi, \tau$ under $\mathbb{Q}$ (as $\mathbb{F}$-intensity of random times) as non-negative $\mathbb{F}$-progressively measurable processes $\gamma^i, \gamma^{(i)}, \gamma^{\tau,\pi}, \gamma^\tau$ given by

$$\begin{aligned}
\gamma_s^i &= f_s^i/1 - F_s^i \\
\gamma^{(i)} &= f_s^{(i)}/1 - F_s^{(i)} \\
\gamma^{\tau,\pi} &= f_s^{\tau,\pi}/1 - F_s^{\tau,\pi} \\
\gamma^\tau &= f_s^\tau/1 - F_s^\tau
\end{aligned}$$

Moreover, the hazard process is continuous if and only if the submartingale $F^\bullet$ follows a continuous processes. For simplicity, we will always consider the Hazard process as continuous.

Under the $(\mathcal{H})$ hypothesis in proposition 4.1.4

$$\forall t, \mathbb{P}(\tau \leq t \mid \mathcal{F}_\infty) = \mathbb{P}(\tau \leq t \mid \mathcal{F}_t) := F_t^\bullet$$

and $F^\bullet$ is increasing and supposed continuous, then the predictable increasing process of the Doob-Meyer decomposition of $G^\bullet = 1 - F^\bullet$ is equal to F and

$$M_t^\bullet = H_t - \int_0^{t\wedge\tau} \frac{dF_s^\bullet}{G_s^\bullet}$$

is a $\mathbb{G}$-martingale

Other useful martingales as in [Bielecki01] are

$$L_t^i = (1 - H_t)e^{\Gamma_t^i} \text{ is a } \mathbb{G} - \text{martingale}$$

$$L_t^{(i)} = (1 - H_t)e^{\Gamma_t^{(i)}} \text{ is a } \mathbb{G} - \text{martingale}$$

and the following proposition to value survival claims:

Proposition 5.1.2. *Suppose X $\mathbb{F}$-martingale and $XL^\bullet$ is $\mathbb{Q}$-martingale then $XL^\bullet$ is a $\mathbb{G}$-martingale*

$$\mathbb{E}^{\mathbb{Q}}[X_t L_t^\bullet \mid \mathcal{G}_s] = X_s L_s^\bullet$$

Back to our multivariate setting, in order to express credit dependence among obligors, we need to link the definitions of processes $F^\bullet$ with credit risk contagion. In the multivariate setting, $\forall \pi \in \Pi$ the first default $\tau^{\pi(1)}$ will potentially have a contagion effect on the default time of the trigger event $\tau = \tau_C \wedge \tau_I \wedge \tau_{Ref}$. This will also imply that the valuation of contingent claims X_t will have to start from the conditional of the event $1_{\{\tau_{\pi(1)}>t\}}$ typically

$$\mathbb{E}^{\mathbb{Q}}[1_{\{\tau_{\pi(1)}>t\}} X_t \mid \mathcal{G}_t]$$

In conclusion, we will need to find ways to express F^i as a function of $F^{(i)}$ or link multivariate distribution with marginal distributions. Alternatively this dependence can also be expressed as a functional of martingales $M^\bullet$.

Valuation via the Martingale Approach

From the **Wanatabe Characterisation**, in [Bremaud81], the default process $H_t^\bullet = 1_{\{\tau_\bullet \leq t\}}$ admits the $\mathbb{F}$-martingale intensity process $\lambda^\bullet$ under the spot martingale measure $\mathbb{Q}$. With Martingale Characterisation of Intensity theorem page 29 in [Bremaud81], the $\mathbb{F}$-martingale intensity $\lambda^\bullet$ of $\tau_\bullet$ is an $\mathbb{F}$-progressively measurable process such that the compensated process $M^\bullet$ is given by

$$M_t^\bullet := H_t^\bullet - \int_0^{t\wedge\tau_\bullet} \lambda_u^\bullet du = H_t^\bullet - \int_0^t \bar{\lambda}_u^\bullet du, \forall t \in \mathbb{R}_+$$

follows a $\mathbb{G}$-martingale under $\mathbb{Q}$ with $\bar{\lambda}_t^\bullet := 1_{\{\tau_\bullet \geq t\}} \lambda_t^\bullet$.

The counterparty equations (4.11) and (4.14) of the price process can be adjusted w.r.t the Jump processes with the integration with respect to the associated intensity measure $\bar{\lambda}_t^\bullet dt$ of the different obligors. We will however focus on the Hazard process approach since to is more difficult to construct correlation of martingales rather than correlate Hazard processes.

5.1.2 Conditional Expectations under multivariate case

Having now introduced how the probabilities associated with identities are represented either through martingales or hazard process, we return to our multivariate framework to specify, using the hazard processes defined in the previous section and rewrite the Credit Valuation Adjustment in propositions 4.1.2 and 4.1.3, page 53-56.

Equations (4.11) and (4.14) require that we express the conditional expectations based on the default or survival of the trigger event $\mathcal{T}$. From Lemma 5.1.2 in [Bielecki01], page 143,

Lemma 5.1.1. *For any $\mathcal{G}$-measurable and $\mathbb{Q}$-integrable random variable X we have*

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}[\sum_{\pi\in\Pi} 1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} 1_{\{\tau\leq T\}} X \mid \mathcal{G}_t] \\ = \quad &\sum_{\pi\in\Pi^\tau} 1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} 1_{\{\tau\leq T\}} \frac{\mathbb{E}^{\mathbb{Q}}[1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} 1_{\{\tau\leq T\}} X \mid \mathcal{F}_t]}{\mathbb{Q}(1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} 1_{\{\tau\leq T\}} \mid \mathcal{F}_t)} \end{aligned} \tag{5.7}$$

Proof. We can establish the proof for a permutation π and generalise with the linearity of expectation for disjoints events for all permutations $\pi \in \Pi$.

First, we consider the the indicator function of the event $1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}}$ and the associated event set $C = \{t < \tau_{\pi(-)} < \tau < \tau_{\pi(+)}\}$ related to the observation of all the default events of obligors in E. From the decomposition property measurable variable, if Y_t is $\mathcal{G}_t$-measurable random variable then there exists y_t an $\mathcal{F}_t$-measurable random variable such that

$$Y_t 1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} = y_t 1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}}$$

This

$$\mathcal{G}_t^* = \{A \in \mathcal{G} : \exists B \in \mathcal{F}_t, A \cap \{t < \tau_{\pi(-)} < \tau < \tau_{\pi(+)}\} = B \cap \{t < \tau_{\pi(-)} < \tau < \tau_{\pi(+)}\}\} \tag{5.8}$$

is a σ-algebra and

- for $\mathcal{H}_t = \bigvee_{i=1}^m \mathcal{H}_t^i \subset \mathcal{G}_t^* : A = \{\tau_i \leq u, \forall i \in E\}$ for $u \leq t$ and $B = \emptyset$, and
- For $\mathcal{F}_t \subset \mathcal{G}_t^* : A = B$.

so that $\mathcal{G}_t \subset \mathcal{G}_t^*$. Using the monotone class theorem and if $Y_t = 1_A$ with $A \in \mathcal{G}_t, \exists B \in \mathcal{F}_t$ such that $1_{A\cap\{\tau>t\}} = 1_{B\cap\{\tau>t\}}$.

Second, we take the expectation of Y under $\mathcal{G}_t$

$$\begin{aligned}
\textstyle\int_A 1_C Y \mathbb{P}(C \mid \mathcal{F}_t) d\mathbb{P} &= \int_{A\cap C} Y \mathbb{P}(C \mid \mathcal{F}_t) d\mathbb{P} = \int_{B\cap C} Y \mathbb{P}(C \mid \mathcal{F}_t) d\mathbb{P} \\
&= \int_B 1_C Y \mathbb{P}(C \mid \mathcal{F}_t) d\mathbb{P} = \int_B \mathbb{E}(1_C Y \mid \mathcal{F}_t) \mathbb{P}(C \mid \mathcal{F}_t) d\mathbb{P} \\
&= \int_B \mathbb{E}(1_C \mathbb{E}(1_C Y \mid \mathcal{F}_t) \mid \mathcal{F}_t) d\mathbb{P} = \int_{B\cap C} \mathbb{E}(1_C Y \mid \mathcal{F}_t) d\mathbb{P} \\
&= \int_{A\cap C} \mathbb{E}(1_C Y \mid \mathcal{F}_t) d\mathbb{P} = \int_A 1_C \mathbb{E}(1_C Y \mid \mathcal{F}_t) d\mathbb{P}
\end{aligned}$$

which gives the above formula. □

The equation (5.7) with the event $\{t < \tau_{\pi(-)} < \tau < \tau_{\pi(+)}\}$ will be of interest in the Credit Valuation Adjustment to express the claim represented by the random variable X under the filtration $\mathcal{F}_t$ weighted by the probability $\mathbb{Q}(1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} 1_{\{\tau\leq T\}} \mid \mathcal{F}_t)$. We can consider that from a valuation point of view that at the beginning of the simulation this probability weight is equal to one.

Additionally, the same reasoning can be applied to the event $\{\tau_{1(\pi(-))} < t < \tau_{2(\pi(-))} < \tau < \tau_{\pi(+)}\}$ where the pre-trigger permutation $\tau_{\pi(-)}$ is split according to the event t as $\tau_{1(\pi(-))} < t < \tau_{2(\pi(-))}$. This, will enable to express the Credit Valuation Adjustment under the filtration $\mathcal{F}_t$ and the probability weight $\mathbb{Q}(1_{\{\tau_{1(\pi(-))}<t<\tau_{2(\pi(-))}<\tau<\tau_{\pi(+)}\}} \mid \mathcal{F}_t) < 1$. Note that in this case the defaults in the permutation $\tau_{1(\pi(-))}$ will have impacted though dependence models the default events posterior to time t.

Or expressed in the case where the random variable defines a defaultable claim like (3.3), page 36, using the separability of measurable processes along the permutation of defaults.

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}[\sum_{\pi\in\Pi} 1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} 1_{\{\tau\leq T\}} \beta_\tau LGD_\tau \int_t^T \beta_u D_u^{d,n+1} \mid \mathcal{G}_t] \\
= \quad & \sum_{\pi\in\Pi^\tau} 1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} 1_{\{\tau\leq T\}} \frac{\mathbb{E}^{\mathbb{Q}}[\ \int_t^T 1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} 1_{\{\tau\leq T\}} \beta_u D_u^{d,n+1} du \mid \mathcal{F}_t]}{\mathbb{P}(1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} 1_{\{\tau\leq T\}} \mid \mathcal{F}_t)}
\end{aligned}
$$

The equation (5.7) adjusted for the different permutations and occurring events within a permutation enables to decompose the Credit Valuation Adjustment with a simulation of the discounted value of the claim in the future under filtration $\mathcal{F}_t$ and weighted by the probability of the default event under consideration.

Moreover, the previous Lemma 5.1.1 can be expressed using the hazard processes for a random variable X that is $\mathcal{F}_T$-measurable as in the case of vulnerable claims. Thus,

Lemma 5.1.2. *For any $\mathcal{F}_T$-measurable and $\mathbb{Q}$-integrable random variable X we have*

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[\sum_{\pi\in\Pi} 1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} 1_{\{\tau>T\}} X \mid \mathcal{G}_t] &= 1_{\{\tau>t\}} \frac{\mathbb{E}^{\mathbb{Q}}[\sum_{\pi\in\Pi} 1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} 1_{\{\tau>T\}} X \mid \mathcal{F}_t]}{\mathbb{Q}(\sum_{\pi\in\Pi} 1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} 1_{\{\tau>t\}} \mid \mathcal{F}_t)} \\
&= 1_{\{\tau>t\}} \mathbb{E}^{\mathbb{Q}}[X e^{-\int_t^T \gamma_u^\tau du} \mid \mathcal{F}_t]
\end{aligned}
$$

The lemma 5.1.2, typically, will be used in the context of the CVA formula with the random variable X representing counterparty exposure and an expression of the default of the trigger under a multivariate framework.

Proof. Since $1_{\{\tau>t\}} \subset 1_{\{\tau>T\}}$, we have

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[\sum_{\pi\in\Pi} 1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} 1_{\{\tau>T\}} X \mid \mathcal{F}_t] &= \mathbb{E}^{\mathbb{Q}}[X \mathbb{E}^{\mathbb{Q}}[\sum_{\pi\in\Pi} 1_{\{t<\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} 1_{\{\tau>T\}} \mid \mathcal{F}_T] \mid \mathcal{F}_t] \\
&= \mathbb{E}^{\mathbb{Q}}[X G^\tau(T) \mid \mathcal{F}_t]
\end{aligned}
$$

□

The Hazard processes introduced for a multivariate context in the previous section 5.1.1, page 63, can express the probability event of defaults and, thus, we can adjust the expectation in (4.11) and (4.14).

Lemma 5.1.3. *If h is a $\mathbb{F}$-predictable bounded process, then $\forall t \leq s < \infty$ where τ represents the trigger event with f the corresponding marginal density function*

$$\mathbb{E}^{\mathbb{Q}}[h_\tau 1_{\{s \geq \tau > t\}} \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}\left[\sum_{\pi \in \Pi} \int_t^s h_u f_u^{\tau,\pi} du \mid \mathcal{F}_t\right] = \sum_{\pi \in \Pi} \mathbb{E}^{\mathbb{Q}}\left[\int_t^s h_u f_u^{\tau,\pi} du \mid \mathcal{F}_t\right]$$

Proof. We can prove by approximating h by using a stepwise bounded $\mathbb{F}$-predictable processes. Consider a permutation of defaults $\forall \pi \in \Pi$ such as

$$t < \tau_{\pi(-)} < \tau < \tau_{\pi(+)}$$

and $\tau_{(1)} \in \tau_{\pi(-)}$ with time $s \geq \tau$. We can break the time interval $(t, s]$ according to the time of successive defaults for the permutation π. According to proposition 4.1.1 each measurable function h^i on the sub intervals $(\tau_{\pi(i)}, \tau_{\pi(i+1)}]$ can be further approximated via a partition.

$\forall i \in \pi$ define a partition $n^i = \{t_n^{i-1} = \tau_{\pi(i)} = t_0^i, \ldots, t_n^i = \tau_{\pi(i+1)}\}$ and independent of $\tau_{\pi(i)}$ and thus

$$h_u^i = \sum_{j=0}^{n-1} h_{t_i}^i 1_{t_j^i < u \leq t_{j+1}^i} \tag{5.9}$$

Then

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[h_\tau 1_{\{s \geq \tau > t\}} \mid \mathcal{F}_t] &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{\pi \in \Pi} \sum_{i=0}^{n-1} h_{\tau(i)} 1_{\tau(i) < u \leq \tau(i+1)} \mid \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\sum_{\pi \in \Pi} \sum_{i=0}^{n-1} h_u^i 1_{\tau(i) < u \leq \tau(i+1)} \mid \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\sum_{\pi \in \Pi} \sum_{i=0}^{n-1} \sum_{j=n_0^i}^{n_{n-1}^i} h_u^j 1_{t_j < u \leq t_{j+1}} \mid \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\sum_{\pi \in \Pi} \sum_{i=0}^{n-1} \sum_{j=n_0^i}^{n_{n-1}^i} \mathbb{E}^{\mathbb{Q}}[h_u^j 1_{t_j < u \leq t_{j+1}} \mid \mathcal{F}_{t_j}] \mid \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\sum_{\pi \in \Pi} \sum_{i=0}^{n-1} \mathbb{E}^{\mathbb{Q}}\left[\int_{\tau_i}^{\tau_{j+1}} h_u f_u du \mid \mathcal{F}_{\tau_i}\right] \mid \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\sum_{\pi \in \Pi} \mathbb{E}^{\mathbb{Q}}\left[\int_t^s h_u f_u du \mid \mathcal{F}_t\right] \mid \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\sum_{\pi \in \Pi} \int_t^s h_u f_u du \mid \mathcal{F}_t\right] = \sum_{\pi \in \Pi} \mathbb{E}^{\mathbb{Q}}\left[\int_t^s h_u f_u du \mid \mathcal{F}_t\right]
\end{aligned}$$

This requires that the impact of prior defaults to τ impact $f_u, u \in \mathbb{R}_+$ instantaneously and that the density function of any trigger obligor (and by extension any other obligor in E) is defined on the interval $(t, s], \forall \pi \in \Pi$. We use as well the linearity of expectation under the separability of the permutation of the obligors. $\square$

This formula will be of interest in the case of the valuation of non-defaultable claims under counterpart credit risk with the trigger event being either a counterpart C default or the investor I default and their corresponding marginal density function.

We define τ as the default time related to the trigger event in our multivariate setting. [Protter90] in theorem 11, page 159, demonstrate using the Dominated Convergence theorem that

Theorem 5.1.2. *Let X be a square integrable martingale and H be a predictable bounded process. Then $H \cdot X$ is a square integrable Martingale.*

In the multivariate setting, we state that,

Proposition 5.1.3 (Martingale Representation theorem)**.** *The martingale*

$$M_t^h = \mathbb{E}^{\mathbb{Q}}[h_\tau \mid \mathcal{G}_t] = \mathbb{E}^{\mathbb{Q}}[\sum_{\pi \in \Pi} 1_{\{\tau_{\pi(-)} < \tau < \tau_{\pi(+)}\}} h_\tau \mid \mathcal{G}_t]$$

where h is an $\mathcal{F}$-predictable process such that $\mathbb{E}^{\mathbb{Q}}[\mid h_\tau \mid] < \infty$ admits the following decomposition

$$M_t^h := m_0^h + \int_0^t (1 - H_u) e^{\Gamma_u} dm_u^h + \int_{(0,t]} (h_u - M_{u-}^h) dM_u \tag{5.10}$$

where

$$m_t^h = \mathbb{E}^{\mathbb{Q}}[\int_0^\infty h_u f_u du \mid \mathcal{F}_t]$$

Proof. Again, given the trigger event and its associated default event $\tau = \min\{\tau_C, \tau_I, \tau_{Ref}\}$ $\forall \pi \in \Pi, \exists \tau \in \pi, h_\tau$ is a stopped $\mathcal{F}$-predictable process under the proposition 4.1.1

$$M_t^h = \sum_{\pi \in \Pi} 1_{\{\tau_{\pi(-)} < \tau < \tau_{\pi(+)}\}} h_\tau 1_{\{\tau \le t\}} + \mathbb{E}^{\mathbb{Q}}[h_\tau \sum_{\pi \in \Pi} 1_{\{\tau_{\pi(-)} < \tau < \tau_{\pi(+)}\}} 1_{\{\tau > t\}} \mid \mathcal{G}_t] \tag{5.11}$$

with h_τ G_τ-measurable and $1_{\{\tau \le t\}} h_\tau$ G_τ-mesurable. The second term can be rewritten as

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[h_\tau \sum_{\pi \in \Pi} 1_{\{\tau_{\pi(-)} < \tau < \tau_{\pi(+)}\}} 1_{\{\tau > t\}} \mid \mathcal{G}_t] &= 1_{\{\tau > t\}} \frac{\mathbb{E}^{\mathbb{Q}}[\sum_{\pi \in \Pi} 1_{\{\tau_{\pi(-)} < \tau < \tau_{\pi(+)}\}} 1_{\{\tau > t\}} h_\tau \mid \mathcal{F}_t]}{\mathbb{Q}(\sum_{\pi \in \Pi} 1_{\{\tau_{\pi(-)} < \tau < \tau_{\pi(+)}\}} 1_{\{\tau > t\}} \mid \mathcal{F}_t)} \\
&= 1_{\{\tau > t\}} \frac{\mathbb{E}^{\mathbb{Q}}[\sum_{\pi \in \Pi} 1_{\{\tau_{\pi(-)} < \tau < \tau_{\pi(+)}\}} 1_{\{\tau > t\}} h_\tau \mid \mathcal{F}_t]}{G_t^\tau}
\end{aligned}$$

The previous equation can be expressed using the lemma 5.1.3 as

$$1_{\{\tau>t\}}e^{\Gamma_t^\tau}\mathbb{E}^{\mathbb{Q}}[\sum_{\pi\in\Pi} 1_{\{\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}}1_{\{\tau>t\}}h_\tau \mid \mathcal{F}_t] = 1_{\{\tau>t\}}e^{\Gamma_t^\tau}\mathbb{E}^{\mathbb{Q}}[\int_t^\infty h_u f_u^\tau du \mid \mathcal{F}_t]$$

with $h_u \sum 1_{\{u>t\}}$ being an $\mathcal{F}$-predictable process.

Set

$$\begin{aligned}
J_t := e^{\Gamma_t^\tau}\mathbb{E}^{\mathbb{Q}}[\int_t^\infty h_u f_u^\tau du \mid \mathcal{F}_t] &= e^{\Gamma_t^\tau}\mathbb{E}^{\mathbb{Q}}[\sum_{\pi\in\Pi} 1_{\{\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}}\int_t^\infty h_u f_u^{\tau,pi} du \mid \mathcal{F}_t] \\
&= e^{\Gamma_t^\tau}\left(M_t^h - \int_0^t h_u f_u^\tau\right)
\end{aligned}$$

since $F_t^\tau = \int_0^t f_u^\tau du = \sum_{\pi\in\Pi} 1_{\{\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}}\int_0^t f_u^{\tau,\pi} du$.

Thus,

$$\begin{aligned}
dJ_t &= e^{\Gamma_t^\tau} dm_t^h + J_t\gamma_t^\tau dt - e^{\Gamma_t^\tau} h_t f_t^\tau dt \\
&= e^{\Gamma_t^\tau} dm_t^h + J_t\gamma_t^\tau dt - \gamma_t^\tau h_t dt \\
&= e^{\Gamma_t^\tau} dm_t^h + (J_t - h_t)\gamma_t^\tau dt
\end{aligned}$$

Moreover, from equation (5.11)

$$\begin{aligned}
M_t^h &= h_t dH_t^\tau + d((1-H_t^\tau)J_t) \\
&= h_t dH_t^\tau + (1-H_t^\tau)dJ_t - J_t dH_t^\tau \\
&= h_t dH_t^\tau + (1-H_t^\tau)e^{\Gamma_t^\tau} dm_t^h + (1-H_t^\tau)(J_t-h_t)\gamma_t^\tau dt - J_t dH_t^\tau
\end{aligned}$$

On the event $\{\tau > t\}$ we have the equality $J_t = M_t^h$, thus

$$\begin{aligned}
M_t^h &= h_t dH_t^\tau + (1-H_t^\tau)e^{\Gamma_t^\tau} dm_t^h + (1-H_t^\tau)(M_t^h-h_t)\gamma_t^\tau dt - J_t dH_t^\tau \\
&= (1-H_t^\tau)e^{\Gamma_t^\tau} dm_t^h + (h_t - M_t^h)dM_t^\tau
\end{aligned}$$

with $M_t^\tau = H_t^\tau - \int_0^t (1-H_u^\tau)\gamma_u^\tau du$

$\square$

Those previous proposition will enable the valuation of non defaultable claim under counterparty credit risk under the multivariate framework with the $(\mathcal{H})$-hypothesis enabling the valuation of the risk-free part under the filtration $\mathbb{F}$. (i.e not requiring the input of credit desks.) Now, turning to the case of the defaultable claim, we need first to consider the case of survival claims.

Survival Claims We consider the payoff $X1_{\tau>T}$ at maturity T where X is a $\mathcal{F}_T$-measurable $\mathbb{Q}$-integrable random variable. Under the multivariate setting,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\sum_{\pi\in\Pi} 1_{\{\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}}e^{-\int_t^T r_s ds}X1_{\tau>T} \mid \mathcal{G}_t\right] &= 1_{\tau>t}\mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_s+\gamma^\tau ds}X \mid \mathcal{F}_t] \\
&= 1_{\tau>t}(G_t^\tau)^{-1}\mathbb{E}^{\mathbb{Q}}[\beta_T G_T^\tau X \mid \mathcal{F}_t] \qquad (5.12)
\end{aligned}$$

General Defaultable Claims

Proposition 5.1.4. *Consider the $(\mathcal{H})$-hypothesis holds and define the claim $\Pi(t,T) = (X, A, Z, \tau)$ as before with τ defining the default event of the reference in the trigger set $\mathcal{T}$, X a $\mathcal{F}_T$-measurable random variable paid at T, A a $\mathbb{F}$-predictable dividend process and Z a $\mathbb{F}$-predictable recovery process paid at default time.*

The dividend process decomposed as

$$\begin{aligned} D_t &= \sum_{\pi\in\Pi} 1_{\{\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} X 1_{\{\tau_{Ref}>T\}} \\ &+ \sum_{\pi\in\Pi} 1_{\{\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} \int_0^{t\wedge T} (1 - H_u^{Ref}) dA_u \\ &+ \sum_{\pi\in\Pi} 1_{\{\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} \int_0^{t\wedge T} 1_{\{\tau_{Ref}<u\}} Z_u du \end{aligned}$$

with

$$S_t^{n+1} = \beta_t^{-1} \mathbb{E}^{\mathbb{Q}}[\int_{]t,T]} \beta_u dD_u \mid \mathcal{G}_t]$$

can be expressed as

$$S_t^{n+1} = 1_{\{\tau_{Ref}>t\}} \beta_t^{-1} (G_t^{Ref})^{-1} \mathbb{E}^{\mathbb{Q}}[\beta_T G_T^{Ref} X + \beta_T Y_T G_T^{Ref} + \int_t^T R_u f_u^{Ref} du \mid \mathcal{F}_t] \tag{5.13}$$

Proof.

$$S_t^{n+1} = \beta_t^{-1} \mathbb{E}^{\mathbb{Q}}[\sum_{\pi\in\Pi} 1_{\{\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} (\beta_T X 1_{\{\tau_{Ref}>T\}} + Y_T 1_{\{\tau_{Ref}>T\}} + R_{\tau_{Ref}} 1_{\{t<\tau_{Ref}\le T\}}) \mid \mathcal{G}_t]$$

where

$$Y_u = \int_t^u \beta_s dA_s$$

and

$$R_u = \beta_u Z_u + Y_u$$

The first two terms can be valued with the survival claim equation (5.12), the last term is given by

$$\mathbb{E}^{\mathbb{Q}}[\sum_{\pi\in\Pi} 1_{\{\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} R_{\tau_{Ref}} 1_{\{t<\tau_{Ref}\le T\}} \mid \mathcal{G}_t] = 1_{\{\tau_{Ref}>t\}} (G_t^{Ref})^{-1} \mathbb{E}^{\mathbb{Q}}[\int_t^T R_u f_u^{Ref} du \mid \mathcal{F}_t]$$

Thus, the formula of the defaultable claim can be rewritten in the multivariate context as

$$S_t^{n+1} = 1_{\{\tau_{Ref}>t\}} \beta_t^{-1} (G_t^{Ref})^{-1} \mathbb{E}^{\mathbb{Q}}[\beta_T G_T^{Ref} X + \beta_T Y_T G_T^{Ref} + \int_t^T R_u f_u^{Ref} du \mid \mathcal{F}_t]$$

Note that at valuation time t, $\beta_t^{-1} = 1$ $\square$

Single-name credit default swaps in multivariate context: Now, we can express directly the single-name credit default swap in the multivariate framework with $X = 0, dAt = \kappa dt$ and $Z = LGD$. the value of the payer CDS is thus

$$S_t(\kappa) = 1_{\{\tau_{Ref}>t\}}(G_t^{Ref})^{-1}\mathbb{E}^{\mathbb{Q}}[-\int_t^T \beta_u G_u^{Ref}\kappa du + \int_t^T \beta_y Z_u \gamma_u^{Ref} G_u^{Ref} du \mid \mathcal{F}_t] \tag{5.14}$$

The market spread is the spread quoted such that $S_t(\kappa(t,T)) = 0$, thus by definition the pre-default market spread CDS at time t is given by

$$\kappa(t,T) = \frac{\mathbb{E}^{\mathbb{Q}}[\int_t^T \beta_y Z_u \gamma_u^{Ref} G_u^{Ref} du \mid \mathcal{F}_t]}{\mathbb{E}^{\mathbb{Q}}[\int_t^T \beta_u G_u^{Ref} du \mid \mathcal{F}_t]}$$

After inception the simulated value of the price $S_t(\kappa)$ of market CDS of maturity T initiated at 0 at rate κ is given by

$$S_t(\kappa) = 1_{\{\tau_{Ref}>t\}}(\kappa(t,T) - \kappa)\mathbb{E}^{\mathbb{Q}}[\int_t^T \beta_s G_s^{Ref} ds \mid \mathcal{F}_t]$$

This equation will enable by simulating the single-name CDS spread to compute the Market value of the position entered at inception and generate the metrics such as EPE and PFE. The alternative is to simulate the survival probability of the reference of the CDS an infer the “fair spread”of the claim.

The CDS price process dynamics of the claim can be expressed as

Proposition 5.1.5 (CDS price dynamics)**.** *The (ex-dividend) CDS price process is*

$$dS_t(\kappa) = -S_{t-}(\kappa)dM_t^{Ref} + (1 - H_t^{Ref})(r_t S_t + \kappa - \gamma_t^{Ref} LGD^{Ref}(t))dt + (1 - H_t^{Ref})(\beta_t G_t^{Ref})^{-1} dm_t^{Ref}$$

with m_t^{Ref} an $\mathbb{F}$-continuous martingale with

$$m_t^{Ref} = \mathbb{E}^{\mathbb{Q}}[\int_0^T \beta_s G_s^{Ref}(LGD^{Ref}(s)\gamma_s^{Ref} - \kappa)ds \mid \mathcal{F}_t]$$

Proof. We have

$$S_t(\kappa) = L_t^{Ref}\beta_t^{-1}\left(m_t^{Ref} - \int_0^t \beta_s G_s^{Ref}(LGD^{Ref}(s)\gamma_s^{Ref} - \kappa)ds\right)$$

using the product rule

$$\begin{aligned}
dS_t(\kappa) &= L_t^{Ref}\beta_t^{-1}\left(dm_t^{Ref} - \beta_t G_t^{Ref}(LGD^{Ref}(t)\gamma_t^{Ref} - \kappa)dt\right) \\
&+ r_t\beta_t L_t^{Ref}\left(m_t^{Ref} - \int_0^t \beta_s G_s^{Ref}(LGD^{Ref}(s)\gamma_s^{Ref} - \kappa)ds\right)dt \\
&- \beta_t L_t^{Ref}\left(m_t^{Ref} - \int_0^t \beta_s G_s^{Ref}(LGD^{Ref}(s)\gamma_s^{Ref} - \kappa)ds\right)dM_t^{Ref}
\end{aligned}$$

$\square$

In the case of a CDS we can also value under the case $\mathbb{F} = \emptyset$ thus, $\mathbb{G} = \bigvee_{i=1}^{m} \mathbb{H}^i$. This is equivalent to the Hazard function approach in the litterature and where most of the implied default are extracted for CDS.

k^{th}-to-default swaps in multivariate context: Similarly, we can value the k^{th}-to-default swap where the protection seller is exposed to the k^{th} entity to default within a portfolio of credit sensitive instruments. Note at this point that in the multivariate framework, the i^{th} default in E is not necessiraly the i^{th} default in our k^{th}-to-default swap. For simplicity we assume that the i^{th} default in E corresponds to the k^{th} default triggering the event set $\mathcal{T}$. In this case the defaults prior to i will have an impact on i^{th} propensity to default. The contract is unwound immediately after the i^{th} credit event (Recall that this is not the case of indices or CDO.) Then if $1_{\{\tau_{(i)}=\tau_{Ref}\leq T\}}$ the payments is made on the name $j \in E$ with $\tau_{(i)} = \tau_j$ under the permutation π that is realised. Thus, in the multivariate framework with $X = 0, dAt = \kappa^{(i)}dt$ and $Z = LGD$. the value of the payer CDS is thus

$$S_t(\kappa^{(i)}) = 1_{\{\tau_{Ref}>t\}}(G_t^{Ref})^{-1}\mathbb{E}^{\mathbb{Q}}[-\int_t^T \beta_u G_u^{Ref}\kappa^{(i)}du + \int_t^T \beta_y LGD^{Ref}(u)\gamma_u^{Ref}G_u^{Ref}du \mid \mathcal{F}_t] \quad (5.15)$$

with the notation

$$\begin{aligned}
(G_t^{Ref}) = (G_t^{(i)}) &= \mathbb{Q}(\sum_{\pi\in\Pi}\{\tau_{\pi(-)} < \tau < \tau_{\pi(+)}\} \cap \{\tau = \tau_{(i)} > t\}) \\
(Z_t^{Ref}) &= (Z_t^{(i)})
\end{aligned}$$

Now, having expressed conditional events in the equations (4.11) and (4.14) and the claims under no-counterparty exposure in a multivariate context, it is necessary to focus on the multivariate aspect of the Hazard processes of obligors in E and try to express the inherent dependence of credit default that is contained in the permutations.

Finally, those previous formulas can be used to value the CVA part of the claims when time t is inside the permutation i.e. all the events of the type

$$\sum_{\pi\in\Pi} 1_{\{\tau_{\pi_1(-)}<t<\tau_{\pi_2(-)}<\tau<\tau_{\pi(+)}\}}1_{\{\tau\leq T\}}$$

where some of the defaults have happened but the default time of the trigger event is yet to happen. This will be useful for the valuation of the CVA through Monte-Carlo simulations. This knowledge is required for Expected Positive Exposure (EPE) as this metric is built on the distribution of the claim at each point in the future.

To complete the valuation of equations (4.11) and (4.14) we need now to review the models of credit risk dependence to express the probability of the permutations Π.

5.1.3 Counterparty Credit Risk under multivariate case

The next two proposition of valuing the two categories of claims, non defaultable and defaultable, will be used in the reminder of this document as the equations that link the multivariate framework hazard processes with credit risk dependence and assess its impact on the Credit Valuation Adjustment in case the obligors in E exhibit credit risk contagion.

Non defaultable claim

We rewrite the proposition 4.1.2, page 53, of non defaultable claim in a multivariate setting using the hazard processes defined in equation 5.2 and under the $(\mathcal{H})$-hypothesis and the separability hypothesis:

Proposition 5.1.6 (Bilateral CVA for non-defaultble claim in multivariate setting)**.** *Consider an $\mathcal{F}_T$-measurable non-defaultable claim X defined in (3.1) under bilateral counterpart credit risk as*

$$
\begin{aligned}
\mathbb{E}_t(\beta_t X(t,T)^d) &= \mathbb{E}\left(1_{\{\tau>t\}}\int_t^T \beta_u dX_u \middle| \mathcal{F}_t\right) \\
&- 1_{\{\tau_C>t\}}\frac{\mathbb{E}(\sum_{\pi\in\Pi^C} 1_{\{\tau_{\pi(-)}<\tau_C<\tau_{\pi(+)}\}}1_{\{\tau_C\leq T\}}\beta_{\tau_C} LGD_C X(\tau_C)^+ \mid \mathcal{F}_t)}{\mathbb{E}(\sum_{\pi\in\Pi^C} 1_{\{\tau_{\pi_1(-)}<t<\tau_{\pi_2(-)}<\tau_C<\tau_{\pi(+)}\}} \mid \mathcal{F}_t)} \\
&+ 1_{\{\tau_I>t\}}\frac{\mathbb{E}(\sum_{\pi\in\Pi^I} 1_{\{\tau_{\pi(-)}<\tau_I<\tau_{\pi(+)}\}}1_{\{\tau_I\leq T\}}\beta_{\tau_I} LGD_I X(\tau_I)^- \mid \mathcal{F}_t)}{\mathbb{E}(\sum_{\pi\in\Pi^I} 1_{\{\tau_{\pi_1(-)}<t<\tau_{\pi_2(-)}<\tau_I<\tau_{\pi(+)}\}} \mid \mathcal{F}_t)} \\
&= X(t) - BCVA_t
\end{aligned} \tag{5.16}
$$

with

$$
\begin{aligned}
BCVA_t &:= CVA_t - DVA_t \\
&:= 1_{\{\tau_C>t\}}\frac{\mathbb{E}(\sum_{\pi\in\Pi^C} 1_{\{\tau_{\pi(-)}<\tau_C<\tau_{\pi(+)}\}}1_{\{\tau_C\leq T\}}\beta_{\tau_C} LGD_C X(\tau_C)^+ \mid \mathcal{F}_t)}{\mathbb{E}(\sum_{\pi\in\Pi^C} 1_{\{\tau_{\pi_1(-)}<t<\tau_{\pi_2(-)}<\tau_C<\tau_{\pi(+)}\}} \mid \mathcal{F}_t)} \\
&- 1_{\{\tau_I>t\}}\frac{\mathbb{E}(\sum_{\pi\in\Pi^I} 1_{\{\tau_{\pi(-)}<\tau_I<\tau_{\pi(+)}\}}1_{\{\tau_I\leq T\}}\beta_{\tau_I} LGD_I X(\tau_I)^- \mid \mathcal{F}_t)}{\mathbb{E}(\sum_{\pi\in\Pi^I} 1_{\{\tau_{\pi_1(-)}<t<\tau_{\pi_2(-)}<\tau_I<\tau_{\pi(+)}\}} \mid \mathcal{F}_t)} \\
&:= 1_{\{\tau>t\}}\frac{\mathbb{E}(\sum_{\pi\in\Pi^I\cup\Pi^C} 1_{\{\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}}1_{\{\tau\leq T\}}\beta_{\tau}[ED(\tau)] \mid \mathcal{F}_t)}{\mathbb{E}(\sum_{\pi\in\Pi^I\cup\Pi^C} 1_{\{\tau_{\pi_1(-)}<t<\tau_{\pi_2(-)}<\tau<\tau_{\pi(+)}\}} \mid \mathcal{F}_t)}
\end{aligned} \tag{5.17}
$$

Note that the event $1_{\{\tau>t\}}$ in simulation started at time t is not relevant.

Defaultable claim

We rewrite the proposition 4.1.3, page 56, of defaultable claim in a multivariate setting using the hazard processes defined in equation 5.2

Proposition 5.1.7 (Bilateral CVA for defaultble claim in multivariate setting)**.** *Consider an $\mathcal{F}_T$-measurable non-defaultable claim X defined in (3.1) under bilateral counterpart credit risk as*

$$
\begin{aligned}
\mathbb{E}_t(\beta_t S_t^{n+1,d}) &= 1_{\{\tau_{Ref}>t\}}(\mathbb{E}(\sum_{\pi\in\Pi^{Ref}} 1_{\{\tau_{\pi_1(-)}<t<\tau_{\pi_2(-)}<\tau_{Ref}<\tau_{\pi(+)}\}} \mid \mathcal{F}_t))^{-1} \\
& \quad \mathbb{E}\left(\int_t^T \sum_{\pi\in\Pi^{Ref}} 1_{\{\tau_{\pi(-)}<\tau_{Ref}<\tau_{\pi(+)}\}} 1_{\{\tau_{Ref}\le s\}}\beta_s dA_s\right. \\
&+ \sum_{\pi\in\Pi^{Ref}} 1_{\{\tau_{\pi(-)}<\tau_{Ref}<\tau_{\pi(+)}\}} 1_{\{\tau_{Ref}\le T\}}\beta_{\tau_R} Z_{\tau_R} \\
&+ \left.\sum_{\pi\in\Pi^{Ref}} 1_{\{\tau_{\pi(-)}<\tau_{Ref}<\tau_{\pi(+)}\}} 1_{\{\tau_{Ref}> T\}}\beta_T X_t^d \middle| \mathcal{F}_t\right) \\
&- 1_{\{\tau_C>t\}}\frac{\mathbb{E}(\sum_{\pi\in\Pi^C} 1_{\{\tau_{\pi(-)}<\tau_C<\tau_{\pi(+)}\}} 1_{\{\tau_C\le T\}}\beta_{\tau_C} LGD_C X(\tau_C)^+ \mid \mathcal{F}_t)}{\mathbb{E}(\sum_{\pi\in\Pi^C} 1_{\{\tau_{\pi_1(-)}<t<\tau_{\pi_2(-)}<\tau_C<\tau_{\pi(+)}\}} \mid \mathcal{F}_t)} \\
&+ 1_{\{\tau_I>t\}}\frac{\mathbb{E}(\sum_{\pi\in\Pi^I} 1_{\{\tau_{\pi(-)}<\tau_I<\tau_{\pi(+)}\}} 1_{\{\tau_I\le T\}}\beta_{\tau_I} LGD_I X(\tau_I)^- \mid \mathcal{F}_t)}{\mathbb{E}(\sum_{\pi\in\Pi^I} 1_{\{\tau_{\pi_1(-)}<t<\tau_{\pi_2(-)}<\tau_I<\tau_{\pi(+)}\}} \mid \mathcal{F}_t)} \\
&= S_t^{n+1} - BCVA_t
\end{aligned} \tag{5.18}
$$

with

$$
\begin{aligned}
BCVA_t &:= CVA_t - DVA_t \\
&:= 1_{\{\tau_C>t\}}\frac{\mathbb{E}(\sum_{\pi\in\Pi^C} 1_{\{\tau_{\pi(-)}<\tau_C<\tau_{\pi(+)}\}} 1_{\{\tau_C\le T\}}\beta_{\tau_C} LGD_C X(\tau_C)^+ \mid \mathcal{F}_t)}{\mathbb{E}(\sum_{\pi\in\Pi^C} 1_{\{\tau_{\pi_1(-)}<t<\tau_{\pi_2(-)}<\tau_C<\tau_{\pi(+)}\}} \mid \mathcal{F}_t)} \\
&- 1_{\{\tau_I>t\}}\frac{\mathbb{E}(\sum_{\pi\in\Pi^I} 1_{\{\tau_{\pi(-)}<\tau_I<\tau_{\pi(+)}\}} 1_{\{\tau_I\le T\}}\beta_{\tau_I} LGD_I X(\tau_I)^- \mid \mathcal{F}_t)}{\mathbb{E}(\sum_{\pi\in\Pi^I} 1_{\{\tau_{\pi_1(-)}<t<\tau_{\pi_2(-)}<\tau_I<\tau_{\pi(+)}\}} \mid \mathcal{F}_t)} \\
&:= 1_{\{\tau>t\}}\frac{\mathbb{E}(\sum_{\pi\in\Pi^I\cup\Pi^C} 1_{\{\tau_{\pi(-)}<\tau<\tau_{\pi(+)}\}} 1_{\{\tau\le T\}}\beta_{\tau} [ED(\tau)] \mid \mathcal{G}_t)}{\mathbb{E}(\sum_{\pi\in\Pi^I\cup\Pi^C} 1_{\{\tau_{\pi_1(-)}<t<\tau_{\pi_2(-)}<\tau<\tau_{\pi(+)}\}} \mid \mathcal{F}_t)}
\end{aligned} \tag{5.19}
$$

The propositions 5.1.6 and 5.1.7 above require the expression of the dependence modelling of credit risk through $G_\cdot^{\tau_C}, G_\cdot^{\tau_I}$ and $G_\cdot^{\tau_{Ref}}$ among the ordering of defaults with the identification of the default probability of obligors in the trigger set $\mathcal{T}$. Those propositions highlight a fact mentioned in [Brig12], that the CVA and the BVA parts cannot be valued separately and combined to form the BVCA formula due to the *first-to-default* time that triggers the valuation in BCVA. Thus, we believe that the equations 5.17 and 5.19 contain naturally the ordering of the defaults in the set $\mathcal{T}$. (Note that in a full contagion setting even defaults in τ_π will influence the *first-to-default* time)

Note: Similarly the reference to S_t^{n+1} accounts to a difference with the probability $G_\cdot^{\tau_{Ref}}$ postpones the default time of C and I after the default time of Ref. This is different from a classic valuation of non-defaultable claim which only considers the set $Reg = f$.

Typically, the models of $G_\cdot^{\tau_C}, G_\cdot^{\tau_I}$ and $G_\cdot^{\tau_{Ref}}$ tend to be not of closed-form formulas type except in the case of the Gaussian copula and tend to rely on numerical analysis. We will first review models of dependency and how this dependence is realised in a dynamic or a static way. Then we will introduce dependency models that use Markov chains with a presentation of a Markov chain model called the multivariate phase-type distribution.

5.2 Modelling credit risk dependence

As figures 5.1 , 5.2 and 5.3, page 80-82 illustrate, the US Senate Committee on Finance [Senate10] analysed the protection selling and buying of Goldman Sachs w.r.t AIG. This content highlights a net protection buying of \$+1,711m with 148 different counterparties and, thus, is a natural and direct example of the level of interdependence and preferential channels for contagion in the case of insolvability of a protection seller after a default of the reference entity AIG.

Interestingly, [Vitali11], a recent study using network topology, assessed the structure of the control network of transnational corporations (TNCs) to unveil its effects on global market competition and financial stability. [Vitali11] found that similar to the world wide web, the TNCs network of control has a bow tie structure with its core very small compared to the other sections of the bow-tie and holding collectively a large fraction of the total network control. Nearly 4/10 of the control over the economic value of TNCs in the world is held via a complicated web of ownership relations, by a group of 147 TNCs in the core which has almost full control over itself. Implication for the financial stability is that 3/4 of this core are financial intermediaries.

The dependency structure is also presenting the interesting feature of always evolving as the 2011 article [Xydias11] about Synthetic Exchange Traded Funds (ETF) shows: *"ETFs that use swaps to clone stock, bond or currency returns have been criticised by regulators and firms including Fidelity Investors, which say clients risk losing money should the banks writing the derivatives become insolvent."* Due to the European sovereign bond crisis and its impact of the credit worthiness of European based banks, investors pulled the most money in at least two years from European Exchange-Traded Funds that use derivatives to track asset performance, moving record amounts into ETFs backed by physical bonds and shares instead. The rise of those "synthetic"exposure claims represent an ever changing landscape of credit contagion and justify the rationale of modelling dependence structures.

Thus, Counterparty credit risk through Credit Valuation Adjustment imply the necessity of modelling credit risk dependence among obligors entity such as the investor I, the counterpart C and the reference obligors Ref.

5.2.1 Dependence modelling

Structured-form models

Dependence modelling with structural models is either expressed in the correlation of the barriers or the correlation of the geometric Brownian motion of the underlying firm process $V_i, i = 1, 2$. [Haworth06] proposes an analytical model for corporate bond yields in presence of default contagion and two-firm credit default swap baskets. Extending to a basket of three firms [Haworth06] establishes the joint probability distributions as a function of the running minimum i.e. the first to default name as a Bessel process. Additionally, [Lipton09] proposes a structural numerical method for computing the Credit Valuation Adjustment by introducing jumps in the evolution of the asset value V_t. However most of the structural models are limited in their ease of calibration from market data and in their handling of an increase of obligors under consideration in the contagion obligor sets.

CONFIDENTIAL TREATMENT REQUESTED BY GOLDMAN SACHS
AIG External CDS Notional by Counterparty as-of 9/15/08

Counterparty	Buy	Sell	Net
CITIBANK, N.A.	702,800,000	(300,554,000)	402,246,000
CREDIT SUISSE INTERNATIONAL	632,080,000	(322,350,000)	309,730,000
MORGAN STANLEY CAPITAL SERVICES INC.	808,000,000	(565,500,000)	242,500,000
JPMORGAN CHASE BANK N.A.-LONDON BRANCH	413,940,000	(197,900,000)	216,040,000
LEHMAN BROTHERS SPECIAL FINANCING, INC	713,261,082	(538,481,000)	174,780,082
SWISS RE FINANCIAL PRODUCTS CORPORATION	168,000,000	(35,900,000)	132,100,000
PIMCO FUNDS-TOTAL RETURN FUND	135,000,000	(15,000,000)	120,000,000
DEUTSCHE BANK AG-LONDON BRANCH	1,656,846,700	(1,569,600,000)	87,246,700
KBC FINANCIAL PRODUCTS (CAYMAN ISLANDS) LTD.	143,050,000	(58,400,000)	84,650,000
ROYAL BANK OF CANADA-LONDON BRANCH	108,000,000	(32,000,000)	76,000,000
PIMCO FUNDS-LOW DURATION FUND	70,200,000		70,200,000
SOCIETE GENERALE	117,050,000	(54,770,000)	62,280,000
WACHOVIA BANK, NATIONAL ASSOCIATION	122,214,000	(62,000,000)	60,214,000
NATIXIS FINANCIAL PRODUCTS INC.	76,345,000	(20,000,000)	56,345,000
MERRILL LYNCH INTERNATIONAL	758,400,000	(716,965,000)	41,435,000
NATIXIS	65,580,000	(28,515,600)	37,064,400
BANK OF NOVA SCOTIA (THE)	43,900,000	(7,735,000)	36,165,000
CREDIT AGRICOLE CORPORATE AND INVESTMENT BANK	82,300,000	(47,500,000)	34,800,000
BNP PARIBAS	31,500,000		31,500,000
DRESDNER BANK AG-LONDON BRANCH	84,100,000	(54,990,000)	29,110,000
ALPHADYNE INTERNATIONAL MASTER FUND, LTD.	48,757,000	(20,986,000)	27,771,000
BANK OF AMERICA, NATIONAL ASSOCIATION	269,200,000	(244,130,000)	25,070,000
MBIA INC.	25,000,000		25,000,000
BANK OF MONTREAL-LONDON BRANCH	25,000,000		25,000,000
COMMERZBANK AKTIENGESELLSCHAFT	25,000,000		25,000,000
LYXOR STARWAY SPC-LYXOR STARWAY ALPHADYNE SEGREGATED PFLO	47,743,000	(25,014,000)	22,729,000
UNICREDIT BANK AG	20,000,000		20,000,000
GOVERNMENT OF SINGAPORE INVESTMENT CORPORATION PTE LTD	20,000,000		20,000,000
BANCO FINANTIA SA	20,000,000		20,000,000
BANK OF MONTREAL-CHICAGO BRANCH	18,000,000		18,000,000
WICKER PARK CDO I, LTD.	17,500,000		17,500,000
BLUECORR FUND, LLC	34,000,000	(18,400,000)	15,600,000
SUTTONBROOK CAPITAL PORTFOLIO LP	15,000,000		15,000,000
CITIBANK, N.A.-LONDON BRANCH	12,500,000		12,500,000
BLUEMOUNTAIN TIMBERLINE LTD.	24,900,000	(12,900,000)	12,000,000
PIMCO GLOBAL CREDIT OPPORTUNITY MASTER FUND LDC (PIMCO 4810)	12,000,000		12,000,000
AQR ABSOLUTE RETURN MASTER ACCOUNT L.P.	11,750,000		11,750,000
MOORE MACRO FUND, L.P.	10,000,000		10,000,000
NORGES BANK	10,000,000		10,000,000
JPMORGAN CHASE BANK, NATIONAL ASSOCIATION	547,997,000	(538,751,000)	9,246,000
FORTIS BANK	8,000,000		8,000,000
PIMCO COMBINED ALPHA STRATEGIES MASTER FUND LDC (PIMCO 4863)	8,000,000		8,000,000
WESTLB AG-LONDON BRANCH	8,000,000		8,000,000
AQR GLOBAL ASSET ALLOCATION MASTER ACCOUNT, L.P.	7,750,000		7,750,000
CITADEL EQUITY FUND LTD.	7,400,000		7,400,000
ALLIANZ GLOBAL INVESTORS KAG-ALLIANZ PIMCOMOBIL-FONDS-520004	7,000,000		7,000,000
BARCLAYS BANK PLC	790,000,000	(783,910,000)	6,090,000
PIMCO COMBINED ALPHA STRATEGIES MASTER FUND LDC (PIMCO 4866)	6,000,000		6,000,000
ARROWGRASS MASTER FUND LTD	15,500,000	(10,000,000)	5,500,000
MIZUHO INTERNATIONAL PLC	5,400,000		5,400,000
RABOBANK INTERNATIONAL-LONDON BRANCH	5,000,000		5,000,000
STANDARD CHARTERED BANK-SINGAPORE BRANCH	5,000,000		5,000,000
MILLENNIUM PARK CDO I, LTD.	5,000,000		5,000,000
III RELATIVE VALUE CREDIT STRATEGIES HUB FUND LTD.	5,000,000		5,000,000

Figure 5.1: Exposure of Goldman Sachs through CDS Counterparties with the company AIG as a reference name - Page (1/3) - Source: [Senate10]

INTERNATIONALE KAG MBH-INKA B	4,500,000		4,500,000
GOLDENTREE MASTER FUND, LTD.	4,480,000		4,480,000
NATIONAL BANK OF CANADA	5,000,000	(2,000,000)	3,000,000
LOOMIS SAYLES MULTI-STRATEGY MASTER ALPHA, LTD.	3,250,000	(250,000)	3,000,000
PIMCO VARIABLE INSURANCE TRUST-LOW DURATION BOND PORTFOLIO	2,700,000		2,700,000
TIDEN DESTINY MASTER FUND LIMITED	2,500,000		2,500,000
STICHTING PENSIOENFONDS OCE	2,450,000		2,450,000
INTESA SANPAOLO SPA	2,000,000		2,000,000
PIMCO GLOBAL CREDIT OPPORTUNITY MASTER FUND LDC (PIMCO 4807)	2,000,000		2,000,000
DCI UMBRELLA FUND PLC-DIVERSIFED CRED INVESTMENTS FD THREE	2,000,000		2,000,000
HALBIS DISTRESSED OPPORTUNITIES MASTER FUND LTD.	2,000,000		2,000,000
UBS FUNDS (THE)-UBS DYNAMIC ALPHA FUND	1,250,000		1,250,000
GOLDENTREE MASTER FUND II, LTD.	1,180,000		1,180,000
RP RENDITE PLUS-MULTI STRATEGIE INVESTMENT GRADE (MSIG)	1,100,000		1,100,000
CAIRN CAPITAL STRUCTURED CREDIT MASTER FUND LIMITED	1,000,000		1,000,000
ALLIANZ GLOBAL INV KAG MBH-DBI PIMCO GLBL CORP BD FDS-551416	1,000,000		1,000,000
PIMCO FUNDS: PACIFIC INVESTMENT MNGMT SER-FLOATING INCOME FD	800,000		800,000
UBS DYNAMIC ALPHA STRATEGIES MASTER FUND LTD.	750,000		750,000
ALLIANZ GLOBAL INVESTORS KAG MBH-DIT FDS VICTORIA DFS 558513	600,000		600,000
PIMCO FUNDS: GLOBAL INVESTORS SERIES PLC-LOW AVE DURATION FD	600,000		600,000
INTERNATIONALE KAPITALANLAGEGESELLSCHFT MBH-PKMF INKA-556490	550,000		550,000
BFT VOL 2	500,000		500,000
PIMCO FUNDS-LOW DURATION FUND II	500,000		500,000
GOLDENTREE CREDIT OPPORTUNITIES MASTER FUND, LTD.	340,000		340,000
EMBARQ SAVINGS PLAN MASTER TRUST	300,000		300,000
RUSSELL INVESTMENT COMPANY-RUSSELL SHORT DURATION BOND FUND	300,000		300,000
PIMCO FUNDS-LOW DURATION FUND III	300,000		300,000
EQUITY TRUSTEES LIMITED-PIMCO AUSTRALIAN BOND FUND	300,000		300,000
PUBLIC EDUCATION EMPLOYEE RETIREMENT SYSTEM OF MISSOURI	200,000		200,000
PIMCO BERMUDA TRUST II-PIMCO JGB FLOATER FOREIGN STRATEGY FD	200,000		200,000
D.B. ZWIRN SPECIAL OPPORTUNITIES FUND, LTD.	29,047,250	(28,945,750)	101,500
PIMCO BERMUDA TRUST II-PIMCO BERMUDA JGB FLOATER US STRA FD	100,000		100,000
FRANK RUSSELL INVESTMENT COMPANY-FIXED INCOME II FUND	100,000		100,000
D.B. ZWIRN SPECIAL OPPORTUNITIES FUND, LLC	7,777,750	(7,949,250)	(171,500)
SEI INSTITUTIONAL INVESTMENTS TRUST-ENHANCED LIBOR OPP FUND		(500,000)	(500,000)
WMP LIBOR PLUS TRADING LIMITED		(570,000)	(570,000)
WELLINGTON TRUST CO, MULT CTF TR-LIBOR PLUS HIGH QUALITY PTF		(655,000)	(655,000)
CITIGROUP GLOBAL MARKETS LIMITED		(700,000)	(700,000)
SEI INSTITUTIONAL MANAGED TRUST-ENHANCED INCOME FUND		(710,000)	(710,000)
WELLINGTON TRUST COMPANY, NA MULT CIF TR II-US EQ IDX PLS I		(800,000)	(800,000)
SEI DAILY INCOME TRUST-ULTRA SHORT BOND FUND		(1,360,000)	(1,360,000)
BAUPOST VALUE PARTNERS, L.P.-III		(1,385,000)	(1,385,000)
PENSION BENEFIT GUARANTY CORP (WELLNGTN 6334 PG01 GL BL PTF)		(1,700,000)	(1,700,000)
YB INSTITUTIONAL LIMITED PARTNERSHIP		(1,765,000)	(1,765,000)
OIL INVESTMENT CORPORATION LTD.		(1,905,000)	(1,905,000)
PIMCO CAYMAN TRUST-PIMCO CAYMAN GL AG EX-JPN BD (PIMCO 2763)		(2,000,000)	(2,000,000)
INTEL CORPORATION PROFIT SHARING RETIREMENT PLAN		(2,000,000)	(2,000,000)
GPC LXIV, LLC		(2,094,000)	(2,094,000)
IBM PERSONAL PENSION PLAN TRUST (PIMCO 2642)		(2,100,000)	(2,100,000)
BANCO SANTANDER, S.A.	63,500,000	(66,000,000)	(2,500,000)
PB INSTITUTIONAL LIMITED PARTNERSHIP		(2,520,000)	(2,520,000)
STRUCTURED INVT HOLDINGS IV SPC-TREESDALE CORP CREDIT A SEG		(2,760,000)	(2,760,000)
PIMCO FUNDS-LONG TERM US GOVERNMENT		(2,900,000)	(2,900,000)
ASHLAND INC. EMPLOYEE SAVINGS PLAN TRUST		(3,000,000)	(3,000,000)
PIMCO BERMUDA TR II-PIMCO BERMUDA GLOB AGGR EX-JAP BD FD (M)		(3,000,000)	(3,000,000)
BAUPOST VALUE PARTNERS, L.P.-I		(3,010,000)	(3,010,000)
STAPLE STREET AVIATION (MASTER), L.P.		(3,500,000)	(3,500,000)
STICHTING PENSIOENFONDS UWV		(3,600,000)	(3,600,000)
RAVEN CREDIT OPPORTUNITIES MASTER FUND, LTD.		(5,000,000)	(5,000,000)

Figure 5.2: Exposure of Goldman Sachs through CDS Counterparties with the company AIG as a reference name - Page (2/3) - Source: [Senate10]

PIMCO FUNDS-PRIVATE ACCOUNT PORTFOLIO SERIES:INV GRADE CORP		(5,000,000)	(5,000,000)
LEHMAN BROTHERS INTERNATIONAL (EUROPE)		(5,000,000)	(5,000,000)
GREYWOLF STRUCTURED PRODUCTS MASTER FUND, LTD.		(5,000,000)	(5,000,000)
DEPFA BANK PUBLIC LIMITED COMPANY		(5,000,000)	(5,000,000)
AUTONOMY MASTER FUND LIMITED		(5,000,000)	(5,000,000)
SPV UNO, LLC		(5,000,000)	(5,000,000)
NORDEA BANK FINLAND PLC	3,000,000	(8,000,000)	(5,000,000)
HB INSTITUTIONAL LIMITED PARTNERSHIP		(5,325,000)	(5,325,000)
CLAREN ROAD CREDIT OPPORTUNITIES MASTER FUND, LTD.		(6,000,000)	(6,000,000)
LISPENARD STREET CREDIT (MASTER), LTD.		(6,500,000)	(6,500,000)
CQS CAPITAL STRUCTURE ARBITRAGE MASTER FUND LIMITED		(7,000,000)	(7,000,000)
DBS BANK LTD.	5,000,000	(12,500,000)	(7,500,000)
STICHTING SHELL PENSIOENFONDS		(7,500,000)	(7,500,000)
ZUERCHER KANTONALBANK		(10,000,000)	(10,000,000)
TEMPO MASTER FUND L.P.		(10,000,000)	(10,000,000)
UNICREDIT BANK AUSTRIA AG		(10,000,000)	(10,000,000)
OCH-ZIFF CAPITAL STRUCTURE ARBITRAGE MASTER FUND, LTD		(10,000,000)	(10,000,000)
THE ROYAL BANK OF SCOTLAND PUBLIC LIMITED COMPANY	182,230,000	(193,800,000)	(11,570,000)
BLUECREST MULTI STRATEGY CREDIT MASTER FUND LIMITED	6,300,000	(20,200,000)	(13,900,000)
PRESIDENT AND FELLOWS OF HARVARD COLLEGE		(15,000,000)	(15,000,000)
BLUE MOUNTAIN CREDIT ALTERNATIVES MASTER FUND L.P.	566,100,000	(581,300,000)	(15,200,000)
DZ BANK AG DEUTSCHE ZENTRAL-GENOSSENSCHAFTSBANK		(20,000,000)	(20,000,000)
FRONTPOINT RELATIVE VALUE OPPORTUNITIES FUND, L.P.		(22,840,000)	(22,840,000)
NOMURA INTERNATIONAL PLC	5,000,000	(30,000,000)	(25,000,000)
WESTPAC BANKING CORPORATION		(40,000,000)	(40,000,000)
ROYAL BANK OF CANADA		(43,000,000)	(43,000,000)
BNP PARIBAS-LONDON BRANCH	212,600,000	(257,425,000)	(44,825,000)
SOCIETE GENERALE-NEW YORK BRANCH		(50,000,000)	(50,000,000)
BANCA IMI S.P.A.		(50,000,000)	(50,000,000)
BANK OF TOKYO-MITSUBISHI UFJ, LTD.-NEW YORK BRANCH		(50,000,000)	(50,000,000)
CLAREN ROAD CREDIT MASTER FUND, LTD.		(59,146,000)	(59,146,000)
THE ROYAL BANK OF SCOTLAND N.V.-LONDON BRANCH	55,000,000	(124,500,000)	(69,500,000)
OZ MASTER FUND LTD.	13,000,000	(115,000,000)	(102,000,000)
UBS AG-LONDON BRANCH	543,639,000	(657,300,000)	(113,661,000)
HSBC BANK USA, NATIONAL ASSOCIATION	183,500,000	(350,100,000)	(166,600,000)
Total	**10,950,507,782**	**(9,239,366,600)**	**1,711,141,182**

Figure 5.3: Exposure of Goldman Sachs through CDS Counterparties with the company AIG as a reference name - Page (3/3) - Source: [Senate10]

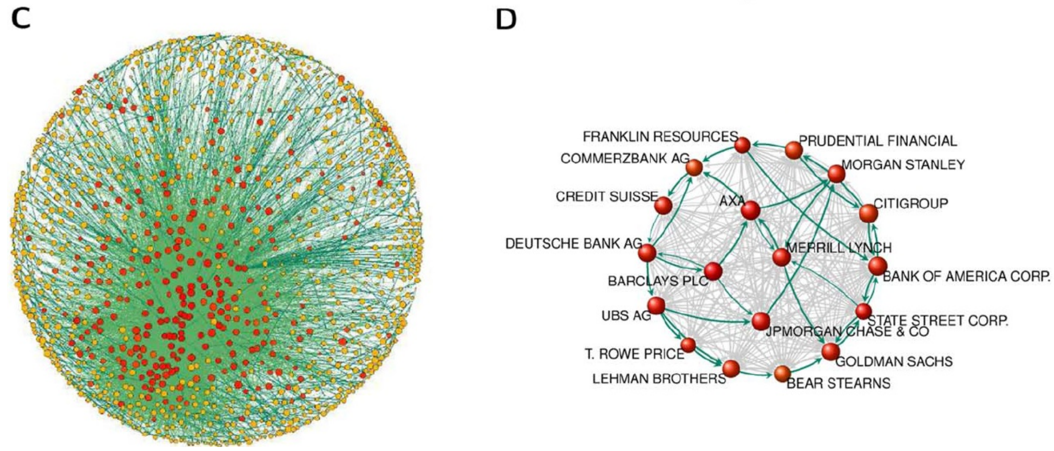


Figure 5.4: Corporate Network Dependence Illustration through cross capital exposure- Source: [Vitali11]

Reduced-form models

Multivariate point process A first way of realising dependence modelling in a reduced-form setting requires introducing dependence inside multivariate point process.

From the previous univariate point process, let $(T_n)_{n=0}^{\infty}$ be a univariate point process and let $(Z_n)_{n=1}^{\infty}$ be a sequence of random variables with $Z_n \in \{1, \ldots, m\}$. For $i = 1, \ldots, m$ we define N_t^i as

$$N_t = \sum_{n=1}^{\infty} 1_{\{T_n \le t\}} 1_{\{Z_n = i\}}$$

the process N_t^i is also a point process, which have been obtained by thinning the univariate process $(T_n)_{n=0}^{\infty}$ by picking out those jumps T_n where $Z_n = i$. We call the m-dimensional stochastic process $(N_t^1, N_t^2, \ldots, N_t^m)$ a m-variate point process $(T_n, Z_n)_{n=1}^{\infty}$. This is sometimes called a marked point process, MPP, with mark space $\{1, 2, \ldots, m\}$ with mark space E as introduced in Chapter 8 in [Bremaud81] and is defined as:

Theorem 5.2.1. *Let $(T_n, Z_n)_{n=1}^{\infty}$ be a MPP with mark-space $\{1, \ldots, m\}$ and define $(N_t^1, \ldots, N_t^m)$ as previously. Let $\mathcal{F}_t$ be the filtration given by*

$$\mathcal{F}_t = \mathcal{F}_0 \vee \mathcal{F}_t^N$$

where

$$\mathcal{F}_t^N = \bigvee_{i=1}^{m} \sigma(N_s^i; s \le t)$$

Assume that N_t^i admits the $\mathcal{F}_t$ intensities λ_t^i for $i = 1, 2, \ldots, m$. Then the processes λ_t^i for $i = 1, 2, \ldots, m$ is uniquely determine by the distribution $(T_n, Z_n)_{n=1}^{\infty}$

In the case of an m-variate point process $(N_t^1, \dots, N_t^m)$ where Λ_t^i is the compensator to N_t^i for $i = 1, 2, \dots, m$ and if $N_t = \sum_{i=1}^m N_t^i$ then it is easy to show that $\Lambda_t = \sum_{i=1}^m \Lambda_t^i$ is the compensator to N_t. [Norros86] uses the compensator to generate a set of correlated default times through the "total hazard"method. Consider $N_t^i = 1_{\{\tau_i \le t\}}$ a cádlág and increasing sub-martingale process. By the Doob-Meyer decomposition theorem, there exists an increasing $\mathcal{F}_t^N$-predictable compensator processes Λ_t^i where $\Lambda_t^i = 0$ and uniformly integrable martingale M_t^i such that

$$N_t^i = \Lambda_t^i - M_t^i$$

Suppose now that the probability space is large enough to endow the i.i.d random variables $E_1, E_2, \dots, E_m$ with $E_1 =^d Exp(1)$ and defined as

$$E_i = \Lambda_{\tau_i}^i,\ i = 1, \dots, m$$

With $\mathcal{F}_t^N$ generated by $(N_n, T_n)_{n=1}^m$, [Norros86] defines Λ_t^i in recursive form

$$\Lambda_t^i = \sum_{n=0}^{m-1} 1_{\{T_n < t \le T_{n+1}\}} \Lambda_t^i(n; T_1, Z_1, \dots, T_n, Z_n)$$

with the simulation of a random time τ linked to the inverse functions

$$(\Lambda_t^i)^{-1}(x) = \inf\{t \ge 0 : \Lambda_t^i \ge x\},\ i = 1, 2, \dots, m,\ x > 0$$

Using this framework, [Shaked87] presented an algorithm called the total hazard construction that linked the compensators $\Lambda_t^i, \forall i \in E$ and the multivariate random times $\tau_1, \dots, \tau_m$

$$\hat{\tau}_{j_k} = \hat{\tau}_{j_{k-1}} + \Lambda_{j_k}^{-1}(E_{j_k} - \Lambda_{\tau_{j_1}}^i - \sum_{n=2}^{k-1} \Lambda_{\tau_{j_n} - \tau_{j_{n-1}}}^i (\tau_{J_{n-1}} = t_{J_{n-1}}) \mid \tau_{j_{k-1}} = \hat{\tau}_{j_{k-1}})$$

where j_k is the ordering of default in the set E per simulation and $\hat{\tau}_{j_k}$ its associated simulated default time.

Furthermore, let $X_t = (X_t^1, \dots, X_t^d)$ be a d-dimensional stochastic process generated by observable market quantities, such as interest rates with the associated filtration $\mathcal{F}_t^X = \sigma(X_s^1, \dots, X_s^d; s \le t)$, then a direct way of expressing dependence is through the correlation of intensity processes. For example, [Duffie99] and [Duffie03] introduce dependence among m obligors by assuming that the processes λ_t^i are correlated as in

$$\lambda_t^i = \hat{\lambda}_t^i + X_c$$

with $\hat{\lambda}_t^i, X_c$ being respectively an independent process and a basic process containing some macroeconomic factors that affect all the obligors. However, the default correlations that are obtained in this setting are too low. Other studies contend that this is wrong and that enough correlation may be obtained if one makes a proper choice of the underlying factors that drive X_c. A further natural extension, is to had common jump process to achieve higher correlation like in [Duffie03] and [Duffie01]. This method is particularly used for “correlation ”sensitive products like collateralised debt obligation as in [Chapovsky06]. Another approach in the intensity family, is found in [Lindskog] where instead of allowing for correlation or jumps in obligor intensities, the authors consider the possibility of simultaneous defaults and allowing those “block ”defaults specific intensities. This framework called the Common Poisson Shock (CPS) results in a Markshall-Olkin copula linking the first default times of single-names. This model leads to portfolio loss dynamics that are consistent with traded index products but at the expense of repeated defaults of obligors accross defaulted groups of obligors.

The Copula approach: A common way to introduce dependence in credit risk is to use copulas as introduced in [Li00]. Copulas is a well-known concept from multivariate statistics and used for example in survival analysis and actuarial statistics. By definition, a copula is a multivariate distribution function such that its marginal distributions are uniformly distributed $\mathcal{U}_{[0,1]}$.

Let $X_1, X_2, \ldots, X_m$ be m real valued random variables with distribution functions $F_i(x) = \mathbb{P}[X_i \leq x]$ for $i = 1, 2, \ldots, m$. and X be the m-dimensional vector $X = (X_1, X_2, \ldots, X_m)$ with the associated the joint distribution $F(x) = \mathbb{P}[X_1 \leq x_1, X_2 \leq x_2, \ldots, X_m \leq x_m]$ where $x = (x_1, x_2, \ldots, x_m)$.

The main reason for using copulas in applied multivariate analysis is that the copula completely describes the joint distribution of $X_1, X_2, \ldots, X_m$, via the marginal distributions $F_i, i = 1, 2, \ldots, m$. This is done using the following theorem in [Nelsen99]

Theorem 5.2.2 (Sklars Theorem)**.** *Let $X_1, X_2, \ldots, X_m$ be m real valued random variables $F_i(x) = \mathbb{P}[X_i \leq x]$ for $i = 1, 2, \ldots, m$ with distribution functions and joint distribution F. Then there exists an m-dimensional copula C such that*

$$F(x) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n))$$

where $x = (x_1, x_2, \ldots, x_m)$. If F_i are continuous for all $i = 1, \ldots, m$ then the copula C is unique.

In a portfolio context, static dependence is classically modelled trough a covariance matrix using a copula framework for tractability with the value of the correlation being represented by a single number. C is called a Archemidean “gaussian”copula with F the bivariate normal distribution with correlation parameter ρ. A commonly used alternative is the *t-copula* with F being a t-Student distribution. In general, joint distributions and copulas have some “contagion”already built into the copula concept but the likelihood of simultaneous credit events tends to be underestimated due to the ”fat tail”behaviour of the normal distribution. The coefficient of lower tail dependence is defined, for a bivariate random variable $X = (X_1, X_2)$, as

$$\lambda = \lim_{\alpha \to 0} \mathbb{Q}[X_2 < q_2(\alpha) | X_1 < q_1(\alpha)] = \lim_{u \to 0} \frac{C(u,u)}{u} \tag{5.20}$$

where $q_i(\alpha)$ is the α'th quantile of X_i. However, the gaussian Copula has zero tail dependence $\lambda = 0$ independently of the correlation and thus fails to capture contagion at the lower tail distribution.

In risk management applications, the t-Student distribution is often preferred because of its *tail dependence* properties given by

$$\lambda = 2t_\nu \left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}} \right) \tag{5.21}$$

with ν degrees of freedom.

While copula-based models are widely used, they are essentially restricted to a one-period setting but even within a one-period setting copulas have some drawbacks: for example, the copula changes in a none-too-easily understood way when moving from initial joint default distribution to a conditional distribution given survival up to some time t. When the density function F is known, the related copulas can just be obtained by means of the formula

$$\mathbb{Q}[U_1 \leq u_1, \ldots, U_m \leq u_m] = C(u_1, u_2, \ldots, u_n) = F(F_1^{-1}(x_1), F_2^{-1}(x_2), \ldots, F_n^{-1}(x_n))$$

with the latent variables $U_1, U_2, \ldots, U_m \in [0,1]$.

Conditional independence: It should be clear that there are some risk factors common to obligor $i = 1, 2, \ldots, m$ such as interest rates, inflation etc. Those factors are defined as exogenous risk factors. However every firm i is also subject to individual risk factors, not necessarily shared with the other firms j, for $j \neq i$. These are called idiosyncratic risk factors. The concept of conditional independence states that once we have conditioned on the common information $\mathcal{G}_t$, then the idiosyncratic risk factors are independent, i.e. $\tau_1, \ldots, \tau_m$ are independent. More formally as defined in [Bielecki01]:

Definition 5.2.1. *Let $\tau_1, \tau_2, \ldots, \tau_m$ be the default times for obligor $i = 1, 2, \ldots, m$. Then $\tau_1, \tau_2, \ldots, \tau_m$ are said to be conditionally independent with respect to the filtration $(\mathcal{G}_t)_{t \geq 0}$ if for any $T > 0$ and any $t_1, \ldots, t_n \in [0, T]$*

$$\mathbb{P}[\tau_1 > t_1, \ldots, \tau_m > t_m \mid \mathcal{G}_T] = \prod_{i=1}^{m} \mathbb{P}[\tau_i > t_i \mid \mathcal{G}_T]$$

If it is assumed that the default times are independent with respect to some suitably choosen filtration $\mathcal{G}_t$, it is rather straightforward to derive risk neutral prices for different kind of credit derivatives of basket types such as first-to-default swaps or more generally n-to-default swaps on a credit portfolio. Example of this method can be found in [Kijima00] and [Lando00].

So far, most of those methods of dependency are characterised by a structural feature that tries to replicate a credit risk link between the m obligors in E. This has led studies to categorise that link between a static vs dynamic divide line.

Dynamic dependence modelling:

According to [Davis09] and [Schőnbucher03], these models are in three categories for modelling the dependence structure: factor models, frailty models and contagion models.

- A *factor model* is one where the hazard rate

$$h_i(t) = h_i(X_t, t, w)$$

is a function of some common factor X_t. This factor is observable or not and typically represents the macro-economy or a specific sector. To some extent, it is similar in logic with CAPM or Fama-French factor models to express excess returns with factors like market returns, small capitalisation - big capitalisation distinction. The factors tend to have a connection with hazard rate or return they tend to explain. This was the case of [Duffie99] and [Duffie03] quoted earlier.

- A *frailty model* is similar in form to a factor model but some of the factors are abstract statistical factors that are not related to economic variables. Those type of models were introduced in [Schőnbucher03] but the lack of economic link of some factors limit the field of applications. They tend typically to be used in augmented factor VAR where several factors each having a limited by non negligible explanatory power are merged into on single parameters like PCA usually with dummy values.

- A *contagion models* is one where the hazard rate of obligor i :

$$h_i(t)\mathbb{1}_{\{\tau_i > t\}} = h_i(S_t^i, t, w)$$

where

$$S_t^i = \{\tau_j : \tau_j < t \wedge \tau_i\}$$

is directly affected by defaults of other obligors. This is the category that represents the most dynamic aspect of dependence.

Transition from factor models to contagion models has been introduced first by [Davis99] with an extension of correlation effects to the Binomial Expansion Technique (EBT) used by Moody's to evaluate Credit portfolio structure like in CBO/CLO claims. The key element in contagion models is the definition of the evolutionary structure of the relationship between obligors. [Davis99] and [Zheng10] present structures that model default interaction in intuitively credible ways where Markov chain provide an effective modelling framework because of computational efficiency. Building on this concept, [Crepey09] , [Bielecki09] and [Bielecki07] consider a Markovian model of credit risk in which simultaneous defaults are possible.

At this point, we could mention that we have mainly focused on so-called "bottom-up"models where each obligor in a portfolio is modelled and the portfolio is modelled through the specificities of its constituents. That is mainly because we are interested in contagion in k^{th}-to-default basket products where reference name are typically less than a dozen. However, another class of models, called "top-down"models tend to models the portfolio properties as a whole and then isolate single entity behaviour. Those products are mainly designed in modelling credit dependency among large portfolio and are used in the case of index credit derivatives or collateralised debt obligations.

The book [Brig11c] provides an interesting account of those dependency issues in portfolio context beyond the previously mentioned Copula model. In a nutshell, the "top-down"models try to model parameters based on the loss distribution L_t used in valuing the index credit derivatives products in section 1.1.2 and 1.1.2. Instead of being the results of the single obligors defaults in the "bottom-up"models, the process L_t is more viewed as an input. This input aspect is described in [Brig07b] through the "Expected Tranche Loss"which represented the process L_t^γ for a tranche γ. Along a better dynamics process, the challenge is to recoup the evolution of the identified obligors default time to enable hedging through single-name CDS, [Brig07b] and [Brig07c], use a framework called Generalised-Poisson Loss that is close to [Lindskog] but identify which obligors have defaulted in cluster thus avoiding the repeated default and assigning to each cluster an independent intensity. The process modelled is the loss L_t rather than default counting process in [Lindskog]. Alternatively, another approach in [Bielecki13] establishes the top to down link by conserving the markovian property of the multivariate process to the markovianity of the single obligor processes via a constraint called the "Markov copula "property as introduced in [Bielecki06b] thus limiting the complexity of the model. However, the rationale of the loss process is not so much the topic of this document. We are more interested in identifying the specific obligor that is impacted by other obligor default time. The main reason being that we identify the counterparty C and the investor I specifically as well as the reference obligors in defaultable claims. Under this constraint, the "bottom-up"contagion models presented above seem to be the correct modelling way.

Introduced in [Davis99], the concept of default contagion or infectious default is central in the rest of this document. Previously, we have shown how to introduce dependence, for example by using affine process, or through copulas. These concepts are direct results of the intuitive idea that obligors are subjected to common underlying so called exogenous risk factors, such as the vector X_t. Furthermore, we also studied the concept of conditionally independent default times which lead to tractable formulas for many important basket credit derivatives. However, it is commonly believed that at the default of one firm, say firm j, the intensity λ_t^i for say $i \neq j$ will jump, normally upwards (like in [Jarrow01]).

The Jarrow-Yu model: In order to solve the problems of how to simulate i, compute $\mathbb{Q}[i > t]$ when the obligors $i = 1, 2, \ldots, m$ undergo default contagion, [Jarrow01] postulate the information structure is asymmetric.

Definition 5.2.2. *Let the some of the obligors $\{1, ..., m\}$ undergo default contagion. Then there exists disjoint sets $S_1 = \{1, \ldots, k\}, S_2$ called the set of primary firms and the set of secondary firms, such that $S_1 \cup S_2 = \{1, \ldots, m\}$ and*

- *for any firm from the set S_1 of primry firms, the default intensity depends only on $\mathbb{F}$*
- *the default intensity of each firm from the set S_2 of secondary firms may depend not only on the filtration $\mathbb{F}$ but also on the status (defaulted or not) of the primary firms.*

The intuitive idea is that the default-intensity for primary obligors is only dependent of the common macroeconomic market variables X_t while the default intensity for secondary obligors could depend on X_t as well as the default status of the primary firms.

Thus, with $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ the enlarged filtration, the filtration generated by the macroeconomic factors and the observations of default of the secondary firms is

$$\tilde{\mathbb{F}} = \mathbb{F} \vee \mathbb{H}^{k+1} \vee \ldots \vee \mathbb{H}^m$$

Additionally, in intensity processes $\lambda^i, i = k+1, \ldots, m$ are now given the "contagion intensity process"

$$\lambda_t^i = \mu_t^i + \sum_{l=1}^{k} \nu_t^{i,l} 1_{\{\tau_l \leq t\}}, \tag{5.22}$$

where $\mu_t^i, \nu_t^{i,l}$ are $\mathbb{F}$-adapted stochastic processes. Then

- the default times $\tau_1, \ldots, \tau_k$ of primary firms are no longer conditionally independent when we replace the filtration $\mathbb{F}$ by $\tilde{\mathbb{F}}$,
- the intensity of default for a primary firm with respect to $\tilde{\mathbb{F}}$ differs from the corresponding intensity with respect to $\mathbb{F}$,

[Jarrow01] establishes in a 2 obligor setting the closed-form solutions for the arbitrage-free price of defaultable bonds, and default swaps when the intensities are set constant. The equation (5.22) will be used in this document as it is a excellent way to link dependence between obligors in a natural way of "contagion ".

We now focus on a class of process called "markov chains"that could handle dependency modelling in a multivariate framework.

Chapter 6

Modelling dependence through Markov chains

As presented in [Davis99] and in the previous section, the need to express credit contagion will requite to built models that incorporate a dynamical aspect in their sensitivity to other obligor default and also macro risk factors driving the value of contingent claims. A Model class that exhibit such behaviour are using Markov jump processes to express the contagion between the obligors. We provide first the presentation about Markov jump process as in [Bielecki01] based on [Last95] and [Rogers00]. Additionally, [Asmu03] provides an introduction to a Markov jump process called phase-type distribution that can provide a flexible framework for generating dynamic credit risk evolution. We will apply this methodology to the valuation of Credit Valuation Adjustment to the financial claims of interest, defaultable and non-defaultable ones.

6.1 Markov Jump Process definition

As previously, we assume the underlying probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ as well as a finite set $\mathcal{K} = \{1, \ldots, K\}$ which plays the role of the state space for a Markov chain of interest. Since the state space is finite, it is clear that any function $h : \mathcal{K} \rightarrow \mathbb{R}$ is bounded or measurable provided that we endow the state space with the σ-field of all its subsets.

In credit risk setting, the states of obligor in the set E will be represented in the state space $\mathcal{K}$ of the Markov jump process. Thus the cardinality of the permutations or ordering from E has to correspond to the maximum number of states K of the Markov chain. Thus,

$$K = \mid E \mid$$

Let $C_t, t \in \mathbb{R}^+$ be a right continuous stochastic process on $(\Omega, \mathcal{G}, \mathbb{Q})$ with values in the finite set $\mathcal{K}$ and let $\mathbb{F}^C$ be the filtration generated by this process. Also let $\mathbb{G}$ be some filtration such that $\mathbb{F}^C \subseteq \mathbb{G}$. we provide here the general context definitions w.r.t the **Markov chain** and its associated **probability matrix**:

Definition 6.1.1. *A process C is a continuous time $\mathbb{G}$-Markov chain if for an arbitrary function $h : \mathcal{K} \to \mathbb{R}$ $\forall s, t \in \mathbb{N}$ we have*

$$\mathbb{E}_{\mathbb{Q}}(h(C_{t+s}) \mid \mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(h(C_{t+s}) \mid C_t) \tag{6.1}$$

A continuous time $\mathbb{G}$-Markov chain C is said to be time-homogeneous if, in addition, $\forall s, t, u \in \mathbb{N}$ we have

$$\mathbb{E}_{\mathbb{Q}}(h(C_{t+s}) \mid C_t) = \mathbb{E}_{\mathbb{Q}}(h(C_{u+s}) \mid C_u) \tag{6.2}$$

Definition 6.1.2. *A two-parameter family $\mathcal{P}(t, s), t, s \in \mathbb{R}_+, t \leq s$ of stochastic matrices is called the family of transition probability matrices for the $\mathcal{G}$-markov chain C under $\mathbb{Q}$ if $\forall t, s \in \mathbb{R}_+, t \leq s$,*

$$\mathbb{Q}\{C_s = j \mid C_t = i\} = p_{i,j}(t, s) \;, \forall t, s \in \mathcal{K} \tag{6.3}$$

In particular the equality $\mathcal{P}(t, t) = Id$ is satisfied $\forall i \in \mathbb{R}_+$

6.1.1 Time-homogeneous markov chain

Definitions

For a time-homogeneous Markov chain, the transition probability matrix (6.3) is defined as

Definition 6.1.3. *The one parameter family $\mathcal{P}(t),\ t \in \mathbb{R}_+$ of stochastic matrices is called the family of transition probability matrices for the time-homogeneous $\mathbb{G}$-markov chain C under $\mathbb{Q}$ if for every $t, s \in \mathbb{R}_+$*

$$\mathbb{Q}\{C_{s+t} = j \mid C_s = i\} = p_{i,j}(t) \;, \forall i, j \in \mathcal{K} \tag{6.4}$$

If $\mathcal{P}(t),\ t \in \mathbb{R}_+$ is the family of transition matrices for C then for any subset $A \subseteq \mathcal{K}$ we have

$$\mathbb{Q}\{C_{s+t} \in A \mid C_t\} = \sum_{j \in A} p_{C_t,j}(s) \;, \forall t, s \in \mathbb{R}_+ \tag{6.5}$$

Properties

Moreover, the **Chapman-Kolmogorov** equation is satisfied, namely,

$$\mathcal{P}(t + s) = \mathcal{P}(t)\mathcal{P}(s) = \mathcal{P}(s)\mathcal{P}(t) \;, \forall t, s \in \mathbb{R}_+ \tag{6.6}$$

Equivalently, $\forall i, j \in \mathcal{K}$ and $\forall t, s \in \mathbb{R}_+$

$$p_{i,j}(t + s) = \sum_{k=1}^{K} p_{i,k}(t)p_{k,j}(s) = \sum_{k=1}^{K} p_{i,k}(s)p_{k,j}(t) \tag{6.7}$$

Let the K-dimensional row vector $\mu_t = [\mu_t(i)]_{1 \leq i \leq K} = [\mathbb{Q}\{C_t = i\}]_{1 \leq i \leq K}$ denote the probability distribution at time t of the $\mathbb{G}$-markov chain C under $\mathbb{Q}$. The probability distribution at time $t + s$ is given by

$$\mu(t+s) = \mu_0 \mathcal{P}(t+s) = \mu_t \mathcal{P}(s) = \mu_s \mathcal{P}(t) \ , \forall t, s \in \mathbb{R}_+ \tag{6.8}$$

An important condition on the family $\mathcal{P}(\cdot)$ that it is right-continuous at time $t = 0$ and thus with the Chapman-Kolmogorov equation this implies that

$$\lim_{s \downarrow 0} \mathcal{P}(t+s) = \mathcal{P}(t) \ , \forall t > 0$$

and thus,

$$\lim_{s \downarrow 0} \mathbb{Q}\{C_{s+t} = j \mid C_s = i\} = \delta_{i,j} \ , \forall i, j \in \mathcal{K}, t > 0$$

It is a well known fact that the right continuity of the family $\mathcal{P}(\cdot)$ at time $t = 0$ implies the right-hand side differentiability at time $t = 0$ of this family (Theorem 8.1.2 in [Rolski98]). The following finite limits exists at time $t = 0$

$$\lambda_{ij} := \lim_{t \downarrow 0} \frac{p_{i,j}(t) - p_{ij}(0)}{t} = \lim_{t \downarrow 0} \frac{p_{i,j}(t) - \delta_{ij}}{t}, \ \forall i, j \in \mathcal{K} \tag{6.9}$$

with $\lambda_{ij} > 0$ and $\lambda_{ii} = -\sum_{j=1, j \neq i}^{K} \lambda_{ij}$ for every $i \neq j$.

The matrix

$$\Lambda := [\lambda_{i,j}]_{1 \leq i,j \leq K}$$

is called the **infinitesimal generator matrix** for a Markov chain associated with the family $\mathcal{P}$. Since each entry $\lambda_{i,j}$ of the matrix Λ can be shown to represent the intensity of a transition from the state i to the state j the infinitesimal generator matrix Λ is also called the **intensity matrix**.

Invoking the Chapman Kolmogorov equation and the limit of the transition intensity, one may derive the backward Kolmogorov equation

$$\frac{d\mathcal{P}(t)}{dt} = \Lambda \mathcal{P}(t), \ \mathcal{P}(0) = Id \tag{6.10}$$

the forward Kolmogorov equation

$$\frac{d\mathcal{P}(t)}{dt} = \mathcal{P}(t)\Lambda, \ \mathcal{P}(0) = Id \tag{6.11}$$

where at time $t = 0$ we take the right-hand side derivatives. As detailed in [Last95] and [Rogers00], it is well known that both these equations have the same unique solution through en exponential matrix

$$\mathcal{P}(t) := e^{t\Lambda} := \sum_{n=1}^{\infty} \frac{\Lambda^n t^n}{n!}, \ t \in \mathbb{R}_+ \tag{6.12}$$

We say that a state $k \in \mathcal{K}$ is **absorbing** for a time homogeneous $\mathbb{G}$-Markov chain $C_t, t \in \mathbb{R}_+$ if the following holds

$$\mathbb{Q}\{C_s = k \mid C_t = k\} = 1, \forall t, s \in \mathbb{R}_+, t < s$$

If a state $k \in \mathcal{K}$ is absorbing $\lambda_{kj} = 0$ for every $j = 1, \ldots, K$. We will postulate K is absorbing, representing default of obligors. We associate τ the random variable of absorption at K, i.e., $\tau = \inf\{t > 0 : C_t = K\}$. Assume that $\tau < \infty, \mathbb{Q}$- a.s. thus implies that the state K is the only state recurrent state for C.

Martingale characterisation

The following important result provides a martingale characterisation of a time-homogeneous Markov chain C in terms of its infinitesimal generator.

For any state $i \in \mathcal{K}$ and for any function $h; \mathcal{K} \to \mathbb{R}$ we denote

$$(\Lambda h)(i) = \sum_{j=1}^{K} \lambda_{ij} h(j)$$

Proposition 6.1.1. *A process C is a time-homogeneous $\mathbb{G}$-Markov Chain under $\mathbb{Q}$, with the initial distribution μ_0 and the infinitesimal generator matrix Λ, if and only if the following conditions are satisfied:*

- $\mathbb{Q}\{C_0 = i\} = \mu_0(i)$ *for every* $i \in \mathcal{K}$,
- *for any function $h : \mathcal{K} \to \mathbb{R}$ the process M^h defined by the formula*

$$M_t^h = h(C_t) - \int_0^t (\Lambda h)(C_u) du, \ t \in \mathbb{R}_+ \tag{6.13}$$

  follows a $\mathbb{G}$-martingale under $\mathbb{Q}$.

Typically, we take $h(\cdot) = 1_i(\cdot)$, thus

$$M_t^h = H_t^i - \int_0^t \lambda_{C_u,i} du, \ t \in \mathbb{R}_+ \tag{6.14}$$

follows a $\mathbb{G}$-martingale under $\mathbb{Q}$.

The proof of this proposition can be found in [Rogers00], [Last95] or [Daley02]. The Martingale characterisation of the Markov chain C_t in proposition 6.1.1, page 93, is similar to the Martingale characterisation of the default process $H_t^\bullet = 1_{\{\tau_\bullet \leq t\}}$ in section 5.1.1, page 67. Thus, the default time τ of can be represented by a Markov chain with default occurring when the chain exhibits a particular jump.

Probability Distribution of the Absorption Time

More explicit formulae for the conditional expectations with respect to the σ-field can be obtained if the knowledge of conditional laws of C. For every $0 \leq t \leq s$

$$\mathbb{Q}\{\tau > s \mid C_t = i\} = 1 - \mathbb{Q}\{C_s = K \mid C_t = i\} = 1 - p_{iK}(s-t)$$

or

$$\mathbb{Q}\{\tau > s \mid \mathcal{G}_t\} = 1_{\{s \leq t\}} 1_{\{\tau > s\}} + 1_{\{s > t\}} \sum_{i=1}^{K-1} H_t^i (1 - p_{iK}(s-t))$$

Martingales Associated with Transitions

(Covered in [Bielecki01] with details in [Bremaud81], [Last95] and [Rogers00])

We quote here important examples of martingales associated with the absorption time τ and with the number of transition. Those will be of particular interest to capture obligor defaults in the case of a multivariate setting and will be central in valuing credit derivatives with multiple obligor references.

For any fixed $i \neq j \in E$, let H_t^{ij} stand for the number of jumps of the process C from i to j in the interval $(0, t]$. Formally for any $i \neq j$ we set

$$H_t^{ij} = \sum_{0 < u \leq t} H_{u-}^i H_u^j, \ \forall t \in \mathbb{R}_+ \tag{6.15}$$

Lemma 6.1.1. *For every $i, j \in \mathcal{K}, i \neq j$ the processes*

$$M_t^{ij} = H_t^{ij} - \int_0^t \lambda_{ij} H_u^i du = H_t^{ij} - \int_0^t \lambda_{C_u j} H_u^i du$$

and

$$H_t^K = H_t - \int_0^t \sum_{i=1}^{K} \lambda_{iK} H_u^i du = H_t^{ij} - \int_0^t \lambda_{C_u j} (1 - H_u) du$$

follow $\mathbb{G}$-martingales (and $\mathbb{F}^C$ martingales)

Change of a Probability Measure

(Covered in [Bielecki01] with details in [Bremaud81], [Last95] and [Rogers00])

Since counterparty credit risk is valued in pricing context with CVA adjustments and in a risk context with EPE metrics, we need to ask how the Markov property and the generator Λ of the time homogeneous Markov chain C are affected by a change of the reference probability measure $\mathbb{Q}$ to an equivalent probability measure $\mathbb{Q}^*$ on $(\Omega, \mathcal{G}_T)$ for some fixed T. (We do not need K absorbing here.)

Consider a family $\tilde{\kappa}^{kl}, k, l \in \mathcal{K}, k \neq l$ of bounded $\mathbb{F}^C$-predictable, real-valued processes such that $\tilde{\kappa}^{kl} > -1$ and $\tilde{\kappa}^{kk} = 0$. Let us define an auxiliary $\mathbb{G}-$martingale M (which is also an $\mathbb{F}^C$ martingale) by setting

$$M_t = \int_{]0,t]} \sum_{k,l=1}^{K} \tilde{\kappa}_u^{kl} dM_u^{kl}$$

and a $\mathbb{G}$-martingale $\eta_t, t \in [0, T]$ with

$$\eta_t = 1 + \int_{]0,t]} \sum_{k,l=1}^{K} \eta_{u-} \tilde{\kappa}_u^{kl} dM_u^{kl}$$

[Last95] establishes the following proposition for $\tilde{\kappa}_t^{kl} = \kappa_{kl}(t) : \mathbb{R}_+ \to (-1, \infty)$ a Borel measurable and bounded function with $\kappa_{kk} = 0$ that provides sufficient conditions for a $\mathbb{G}$-markov chain C to remain a (time-inhomogeneous in general) $\mathbb{G}$-markov chain under $\mathbb{Q}^*$

Proposition 6.1.2. *Let the probability measure $\mathbb{Q}^*$ be defined before with the Radon-Nikodym density η_T given earlier. Then*

- *the process $C_t, t \in [0, T]$ is a $\mathbb{G}$-markov chain under $\mathbb{Q}^*$*
- *the infinitesimal generator $\Lambda^*(t) = [\lambda^*_{i,j}(t)]_{1 \leq i,j \leq K}$ for C under $\mathbb{Q}^*$ satisfies for $i \neq j$*

$$\lambda^*_{i,j}(t) = (1 + \kappa_{ij}(t))\lambda_{i,j}, \ \forall t \in [0, T]$$

and

$$\lambda^*_{i,i}(t) = -\sum_{j=1, i \neq j}^{K} \lambda^*_{i,j}(t), \ \forall t \in [0, T]$$

- *the two parameter family $\mathcal{P}^*(t, s), 0 \leq s \leq t \leq T$ of transition matrices for C relative to $\mathbb{Q}^*$ satisfies the forward Kolmogorov equation*

$$\frac{d\mathcal{P}^*(t, s)}{ds} = \mathcal{P}^*(t, s)\Lambda^*(s), ; \mathcal{P}^*(t, t) = Id \tag{6.16}$$

and the backward Kolmogorov equation

$$\frac{d\mathcal{P}^*(t, s)}{dt} = \Lambda^*(t)\mathcal{P}^*(t, s), ; \mathcal{P}^*(s, s) = Id \tag{6.17}$$

Note that equations (6.16) and (6.17) are defined as the "Peano-Baker series"or "Magnus expansion"with convergence and existence rules defined in [Higham05] and [Dacunha05]. Let's now review the existence of the solution (6.12) of the Kolmogorov equation (6.10) and its application to credit risk modelling.

Markov jump in a credit risk context

Existence of probability distribution: To ensure that the family $\mathcal{P}(\cdot)$ associated with its infinitesimal generator $\boldsymbol{Q}$ defines a probability distribution like equation (6.8), page 92, such as $\mu(t+s) = \mu_t \mathcal{P}(s),\ \forall t, s \in \mathbb{R}_+$ will requite that $\boldsymbol{Q}$ is a non-negative square matrix.

For non-negative square matrices, the main result is the theorem of Perron-Frobenius which states that a non-negative square matrix has a maximal non-negative eigenvalue which is not exceeded in absolute value by any other eigenvalue and corresponding to which there is a non-negative eigenvector. In [Cox77], the *Perron-Forbenius theorem* states that

Theorem 6.1.1 (Perron-Frobenius theorem)**.** *Suppose $\boldsymbol{A} \geq 0$ (positive matrix) and irreductible then*

- *$\boldsymbol{A}$ has a real positive eigenvalue λ_1 with the following properties:*
- *corresponding to λ_1 there is an eigenvector $\boldsymbol{x}$ all of whose elements may be taken as positive i.e. there exists a vector $\boldsymbol{x} > 0$ such that*

$$\boldsymbol{A}\boldsymbol{x} = \lambda_1 \boldsymbol{x}$$

- *if α is another eigenvalue of $\boldsymbol{A}$ then*

$$\alpha \leq \lambda_1$$

- *λ_1 increases when any element of $\boldsymbol{A}$ increases*
- *λ_1 is a simple root of the determinental equation*

$$\mid \lambda_1 \boldsymbol{I} - \boldsymbol{A} \mid = 0$$

Remark: If λ_1 itself is the only eigenvalue of modulus λ_1 then $\boldsymbol{A}$ is said to be *primitive*.

This theorem ensures that the limiting behaviour of $\boldsymbol{A}^n$ is defined by its eigenvectors and is positive. Applied to Markov chain it helps specify the conditions of the ergodicity of finite Markov chains.

If the Markov chain is ergodic then its transition matrix is primitive and irreducible, i.e $\boldsymbol{P} \in \mathcal{P}(\cdot)$ has a simple eigenvalue 1 which exceeds all other eigenvalues in modulus. Conversely, if $\boldsymbol{P}$ is primitive and irreducible then the system is ergodic. The eigenvalue $\lambda_1 = 1$ is simple and all other eigenvalues are strictly less than 1 in modulus.
According to the theorem of Perron and Frobenius there is a positive row eigenvector $\pi = (\pi_j)$ satisfying $\pi \boldsymbol{P} = \pi$ and we can normalise this vector so that $\sum \pi_j = 1$. From [Cox77] we have the property

Proposition 6.1.3. *If $\boldsymbol{P}$ is primitive and irreductible then the system is ergodic and*

$$\lim_{n \to \infty} p_{jk}^{(n)} = \pi_k > 0$$

the limit being approached geometrically fast and uniformly for all j and k. Conversely if the system is ergodic then $\boldsymbol{P}$ is primitive and irreducible.

The case of Markov processes in continuous time for processes with a finite number m of states can be found in [Bellman60], page 192. For a fixed initial state i the probabilities $p_{ij}(t)(j = 1, \ldots, m)$ satisfy a set of m simultaneous linear differential equations with constant coefficients determined by $\boldsymbol{Q}$ known as the Chapman Kolmogorov equations presented earlier. It is well known that the solution of such equations is given by the linear combinations of exponential terms, the coefficients of t in the exponents being the eigenvalues of $\boldsymbol{Q}$. There are minor complications if $\boldsymbol{Q}$ has multiple eigenvalues. We already know that $\boldsymbol{Q}$ has a zero eigenvalue and it is extremely plausible that the remaining eigenvalues have negative real parts. We, then, have an equilibrium distribution determined as a limit as t tends to infinity from the contribution of the zero eigenvalue plus a transient part dying away exponentially as t increases.

Thus with the spectral resolution of $\boldsymbol{Q}$ we have

$$\boldsymbol{Q} = \boldsymbol{B}\mathrm{diag}(\lambda_1, \ldots, \lambda_m)\boldsymbol{C^t}$$

where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of $\boldsymbol{Q}$ and $\lambda_1 = 0$ and $\boldsymbol{BC^t} = \boldsymbol{I}$. The matrices $\boldsymbol{B}$ and $\boldsymbol{C^t}$ are respectively formed from the right and left eigenvalues of $\boldsymbol{Q}$. The representation requires that the eigenvalues are distinct, which we assume for simplicity. Then, we have

$$\boldsymbol{P}(t) = \boldsymbol{B}\mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_m t})\boldsymbol{C^t}$$

We assume that all states of $\boldsymbol{Q}$ inter-communicate. Thus, the diagonal matrix tends to $\mathrm{diag}(1, 0, \ldots, 0)$, the rate of approach to the limit being exponential at a rate governed asymptotically by the largest non-zero eigenvalue of $\boldsymbol{Q}$. Further, the limiting form of $\boldsymbol{P}(t)$ is

$$\boldsymbol{bc}^t = (b_{11}, \ldots, b_{m1})^t(c_{11}, \ldots, c_{1m})$$

(corresponding to $\lambda_1 = 0$).

Computation of the exponential matrix: Since the Perron-Frobenius theorem is mainly a way to express the ergodicity of the Makov jump process as a function of its eigenvalues, the probabiltity vector

$$\mathcal{P}(t) := e^{t\boldsymbol{Q}} := \sum_{n=1}^{\infty} \frac{\boldsymbol{Q}^n t^n}{n!},\ t \in \mathbb{R}_+$$

will still require the compute easily $e^{t\boldsymbol{Q}}$ along time t. The paper [Moler03] famously presents the methods to compute the exponential matrix which are detailed in appendix B.4, page 167. We present here the methods that might apply in our context of multivariate credit risk i.e. when the matrix $\boldsymbol{Q}$ exhibit an **upper triangular structure**. Also, it will be required to scale up and down in dimension easily as the inclusion of several obligors in the case of valuing different k^{th}-to-default swaps.

The exponential matrix is mainly found in Control theory, for the following state equation

$$\frac{\partial x(t)}{\partial t} = \boldsymbol{Q}x(t)$$

with $\boldsymbol{Q}$ a given fixed, real or complex n-by-n matrix, $x(t)$ a column vector and the initial condition

$$x(0) = x_0$$

The solution is given by

$$x(t) = e^{\boldsymbol{Q}t}x_0$$

with the exponential matrix $e^{\boldsymbol{Q}t}$ defined by the power series

$$\exp(\boldsymbol{Q}t) = \boldsymbol{I} + t\boldsymbol{Q} + \frac{\boldsymbol{Q}^2t^2}{2!} + \frac{\boldsymbol{Q}^3t^3}{3!} + ...$$

There main ways to compute the exponential of a matrix are :

- Series methods revolving around the Taylor series expansion or the characteristic polynomial of $\boldsymbol{Q}$: The main problem with those approaches is the error that is due when raising power of the elements of the matrix $\boldsymbol{Q}$ might produce rounding errors.
- Ordinary Differential Equations methods by solving iteratively,

  $$f(x,t) = \frac{\partial x(t)}{\partial t} - \boldsymbol{Q}x(t)$$

  using numerical optimisation methods with initial condition $x(0) = x_0$. The process is cumbersome with dimension greater than 2 with the possibility of eigenvalues that can turn complex in the iterative process.
- The second Series method is the **Pade Approximation** based on the ratio of two rational functions of the matrix $\boldsymbol{Q}$. [Higham05] explains the method of Pade approximation which is based on a limited polynomial exponential to approximate the matrix exponential. The (p,q) Pade Approximation to $\exp(\boldsymbol{Q}t)$ is defined by

  $$R_{pq}[\boldsymbol{Q}] = [D_{pq}[\boldsymbol{Q}]]^{-1}N_{pq}[\boldsymbol{Q}]$$

  with

  $$N_{pq}[\boldsymbol{Q}] = \sum_{j=0}^{p} \frac{(p+q-j)!p!}{(p+q)!j!(p-j)!}\boldsymbol{Q}^j$$

  and

  $$D_{pq}[\boldsymbol{Q}] = \sum_{j=0}^{q} \frac{(p+q-j)!q!}{(p+q)!j!(q-j)!}\boldsymbol{Q}^j$$

  where non-singularity of $D_{pq}[\boldsymbol{Q}]$ is assured if p and q are large enough or if the eigenvalues λ_i of $\boldsymbol{Q}$ are negative. This method is the method of choice in widely popular vector software like Matlab or R.

- Matrix decomposition methods using the properties of the matrix. The matrix decomposition methods are likely to be most efficient for problems involving large matrices and repeated evaluation based on factorisations and decompositions of the matrix $\boldsymbol{Q}$.

  If $\boldsymbol{Q}$ happens to be symmetric, the matrix decomposition are based on the transformation of the form

  $$\boldsymbol{Q} = \boldsymbol{S}\boldsymbol{R}\boldsymbol{S}^{-1}$$

  thus,

  $$e^{\boldsymbol{Q}t} = \boldsymbol{S}e^{\boldsymbol{R}t}\boldsymbol{S}^{-1}$$

  The natural decomposition is to take $\boldsymbol{S}$ to be the matrix whose columns are eigenvectors of $\boldsymbol{Q}$ with

  $$e^{\boldsymbol{Q}t} = \boldsymbol{V}e^{\boldsymbol{D}t}\boldsymbol{V}^{-1}$$

  in case $\boldsymbol{V}$ is non singular and $\boldsymbol{D}$ is the diagonal matrix with the eigenvalues. The difficulty in that case is that $\boldsymbol{Q}$ may be close to singular which means leading to non convergence.

  [Golub83] describes the case when the matrix $\boldsymbol{Q}$ is **upper-triangular**. Thus, the product of upper-triangular matrices is an upper triangular and the power series, i.e the exponential is upper triangular. Furthermore, the eigenvalues of an upper triangular matrix are the values on the diagonal. If none of the diagonal value is zero the matrix is also invertible. Additionally the exponential of a diagonal matrix is also a diagonal matrix with the value being the exponential of diagonal value. Thus, if $\boldsymbol{Q} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}$ where matrix $\boldsymbol{D}$ is a diagonal matrix with the eigenvalues of matrix $\boldsymbol{Q}$ and matrix $\boldsymbol{U}$ is an upper matrix representing the eigenvectors of $\boldsymbol{Q}$ then $\exp^{\boldsymbol{T}t} = \boldsymbol{U}\exp^{\boldsymbol{D}t}\boldsymbol{U}^{-1}$. The computation of the matrix $\boldsymbol{U}$ is obtained with the iterative process called the backward substitution for upper triangular or forward substitution for lower triangular matrices.

The preferred choice of the algorithm in terms of stability and efficiency will typically be a function of the distribution of the eigenvalues (confluent or not, situated in the left pane) and the structure of the matrix $\boldsymbol{Q}$. Numerically, the Pade Approximation is **scalable** and can be applied in general case while the decomposition method can be applied only when the structure is upper triangular.

Having conditions on the infinitesimal generator $\boldsymbol{Q}$, we now move to a Markov jump process called phase-type distribution that we believe presents interesting feature to capture credit contagion in a multivariate setting. We transpose the main previous results to this model and then will apply it to credit modelling in the next chapter.

6.1.2 Phase-type distribution structure in credit contagion setting

Phase-type distribution definition

A class of Markov chain that presents an interest in credit risk modelling is the so-called phase type distribution introduced by [Neuts78] as an homogeneous Markov jump process and characterised by a unique absorption state that is accessed through an exit vector positioned at right-hand side of the transition matrix. Thus, under those specificities, all previous results about Markov jump process $(C_t)_{t\geq 0}$ are still valid. Recalling the presentation in [Asmu03], consider a phase-type distribution $(Y_t)_{t\geq 0}$, being a Markov jump process based on the finite space $\mathcal{K} = \mathbf{E} \cup \Delta$ where Δ is an absorbing state and $\mathbf{E}$ consist of $K-1$ different transient states with $(\mid \mathbf{E} \mid) = K-1$.

The phase-type distribution introduces the following definitions from Chapter 2.2 page 44 in [Neuts78]:

Definition 6.1.4 (Phase-type distribution)**.** *Consider $(Y_t)_{t\geq 0}$ a Markov jump process defining a phase-type distribution on $\mathbb{R}_+$ as in [Neuts78]. The following points characterise the chain Y_t:*

- *Let α be the initial distribution m-dimension vector and define τ as the time to absorption, that is*

$$\tau = \inf\{t > 0 : Y_t = \Delta\} \tag{6.18}$$

- *$F_\alpha(t)$ be the cumulative distribution function of the time to absorption τ given the initial distribution α*

- *Let $\boldsymbol{Q}$ be the intensity matrix of $(Y_t)_{t\geq 0}$ such that*

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{T} & | & \boldsymbol{t} \\ -- & | & -- \\ \boldsymbol{0} & | & 0 \end{pmatrix} \tag{6.19}$$

  where $\boldsymbol{T}$ is a $K-1 \times K-1$ intensity matrix for transition between the states in $\mathbf{E}$ and $\boldsymbol{t}$ is the absorption vector towards Δ. The matrix $\boldsymbol{T}$ is called the phase generator and $\boldsymbol{t}$ the exit vector.

Typically, the chain starts in a non-absorbing state, so that $\mathbb{P}[Y_0 = \Delta] = 0$ and as, for Markov jump processes, all rows in $\boldsymbol{Q}$ must sum to zero and thus $\boldsymbol{t} = -\boldsymbol{T}\boldsymbol{1}$.

By definition the absorbing state Δ is the only recurrent state all the other states being transient states. This lead to the important Lemma 2.2.1 page 45 in [Neuts78]:

Lemma 6.1.2. *The states $1, \ldots, K-1$ are transient if and only if the matrix $\boldsymbol{T}$ is non singular.*

[Neuts78] originally defined the triplet $(\boldsymbol{E}, \alpha, \boldsymbol{T})$ as a phase-type distribution where F is the distribution of the time to absorption τ with the following closed-form formula properties from Lemma 2.2.2 in [Neuts78]:

$$\begin{aligned} F(t) = \mathbb{P}[\tau < t] = 1 - \alpha e^{\boldsymbol{T}t\boldsymbol{1}} \\ f(t) = \alpha e^{\boldsymbol{T}t\boldsymbol{1}} \end{aligned} \tag{6.20}$$

with

$$e^{\boldsymbol{T}t} = \sum_{n=0}^{\infty} \frac{1}{n!} \boldsymbol{T}^n t^n$$

where $f(t)$ is the density of $F(t)$ and $\boldsymbol{1} = (1, \dots, 1)^t \in \mathbb{R}^{K-1}$. $e^{\boldsymbol{T}t}$ is the matrix exponential which has a closed-form expression in terms if the eigenvalue decomposition of $\boldsymbol{T}$.

Credit obligor definitions

[Herb10] builds on the work of [Asmu03] and extends the work of [Assaf84] by using multivariate phase-type distribution in a context of credit contagion in a set of m obligors. As we will also focus on the multivariate aspect of the phase-type distribution where we detail here some key aspects. The markov chain Y_t describes per each state the status vector of the m obligors being either defaulted or not. [Assaf84] identifies that in the multivariate context, the distribution F of the time to absorption τ designates the time where all obligors would have defaulted. Thus, by identifying all the states where obligor i has defaulted with Δ_i the final absorbing state for Y_t is defined as:

$$\Delta = \bigcap_{i=1}^{m} \Delta_i, \ \Delta \in \mathcal{K}, \ \Delta_i \in \mathcal{K} \tag{6.21}$$

[Asmu03] and [Herb10] build on this property to express multivariate probability default in a closed-form formula format using formula (6.20) and identifying with state corresponds to defaults of any particular obligor.

Thus, consider m obligors in E and τ_i be the time of default for obligor i, with $i \in E = \{1, 2, \dots, m\}$, and the corresponding default point processes

$$H_t^i = 1_{\{\tau_i < t\}}, \ i = 1, 2, \dots, m.$$

contain any information regarding the default times $\tau_1, \dots, \tau_m$ with the filtrations

$$\mathcal{H}_t^i = \sigma(H_t^i; s \leq t)$$
$$\mathcal{H}_t = \bigvee_{i=1}^{n} \mathcal{H}_t^i$$

From [Asmu03], extending the work of [Assaf84],

Proposition 6.1.4. *There exists a markov Jump process $(Y_t)_{t \geq 0}$ with infinitesimal generator $\boldsymbol{Q} := [\lambda_{i,j}]_{1 \leq i,j \leq K}$ on a finite state space E and a family of absorption sets $\{\Delta_i\}_{i=1}^m$ such that the stopping times*

$$\tau_i = \inf\{t > 0 : Y_t \in \Delta_i\}, \ i \in E, \Delta_i \in \mathcal{K}$$

define a joint distribution of default times $(\tau_1, \dots, \tau_m)$

The joint distribution of $(\tau_1, \ldots, \tau_m)$ is called a **multivariate phase-type distribution** or MPH.

We can now connect the states set $\mathcal{K}$ of the chain Y_t in a multivariate setting to the obligors in E and their default ordering as presented in section 4.1, page 47. Since the state Δ represents all obligors having defaulted, from the initial state, each state Y_t corresponds to a set of consecutive defaulted obligors that defined at time t a part of the permutation $\{\tau_{\pi(-)} < \tau < \tau_{\pi(+)}\}$. Thus, the filtration $\mathcal{H}_t$ representing the observation of the default times of the permutations $\pi \in \Pi$ is characterised by the observation of the states of the Markov chain Y_t with

$$\mathcal{H}_t = \mathcal{F}_t^Y = \sigma(Y_t, t \in \mathbb{R}_+)$$

We will in the reminder of the document when quoting the filtration $\mathcal{H}_t$ imply its equivalent one $\mathcal{F}_t^Y$.

Additionally, since the states of the markov chain Y_t represent the evolution of the credit status of the multivariate obligors $(X_1, \ldots, X_m)$, starting from a non defaulted status represented by the initial vector $\alpha = (1, 0, \ldots, 0)$ and the states will be ordered such that the infinitesimal generator $\boldsymbol{Q}$ will exhibit an upper-triangular structure.

Markov copula Property in a phase-type setting

Since the multivariate phase-type distribution $(\alpha, \boldsymbol{E}, \boldsymbol{T})$ represented by the Markov chain Y_t has the markovian property (6.1), page 91, for the vector $(X_1, \ldots, X_m)$, it will be interesting to have the same property for each the marginal vectors $X_i, i \in E$.

Historically, as mentioned before, two major families of dynamic models have been developed: one is known as *bottom-up* and the other is known as *top-down*. Bottom-up models describe the evolution of each individual default process and are therefore driven by the information generated by the underlying pool of obligations. Top-down models describe the evolution of the portfolio loss process (or functionals) and can be viewed as reduced information models since only the information about the sum of the defaults and functionals is used. Top-down are computationally efficient by fail to incorporate the marginal information that could produce sensible hedging results. The idea of the **Markov copula** approach is to combine the advantage of a copula model with a dynamic bottom-up approach.

We assess this potentiality through the concept of **Markov Copula** introduced first in [Bielecki08] (A more general account of the theory of the copula can be found in [Bielecki15a] and [Bielecki15b]) with example for markov jump processes and Markov jump diffusion processes. [Bielecki08] study the dependence between components of multivariate processes and the conditions such that the multivariate process and its embedded marginal ones share the markovian property. Most of the applications in [Bielecki08] has been focused on Markov Jump diffusion processes representing the intensity modulated by the chain values. By enabling simultaneous default and ensuring the Markov Copula properties, [Bielecki08] and [Bielecki13] can calibrate separately each obligors to CDS and then simulate loss processes L_t to replicate CDO or index Credit derivatives. Another approach in Markov copula is presented in [Bielecki11] is to construct a multivariate chain with components that are given Markov chain. This approach is directly linked to the Markov copula property of the MPH. The property linking the intensity of default of each obligors with the multivariate intensity is found in Theorem 1.1 and Theorem 1.2 in [Bielecki11] that we will quote below and use in our framework.

Consider a multivariate Markov Chain $Z = (X_1, \ldots, X_m)$ of dimension $\mid E \mid$ which is a finite Markov chain w.r.t its natural filtration $\mathbb{F}^Z = \bigvee_{i=1}^{m} \mathbb{F}^{X_i}$. X_i is defined as chain representing the states of obligor $i \in E$. The conditions linking the infinitesimal generators and Markov chains of Z with X_i's can be expressed in both ways.

- Condition 1: Sufficient and necessary conditions on the infinitesimal generator of Z so that the components X_i are markov chains with respect to their natural filtrations $\mathbb{F}^{X_i}$.
- Condition 2: Conditions on the construction of a mutlivariate markov chain Z whose components X_i's are themselves markov chains w.r.t their natural filtrations $\mathbb{F}^{X_i}$ and have desired infinitesimal characteristics.

The conditions are stated in the case of two dimensions:

Consider $\mathcal{S}$ and $mathcalO$ a finite sets, with the markov chain $Z = (X, Y)$ defined on $\mathcal{Z} = \mathcal{S} \times \mathcal{O}$ with generator function $A^Z(t) = [\lambda_{jk}^{ih}(t)]_{i,j \in \mathcal{S}, k,h \in \mathcal{O}^m}$.

The condition 1 is equivalent to

$$\sum_{k \in \mathcal{O}} \lambda_{jk}^{ih}(t) = \sum_{k \in \mathcal{O}} \lambda_{jk}^{ih'}(t), \ \forall h, h' \in \mathcal{O}, \forall i, j \in \mathcal{S}, i \neq j$$

and

$$\sum_{j \in \mathcal{S}} \lambda_{jk}^{ih}(t) = \sum_{j \in \mathcal{S}} \lambda_{jk}^{i'h}(t), \ \forall i, i' \in \mathcal{S}, \forall h, k \in \mathcal{O}, h \neq k$$

with

$$f_j^i(t) := \sum_{k \in \mathcal{O}} \lambda_{jk}^{ih}(t) i, j \in \mathcal{S}, i \neq j, \ f_i^i(t) := -\sum_{j \in \mathcal{S}, i \neq j} f_j^i(t) \forall i \in \mathcal{S},$$

and

$$g_k^h(t) := \sum_{j \in \mathcal{S}} \lambda_{jk}^{ih}(t) k, h \in \mathcal{O}, h \neq k, \ g_h^h(t) := -\sum_{k \in \mathcal{O}, h \neq k} g_k^h(t) \forall h \in \mathcal{O},$$

where X , Y are Markov Chains with generator $A^X(t) = [f_j^i(t)]_{i,j \in \mathcal{S}}$ and $A^Y(t) = [g_k^h(t)]_{h,k \in \mathcal{O}}$ respectively

In Addressing Condition 2, [Bielecki11] states the following constraints on intensities:

Proposition 6.1.5. *two Markov chains X and Y and their filtrations and generator $A^X(t) = [\alpha^i_j(t)]_{i,j\in\mathcal{S}}$ and $A^Y(t) = [\beta^h_k(t)]_{h,k\in\mathcal{O}}$. Define System of equations in unknowns $\lambda^{ih}_{jk}(u)$ where $i, j \in \mathcal{S}, h, k \in \mathcal{O}, (i, h) \neq (j, k)$*

$$\sum_{k\in\mathcal{O}} \lambda^{ih}_{jk}(t) = \alpha^i_j(t),\ \forall h \in \mathcal{O}, \forall i, j \in \mathcal{S}, i \neq j$$

$$\sum_{j\in\mathcal{S}} \lambda^{ih}_{jk}(t) = \beta^h_k(t),\ \forall i \in \mathcal{S}, \forall h, k \in \mathcal{O}, h \neq k$$

Suppose system of equations admits solutions such that the matrix function $A(t) = [\lambda^{ih}_{jk}(t)]_{i,j\in\mathcal{S},h,k\in\mathcal{O}}$ with

$$\lambda^{ih}_{ih}(t) = -\sum_{(j,k)\in\mathcal{S}\times\mathcal{O},(j,k)\neq(i,h)} \lambda^{ih}_{jk}(t)$$

properly defines an infinitesimal generator.

Condition 1 and Condition 2 in order to build a MPH Markov copula are equivalent to the following proposition:

Proposition 6.1.6 (No simultaneous default Markov copulae)**.** *Under the no-simultaneous default assumption, the Markov copulas conditions either when Z is a Markov chain or when all X_i are Markov chains imply a no-contagion condition between those obligors identified by any X_i.*

Proof. We consider Markov chains with an absorption state and in a credit risk context i.e. $\mathcal{S} = \{0, 1\}$. Under the assumption of no-simultaneous default and an upper-triangular structure of the infinitesimal generator (post non-default impossibility), the above conditions imply Condition 1:

$$\lambda^0_p = \lambda^{|1|}_{|1|,p} = \lambda^{|2|}_{|2|,p} = \ldots = \lambda^{|m-1|}_{\Delta} \tag{6.22}$$

where $\mid k \mid, k \in E$ represents the number of obligor having already defaulted in Z with the exception of $p \in E$, $\{\mid k \mid, p\}$ identifies Z with $k + 1$ defaults including p

Comment: With this condition, obligor p default is insensitive with the previous defaults and . an the intensity of default of obligor p is equal to,

$$\alpha^0_1 = \lambda^0_p + \lambda^{|1|}_{|1|,p} + \lambda^{|2|}_{|2|,p} + \ldots + \lambda^{|m-1|}_{\Delta}$$

Condition 2: The condition starting from the markov chain Z is equal to

$$\alpha^0_1 = \lambda^0_p = \lambda^{|1|}_{|1|,p} = \lambda^{|2|}_{|2|,p} = \ldots = \lambda^{|m-1|}_{\Delta}, \tag{6.23}$$

Both structure either starting from Z or starting for X_i imply that under no-simultaneous default it is impossible to add credit contagion and build a Markov copulae structure for multivariate chains. [Crepey09] uses the 2 obligors Markov jump process with a Markov copulae structure to value the counterparty risk of a CDS. Under the previous *Markov consistency* joint defaults were necessary. Since our multivariate setting in section 4.1 does not allow for simulatenous default as per assumption 3.1.1 we will not work under the Markov Copula framework. Thus, we need to express multivariate probability distribution linked to the intensity matrix $\boldsymbol{Q}$.

□

Probability distribution definitions

[Herb10] established the following results for probability distributions under a phase-type distribution framework:

In the setting with m obligors, the probability of reaching the sequence $\mathbf{j} = \{j_1, .., j_m\}$ representing the ordering of default of obligors with the $\mid K-1 \mid \times \mid K-1 \mid$ transition matrix is given by

$$\mathbb{P}(Y_t = \mathbf{j}) = \alpha \exp^{\boldsymbol{Q}t} \boldsymbol{e_j} \quad \text{for} \quad \boldsymbol{e_j} \in \mathbb{R}^{|K-1|} \tag{6.24}$$

with $\boldsymbol{j} \in \boldsymbol{E}$ and $\boldsymbol{e_j}$ denote a column vector in $\mathbb{R}^{|\boldsymbol{E}|}$ where the entry at position $\boldsymbol{j}$ is 1 and the other entries are zero.

We now list some of the closed-form formula for multivariate default and survival distributions, marginal distributions.

The joint survival distribution: The following proposition was stated in [Assaf84]

Proposition 6.1.7. *Consider m obligors and the ordered vector $(t_1, \ldots, t_m) \in \mathbb{R}_+^m$ with $t_{i_1} < \ldots < t_{i_m}$ its ordering where $(i_1, \ldots, i_m)$ is permutation of $(t_1, \ldots, t_m)$, Then the joint survival distribution is given by*

$$\mathbb{Q}[\tau_1 > t_1, ..., \tau_m > t_m] = \alpha \left(\prod_{k=1}^{m} e^{\boldsymbol{Q}(t_{i_k} - t_{i_{k-1}})} \boldsymbol{G}_{i_k} \right) \quad \textit{where} \quad t_{i0} = 0 \tag{6.25}$$

with the status matrix $\boldsymbol{G}_i$ be $\mid K-1 \mid \times \mid K-1 \mid$ diagonal matrix (Recall $\boldsymbol{E}= K-1$), i.e. $(\boldsymbol{G}_i)_{\boldsymbol{j},\boldsymbol{j}'} = 0$ for $\boldsymbol{j} \neq \boldsymbol{j}'$ and

$$(\boldsymbol{G}_i)_{\boldsymbol{j},\boldsymbol{j}'} = \begin{cases} 1 & \text{if} \quad \boldsymbol{j} \in \boldsymbol{E}_i \\ 0 & \text{if} \quad \boldsymbol{j}' \notin \boldsymbol{E}_i \end{cases} \tag{6.26}$$

where $\boldsymbol{E}_i$ describes the state in matrix $\boldsymbol{Q}$ where obligor i is non-defaulted.

The framework in [Assaf84] and expanded in [Herb10] enables to formulate multivariate probability distributions via Markov theory and equation (6.24) through the identification of Status matrices like (6.26) are used to identify the permutation cases that are in relation with the various distributions expressing default or survival ordering scenarios.

We state the expressions from [Herb10] for the marginal survival distributions $\mathbb{Q}(\tau_i > t)$, $\mathbb{Q}(\tau^{(k)} > t)$, and $\mathbb{Q}(\tau^{(k)} > t, \tau^{(k)} = \tau_i)$ using the ordered sequence of defaults $(\tau^{(k)})_{k=1}^m$ for a set of m obligors that corresponds to our permutation setting like in section 4.3.

$$\tau^{(0)} = 0 \tag{6.27}$$

$$\tau^{(k)} = \inf\{\tau_i : 1 \leq i \leq m; \tau_i > \tau^{(k-1)}\}, \ k = 1, \ldots, m \tag{6.28}$$

The marginal survival distributions $\mathbb{Q}(\tau_i > t)$ are needed to compute single-name CDS while the ordered marginal survival distributions and related quantities, $F_k(t) = \mathbb{Q}[\tau^{(k)} > t]$ and $F_{k,i}(t) =$

$\mathbb{Q}[\tau^{(k)} > t, \tau^{(k)} = \tau_i]$ for $i = 1, \ldots, m$, are needed to price basket default swaps such as k^{th}-to default swaps.

The marginal survival distributions:

Lemma 6.1.3. *Consider the m obligors in E whose default time τ_i are defined as in proposition 6.1.4. Then the marginal survival distribution and the ordered marginal survival distribution are given by,*

$$\mathbb{Q}[\tau_i > t] = \alpha e^{\boldsymbol{Q}t} \boldsymbol{g}^{(i)} \quad \textit{and} \quad \mathbb{Q}[\tau^{(k)} > t] = \alpha e^{\boldsymbol{Q}t} \boldsymbol{m}^{(k)} \tag{6.29}$$

where the column vectors $\boldsymbol{g}^{(i)}, \boldsymbol{m}^{(k)}$ of length $\mid K \mid$ are defined as

$$\boldsymbol{g}^{(i)_j} = 1_{\{\boldsymbol{j} \in (\Delta_i)^C\}} \quad \textit{and} \quad \boldsymbol{m}_{\boldsymbol{j}}^{(k)} = 1_{\{\boldsymbol{j} \in \cup_{n=0}^{k-1} E_n\}} \tag{6.30}$$

where $(\Delta_i)^C$ is the set in which obligor i is not defaulted and E_n is the set consisting of n elements of $\{1, \ldots, m\} having defaulted$.

In this case the status matrix of the marginal survival distribution $\boldsymbol{g}^{(i)}$ represents an identification of all the states in K where obligor i is non defaulted. The ordered marginal survival distribution status matrix $\boldsymbol{m}^{(k)}$ represents the states in K where there has been less than k defaults.

The marginal ordered k^{th} survival distribution: The marginal k^{th} survival distribution will be of interest for the multi-name credit derivatives products since those products are more exposed to the contagion impact by being more senior in the capital structure and having embedded leverage. However it will be of interest to identify which obligor $i \in E$ is the k^{th} entity to default. Thus, the closed-form formula $\mathbb{Q}(\tau^{(k)} > t, \tau^{(k)} = \tau_i)$ will be key to capture cases of wrong-way risk and k^{th}-to-default spreads where the interest of having credit protection is increasing at the same time the counterparty credit default becomes more likely.

Lemma 6.1.4. *Consider the m obligors in E whose default time τ_i are defined as in proposition 6.1.4. Then the the marginal ordered k^{th} survival distribution which is the probability that the k^{th} default is identified by obligor i and does not occur before t is expressed as,*

$$\begin{aligned}
\mathbb{Q}(\tau^{(k)} > t, \tau^{(k)} = \tau_i) &= \alpha e^{\boldsymbol{Q}t} \left(\sum_{l=0}^{k-1} \left(\prod_{p=l}^{k-1} \boldsymbol{G}^{i,p} \boldsymbol{P} \right) \boldsymbol{h}^{i,k} \right) \\
&= \alpha e^{\boldsymbol{Q}t} \left(\sum_{l=0}^{k-1} \left(\prod_{p=l}^{k-1} \boldsymbol{G}^{i,p} \boldsymbol{P} \right) \boldsymbol{H}^{i,k} \boldsymbol{1} \right) \\
&= \alpha e^{\boldsymbol{Q}t} \boldsymbol{R}^{i,k} \boldsymbol{H}^{i,k} \boldsymbol{1}
\end{aligned} \tag{6.31}$$

for $k = 1, ..., m$, where

$$\boldsymbol{P}_{j,j'} = \frac{\boldsymbol{Q}_{j,j'}}{\Sigma_{k \neq j} \boldsymbol{Q}_{j,k}} \qquad j, j' \in \boldsymbol{E}$$

and

$$\boldsymbol{R}^{i,k} = \sum_{l=0}^{k-1} \left(\prod_{p=l}^{k-1} \boldsymbol{G}^{i,p} \boldsymbol{P} \right)$$

$$\textit{with} \quad \boldsymbol{h}_j^{i,k} = 1_{\{j \in (\Delta_i) \cap \boldsymbol{E}_k\}} \ \ \textit{and} \ \ \boldsymbol{G}_{j,j}^{i,k} = 1_{\{j \in (\Delta_i)^C \cap \boldsymbol{E}_k\}} \ \ \textit{and} \ \ \boldsymbol{G}_{j,j'}^{i,k} = 0 \ \ if \ \ j \neq j'$$

$$\textit{and} \quad \boldsymbol{H}_{j,j}^{i,k} = 1 \ \ \textit{if} \ \ j \in \Delta_k^i \ \ , \boldsymbol{H}_{j,j}^{i,k} = 0 \ \ \textit{if} \ \ j \neq \Delta_k^i \ \ \textit{and} \ \ \boldsymbol{H}_{j,j'}^{i,k} = 0 \ \ \textit{if} \ \ j \neq j'$$

where $\boldsymbol{h}^{i,k}$ *is a column vectors of length* $\mid K \mid$ *and* $\boldsymbol{G}^{i,k}, \boldsymbol{H}^{i,k}$ *are* $\mid K \mid \times \mid K \mid$ *diagonal matrices.*

The status matrix $\boldsymbol{H}_{j,j'}^{i,k}$ or $\boldsymbol{h}_j^{i,k}$ represent an identification of all the states in K where k obligors have defaulted and obligor i is not defaulted. The status matrix $\boldsymbol{G}^{i,k}$ is similar in structure but represent the case where obligor i has defaulted

Proof. The proof in [Herb10] is established by regrouping the states of $\boldsymbol{Q}$ by considering a sub-Markov Chain $(Y_n)_{t \geq 0}$ representing the progress of a chain representing the number of obligors having defaulted. Using the conditional probabilities of transition properties

$$\mathbb{Q}\{C_{t_n} = j \mid C_{t_{n-1}} = i\} = p_{ij} := -\frac{\lambda_{ij}}{\lambda_{ii}}, \ \forall i, j \in \mathcal{K}, i \neq j$$

the intensity of states are rewritten according to the matrix of probabilities

$$\boldsymbol{P}_{j,j'} = \frac{\boldsymbol{Q}_{j,j'}}{\Sigma_{k \neq j} \boldsymbol{Q}_{j,k}} \qquad j, j' \in \boldsymbol{E}$$

□

With the above distributions, we derive now the closed-form solutions for single-name CDS spreads and k^{th}-to-default swaps.

The k^{th}-to-default swap spread and the single-name CDS spread: For the obligor set E, the fair spread $\kappa^{(k)}(T)$ at time 0 of a k^{th}-to-default swaps of maturity T is defined for $n-1$ payment dates over $[0, T]$, for m obligors as

$$\kappa^{(k)}(T) = \frac{\sum_{j=1}^{m} LGD_j \int_0^T \beta_s dF_{k,j}(s)}{\sum_{j=1}^{n} \left(D(0, t_j)\theta_j \mathbb{Q}[T_k > t_j] + \int_{t_{j-1}}^{t_j} \beta_s (s - t_{j-1}) dF_k(s) \right)} \tag{6.32}$$

using the previous expressions for marginal distributions:

$$F_k(s) = \mathbb{Q}[\tau^{(k)} > t] = \alpha e^{\boldsymbol{T}t} \boldsymbol{M}_k \boldsymbol{1}$$

and

$$F_{k,j}(t) = \mathbb{Q}[\tau^{(k)} \leq t, \tau^{(k)} = \tau_i] = \alpha e^{\boldsymbol{Q}t} \boldsymbol{R}^{i,k} \boldsymbol{H}^{i,k} \boldsymbol{1}$$

[Herb10] expressed the following integrals

$$\int_0^T \boldsymbol{Q} e^{(\boldsymbol{Q}-r\boldsymbol{I})t} dt \quad \text{and} \quad \int_0^T t\,\boldsymbol{Q} e^{(\boldsymbol{Q}-r\boldsymbol{I})t} dt$$

in the multivariate phase type distribution, leading to the proposition

Proposition 6.1.8. *Consider the m obligors in E whose default time τ_i are defined as in proposition 6.1.4 and assume that the interest rate r is constant, then $\kappa^{(k)}(T)$, the k^{th} to default swap premium, is expressed in the multi-variate phase type distribution framework as :*

$$\kappa^{(k)}(T) = \frac{\alpha(\boldsymbol{A}(0) - \boldsymbol{A}(T))\boldsymbol{\Phi_k 1}}{\alpha\left(\sum_{j=1}^{n}\left(\theta_j e^{(\boldsymbol{Q}-r\boldsymbol{I})t_j} + \boldsymbol{C}(t_{j-1}, t_j)\right)\right)\boldsymbol{M}_k \boldsymbol{1}} \tag{6.33}$$

where

$$\boldsymbol{\Phi_k} = \sum_{i=1}^{m}(LGD_i)\boldsymbol{R}^{i,k}\boldsymbol{H}^{i,k}$$

$$\boldsymbol{C}(s,t) = s(\boldsymbol{A}(t) - \boldsymbol{A}(s)) - \boldsymbol{B}(t) + \boldsymbol{B}(s) \text{ for } \boldsymbol{A}(t) = \boldsymbol{V}(\boldsymbol{I} + r\,\boldsymbol{W})^{-1})e^{\boldsymbol{W}t}\boldsymbol{V}^{-1}$$

$$\boldsymbol{B}(t) = \boldsymbol{V}(\boldsymbol{W} + r\boldsymbol{I})(t\boldsymbol{I} - \boldsymbol{W}^{-1})\boldsymbol{W}^{-1}e^{\boldsymbol{W}t}\boldsymbol{V}^{-1}$$

and

$$e^{(\boldsymbol{Q}-r\boldsymbol{I})t} = \boldsymbol{V}e^{\boldsymbol{W}t}\boldsymbol{V}^{-1}$$

where $\boldsymbol{W}$ is a diagonal matrix where the diagonal elements are the eigenvalues of $\boldsymbol{Q} - r\boldsymbol{I}$ and the columns of the matrix $\boldsymbol{V}$ are the corresponding eigenvectors.

Similarly since the single-name CDS is a k^{th}-to-default swaps with $k = 1$ we can rewrite the status matrix

Proposition 6.1.9. *Consider the m obligors in E whose default time τ_i are defined as in proposition 6.1.4 and assume that the interest rate r is constant. Then $S_i(T)$, the k^{th} to single-name credit default swap spread for obligor i, is expressed in the multi-variate phase type distribution framework as :*

$$S_i(T) = \frac{(1-\phi_i)\alpha(\boldsymbol{A}(0) - \boldsymbol{A}(T))\boldsymbol{g}^{(i)}}{\alpha\left(\sum_{n=1}^{n_T}\left(\theta_n e^{\boldsymbol{Q}t_n}e^{-rt_n} + \boldsymbol{C}(t_{n-1}, t_n)\right)\right)\boldsymbol{g}^{(i)}} \tag{6.34}$$

where

$$\boldsymbol{C}(s,t) = s(\boldsymbol{A}(t) - \boldsymbol{A}(s)) - \boldsymbol{B}(t) + \boldsymbol{B}(s)$$

$$\boldsymbol{A}(t) = e^{\boldsymbol{Q}t}(\boldsymbol{Q} - r\boldsymbol{I})^{-1}\boldsymbol{Q}e^{-rt}$$

$$\boldsymbol{B}(t) = e^{\boldsymbol{Q}t}(t\boldsymbol{I} + (\boldsymbol{Q} - r\boldsymbol{I})^{-1})(\boldsymbol{Q} - r\boldsymbol{I})^{-1}\boldsymbol{Q}e^{-rt}$$

The propositions 6.1.8 and 6.1.9 will be useful in extracting from quoted spreads in the market the matrix $\boldsymbol{Q}$ where the intensities $[\lambda_{i,j}]_{1\le i,j\le K}$ will be structured to identify the ordering of defaulted obligors. Let's review now how to encompass credit dependency in the $\lambda_{i,j}$ such that obligors exhibit credit contagion.

Credit contagion definitions

Until now, we haven't made any specific assumptions on the functional aspect of the intensities $[\lambda_{i,j}]_{1\le i,j\le K}$ of $\boldsymbol{Q}$ except the upper-triangular shape due to the fact that Y_t evolves from no default to the absorption state Δ. Going back to section 5.2.1, page 79, on dependence modelling, [Jarrow01] introduced a contagion intensity model that seems the most interesting in capturing credit contagion in a credit basket claim. In a identical approach as [Jarrow01], [Herb10] categorises the obligors into infector and infected sets with the difference that each obligors can be in both sets. We say that the obligors $i = 1, 2, \ldots, m$ undergo default contagion if for every $i \in \{1, \ldots, m\}$ at least one of the following conditions holds

- $\exists$ set $I_i \neq \emptyset$ and $I_i \subseteq \{1, \ldots, m\}/\{i\}$, $\Delta\lambda^i_{\{t=\tau_j\}} \neq 0 \; \forall j \in I_i$

- $\exists$ set $F_i \neq \emptyset$ and $F_i \subseteq \{1, \ldots, m\}/\{i\}$, $\Delta\lambda^j_{\{t=\tau_i\}} \neq 0 \; \forall j \in F_i$

where I_i is the i-infector set and F_i is the i-infected set.

This will fit well our permutation model. Let $\Pi(m)$ be the set of permutation $\{\pi_1, \ldots, \pi_m\}$ of $\{1, \ldots, m\}$ that will identify the order of the defaults of the entities of the obligor set while the Markov Jump process $(Y_t)_{t\ge 0}$ evolves towards the absorption state Δ.

Each identified permutation $\pi \in \Pi(m), \pi = \{\pi_1, \ldots, \pi_m\}$ with the set of corresponding ordered defaults $\{\tau_{\pi_1} < \tau_{\pi_2} < \ldots < \tau_{\pi_m}\}$ will correspond to the multiple defaults event

$$1_{\{\tau_{\pi_1} < \tau_{\pi_2} < \ldots < \tau_{\pi_m}\}}$$

Note that simultaneous defaults in the set E are also excluded.

Ordered versus non-ordered default contagion: We say that the obligors $i = 1, 2, \ldots, m$ undergo ordered default contagion if they undergo default contagion and if the following conditions hold, for at least one obligor $i \in \{1, \ldots, m\}$

$$\exists\, \pi, \pi^{'} \in \Pi(m), \exists\, j \in I_i,\ \Delta\lambda^i_{\{t=\tau_j\}} 1_\pi \neq \Delta\lambda^i_{\{t=\tau_j\}} 1_{\pi'}$$

The intuitive meaning of ordered default contagion for a set of obligors is that the obligors not only affect each other through their defaults, but also the order in which the obligors default. Equivalently, obligors undergo simple default contagion if the order of defaults of obligors is irrelevant, thus reducing the combinatronics of the permutation set $\Pi(m)$.

It is commonly believed that at the default of one firm, say firm j, the intensity λ^i_t, $i \neq j$ will jump, normally upwards. It is this feature that is captured in the default intensity definition:

The typical cardinality of phases will be

- $\boldsymbol{K} = \sum_{n=0}^{m-1} n! C^m_n$ for inhomogeneous portfolio with ordered defaults,

- $\boldsymbol{K} = 2^m - 1$ for inhomogeneous portfolio with unordered defaults,

- $\boldsymbol{K} = m + 1$ for an homogeneous portfolio.

Default intensity definitions

The model studied in [Herbert05] is specified by requiring that the default intensities have the following form:

$$\lambda_{t,i} = a_i + \sum_{j \neq i} b_{i,j} \mathbb{1}_{\{\tau_j \leq t\}}, \qquad t \leq \tau_i, i, j \in E \tag{6.35}$$

with

- $\lambda_{t,i} = 0 \; \forall t > \tau_i$, and,
- $a_i, b_{i,j} \geq 0$ constant such that $\lambda_{t,i}$ is non negative.

The financial interpretation of (6.35) is that the default intensities are constant, except at the times when default occur: then the default intensity for obligor i jumps by an amount $b_{i,j}$ if it is obligor j which has defaulted. $b_{i,j}$ represents the contagion aspect by encapsulating the impact on the default intensity of i of an earlier default of obligor j. Thus, a positive $b_{i,j}$ means that obligor i is put at higher risk by the default of obligor j, while a negative $b_{i,j}$ means that obligor i in fact benefits from the default of j, and finally $b_{i,j}$=0 if obligor i is unaffected by the default of j. This framework was first developed in [Assaf84] and [Herb08b] . In the remainder of the thesis,

- a_i will define the base intensity,
- $b_{i,j}$ will define the jump intensity.

We will consider the same intensities as in equation (6.35) since this model captures well the contagion with a distinct contagion parameter through the jump intensities $b_{i,j}, \forall i, j \in E$. Thus, the states identified along the matrix $\boldsymbol{Q}$ with $[\lambda_{i,j}]_{1 \leq i,j \leq K}$ have to be identified along the chain Y_t with the corresponding intensity in equation (6.35) as a function of the obligors.

$$\boldsymbol{Q} = [\lambda_{i,j}]_{1 \leq i,j \leq K} = \begin{pmatrix} -a_1 - a_2 & a_1 & a_2 & 0 \\ 0 & -(a_1 + b_{1,2}) & 0 & a_1 + b_{1,2} \\ 0 & 0 & -(a_2 + b_{1,2}) & a_2 + b_{1,2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Computation of the Exponential matrix having further specified the intensities according to equation (6.35), we can adjust accordingly the conditions on the computation of the Exponential matrix. In the case of the multivariate phase type distributions presented earlier, the construction of an upper triangular matrix $\boldsymbol{T}$ with negative elements on the diagonal will ensure that all the eigenvalues are thus negative. The Pade-Approximation method will thus be non singular. Numerically, the choice of p and q will determine prescribed accuracy with

$$cond(D_{pq}[\boldsymbol{Q}]) \simeq cond(e^{\boldsymbol{Q}/2}) \geq e^{(\lambda_1 - \lambda_n)/2}$$

with $\lambda_1 = 1 > ... \geq \lambda_n$ are the real part of the eigenvalues $\boldsymbol{Q}$. This will be the method used in the computation of the exponential matrix for the calibration to default probability embedded in credit derivatives claims. The functions is implemented in Matlab as *'expm()'*.

Additionally, the multivariate phase-type distribution introduced earlier presents non confluent eigenvalues (Confluent eigenvalues occurs when the matrix $\boldsymbol{Q}$ does not have linearly independent vectors.) and are a good cases for matrix decomposition methods. Another improvement can be obtained with the use of triangular estimation of the matrix $\boldsymbol{V}$ of eigenvectors using the QR algorithm. This is the method used in [Herb10].

We will use the Pade-Approximation method since the eigenvalues ensure the convergence of the polynomials and the implementation in vector-type software like Matlab ensures model flexibility. For example, since the calibration is done typically by a numerical optimisation (see next section), the parameter vector and objective function are given by handle functions (usually using a pointer address) and thus allowing to try several models of intensity $\boldsymbol{Q} = [\lambda_{i,j}]_{1 \leq i,j \leq K}$, time-series calibrations or concurrent calibrations. Finally, in section 6.3, page 128, we will model a non upper triangular infinitesimal generator $\boldsymbol{Q}$, thus loosing the capacity of implementation provided by the decomposition method.

Calibration

From the equations (6.33) and (6.34), page 108, all the parameters including the contagion parameters $b_{i,j}$ of the intensity in equation (6.35), page 110, determine the marginal distributions $\mathbb{Q}(\tau_i > t), \mathbb{Q}(\tau^{(k)} > t), \mathbb{Q}(\tau^{(k)} > t, \tau^{(k)} = \tau_i)$ in a single calibration process. By opposition, in the widely used copula model the probability distributions $\mathbb{Q}(\tau_i > t)$ are modelled by idiosyncratic parameters unique for each obligor. Then, the joint dependence is introduced by the copula and its parameters, which are separated from the parameters describing each individual default distribution. Thus, the MPH model presents the functionality of calibrating the idiosyncratic intensity parameter a_i and the dependence or contagion intensity $b_{i,j}$ with $i, j \in E$.

Single name credit default swaps in equation (6.34) provide a direct link between the marginal distribution $\mathbb{Q}(\tau_i > t)$, and the intensities parameters a_i and $b_{i,j}$. Thus, as it is common practice, that the quoted CDS spreads are viewed as proxy of market implied default intensity, thus enabling the extraction of parameters a_i and $b_{i,j}$ under the risk-neutral probability measure $\mathbb{Q}$.

In order to “calibrate”the parameters a_i and $b_{i,j}$, [Cont10] encyclopaedia provides a good overview of all the calibration methods applied in finance, from the typical formula inversion of Black-Scholes formula to iterative numerical minimisation methods. For the matrix analytical closed-form formula in the multivariate phase type distribution framework, numerical minimisation techniques will be required to establish the base intensities a_i and the jump intensities $b_{\{i,j\}}$ that equates the market spread quotes with the model. Most of the methods are presented in [Nocedal06], [Fletcher63],[Berndt74],[More77] and [Marquardt63], and are multivariate versions of the steepest descent method where the direction is set by the Jacobian matrix.

Using the Levenberg-Marquardt non-linear least squares minimisation (Matlab function “lsqnonlin”), we need to estimate the vector $\theta = (\boldsymbol{a}, \boldsymbol{b})$ from CDS quoted spreads for a portfolio of inhomogeneous obligors.

The intensities in a m obligors set, will be computed with:

$$\theta = \text{argmin}_{\hat{\theta}} \sum_{j=1}^{m} [S_t^j(\kappa, T; \hat{\theta}) - S_t^j(\kappa, T)]^2 \tag{6.36}$$

with $S_t^j(\kappa, T)$ being the value of the quoted CDS with spread κ of obligor j and maturity T and $S_t^j(\kappa, T; \hat{\theta})$ being the value of the CDS with the computed spread κ of obligor j and maturity T obtained using the MPH model with parameters $\hat{\theta}$ according to equation (6.34), page 108.

However we can point here that the dimension of vector θ is greater than m leading to an overspecified model where some the parameters have to be fixed or other quoted products have to be included such as a time-series calibration or the calibration process has to be broken or "cascaded"into different steps.

[Herb08b] contains a calibration for a non-homogeneous portfolio in the case of 15 obligor names but the model is "semi-calibrated"for that portfolio. The calibration assumes that the relative dependence structure $b_{i,j}$ in (6.35) are exogenously given, hence the term "semi-calibration"and thus only the base intensities a_i are then obtained by individual calibration to the market CDS spreads. The reason for the "semi-calibration"is that the minimisation technique with a high level of depend parameters a_i and $b_{\{i,j\}}$ take times and is very sensitive to the starting point. The calibration process (6.36) is thus rewritten

$$\mathbf{a} = (a_1, ..., a_m) = \text{argmin}_{\hat{a}} \sum_{j=1}^{m} [S_t^j(\kappa, T, \mathbf{b}; \hat{\mathbf{a}}) - S_t^j(\kappa, T)]^2 \tag{6.37}$$

where the m component of vector $\mathbf{a}$ are calibrated to m CDS spreads. This calibration is implemented in Matlab as previously presented for a set of single-name credit default swaps. We list the results of the "semi-calibrated"calibration in the appendix B.1, page 157. The reproduced spreads of the 10 year tenor are in a range of error negligible ($\approx 10^{-6}\%$) of the quoted spread with intensities typically of the order of $5\ 10^{-3} - 5\ 10^{-2}$.

In order to avoid the "semi-calibration"issue, the calibration can be structured to extract more information from market implied data and use time-series of spreads (5-yr and 10-yr quotes for example) instead of one spread (5-yr) quotes per obligor. The main reason is to enable the calibration of the base intensity a_i and the jump intensity $b_{\{i,j\}}$ at the same time. The calibration process (6.37) is thus rewritten

$$\theta = \text{argmin}_{\hat{\theta}} \sum_{j=1}^{m} [S_t^j(\kappa, T_1; \hat{\theta}) - S_t^j(\kappa, T_1)]^2 + [S_t^j(\kappa, T_2; \hat{\theta}) - S_t^j(\kappa, T_2)]^2 \tag{6.38}$$

with T_1 and T_2 two maturity date. (5-yr and 10-yr quotes in our case).

We implement several variations of equation (6.38), with

- a simultaneous calibration of $\mathbf{a}$ and $\mathbf{b}$,

- an alternate calibration between **a** and **b** at each iteration,
- a successive calibration starting with **a** for a pre-fixed **b**, then calibrate **b** keeping **a** constant.

In terms of numerical errors in the fitting process of all those approaches, the time-series calibration provides a difficult calibration in terms of residual errors and most of the time fails to converge and thus should not be used to calibration base a_i and jump $b_{i,j}$ intensities instead of using the "semi-calibration" process even if the choice of the jump intensity $b_{i,j}$ in the "semi-calibration" will have an impact on the obtained base intensity a_i. The time series calibration is also very difficult for flat or inverted CDS curve and then generate high error of calibration. Conditional to numerical implementation error in our part, we believe one of the main reason of the failure of convergence is linked to the "exponential shape "of the MPH and thus, while this shape is not important in calibrating a single tenor it is a problem in calibrating two tenors at the same time especially if the CDS market curve exhibits a flat or inverted shape. Thus, a time-series calibration would require a piecewise constant intensity vector **a** but would still provide no information about the vector **b**.

Another way of estimating **b** from quoted quotes is to use index credit derivatives like ITRAXX. [Herb08a] adapted the MPH framework where the obligors are homogeneous and, thus, cannot be identified. In this case, the markov chains $(Y_t)_{t\geq 0}$ represents the number of obligors having defaulted towards an absorption state Δ that still represents the state in which all the obligors in E have defaulted. [Herb08a] is able to calibrate a vector $\mathbf{b} = (b_1, \ldots, b_6)$ representing an homogeneous contagion across each tranche of the Index. This method could provide some idea about the contagion level **b** to use in "semi-calibration" process.

Credit dependence via MPH

Since it is difficult to calibrate $\theta = (\mathbf{a}, \mathbf{b})$ simultaneously, it is still possible from the range of values of vector **a** calibrated with equation 6.37 to simulate the impact on the marginal distributions $\mathbb{Q}(\tau_i > t), \mathbb{Q}(\tau^{(k)} > t)$ and $\mathbb{Q}(\tau^{(k)} > t, \tau^{(k)} = \tau_i)$ for a range of values of vector **b**. Thus, the dependency of a_i and $b_{i,j}$ of a set of obligors E can be analysed in a credit contagion with the probability distributions presented in appendix B.2, page 159. This credit dependence will be used in a Counterparty Credit Risk context in the next section.

6.2 Counterparty Credit Risk with Phase type distribution

6.2.1 Credit Valuation Formula

multivariate hazard processes in phase-type distribution

We try now turn to the expression of the probability $\mathbb{Q}[\sum_{\pi\in\Pi} 1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}}1_{\{\tau<T\}} \mid \mathcal{F}_t]$ for the trigger set $\mathcal{T}$ in a closed-form format using the previously introduced multivariate phase-type distributions presented from section 6.1.2, page 100. Additionally, we consider in the reminder of document the loss given default LGD and its associated recovery value REC as constants for each obligor i. Also the probability $\mathbb{Q}[\sum_{\pi\in\Pi} 1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}}1_{\{\tau<T\}} \mid \mathcal{F}_t]$ is independent of the filtration $\mathcal{F}_t$ as we can conceptually consider that interest rates or commodities state variables that are $\mathcal{F}$-measurable do not influence the observations of default times through the filtration $\mathcal{H}$. However, with this simplification it will not be possible to assess the impact of Counterparty Credit Risk of claims under correlation between the interest rates and credit risk.

We know from the marginal survival distribution in equation (6.29), page 106, is a function of the exponential of matrix $\boldsymbol{Q}$ and a status matrix.

$$\mathbb{P}[\tau_i > t] = \mathbb{P}[Y_t \in \boldsymbol{E}_i] = \alpha e^{\boldsymbol{T}t}\boldsymbol{G}_i\boldsymbol{1} = \alpha e^{\boldsymbol{Q}t}\boldsymbol{g}^{(i)}$$

with $g^{(i)}$ a status matrix identical to $\boldsymbol{G}_i$ but incorporating the absorption state in the dimensionality and $\boldsymbol{G}_i$ be $|\boldsymbol{E}| \times |\boldsymbol{E}|$ diagonal matrix, i.e. $(\boldsymbol{G}_i)_{\boldsymbol{j},\boldsymbol{j}'} = 0$ for $\boldsymbol{j}\neq\boldsymbol{j}'$ and

$$(\boldsymbol{G}_i)_{\boldsymbol{j},\boldsymbol{j}'} = \begin{cases} 1 & \text{if} \quad \boldsymbol{j} \in \boldsymbol{E}_i \\ 0 & \text{if} \quad \boldsymbol{j}' \notin \boldsymbol{E}_i \end{cases}$$

where $\boldsymbol{E}_i$ describes the state in matrix $\boldsymbol{Q}$ where obligor i is non-defaulted.

For the non defaultable claim and the defaultable claim, we need to express in the multivariate phase type distribution framework the probability distribution

$$\mathbb{Q}[\sum_{\pi\in\Pi} 1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}}1_{\{\tau<T\}}]$$

of the events in the trigger set $\mathcal{T}$. It is interesting to note here that the event of the permutation set $1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}}$ corresponds to a state in the matrix $\boldsymbol{Q}$ since the matrix states evolve through communicating states according to an ordering of no-defaults towards absorption state Δ. We can write in the case of the counterparty C

$$
\begin{aligned}
\mathbb{Q}[\sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}}1_{\{\tau>t\}}] &= \mathbb{Q}[\tau_C > t] \\
&= \mathbb{Q}[Y_t \in \boldsymbol{E}_C] \\
&= \alpha e^{\boldsymbol{T}t}\boldsymbol{G}_C\boldsymbol{1} \qquad (6.39)
\end{aligned}
$$

with the status matrix $\boldsymbol{G}_C$ be $|\boldsymbol{E}| \times |\boldsymbol{E}|$ diagonal matrix, i.e. $(\boldsymbol{G}_i)_{\boldsymbol{j},\boldsymbol{j}'} = 0$ for $\boldsymbol{j}\neq\boldsymbol{j}'$ and

$$
(\boldsymbol{G}_i)_{\boldsymbol{j},\boldsymbol{j}'} = \begin{cases} 1 & \text{if} \quad \boldsymbol{j} \in \boldsymbol{E}_C \\ 0 & \text{if} \quad \boldsymbol{j}' \notin \boldsymbol{E}_C \end{cases}
$$

where $\boldsymbol{E}_C$ describes the state in matrix $\boldsymbol{Q}$ where obligor C is non-defaulted. Similar matrices $\boldsymbol{G}_I$ and $\boldsymbol{G}_{Ref}$ can be obtained. Thus, we have

$$
\begin{aligned}
\mathbb{Q}[\sum_{\pi\in\Pi^C} 1_{\{\tau(\pi-)<\tau_C<\tau(\pi+)\}}1_{\{t<\tau_C<T\}}] &= \mathbb{Q}[\tau_C > t] - \mathbb{Q}[\tau_C > T] \\
&= \mathbb{Q}[Y_t \in \boldsymbol{E}_C] - \mathbb{Q}[Y_T \in \boldsymbol{E}_C] \\
&= \alpha e^{\boldsymbol{T}t}\boldsymbol{G}_C\boldsymbol{1} - \alpha e^{\boldsymbol{T}T}\boldsymbol{G}_C\boldsymbol{1}
\end{aligned}
$$

with the probabillities $\mathbb{Q}[\sum_{\pi\in\Pi^I} 1_{\{\tau(\pi-)<\tau_I<\tau(\pi+)\}}1_{\{\tau_I<T\}}]$ and $\mathbb{Q}[\sum_{\pi\in\Pi^R} 1_{\{\tau(\pi-)<\tau_R<\tau(\pi+)\}}1_{\tau_R<T}]$ defined in a same way. The probability $\mathbb{Q}[1_{\{t<\tau\leq T\}}\sum_{\pi\in\Pi} 1_{\{\tau(\pi-)<\tau<\tau(\pi+)\}}]$ will be identified with the status matrix $\boldsymbol{G}_{C,I}$ be $|\boldsymbol{E}| \times |\boldsymbol{E}|$ diagonal matrix, i.e. $(\boldsymbol{G}_{C,I})_{\boldsymbol{j},\boldsymbol{j}'} = 0$ for $\boldsymbol{j}\neq\boldsymbol{j}'$ and

$$
(\boldsymbol{G}_{C,I})_{\boldsymbol{j},\boldsymbol{j}'} = \begin{cases} 1 & \text{if} \quad \boldsymbol{j} \in \boldsymbol{E}_{C,I} \\ 0 & \text{if} \quad \boldsymbol{j}' \notin \boldsymbol{E}_{C,I} \end{cases}
$$

where $\boldsymbol{E}_{C,I}$ describes the state in matrix $\boldsymbol{Q}$ where obligor C or I have non-defaulted.

We now express the propositions 5.1.6 and 5.1.7 for valuing the BCVA adjustments of non defaultable and defaultable contingent claims. However to express the counterparty adjustment part since we have closed-form formula for obligors of the set $\mathcal{T}$ for survival distribution but not for its corresponding intensity, we approximate the formula by discretising the interval $(t, T]$ with a time grid $t_0 = t, t_1, \ldots, t_n = T$. For simplicity, in line with numerical simulations, we present the result from time $t = 0$.

Non defaultable claim CCR in phase-type distribution

We rewrite the proposition 5.1.6, page 76 of non defaultable claim in a multivariate setting using the hazard processes defined in equation 5.2.

Proposition 6.2.1 (Bilateral CVA for non-defaultble claim in multivariate setting)**.** *Consider an $\mathcal{F}_T$-measurable non-defaultable claim X defined in (3.1) under bilateral counterpart credit risk as*

$$
\begin{aligned}
X(0,T)^d &= \mathbb{E}\left(\int_0^T \beta_u dX_u\right) \\
&- LGD_C\mathbb{E}(\sum_{i=1}^{n}(\alpha e^{\boldsymbol{T}t_{i-1}}\boldsymbol{G}_C\boldsymbol{1} - \alpha e^{\boldsymbol{T}t_i}\boldsymbol{G}_C\boldsymbol{1})\beta_{\tau_C}X(\tau_C)^+) \\
&+ LGD_I\mathbb{E}(\sum_{i=1}^{n}(\alpha e^{\boldsymbol{T}t_{i-1}}\boldsymbol{G}_I\boldsymbol{1} - \alpha e^{\boldsymbol{T}t_i}\boldsymbol{G}_I\boldsymbol{1})\beta_{\tau_I}X(\tau_I)^-) \\
&= X(T) - BCVA \qquad (6.40)
\end{aligned}
$$

with

$$
\begin{aligned}
BCVA &= CVA - DVA \\
&= LGD_C\mathbb{E}(\sum_{i=1}^{n}(\alpha e^{\boldsymbol{T}t_{i-1}}\boldsymbol{G}_C\boldsymbol{1} - \alpha e^{\boldsymbol{T}t_i}\boldsymbol{G}_C\boldsymbol{1})\beta_{\tau_C}X(\tau_C)^+) \\
&- LGD_I\mathbb{E}(\sum_{i=1}^{n}(\alpha e^{\boldsymbol{T}t_{i-1}}\boldsymbol{G}_I\boldsymbol{1} - \alpha e^{\boldsymbol{T}t_i}\boldsymbol{G}_I\boldsymbol{1})\beta_{\tau_I}X(\tau_I)^-) \\
&= \mathbb{E}(\sum_{i=0}^{n}(\alpha e^{\boldsymbol{T}t_{i-1}}\boldsymbol{G}_{C,I}\boldsymbol{1} - \alpha e^{\boldsymbol{T}t_i}\boldsymbol{G}_{C,I}\boldsymbol{1})\beta_{\tau}[ED(\tau)]) \qquad (6.41)
\end{aligned}
$$

Remark that in the numerical simulation, the first term of equation (6.40) will also be discretised according to $t_0, \ldots, t_n$ with discretisation scheme (Euler scheme, Milstein scheme, etc.) depending on the claim X that is valued.

Defaultable claim CCR in phase-type distribution

We rewrite the proposition 5.1.7, page 76 of non defaultable claim in a multivariate setting using the hazard processes defined in equation 5.2. We suppose here with the discretisation of the interval $[0,T]$ that the processes A, Z and X^d can also be discretised with bucketing, i.e postponing the payments Z_τ or the accumulated payments A_s until τ to the first t_i following τ.

Proposition 6.2.2 (Bilateral CVA for defaultble claim in multivariate setting)**.** *Consider a defaultable claim S_t^{n+1} defined in (3.3) under bilateral counterpart credit risk as*

$$
\begin{aligned}
\mathbb{E}_t(\beta_t S_t^{n+1,d}) &= \mathbb{E}\Bigg(\int_t^T G_s^{\tau_{Ref}}\beta_s dA_s \\
&+ \sum_{i=1}^{n}(\alpha e^{\boldsymbol{T}t_{i-1}}\boldsymbol{G}_{Ref}\boldsymbol{1} - \alpha e^{\boldsymbol{T}t_i}\boldsymbol{G}_{Ref}\boldsymbol{1})\beta_{\tau_{t_i}} Z_{\tau_{t_i}} \\
&+ \alpha e^{\boldsymbol{T}T}\boldsymbol{G}_{Ref}\boldsymbol{1}\beta_T X_t^d\Big) \\
&- LGD_C\mathbb{E}(\sum_{i=1}^{n}(\alpha e^{\boldsymbol{T}t_{i-1}}\boldsymbol{G}_C\boldsymbol{1} - \alpha e^{\boldsymbol{T}t_i}\boldsymbol{G}_C\boldsymbol{1})\beta_{\tau_C}(S_{\tau_C}^{n+1})^+) \\
&+ LGD_I\mathbb{E}(\sum_{i=1}^{n}(\alpha e^{\boldsymbol{T}t_{i-1}}\boldsymbol{G}_I\boldsymbol{1} - \alpha e^{\boldsymbol{T}t_i}\boldsymbol{G}_I\boldsymbol{1})\beta_{\tau_I}(S_{\tau_I}^{n+1})^-) \\
&= S_t^{n+1} - BCVA_t
\end{aligned} \tag{6.42}
$$

with

$$
\begin{aligned}
BCVA_t &= CVA_t - DVA_t \\
&= LGD_C\mathbb{E}(\sum_{i=1}^{n}(\alpha e^{\boldsymbol{T}t_{i-1}}\boldsymbol{G}_C\boldsymbol{1} - \alpha e^{\boldsymbol{T}t_i}\boldsymbol{G}_C\boldsymbol{1})\beta_{\tau_C}(S_{\tau_C}^{n+1})^+) \\
&- LGD_I\mathbb{E}(\sum_{i=1}^{n}(\alpha e^{\boldsymbol{T}t_{i-1}}\boldsymbol{G}_I\boldsymbol{1} - \alpha e^{\boldsymbol{T}t_i}\boldsymbol{G}_I\boldsymbol{1})\beta_{\tau_I}(S_{\tau_I}^{n+1})^-) \\
&= \mathbb{E}(\sum_{i=0}^{n}(\alpha e^{\boldsymbol{T}t_{i-1}}\boldsymbol{G}_{C,I}\boldsymbol{1} - \alpha e^{\boldsymbol{T}t_i}\boldsymbol{G}_{C,I}\boldsymbol{1})\beta_\tau[ED(\tau)])
\end{aligned} \tag{6.43}
$$

Remark that the typical non-defaultable claim like the single-name credit default swaps and the k^{th}-to-default will enable the discretisation of the first term in the equation (6.42). We leave it here in its integral form.

To conclude, the propositions 6.2.1 and 6.2.2 above require the expression of the dependence modelling since $\boldsymbol{G}_i$ are the status matrices of states in $\mid E \mid$ where $i \in \mathcal{T}$.

6.2.2 Numerical Results

Simulation process of Defaultable claims

The proposition 6.2.1, page 115, requires the simulation of value of the claims X_t over $(t,T]$ to generate the pricing metric like CVA or the risk metrics like EPE, ENE and ES (39). The Defaultable claims that we will value are the Interest rate swap and the cross currency swap. Let's first present how this will be implemented with more coding structure presented in appendix B.5, page 174.

QuantLib-based counterparty Monte-Carlo engine implementation We implement the Defaultable products using QuantLib which is a C++ open source library for Quantitative Finance. The main benefit is the object pointer in C++ enabling to call different classes of pricing engine such as Finite Difference methods or Monte-Carlo engine. Thus, it is possible to build Factory classes that will price a generic instrument and use the same code for different type of products. Other functionalities include the time discretisation of the continuous stochastic processes, the inclusion of Day count convention, and the generation of forward curves. Most of the implementation specificities are to be found in [Reiswich10a] [Reiswich10b], [Reiswich10c] and [Ballabio1].

In our context on the probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, the defaultable claims are valued using a $\mathcal{F}_t$-measurable interest rate state variable r_t. The dynamics of the instantaneous short rate r_t under the risk-neutral measure will be given by a square-root process or CIR process,

$$dr_t = \kappa(\mu - r_t)dt + \sigma\sqrt{r_t}dW_t$$

where the parameter vector $\beta = (\kappa, \mu, \sigma, r_0)$ positive deterministic constants with κ is called the mean-reversion speed, μ the long-term rate, and σ the instantaneous volatility. W_t designated the driving Brownian motion. Additionally, r_0 is set at the observed spot rate a time $t = 0$. (We will use the value $\kappa = 2.5, \sigma = 0.05, \mu = 2,5\%$ and $r_0 = 1,75\%$)

For simulation of the Brownian motion W_t we set a seed to a Mersenne-Twister random generator so that the state variables are replicated identically across time for all our product simulations.

We run $500K$ simulations (In the case of the CCS we run the foreign rate as well with different parameters.) of the short rate. The main way of simulating products like the interest-rate swap is to simulate in the future the value of the fair spread and compute the mark-to-market. However due to the lack of accessibility of the volatility of swap spreads on Datastream, we have chosen another approach. With Quantlib, the fair spread is obtained via the Monte-Carlo object that outputs the fair spread using the curve simulated. Since the Time series object generates the curve forward in time, we loop the pricing object in the future t on the forward discount curve to generate a "synthetic fair spread". This would be equivalent to entering a new interest rate swap with maturity $T - t$. Typically the notional is set at 10^6 monetary units, with payment frequency annual or semi-annual, 30/360 convention, and maturity either 7-yr or 15-yr.

Discretisation Scheme: A Monte-Carlo is used to value all the Defaultable claim payoffs considered with a discretisation with a Euler scheme and a weekly step.

The implementation of the CCR for Defaultable claims is presented in appendix B.3, page 161, as it is not directly related to the credit contagion of parties entering a defaultable claim. The numerical simulations behind the illustrations will be used as inputs in the next section when counterparty C and investor I undergo default contagion as specified in equation 6.40, page 116.

6.2.3 Effect of Contagion level on CVA and DVA of an Interest Rate Swap

Having generated in section 6.2.2, page 117, through Monte-carlo simulations, the distribution of interest rate swaps over the interval $(t, T]$, we are interested to assess the impact of the modelling of credit contagion with the phase-type distribution on those claims. In order to do so, with replicate the scenarios of credit contagion that we used in the marginal distribution ($\mathbb{Q}(\tau_i > t)$) illustrations in section B.2, page 159, with the 6 following scenarios:

- counterparty C and investor I are low risk obligors i.e. $a_C = a_I = 0.005$
- counterparty 1 and investor$I$ are normal risk obligors i.e. $a_C = a_I = 0.015$
- counterparty 1 and investor$I$ are high risk obligors i.e. $a_C = a_I = 0.03$
- counterparty 1 is a low risk obligor and investor I is a high risk obligors i.e. $a_C = 0.005,\ a_I = 0.03$
- counterparty 1 is a high risk obligor and investor I is a low risk obligors i.e. $a_C = 0.03,\ a_I = 0.005$
- counterparty 1 is a very high risk obligor and investor I is a low risk obligors i.e. $a_C = 0.04,\ a_I = 0.005$

with a_C, a_I being the respectively the base intensity of counterparty C and investor I. The variation of the contagion level through the jump intensity $b_{\{i,j\}}$ will be as varied on a range of value to assess the impact of contagion between obligors.

We will consider a two payers 15-year and 7-year interest rate swap as in figure B.2, page 162. The valuation using proposition 6.2.1, page 115 is considered under the assumption of independence between the probability distributions $\mathbb{Q}(\tau_i > t), i \in \mathcal{T} = \{I, C\}$ and the $\mathcal{F}_T$-measurable claim X. In this case, the CVA and DVA value in equation 6.41 is adjusted with the separation of the probability and the valuation of the claim X in the future and the BVA are calculated with the discounted distribution of the defaultable claim on the interval $(\tau, T]$ weighted by the probability of default in the trigger set $\mathcal{T}$ over the same interval. We assume also time bucketing by postponing the default time τ to the first t_i following τ.

In terms of implementation, as specified in [Cesari09], it is necessary when aggregating claims with different underluyings into a credit risk systems that the discretisation steps are identical in numbers and in time reference thus to enable consistent calculations and potential netting of risk positions. In accordance with the discretisation in section 6.2.2 for the valuation of the non defaultable claims, we use an identical n-step time-bucketing with a weekly step for the generation of the probability distributions of $\mathbb{Q}(\tau > t)$. Thus, we implement the following equations

$$
\begin{aligned}
BCVA &= CVA - DVA \\
&= LGD_C \sum_{i=1}^{n} (\alpha e^{\boldsymbol{T} t_{i-1}} \boldsymbol{G}_C \boldsymbol{1} - \alpha e^{\boldsymbol{T} t_i} \boldsymbol{G}_C \boldsymbol{1}) \mathbb{E}(\beta_{t_i} X(t_i)^+) \\
&- LGD_I \sum_{i=1}^{n} (\alpha e^{\boldsymbol{T} t_{i-1}} \boldsymbol{G}_I \boldsymbol{1} - \alpha e^{\boldsymbol{T} t_i} \boldsymbol{G}_I \boldsymbol{1}) \mathbb{E}(\beta_{t_i} X(t_i)^-) \\
&= \sum_{i=0}^{n} (\alpha e^{\boldsymbol{T} t_{i-1}} \boldsymbol{G}_{C,I} \boldsymbol{1} - \alpha e^{\boldsymbol{T} t_i} \boldsymbol{G}_{C,I} \boldsymbol{1}) \mathbb{E}(\beta_{t_i} [ED(t_i)]) \qquad (6.44)
\end{aligned}
$$

Thus, the figures 6.1 and 6.2 contain those CVA/DVA absolute values at $t = 0$ and immediately show the sensitivity of the metrics as a function of the intersection of several features such as:

- the maturity of the financial claim,
- the structural exposure (payer vs receiver).

More importantly, the graphs 6.1 and 6.2 highlight the non-linear deterioration of both the DVA and CVA values and, thus, the difficulty in quantifying the direction or the magnitude of the change. However, in the multivariate phase-type as presented in the illustration in section B.2, page 159, the contagion impact is feeding on maturity higher than 10 years thus justifying the higher exposure of the 15-yr swap.

Additionally, the previous figures show that the impact of contagion-correlation or potential wrong-way risk is far wider than the volatility adjustment of $\alpha = 1.4$ required by Basel II.It also confirms the limitation of this fixed volatility adjustment parameter α. This results are consistent with previous work from [Brig07a] that investigates the CVA volatility behaviour through the correlation of the default intensity and the underlying drivers like interest rate. However, [Brig07a] encapsulates a wider contagion scenario describing the state of the world where empirically high period of defaults are consistent with higher interest rates. Same results for exotics like bermudan swaptions and CMS spread options are obtained with [Brig07a] and similar aspects of impact of volatility and correlation for credit and commodities derivatives can be found in [Brig06], [Brig08b], [Brig10b] and [Brig08a].

We believe an overlooked problem in Counterparty Credit Risk is the case of "wrong-way risk"where the credit contagion among a pool of obligor affects at the same time the value of the claim, specifically defaultable ones, and the probability of defaults in the trigger set $\mathcal{T}$ in a feedback loop. Most of the analysis use a correlation as previously of underlying drivers, we hope that the current multivariate framework under permutations $\pi \in \Pi$ will highlight this feedback effect as pointed in the articles [Braithwaite12] and [Carver11].

Comment: In the current section, the implementation is done using $m = 2$ obligors since the probabilities $\mathbb{Q}[\sum_{\pi \in \Pi^C} 1_{t<\{\tau(\pi-)<\tau_C<\tau(\pi+)\}} \mid \mathcal{F}_t]$ is not different than $\mathbb{Q}[1_{t<\tau_C\}} \mid \mathcal{F}_t]$ since, by assumption, the defaultable claim X is not impacted by prior defaults $\tau(\pi-)$ and the impact on

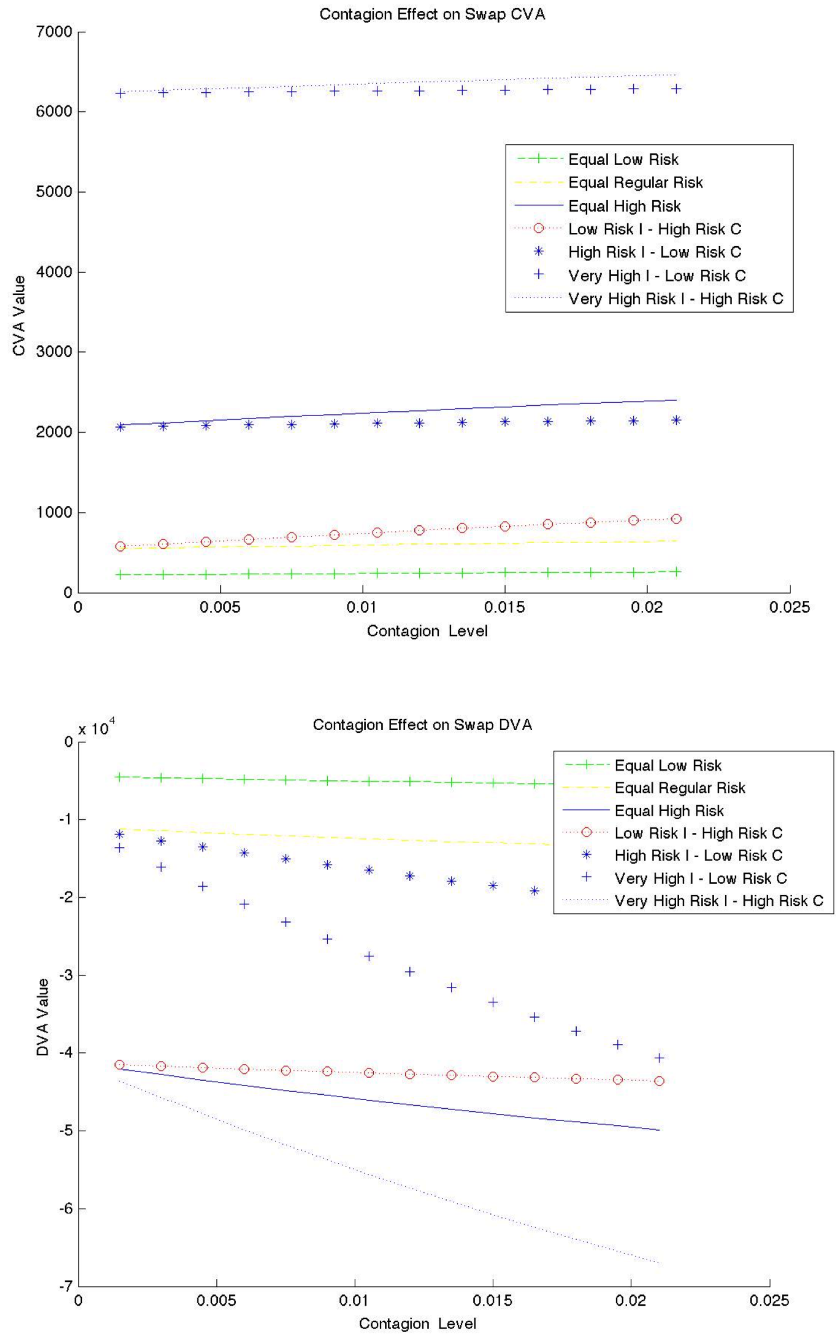
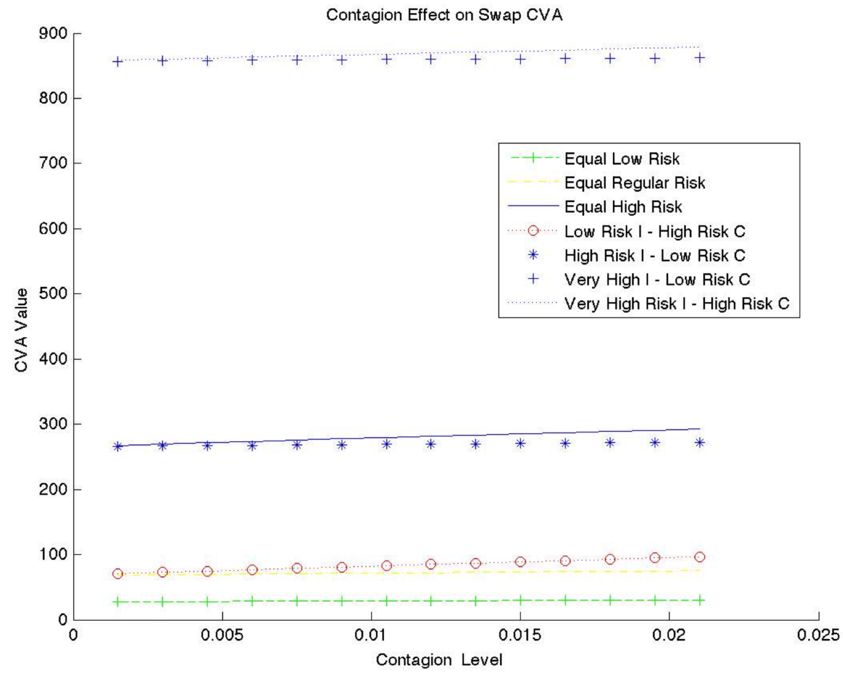


Figure 6.1: Contagion Illustration: CVA and DVA effect for a 15y-payer, semi-annual, interest-rate swap as a function of the credit contagion $b_{I,C}$ for different scenarios of credit riskiness of the counterparty C and the investor I.



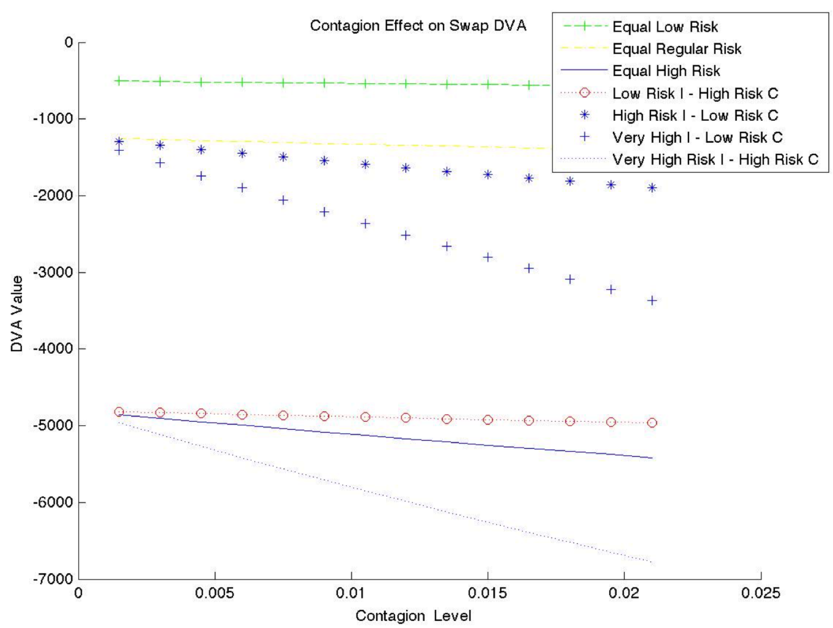


Figure 6.2: Contagion Illustration: CVA and DVA effect for a 7y-payer, semi-annual, interest-rate swap as a function of the credit contagion $b_{I,C}$ for different scenarios of credit riskiness of the counterparty C and the investor I.

τ_C can be viewed as an arbitrary higher intensity a_C. However, in the next section on defaultable claims, this simplification is no longer possible in a multivariate setting.

6.2.4 Effect of contagion level of CVA and DVA on credit derivatives - Illustration of wrong-way risk effect

Credit derivatives transactions are particularly exposed to wrong-way risk when at the time of default of a counterparty selling credit protection is typically a time of higher value of credit protection. With observed default clustering, at the time of default of the counterparty, there is a positive probability of a high defaults spreads environment, in which case, the value of the CDS for a protection buyer is close to the loss given default of the firm. In this setting, wrong-way risk is thus represented in the model by the fact that at the time of default of the counterparty, there is a positive probability that the firm on which the CDS is written defaults too, in which case the loss incurred to the investor is the loss given default of the firm adjusted with the recovery on the counterparty.

In order to value the CVA and DVA on defaultable claims as we did in section 6.2.3 for non defaultable ones, we consider the single-name credit default swap valued in equation 5.14, page 74, and k^{th}-to-default swap valued in equation 5.15, page 75. The valuation of the CVA of those claims will be done using proposition 6.2.2, page 116. Note that in this case the trigger set is $\mathcal{T} = \{C, I, Ref\}$ with $\mid Ref \mid = 6$. We use the intensities a_i calibrated over 10-yr spreads in table B.2, page 158 with the reference set Ref being 6 industrials (Renault, Peugeot, Air Liquide, Sanofi, Total, EDF)and the counteparty C and the investor I being banks i.e. BNP Paribas and Societe Générale. The maturity T of the claims CDS and KTD will correspondingly be set at 10 year.

In accordance with the discretisation in section 6.2.2 for the valuation of the non defaultable claims, we use an identical n-step time-bucketing for the valuation of the with a weekly step for the generation of the probability distributions of $\mathbb{Q}[\sum_{\pi\in\Pi^C} 1_{t<\{\tau(\pi-)<\tau<\tau(\pi+)\}} \mid \mathcal{F}_t]$ for our trigger set $\mathcal{T}$. Thus, we implement the following equations

$$\begin{aligned} BCVA_t &= CVA_t - DVA_t \\ &= LGD_C\mathbb{E}(\sum_{i=1}^{n}(\alpha e^{\boldsymbol{T}t_{i-1}}\boldsymbol{G}_C\boldsymbol{1} - \alpha e^{\boldsymbol{T}t_i}\boldsymbol{G}_C\boldsymbol{1})\beta_{\tau_C}(S_{\tau_C}^{n+1})^+) \\ &- LGD_I\mathbb{E}(\sum_{i=1}^{n}(\alpha e^{\boldsymbol{T}t_{i-1}}\boldsymbol{G}_I\boldsymbol{1} - \alpha e^{\boldsymbol{T}t_i}\boldsymbol{G}_I\boldsymbol{1})\beta_{\tau_I}(S_{\tau_I}^{n+1})^-) \\ &= \mathbb{E}(\sum_{i=0}^{n}(\alpha e^{\boldsymbol{T}t_{i-1}}\boldsymbol{G}_{C,I}\boldsymbol{1} - \alpha e^{\boldsymbol{T}t_i}\boldsymbol{G}_{C,I}\boldsymbol{1})\beta_{\tau}[ED(\tau)]) \end{aligned} \tag{6.45}$$

where in this case the value of the claim X and the default probability G_T^{τ} cannot be separated as in equation 6.44 since they feed each other for any permutation $\pi \in \Pi$. Note that in the multivariate framework, $\boldsymbol{G}_{C,I}$, $\boldsymbol{G}_C$, $\boldsymbol{G}_I$ take into account the fact that Ref is not earlier since it belongs to $\mathcal{T}$.

Thus, the figure 6.3 and the figure 6.4, represents the effects on CVA of the jump intensity $b_{i,j}$ for a single-name credit default swap and k^{th}-to-default swap. The results are presented as the cumulated basis point of the protection buying and not as the absolute value of the CVA since the

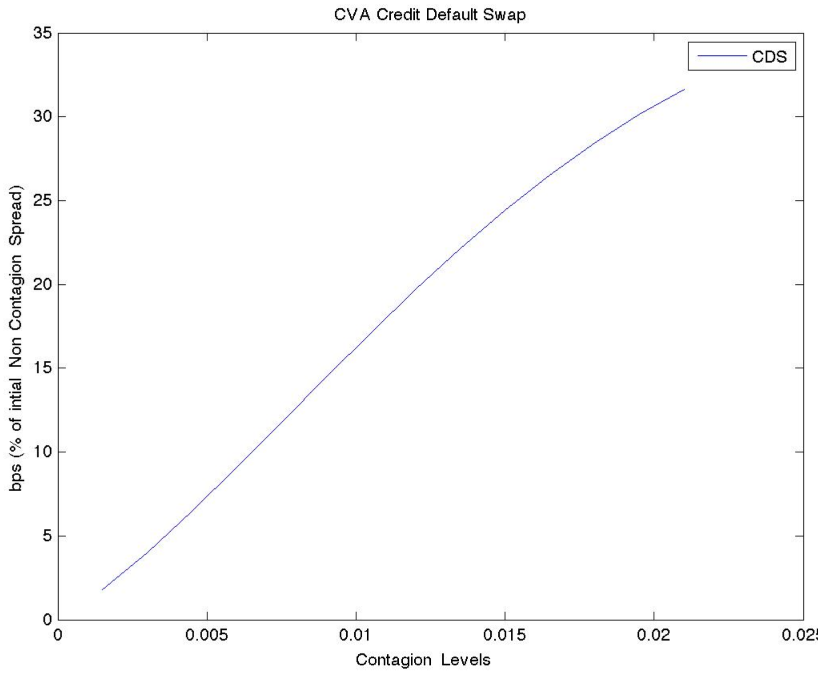


Figure 6.3: Contagion Illustration: CDS spread contagion effect

contagion is located in the CVA value and the default probability weights. Globally, the impact of contagion is much more important in magnitude (except for $k = 1$ as expected by definition) than in the case of interest rate swaps as credit derivatives products are very sensitive to wrong-way risk. Additionally, and as already covered in Basel II, the impact of contagion or wrong-way risk is exponential if the products embed structural credit leverage as k increases in the k^{th}-to-default swaps.

This illustrates that the multi-variate phase-type distribution can provide a way to gauge the impact of contagion for bilateral financial claims and is able to generate tails scenarios that can be beneficial in a risk management concept. Recent press coverage in the financial press in [Braithwaite12], have been focusing on ***"gap option"*** risk which is the *"chance that the counterparty's posted collateral would be wiped out and that the investors would walk away."* [Braithwaite12] mentions the case of Deutsche Bank \$130bn leveraged super-senior portfolio where collateral posted wasn't taking into account scenarios that were *"economically unfeasible"*. The multivariate phase-type distribution applied to levered credit derivatives such as k^{th}-to-default in figure 6.4 captures a similar explosion spreads as the 2007 super-senior spread explosion. The non-linear aspect of the curve for $k \geq 2$ illustrates this strong sensitivity to extreme scenarios of credit leveraged positions and the fact that situations arise when risk are fully non manageable in terms of costs.

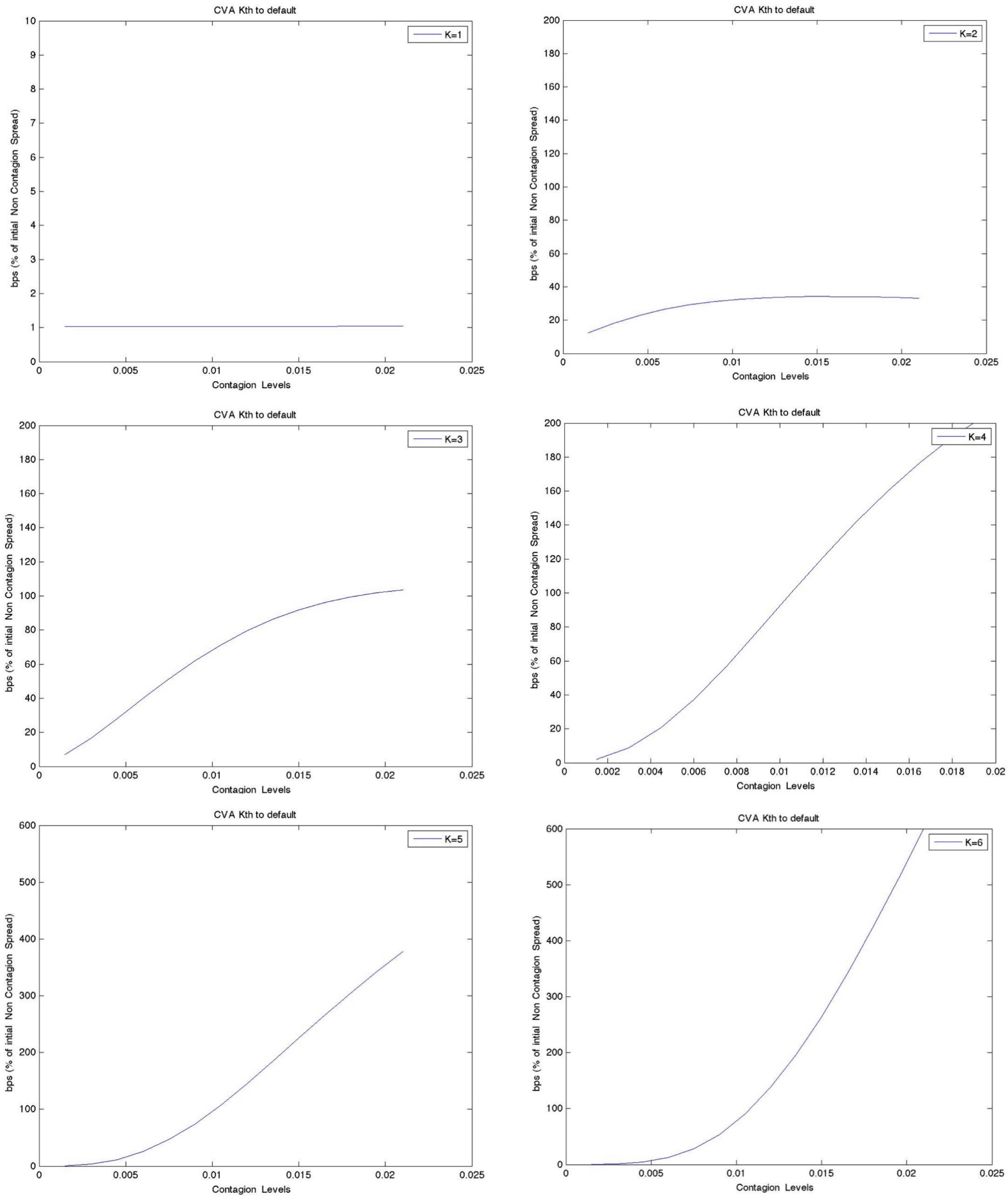


Figure 6.4: Contagion Illustration: K^{th}-to-default spread contagion effect

6.2.5 Limitation Phase

So far, the MPH formula for the marginal survival distribution $F_i(t) = \mathbb{Q}[\tau_i > t]$ and the ordered marginal survival distribution $F_{k,i}(t) = \mathbb{Q}[\tau^{(k)} > t, \tau^{(k)} = \tau_i]$ provide easy calibration of base identity parameters $\boldsymbol{a}$ but prove difficult to calibrate at the same time the base and jump intensity vector $(\boldsymbol{a}, \boldsymbol{b})$. It is also not clear where the information for calibration of vector $\boldsymbol{b}$ should come from: CDS time-series, exogenous "semi-calibration", credit derivatives index tranche spreads, etc.

Beside the calibration limitation for pricing, the MPH framework is useful in risk management context to assess credit contagion scenarios for Counterparty Credit Risk metrics. Section 6.2.4 and section 6.2.3, page 119, highlight the range of variation of CVA and DVA values when counterparties undergo credit risk contagion.

However, figure B.1, in section B.2 page 159, shows that the marginal survival distribution $F_i(t) = \mathbb{Q}[\tau_i > t]$ do not contain randomness parameters likely to generate credit volatility. As in chapter 7 in [Brig10a] and in [Brig09], the computing of the counterparty risk adjustment in the unilateral or the bilateral case is extremely sensitive to joint interaction of default correlation and credit spread volatility. [Brig09] highlights the need of credit spread volatility to capture credit contagion via the measurement of wrong-way risk. As mentioned in [Brig10a] this is necessary to correctly account for the option term in the counterpart valuation formulae (6.41) and (6.43). Moreover, in [Brig14], the Counterparty Credit Risk of a single-name CDS is evaluated in the case of a netted exposure and re-hypothecation of collateral. In this case, the option term in the CCR formaula is represented by the netted exposure on the defaulted exposure and the pre-defaulted collateral account. The author using a doubly stochastic intensity default framework, show that the option term although reduced by the collateral is still sensitive to credit spread volatility and credit contagion. Contagion effects still manage to limit the effectiveness of collateralisation

We will try to keep the benefit of the phase tip framework, i.e identification of first to default scenario among obligors and contagion structure, and account for the credit risk volatility. We do this in the next section by exploring the mixture of phase-type distributions relating to different volatility regimes.

6.3 Regime-shifted Phase-type distribution

6.3.1 Introduction

Building on the concept of phase type distributions as presented in [Asmu03] and [Herb10], we extend the Markov jump process $(Y_t)_{t\geq 0}$ by considering n different regime shifts, each representing identical Markov jump process framework like in the previous Phase document.

Using the work of [Assaf82] and [Assaf85] about the closure properties of phase-type distributions mixtures and their properties, we can augment the dimension of time-homogeneous parameter in order to avoid the deterministic aspect of the closed-form solutions of the phase-type distribution. Thus, we expect the increase of the dimensionality of model parameters to better capture the dynamics of credit risk volatility in survival distribution when obligors undergo credit risk contagion.

Previous similar approach has been attempted to include parameter extension to avoid the non calibration capability of a limited dimension of time-homogeneous model parameters. For example, [Brig05] extends a tractable structural model depending on a random default barrier. This is done by imposing time-constant volatility scenarios where each single scenario imply loosing flexibility in richness of default curves but regain calibration flexibility thanks to the multiple scenarios on otherwise too simple time-constant volatilities.

Thus, in [Brig05], the risk neutral firm value V_t dynamics is expressed as:

$$dV_t = (r_t - q_t)V_t dt + \nu_t V_t dW_t$$

with r_t the risk-free rate, q_t a payout ratio, ν_t the instantaneous volatility and a set of scenarios linking asset barrier and volatility $(H_1, t \to \sigma^1(t))$,..., $(H_N, t \to \sigma^N(t))$ with associated risk-neutral $\mathbb{Q}$ probabilities $p_1, .., p_N$. The price of a payoff Π based on V is obtained through the price of each scenario by weighted iterated expectation:

$$\mathbb{E}[\Pi] = \mathbb{E}\{\mathbb{E}[\Pi|H,\nu]\} = \sum_{i=1}^{N} p_i\{\mathbb{E}[\Pi|H = H_i, \nu = \sigma^i]\}$$

Similarly, since the scenarios are independent, the CDS spread in [Brig05] can be written as:

$$\mathrm{CDS}_{a,b}(t, R, \mathrm{L_{GD}}) = \sum_{i=1}^{N} \mathrm{CDS}_{a,b}(t, R, \mathrm{L_{GD}}; H_i, \sigma^i)p_i$$

This approach enables to identify scenario barriers that are associated with normal market conditions and other barriers with extremely stressed situations while a time-varying barrier will never be rich enough to encapsulates all the possible states of the world in the future. We will in this section try to replicate a similar approach for multivariate phase type distributions.

6.3.2 Proposed Evolution

We define the Markov processes X_t with $X_t \sim \mathbf{Q}(Y_t)$ a phase-type distribution process with $Y_t \sim \mathbf{R}(t)$ a hidden Markov chain defining regime shift R_t. X_t defines the previous Markov jump process - Phase-type distribution with transition between states defining the obligors defaulting or regime change. Y_t defines a hidden Markov chain that defines the n regimes. For example, in the case of $n = 2$ regimes, we could identify R_1 as a *normal regime* and R_2 as a *high regime*. We could expect with the closure properties of phase-type distributions to obtain a phase-type distribution property for the regime phase-type distribution built on regime-independent phase-type distribution.

Conceptually, the Markov chain Y_t expressing the regime chain can be dependent or not of the phase-type transition intensities $\lambda_{i,j}$ of X_t within a set regime. For example, several specificities could define the relation between X_t and Y_t.

- increased defaulted obligors in X_t could increase the probability of shifting the chain in a higher regime in Y_t,
- Y_t can be defined as endogenously or exogenously to the states of X_t. This will result either to the regime transition rate of Y_t to be incorporated or not in the transition rate of X_t.
- possibilities off simultaneous phase transition in X_t and regime transition in Y_t.

6.4 Regime phase-type model

6.4.1 Presentation

Structure of absorption vector $\boldsymbol{t}$

As mentioned in [Neuts78] and in the multivariate case in [Assaf84], phase-type distribution identify a unique absorption state Δ leading to an absorption vector $\mathbf{t}$ on the right-hand side of matrix $\mathbf{Q}$ in order to apply the Phase-type distribution methodology and obtain the closed-form formula (6.20), page 100. In the case of the multivariate obligors $i \in E$, in [Assaf84], an additional requirement is that the absorbing state is defined as

$$\Delta = \bigcap_{i=1}^{m} \Delta_i \tag{6.46}$$

where Δ_i defines the absorption state of obligor i.

In order to work in the same multivariate framework using the same notation as in section 6.1.2, page 100, this condition requires that the absorbing state will be the intersection across obligors and regime as well. Typically this will require to identify a unique absorption vector $\boldsymbol{t}$ on the right hand side to the matrix $\boldsymbol{T}$ thus modifying some $n-1$ absorption vector $\boldsymbol{t}$ of the matrix $\boldsymbol{T}_i$. This will rule-out all the mixture of phase-type distribution which are defined as successive phase processes and not concurrent ones.

Usually the closure properties of the representation $(\alpha_R, \mathbf{T}_R)$ as a function of all the representation $(\alpha_i, \mathbf{T}_i), \forall i \in \{1, \ldots, n\}$ is established though the Laplace Transform to express the transition matrix $\mathbf{T}_R$ and exit vector $\mathbf{t}_R$ as a function of the underlying phase. Theorem 2.2.2 and 2.2.4 in [Neuts78] establish closure properties of $\mathbf{T}_R$ for convolutions and finite mixtures as

Theorem 6.4.1 (Closure of multivariate phase-type distributions)**.** *Let $T = (T_1, \dots, T_n)$ and $S = (S_1, \dots, S_m)$ be independent multi-variate phase-type random vectors. Then the conjunction $(T, S) = (T_1, \dots, T_n, S_1, \dots, S_m)$ is a multi-variate phase-type random vector.*

However most of the applications are concerned with successive phases like theorem [**?**] instead of alternating between phases. We will instead have to assess the singularity of the transition matrix $\mathbf{T}_R$ using Lemma 2.2.1 p45 in [Neuts78].

Lemma 6.4.1. *The states $1, \dots, m$ are transient if and only if the matrix $\boldsymbol{T}$ is singular.*

We will construct the different multivariate phase between regimes using the closure theorem 2.2.1 of absorption state Δ_i of multivariate phase-type distribution in [Assaf84] using communication between phases though regime shifts.

6.4.2 Model presentation

Consider a finite set E containing m obligors and a set R identifying n regimes.

For each regime $j = 1, \dots, n$ the triplet $(\boldsymbol{E}, \alpha_j, \boldsymbol{T}_j)$ is a phase type distribution where F is the distribution of the time to absorption τ with the closed-form formula:

$$F(t) = \mathbb{P}[\tau < t] = 1 - \alpha e^{\boldsymbol{T}_j t \boldsymbol{1}}$$
$$f(t) = \alpha e^{\boldsymbol{T}_j t \boldsymbol{1}}$$

with

$$e^{\boldsymbol{T}_j t} = \sum_{n=0}^{\infty} \frac{1}{n!} {\boldsymbol{T}_j}^n t^n$$

Thus, consider m obligors which we label with $i = 1, 2, \dots, m.$ and τ_i be the time of default for obligor i, with $i = 1, 2, \dots, m$, and the corresponding default point processes

$$H_t^i = 1_{\{\tau_i < t\}},\ i = 1, 2, \dots, m.$$

contain any information regarding the default times $\tau_1, \dots, \tau_m$ with the filtrations

$$\mathcal{H}_t^i = \sigma(H_s^i; s \leq t)$$
$$\mathcal{H}_t = \bigvee_{i=1}^{m} \mathcal{H}_t^i$$

We also define the default times $\tau_1, \dots, \tau_m$ for obligors $i = 1, 2, \dots, m$ and λ_t^i their corresponding default intensities w.r.t $\mathcal{H}_t$. The filtration $\mathcal{H}_t$ does not contain information about the regime the chain Y_t is into at time t. And the previous mathematical setting defined in section 4 and section 6.2 is still valid with a $\boldsymbol{Q}$ matrix composed of n phase type matrices (α_i, T_i)

Consider n regimes labeled $j = 1, \ldots, n$, which defines the hidden Markov chain Y_t that jumps in the set R. The chain is defined purely a transient chain with no absorbing state. We state the following assumption that

Assumption 6.4.1. *No simultaneous regime jump and obligor default can happen at the same time.*

We define the hidden Markov chain Y_t as

Definition 6.4.1. *The one parameter family $\mathcal{P}^t(t),\ t \in \mathbb{R}_+$ of stochastic matrices is called the family of transition probability matrices for the time-homogeneous $\mathbb{G}$-markov chain Y under $\mathbb{Q}$ if for every $t, s \in \mathbb{R}_+$*

$$\mathbb{Q}\{Y_{s+t} = j \mid Y_s = i\} = p^t_{i,j}(t)\ , \forall i, j \in \mathcal{R} \tag{6.47}$$

From Theorem 8.1.2 in [Rolski98] the following finite limits exists at time $t = 0$

$$t_{ij} := \lim_{t \downarrow 0} \frac{p_{i,j^t}(t) - p^t_{ij}(0)}{t} = \lim_{t \downarrow 0} \frac{p^t_{i,j}(t) - \delta_{ij}}{t},\ \forall i, j \in \mathcal{K} \tag{6.48}$$

with $t_{ij} > 0$ and $t_{ii} = -\sum_{j=1, j \neq i}^{K} t_{ij}$ for every $i \neq j$.

The matrix

$$R := [t_{i,j}]_{1 \leq i,j \leq n}$$

is called the **infinitesimal generator matrix** for a Markov chain associated with the family $\mathcal{P}^t$. Since each entry $t_{i,j}$ of the matrix R can be shown to represent the intensity of a transition from the regime i to the regime j the infinitesimal generator matrix R is also called the **regime-transition intensity matrix**., the following finite limits exists at time $t = 0$.

We provide illustrative example of the Matrix $\boldsymbol{T}$ associated with the Matrix $\boldsymbol{Q}$ for n regimes.

6.4.3 Example: $m = 2$ obligors and $n = 2$ regimes:

Considering the two representation $(\alpha_1, \mathbf{T}_1)$ and $(\alpha_2, \mathbf{T}_2)$. The matrix representing the case of two regimes can be identified as the blocks:

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}_1 & | & \mathbf{R}_{12} \\ -- & | & -- \\ \mathbf{R}_{21} & | & \mathbf{T}_2 \end{pmatrix}$$

Same as before, the matrix $\mathbf{T}_1$ and $\mathbf{T}_2$ represent respectively the jump matrix of X_t when Y_t is in state $Y_t = R_1$ and $Y_t = R_2$. The matrix $\mathbf{R}_{12}$ and $\mathbf{R}_{21}$ represent respectively the regime transition matrix respectively, from regime 1 to regime 2 and vice versa.

As previously in, the obligor default set are identified as

$$\{\{0\} = \text{No defaults },\{1\},\{2\},\{1,2\} = \Delta.\}$$

However, the status can also be identified according to Y_t and their regime. Introducing the status $(i,j) = (\textit{obligor},\ \textit{regime})$ with $i \in \{1, m = 2\}$ and $j \in \{1, n = 2\}$, the default set is expanded as

$$\{(\{0\},R_1),(\{1\},R_1),(\{2\},R_1),(\{1,2\},R_1),(\{0\},R_2),(\{1\},R_2),(\{2\},R_2),(\{1,2\},R_2)\}$$

We now give detail of the matrices with the same assumption as in section 6.1.2, page 100 about the intensity of obligors and their default contagion aspect. Thus, for obligor $i \in E$ in regime R_1

$$\lambda_t^i = a_i + \sum_{k\,\in\,I_i} b_{i,k} 1_{\{\tau_k \le t\}},\ i = 1,2,\ldots,m$$

and in regime R_2

$$\lambda_t^{i'} = a_i^{'} + \sum_{k\,\in\,I_i} b_{i,k}^{'} 1_{\{\tau_k \le t\}},\ i = 1,2,\ldots,m$$

$$\mathbf{T}_1 = \begin{array}{r|cccc} & (\{0\},R_1) & (\{1\},R_1) & (\{2\},R_1) & (\{1,2\},R_1) \\ (\{0\},R_1) & -a_1 - a_2 - t_{12|1} & a_1 & a_2 & \\ (\{1\},R_1) & & -a_2 - b_{12} - t_{12|2} & & a_2 + b_{12} \\ (\{2\},R_1) & & & -a_1 - b_{21} - t_{12|3} & a_1 + b_{21} \\ (\{1,2\},R_1) & & & & 0 \end{array}$$

$$\mathbf{T}_2 = \begin{array}{r|cccc} & (\{0\},R_2) & (\{1\},R_2) & (\{2\},R_2) & (\{1,2\},R_2) \\ (\{0\},R_2) & -a_1^{'} - a_2^{'} - t_{21|1} & a_1^{'} & a_2^{'} & \\ (\{1\},R_2) & & -a_2^{'} - b_{12}^{'} - t_{21|2} & & a_2^{'} + b_{12}^{'} \\ (\{2\},R_2) & & & -a_1^{'} - b_{21}^{'} - t_{21|3} & a_1^{'} + b_{21}^{'} \\ (\{1,2\},R_2) & & & & 0 \end{array}$$

$$\mathbf{R}_{12} = \begin{array}{r|cccc} & (\{0\},R_2) & (\{1\},R_2) & (\{2\},R_2) & (\{1,2\},R_2) \\ (\{0\},R_1) & t_{12|1} & & & \\ (\{1\},R_1) & & t_{12|2} & & \\ (\{2\},R_1) & & & t_{12|3} & \\ (\{1,2\},R_1) & & & & 0 \end{array}$$

$$\mathbf{R}_{21} = \begin{array}{r|cccc} & (\{0\},R_1) & (\{1\},R_1) & (\{2\},R_1) & (\{1,2\},R_1) \\ (\{0\},R_2) & t_{21|1} & & & \\ (\{1\},R_2) & & t_{21|2} & & \\ (\{2\},R_2) & & & t_{21|3} & \\ (\{1,2\},R_2) & & & & 0 \end{array}$$

Comment: $t_{12|3}$ Defines the transition for Y_t from regime 2 to regime 1 conditional we are in default state 3 for X_t. Under assumption 6.4.1, in $\mathbf{T}_1$ and $\mathbf{T}_1$ there is either no extra default (intensity on the diagonal of a matrix $\mathbf{T}_i$), a default (intensity on the right side of the diagonal of the matrix $\mathbf{T}_i$) or a transition.

6.4.4 Example - General Case: m obligors n regimes

Thus for $m > 2$ obligors and $n > 2$ regime with representation $(\alpha_i, \mathbf{T}_i), i \in \{1, \ldots, n\}$: This matrix can be identified as the blocks:

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}_{R_1} & \mathbf{R}_{12} & ... & \mathbf{R}_{1n} \\ \mathbf{R}_{21} & \mathbf{T}_{R_2} & ... & \mathbf{R}_{2n} \\ ... & ... & ... & ... \\ \mathbf{R}_{n1} & \mathbf{R}_{n2} & ... & \mathbf{T}_{R_n} \end{pmatrix}$$

Similarly, consider a set of intensities per regime

$$\lambda_t^{i'} = a_i^{R_i} + \sum_{k \,\in\, I_i} b_{i,k}^{R_i} 1_{\{\tau_k \leq t\}}, \ i = 1, 2, \ldots, m$$

In case $m = 2$ obligors in order to have concise matrices with the generator matrix $\mathbf{T}_{R_i}$ in regime R_i defined as:

$$\begin{array}{ccccc} & (\{0\}, R_i) & (\{1\}, R_i) & (\{2\}, R_i) & (\{1,2\}, R_i) \\ (\{0\}, R_i) & -a_1^{R_i} - a_2^{R_i} - \sum_{j \neq i} t_{ij} & a_1^{R_i} & a_2^{R_i} & \\ (\{1\}, R_i) & & -a_2^{R_i} - b_{12}^{R_i} - \sum_{j \neq i} t_{ij} & & a_2^{R_i} + b_{12}^{R_i} \\ (\{2\}, R_i) & & & -a_1^{R_i} - b_{21}^{R_i} - \sum_{j \neq i} t_{ij} & a_1^{R_i} + b_{21}^{R_i} \\ (\{1,2\}, R_i) & & & & 0 \end{array}$$

The Regime matrix $\mathbf{R}_{kl}$ shifting from regime k to regime l is defined as:

$$\begin{array}{ccccc} & (\{0\}, R_l) & (\{1\}, R_l) & (\{2\}, R_l) & (\{1,2\}, R_l) \\ (\{0\}, R_k) \mid & t_{kl|i=\{0\}} & & & \\ (\{1\}, R_k) \mid & & t_{kl|i=\{1\}} & & \\ (\{2\}, R_k) \mid & & & t_{kl|i=\{2\}} & \\ (\{1,2\}, R_k) \mid & & & & 0 \end{array}$$

6.4.5 Regime-phase type distribution existence

We loose the whole upper triangular structure of $\mathbf{T}$ for the phase type process to gain easy existence and calculation for the exponential matrix through the eigenvalues on the diagonal axis. However each of the sub matrix $\mathbf{T}_{R_i}$ has the upper triangular structure and the same of convergence criteria for the exponential.

We need to proceed as in the document phase in [Neuts78] and show

- that $\boldsymbol{T}$ is singular so it is invertible and thus the absorption is certain.
- Using the Perron-Frobenius Theorem 6.5 p26 in [Asmu03] that the eigenvalues are negatives and not null to have an ergodic markov jump process and thus that there exists a distribution.

Proposition 6.4.1. *We consider the general matrix*

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{T}_{R_1} & \boldsymbol{R}_{12} & ... & \boldsymbol{R}_{1n} \\ \boldsymbol{R}_{21} & \boldsymbol{T}_{R_2} & ... & \boldsymbol{R}_{2n} \\ ... & ... & ... & ... \\ \boldsymbol{R}_{n1} & \boldsymbol{R}_{n2} & ... & \boldsymbol{T}_{R_n} \end{pmatrix}$$

where $\boldsymbol{T}_{R_i} = [\lambda_{i,j}]_{1\le i\le j\le K}$ *define upper-triangular matrices with negative eigenvalues with modulus lower than 1 and* $\boldsymbol{R}_{ij} == [t_{i,i}]_{1\le i\le K}$ *define diagonal matrices with positive values and suppose that* $t_{i,i} << \lambda_{i,i}, \forall i \in K$ *then the matrix* $\boldsymbol{T}$ *is singular.*

Remark: As we intend to demonstrate the existence of the phase property for the general case of Markov jump process applied to credit modelling we consider the $\mathbf{T}_{R_i} = [\lambda_{ij}]_{1\le i,j\le K}$and is upper-triangular. We make no specific assumption of credit contagion in the intensities. Addtionnally, the condition $t_{i,i} << \lambda_{i,i}, \forall i \in K$ in our context is equivalent to the fact that the regime shifts occur less often than defaults which seems a sensible assumption.

In [Powell11], we find a method for recursively identify the determinant of a matrix based on the determinant of its N^2 blocks using the Schur complement structure. The theorem is as follows:

Theorem 6.4.2. *Let* S *be an* $(nN)\times(nN)$ *complex matrix, which is partitioned into* N^2 *blocks each of size* $n \times n$

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} & ... & \boldsymbol{S}_{1N} \\ \boldsymbol{S}_{21} & \boldsymbol{S}_{22} & ... & \boldsymbol{S}_{2N} \\ ... & ... & ... & ... \\ \boldsymbol{S}_{N1} & \boldsymbol{S}_{N2} & ... & \boldsymbol{S}_{NN} \end{pmatrix}$$

Then the determinant $\det(S) = \prod_{k=1}^{N} \det(\alpha_{kk}^{(N-k)})$ *where* α^k *are defined by*

$$\alpha_{ij}^{(0)} = S_{ij}$$
$$\alpha_{ij}^{(k)} = S_{ij} - \sigma_{i,N-k+1}^t \tilde{S}_k^{-1} s_{N-k+1,j},\ k \ge 1$$

and the vectors σ_{ij}^t *and* s_{ij} *are*

$$s_{ij} = (S_{ij}\ S_{i+1,j}\ \dots\ S_{N,j})^t,\ \sigma_{ij}^t = (S_{ij}\ S_{i,j+1}\ \dots\ S_{i,N})$$

and

$$\tilde{\boldsymbol{S}}_k = \begin{pmatrix} \boldsymbol{S}_{N-k+1,N-k+1} & \boldsymbol{S}_{N-k+1,N-k+2} & \dots & \boldsymbol{S}_{N-k+1,N} \\ \boldsymbol{S}_{N-k+2,N-k+1} & \boldsymbol{S}_{22} & \dots & \boldsymbol{S}_{N-k+2,N} \\ \dots & \dots & \dots & \dots \\ \boldsymbol{S}_{N,N-k+1} & \boldsymbol{S}_{N,N-k+2} & \dots & \boldsymbol{S}_{N,N} \end{pmatrix}$$

the paper [Powell11] also establishes this proof by induction instead of using the "Baniachewic identity"

Lemma 6.4.2. *Let $\boldsymbol{S}$ be a complex block matrix of the form*

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} & \dots & \boldsymbol{S}_{1N} \\ \boldsymbol{S}_{21} & \boldsymbol{S}_{22} & \dots & \boldsymbol{S}_{2N} \\ \dots & \dots & \dots & \dots \\ \boldsymbol{S}_{N1} & \boldsymbol{S}_{N2} & \dots & \boldsymbol{S}_{NN} \end{pmatrix}$$

and let us define the set of block matrices $\{\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(N-1)}\}$ where $\alpha^{(k)}$ is an $(N-k)\times(N-k)$ block matrix with blocks

$$\alpha_{ij}^{(0)} = S_{ij}$$

$$\alpha_{ij}^{(k+1)} = \alpha_{ij}^{(k)} - \alpha_{i,N-k}^{(k)} \left(\alpha_{N-k,N-k}^{(k)}\right)^{-1} \alpha_{N-k,j}^{(k)},\ k \geq 1$$

Then the determinants of consecutive $\alpha^{(k)}$ are related via

$$\det(\alpha^{(k)}) = \det(\alpha^{(k+1)}) \det(\alpha_{N-k,N-k}^{(k)})$$

with the partition

$$\alpha^{(k)} = \begin{pmatrix} \alpha_{11}^{(k)} & \dots & \alpha_{1,N-k-1}^{(k)} & \alpha_{1,N-k}^{(k)} \\ \vdots & & \vdots & \vdots \\ \alpha_{N-k-1,1}^{(k)} & \dots & \alpha_{N-k-1,N-k-1}^{(k)} & \alpha_{N-k-1,N-k}^{(k)} \\ \alpha_{N-k,1}^{(k)} & \dots & \alpha_{N-k,N-k-1}^{(k)} & \alpha_{N-k,N-k}^{(k)} \end{pmatrix}$$

Proof. From [Powell11], we can use the following theorem

Theorem 6.4.3. *Given a complex block matrix of the form and the matrices $\alpha_{11}^{(k)}$ the determinant of $\boldsymbol{S}$ is given by*

$$\det(S) = \prod_{k=1}^{N} \det(\alpha_{kk}^{(N-k)})$$

Since we know that by definition the matrices $R_{ij}, \forall i,j \in R, i \neq j$ are diagonal matrices with diagonal values representing the transition rate from regime i to regime j and $T_i, i \in R$ is upper-triangular with the transitions rate towards absorption when the Markov chain is in regime i. Thus, we can recursively establish the singularity of matrix $\boldsymbol{T}$ based on its sub-components which determinants are defined as its diagonal values. Recursive Demonstration.

Case $n = 1$. $\mathbf{T}^1 = \mathbf{T}_1$ with $\mathbf{T}_1$ upper-triangular with non null eigenvalues on the diagonal. Thus $\mathbf{T}^1$ non singular.

Case $n = 2$: Suppose

$$\mathbf{T}_2 = \begin{pmatrix} \mathbf{T}^1 & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{T}^2 \end{pmatrix}$$

With $\mathbf{T}^i, \forall i \in R$ upper-triangular non-null eigenvalue structure. All sub-matrices have same dimension. We recall the result

- Block identical
$$\det \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \det(\mathbf{AD} - \mathbf{BC})$$

- Block unidentical
$$\det \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \det(\mathbf{D}) \det(\mathbf{A} - \mathbf{BD}^{-1}\mathbf{C})$$

  with $\mathbf{A}$ a $m \times m$ matrix, $\mathbf{D}$ a $n \times n$ matrix, $\mathbf{B}$ a $n \times m$ matrix and $\mathbf{C}$ a $m \times n$ matrix. (This is the "Baniachewic identity").

Thus we have

$$\det(\mathbf{T}_2) = \det(\mathbf{T}^1\mathbf{T}^2 - \mathbf{R}_{12}\mathbf{R}_{21})$$

Which is a function of the eigenvalue produced by the product 2 upper-triangular matrices minus the product of 2 diagonal matrices. We thus obtain a condition on the eigenvalue of non singularity as

$$\lambda^1_{ii}\lambda^2_{ii} - t^{12}_{ii}t^{21}_{ii} \neq 0, \forall i \in 1, \dots, m \tag{6.49}$$

The matrix $\mathbf{T}_2$ is invertible.

Remark: This would be sufficient for our numerical applications in section 6.4.6, page 138, since we will only consider High and low regime.

Case $n = 3$: Suppose

$$\mathbf{T}_3 = \begin{pmatrix} \mathbf{T}^1 & \mathbf{R}_{12} & \mathbf{R}_{13} \\ \mathbf{R}_{21} & \mathbf{T}^2 & \mathbf{R}_{23} \\ \mathbf{R}_{31} & \mathbf{R}_{32} & \mathbf{T}^3 \end{pmatrix} = \begin{pmatrix} \mathbf{T}_2 & \mathbf{R}_{X3} \\ \mathbf{R}_{3X} & \mathbf{T}^3 \end{pmatrix}$$

with $\det(\mathbf{T}_2) \neq 0$ as previously shown and

$$\mathbf{T}^t_3 = \begin{pmatrix} (\mathbf{T}^3)^t & \mathbf{R}^t_{3X} \\ \mathbf{R}^t_{X3} & \mathbf{T}^t_2 \end{pmatrix} \tag{6.50}$$

By application if the Baniachewic identity, we have

$$\det(\mathbf{T}_3) = \det(\mathbf{T}^3)\det(\mathbf{T}_2 - \mathbf{R}_{X3}(\mathbf{T}^3)^{-1}\mathbf{R}_{3X})$$

we have $\det(\mathbf{T}^3) \neq 0$ but we do not have any specific information about the eigenvalues of the matrix $\mathbf{T}_2$. However with

$$\det(\mathbf{T}_3) = \det(\mathbf{T}_3^t) = \det(\mathbf{T}_2^t)\det((\mathbf{T}^3)^t - \mathbf{R}_{3X}^t(\mathbf{T}_2^t)^{-1}\mathbf{R}_{X3}^t)$$

with $\det(\mathbf{T}_2) \neq 0$ as shown previously with (6.49) valid. The transpose matrices are either lower triangular or diagonal with eigenvalue being the inverse since with (6.49) valid none is null. Thus the second determinant is a derterminant of lower triangular structure with the condition

$$\lambda_{ii}^3 - t_{ii}^{31}t_{ii}^{32}(\lambda_{ii}^2)^{-1}t_{ii}^{13}t_{ii}^{23} \neq 0, \forall i \in 1, \ldots, m \tag{6.51}$$

Suppose now that the case is valid for n with

$$\det(\mathbf{T}_n) = \det(\mathbf{T}_n^t) \neq 0$$

and none of its associated eigenvalue $\theta_{ii}, \forall i \in E$ are null. We need to show that $\mathbf{T}_{n+1}$ is such that

$$\det(\mathbf{T}_{n+1}) = \det(\mathbf{T}_{n+1}^t) \neq 0$$

$$\mathbf{T}_{n+1} = \begin{pmatrix} (\mathbf{T}_n)^t & \mathbf{R}_{Xn} \\ \mathbf{R}_{nX} & \mathbf{T}^{n+1} \end{pmatrix} \tag{6.52}$$

with $\mathbf{T}^{n+1}$ being an upper triangular matrix representing regime $n+1$ and $\mathbf{R}_{Xn}, \mathbf{R}_{nX}$ transition diagonal matrices. Or

$$\det(\mathbf{T}_{n+1}^t) = \det(\mathbf{T}_n^t)\det((\mathbf{T}^{n+1})^t - \mathbf{R}_{nX}^t(\mathbf{T}_n^t)^{-1}\mathbf{R}_{Xn}^t)$$

Thus we need to find that none of the eigenvalues of $\det((\mathbf{T}^{n+1})^t - \mathbf{R}_{nX}^t(\mathbf{T}_n^t)^{-1}\mathbf{R}_{Xn}^t)$ is null. The matrix $(\mathbf{T}^{n+1})^t$ is lower-triangular and $\mathbf{R}_{nX}, \mathbf{R}_{Xn}$ are diagonal matrices. As for our assumptions, for all values in the matrix

$$(\mathbf{T}^{n+1})^t - \mathbf{R}_{nX}^t(\mathbf{T}_n^t)^{-1}\mathbf{R}_{Xn}^t \approx (\mathbf{T}^{n+1})^t$$

thus

$$\det((\mathbf{T}^{n+1})^t) \approx \det((\mathbf{T}^{n+1})^t - \mathbf{R}_{nX}^t(\mathbf{T}_n^t)^{-1}\mathbf{R}_{Xn}^t)$$

We have to condition the fact that

$$\forall \lambda_{ii}, i \in E, \forall t_{ii} i \in E,\ t_{ii}\lambda_{ii} << \lambda_{ii} \tag{6.53}$$

We have thus established the non-singularity of $\boldsymbol{T}$ and thus the fact that all its states are transient. □

Convergence

Having established the non-singularity of the matrix $\boldsymbol{T}$, we need to review the condition that the eigenvalues are lower than 1 to have the convergence of the exponential Matrix according to the Perron-Frobenius theorem. However, the previous proof establishes that the eigenvalues are lower than one albeit to the fact that might not be distinct due to the fact that several regimes might have default intensities that might be identical. This will ensure the Markov chain is Ergodic and thus that there exists a solution that is the probability distribution.

6.4.6 Distribution probabilities

We now specifically consider the case of 2 regimes for computation since the case of n regimes is similar.

We consider the a phase type distribution as in section 6.4.2, page 130 in the case of 2 regimes. The infinitesimal generator is defined as $\boldsymbol{Q}$

$$\mathbf{Q} = \begin{pmatrix} \mathbf{T} & \mathbf{t} \\ \mathbf{0} & 0 \end{pmatrix} \tag{6.54}$$

with

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}_1 & | & \mathbf{R}_{12} \\ -- & | & -- \\ \mathbf{R}_{21} & | & \mathbf{T}_2 \end{pmatrix} \tag{6.55}$$

We suppose for simplicity that the transition between low and high volatility regimes are not linked to the number of defaulted obligors from the set that have yet occurred, i.e

$$t_{12} = t_{12|1} = t_{12|2} = t_{12|3} \text{ and } t_{21} = t_{21|1} = t_{21|2} = t_{21|3}$$

with t_{ij} the transition intensity from regime i to j and $t_{ij|k}$ the transition intensity from regime i to j when X_t is in state $k \in K$.

Remark: In the observation of the time to absorption Δ of the Markov chain $(X_t)_{t\geq 0}$, regime $Y_t = R_1$ or $Y_t = R_2$ is not relevant to observe the transition probabilities of X_t. Thus,

$$\mathbb{Q}(X_t < t) = (1 - \alpha \exp^{\mathbf{Q}t} \mathbf{1}) \tag{6.56}$$

where α is the initial distribution.

Probability distribution definitions

Using the setting in phase type distribution with m obligors, the probability of reaching the sequence $\mathbf{i} = \{i_1, .., i_m\}$ representing the ordering of default of obligors with the $|E| \times |E|$ transition matrix is given by

$$\mathbb{P}(X_t = \mathbf{i}) = \alpha \exp^{\mathbf{Q}t} \mathbf{e_i} \quad \text{for} \quad \mathbf{e_i} \in \mathbb{R}^{|E|} \tag{6.57}$$

where $\mathbf{e_i}$ is the column vector representing the status having identified the states.

Each state is defined as a two dimension process (D, R) with :

- D defines a subset of defaulted (ordered or not) obligors in $\{1, .., m\}$,
- R defines the regime $R \in \{R_1, R_2\}$ under which the obligors of D has been observed.

Note: Each subsets $d \in D$ can be observed either in R_1 low regime or R_2: High regime.

Thus, In the general phase type setting $\boldsymbol{T}$, by identifying the correct states in both regimes (we are in dimension $2m$ compared to the classical phase.) we have

$$\mathbb{Q}[\tau_1 > t_1, ..., \tau_m > t_m] = \alpha \left(\prod_{k=1}^{m} e^{\boldsymbol{Q}(t_{i_k} - t_{i_{k-1}})} \boldsymbol{G}_{i_k} \right) \quad \text{where} \quad t_{i0} = 0 \tag{6.58}$$

Now for a time vector $(t_1,t_2,..., t_m)$ in $\mathbb{R}_+^m$ with the ordering of $(t_1,t_2,..., t_m)$ being

$$\boldsymbol{t}_{i_1} < \boldsymbol{t}_{i_2} < ... < \boldsymbol{t}_{i_m}$$

where $\mathbf{i}=(i_1,i_2,..., i_m)$ is a permutation of the obligor set $(1, 2, ..., m)$, the multivariate survival distribution is expressed as:

$$\begin{aligned}
\mathbb{Q}[\tau_1 > t_1, ..., \tau_m > t_m] &= \mathbb{Q}[\tau_{i_1} > t_{i_1}, ..., \tau_{i_m} > t_{i_m}] \\
&= \mathbb{Q}[X_{i_1} \in E_{i_1}, ..., X_{i_m} \in E_{i_m}] \\
&= \sum_{i_0 \in E} \mathbb{Q}[X_0 = i_0, X_{i_1} \in E_{i_1}, ..., X_{i_m} \in E_{i_m}]
\end{aligned}$$

The set E of the states of X_t can be decomposed in a set belonging to regime R_1 or to regime R_2

$$E = E^{\{Y_t = R_1\}} \cup E^{\{Y_t = R_2\}} = E^{[R_1]} \cup E^{[R_2]} \tag{6.59}$$

with $E^{\{Y_t=R_1\}} \cap E^{\{Y_t=R_2\}} = \emptyset$. Thus,

$$\begin{aligned}
&= \sum_{i_0 \in E^{[R_1]} \cup E^{[R_2]}} \mathbb{Q}[X_0 = i_0, X_{i_1} \in E_{i_1}, ..., X_{i_m} \in E_{i_m}] \\
&= \sum_{i_0 \in E^{[R_1]} \cup E^{[R_2]}} \sum_{i_{i_1} \in E_{i_1}^{[R_1]} \cup E_{i_1}^{[R_2]}} ... \sum_{i_{i_m} \in E_{i_m}^{[R_1]} \cup E_{i_m}^{[R_2]}} \mathbb{Q}[X_0 = i_0, X_{i_1} = i_{i_1}, ..., X_{i_m} = i_{i_m}]
\end{aligned}$$

Since the time-homogeneous property and the Markov property used in the proof are not impacted by the inclusion of regimes, the status matrix $\boldsymbol{G}_{i_k}$ is defined as previously but not is of dimension $2m \times 2m$.

The states of defaults or regime transitions are distinct and perfectly identifiable, thus enables to use the classical previously established framework by adjusting the status matrices or vectors.

Thus, we can express the following survival probabilities:

The marginal survival distribution

$$\mathbb{Q}[\tau_i > t] = \alpha e^{Qt} g^{(i)} \tag{6.60}$$

with $g^{(i)}$ of dimension $n\times \mid E \mid$ with i identifying states of i defaulting independently of regime. Identically, by identifying the regimes, with $j \in R$

$$\mathbb{Q}[\tau_i > t, i \in R_j] = \alpha e^{Qt} g^{(i,R_j)} \tag{6.61}$$

with $g^{(i,R_j)}$ of dimension $n\times \mid E \mid$ with i identifying states of i defaulted in regime j. Naturally, the marginal probability is the same marginal probability in each regime,

$$\mathbb{Q}[\tau_i > t] = \sum_{R_j} \mathbb{P}[\tau_i > t, i \in R_j] = \sum_{R_j} \alpha e^{Qt} g^{(i,R_j)} = \alpha e^{Qt} \sum_{R_j} g^{(i,R_j)} \tag{6.62}$$

The ordered marginal survival distribution

We obtain the same equation for the k^{th} marginal survival distribution with

$$\mathbb{Q}[\tau^{(k)} > t] = \alpha e^{Qt} m^{(k)} \tag{6.63}$$

with $m^{(k)}$ of dimension $n\times \mid E \mid$ with i identifying states of k^{th} default independently of regime. Identically,

$$\mathbb{Q}[\tau^{(k)} > t, i \in R_j] = \alpha e^{Qt} m^{(k,R_j)} \tag{6.64}$$

with $m^{(k,R_j)}$ of dimension $nx \mid E \mid$ with i identifying states of k^{th} default in regime j. Naturally,

$$\mathbb{Q}[\tau^{(k)} > t] = \sum_{R_j} \mathbb{P}[\tau^{(k)} > t, i \in R_j] = \sum_{R_j} \alpha e^{Qt} m^{(k,R_j)} = \alpha e^{Qt} \sum_{R_j} m^{(k,R_j)} \tag{6.65}$$

To illustrate the marginal survival distribution in equation 6.61 with 2 regimes to the same marginal survival distribution under no regimes, we consider two obligors that exhibit both high credit risk with base intensity $a_i = 0.02$ and low jump intensity $b = 0.003$. Arbitrary we set a low regime shift intensity $t = 0.003$ to a regime with high regime with parameters higher than set in the regime case. Figure 6.5, page 141, shows that the regime-case survival curves are lower than in the no regime case which is consistent with what is implemented.

With the same parameters as figure 6.5 the implementation of equation 6.64 can produce the marginal survival distribution per regime. The result are reproduced in figure 6.6, page 142. Note that by convention the markov chain X_t is always started in a low regime position.

Additionally, we can highlight the fact that the regime construction should no impact in terms of the marginal distribution $\mathbb{Q}[\tau_i > t]$. For example, consider the previously high credit risk $m = 2$ obligor set with the same intensity in all regimes, and set a high number of regimes, for example $n = 40$, one should expect that the marginal distribution of each obligor to be identical with or without the introduction of the regime structure. Figure 6.7, page 143, shows that both curves are identical and thus that the regime construction is not technically inconsistent with the multi-variate phase type distribution. This equality will be useful in the case of single-name CDS time-series calibration to extract the intensities vector $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{t})$ for each regime.

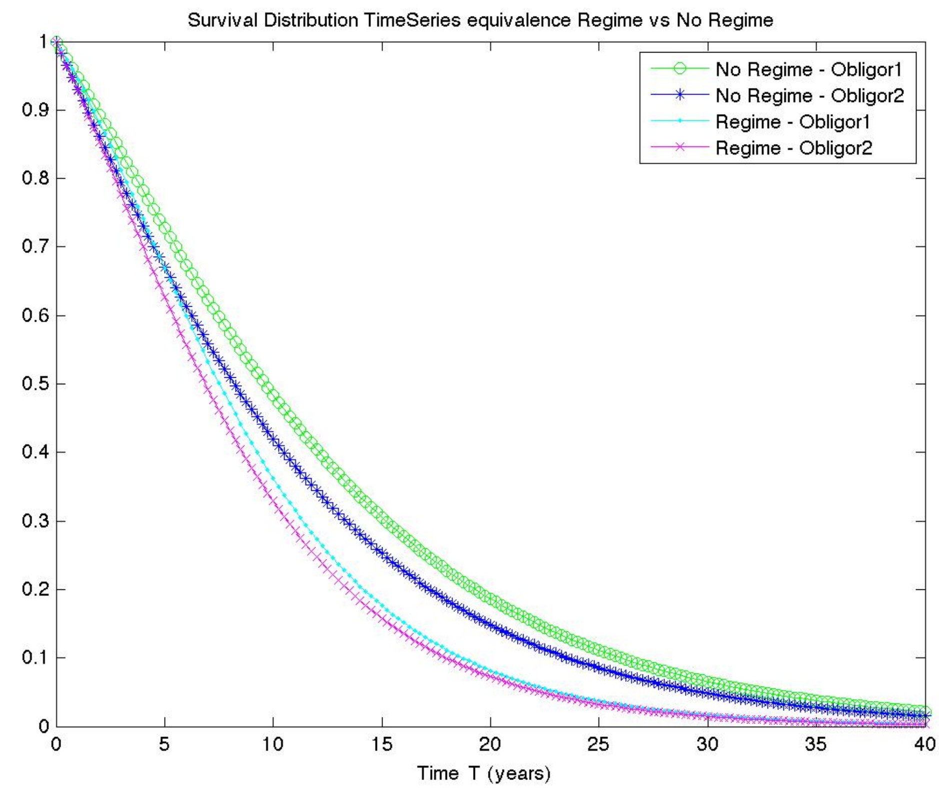


Figure 6.5: Marginal survival distribution $\mathbb{Q}[\tau_i > t]$ for 2 obligors with or without regime shift.

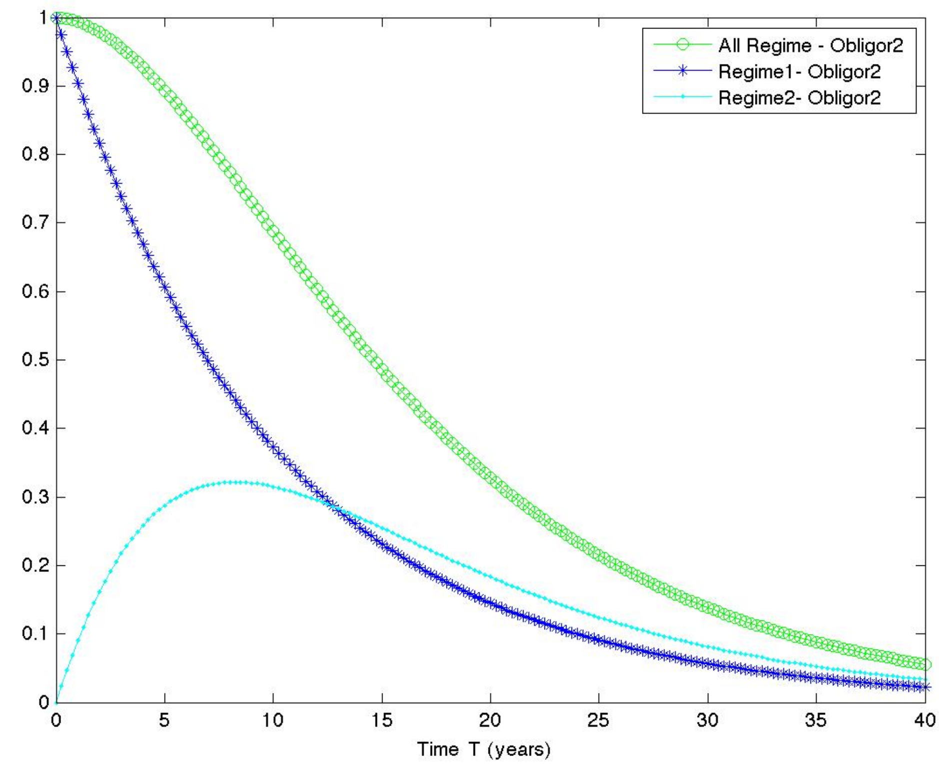


Figure 6.6: Marginal survival distribution $\mathbb{Q}[\tau_1 > t, i \in R_j]$ per regime j for obligor 1 in case of $m = 2$ obligors and $n = 2$ regimes.

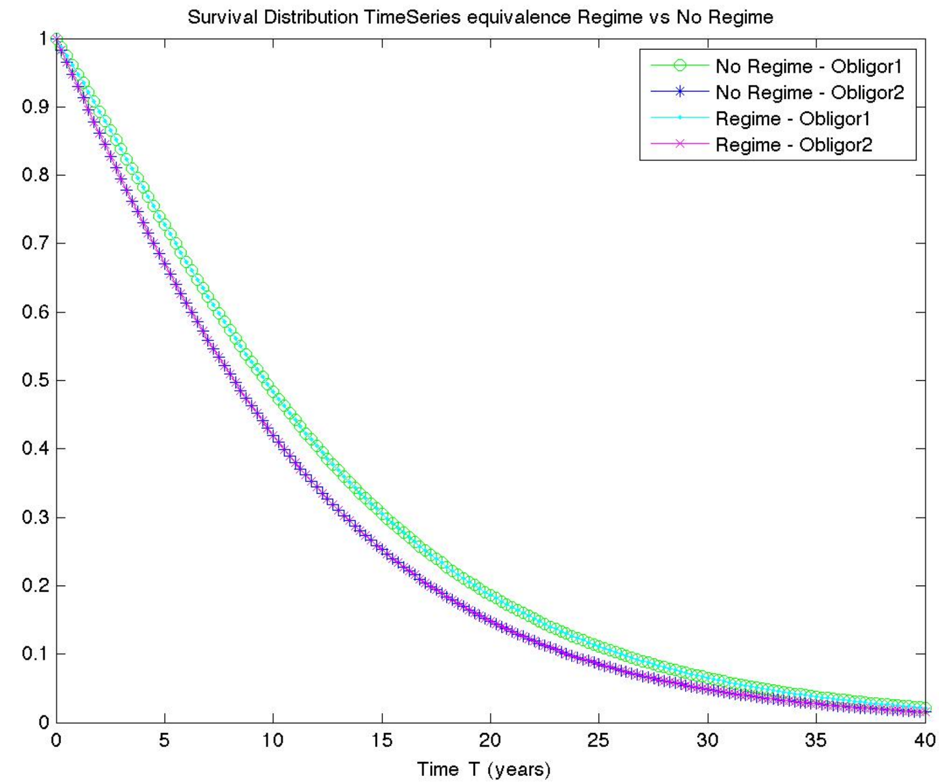


Figure 6.7: Marginal survival distribution $\mathbb{Q}[\tau_i > t]$ for 2 obligors with or without regime shift with identical intensities. Case of n=40 regimes.

Chapter 7

Conclusion

We hope that this thesis has proved useful in addressing the issue of credit risk contagion with the benefit of developing a multivariate method with closed-form formulas to capture the counterparty credit risk under credit risk contagion scenarios. An interesting feature is the capacity to identify and isolate that contagion risk for the benefit of pricing and risk management and should be applicable to all the range of the financial products and major asset classes that are traded by financial institutions.

The model has demonstrated an ease of calibration to extract jump and base intensities thanks to numerical methods that are widely available. Also a notable benefit is the dimension scalability in addressing basket products. However, a current limitation of the model is the fact that it does not have the Markov copula property and in the future the assumption of no simultaneous default should be relaxed to allow decoupling of calibration across CDS and index products. Additionally, the time-series calibration should also be addressed as contagion parameters should benefit from the various credit spread term-structure shapes. So, in the current context, the phase-type distribution is more likely to provide useful results in risk management simulations.

We believe another particular interesting result is related to regulatory requirements associated with financial claims. In both, defaultable or non defaultable case, the sensitivity of the Counterparty Credit Risk to contagion exhibit non linear evolution that question the $\alpha = 1.4$ volatility parameter adjustment. This is particularly the case with k^{th}-to-default swaps, or more generally, credit leveraged products (like LSS). The issue of the previously mentioned calibration is here of importance since this would allow to better link contagion scenario to identified events. The specific result that is interesting is the Counterparty Credit Risk of the k^{th}-to-default swaps in terms of "Gap Risk "when contagion risk goes to level consistent with the financial crisis. The result shows why "Gap Risk "is also termed "walk-away "risk for Leveraged Super Senior products.

To better account for credit spread volatility and capture the optionality term in the Counterparty Credit Risk, the regime-shift phase-type distribution evolution should be further strengthen to calibrate to credit volatility products although the liquidity of those products might question the validity of such approach.

To finish, this document has not covered the new type of risk arising in bilateral claims such as the funding cost with the Funding Valuation Adjustment, the liquidity stress, etc. Those risks have gained more coverage especially in [Brig10a] and point to the fact in its current state the model is structured for credit risk contagion among obligors.

Appendix A

Regulatory concepts on Counterparty Credit Risk

A.1 Economic Capital: measuring and marking Counterparty Credit Risk

As presented in [Picoult05], **Economic Capital** is defined as a measurement of economic risk from an insolvency perspective, i.e. the amount of risk capital that a firm requires to cover the risks (market risk, credit risk, operational risk) that it is running or collecting as a going concern. Financial Institutions hold risk capital as an amount equal at least to Economic Capital. This is different from Regulatory Capital (see section A.2) which is mandatory capital that regulators require to be maintained. Economic Capital is the best estimate of required capital that financial institutions use internally to manage their own risk and to allocate the cost of maintaining Regulatory Capital among different unit within the organisation. The primary distinguishing feature of counterparty risk (also called pre-settlement risk in [Picoult05]) relative to other forms of credit risk is that the magnitude (and the sign) of the credit exposure to a counterparty on any future date is uncertain. It is, thus, more difficult for counterparty credit risk than for a loan for three reasons:

- the uncertainty in the future credit exposure,
- the bilateral nature of counterparty credit exposure,
- the challenge of defining the credit value (defined as **CVA** - Counterparty Valuation Adjustment - see section 2.2.4 for a full definition) to take into account the effect of market spreads on the market value of a derivative portfolio.

The **CVA** can be viewed as the *difference between the risk-free value of a derivative portfolio and the value after taking the counterparty risk into account*. And thus, potential changes in the CVA of each counterparty are key components of the potential loss of the economic value of the portfolio.

The main difference between Lending Risk and Counterparty Credit Risk is encapsulated in the bilateral aspect of CVA depending on the future state of the market. For that reason, Economic Capital for counterparty risk will require a double level of simulation:

- simulation in market rates (covered by desks specific risk management), and,
- simulation of the types of credit event (covered by CVA desks).

Historically, Economic Capital was first intentionally defined for a loan portfolio where Economic Capital as a measure of risk is the potential **Unexpected Loss** (UL) of Economic Value of a portfolio or business over some long time horizon (e.g., one year), at some high confidence level (e.g., 99% confidence level). It can be calculated at different organisational levels of a firm on a standalone or marginal basis. As show in the figure 2.1, Economic Capital is derived from the calculation of the probability distribution of Potential Loss over some time horizon. The two key features of a loss distribution are the **Expected Loss** (EL) and the **Unexpected Loss** where Unexpected Loss is defined as the difference between the **Potential Loss** at a high confidence level (99%) and the Expected Loss. Economic Capital is then the Unexpected Loss measured at a specified confidence level and depends on the shape of the potential loss distribution and the confidence level.

To help define Economic Capital, [Picoult05] introduces also the **Exposure Profile** as the potential exposure to a counterparty at a set of future dates over the life of the portfolio, measured at a specified confidence level which is best represented as an exposure profile rather than a single number because:

- the remaining unsettled cash-flows with a counterparty will contractually change over time as floating rates are set, options expire, etc.
- the potential range or probability distribution of each underlying market rate tend to widen the further out into the future one looks.

An illustration of the time varying feature of the **Exposure Profile** is found on figure 2.2 for an interest rate swap.

Additionally, [Picoult05] develops also the concept **Curent Counterparty Exposure** which is defined as the *immediate exposure to a counterparty representing the current replacement cost of the contracts under an immediate default.* Naively, a VaR measurement might seem the best way to measure this potential exposure although market-factor sensitivities are not normally distributed. The two most common ways of measuring Potential Exposure of a counterparty with multiple transactions are a

- **Simple Transaction** methodology for approximating the exposure, and,
- a more precise and sophisticated portfolio **Simulation Methodology**.

The **Simple Transaction** method defines exposure as the sum of two terms, its current market value and its potential increase in value. The potential increase in value can be expressed either by a time-varying profile over the remaining life of the transaction or as a single number. Typically, standardised tables can be defined to approximate the potential increase in the value of each transaction per unit of notional principal. (This however ignores portfolio effects, the problem of different tenors and is not taking into account offsetting transactions or ignores diversification). So, this method is only applicable for simple end-user with few transactions since this approach comes originally from Economic Capital loans approach.

The **Simulation Methodology** is defined by the following steps:

- simulate scenarios of changes in market factors over time,
- calculate the potential market value of each transaction at each future date of each simulation path,
- calculate the potential exposure of each counterparty at each future date of each simulated path,
- calculate the counterparty's exposure profile at some confidence level at a set of future dates starting with the current exposure calculated today and the **average positive exposure**:

  $$\mathrm{EPE}_t = \mathbb{E}_t(\mathrm{Exposure}_t)^+$$

  (see section 2.2.4 for definition)

A typical simulation result for an Exposure Profile will have the shape presented in figure 2.2 in the case of an Interest Rate Swap to produce Counterparty Credit Risk (CCR) metrics such as:

- **EPE**: Expected Positive Exposure, and,
- **PFE**: Potential Future Exposure.

(see section 2.2.4 for definition):

As we can see through the concepts and definitions of loss, the shape of the potential loss distribution arising from the exposure profile will depend on:

- the type of risk (e.g. market risk, loan portfolio credit risk, counterparty credit risk or operational risk),
- the definition of losses (e.g. economic loss versus accounting loss),
- the time horizon over which the potential loss distribution is simulated (e.g. one year or the life of the portfolio),
- the degree of diversification risk of the underlying portfolio,
- the assumptions underlying the simulation of the future states of the drivers of potential losses.

The influence of this set of variables and parameters has led to several levels of complexity in the computation of the **Economic Capital**:

A.2 Regulatory treatment of Counterparty Credit Risk

Counterparty Risk according to the Bank of International Settlements - Models overview

Common aspects of exposure for CCR:

Key to the implementation required by the regulators is to estimate the **Exposure At Default** (EAD) under the Revised Framework. In summary, from [BIS05], there is three methods that are advanced for calculating the EAD for transactions involving CCR in the banking or the trading book under the Revised Framework:

1. the existing **Current Exposure Method** (CEM).
2. the **Standardised Method** (SM),
3. the **Internal Model Method** (IMM) that uses the concept of the **Expected Positive Exposure** (EPE).

Those methods generate a **Credit Exposure**, which is defined as the cost of replacing the transaction if the counterparty defaults assuming there is no recovery of value. In addition, Credit Exposure depends on one or more underlying market factors and CCR is bilateral by definition. Other typical characteristics of transactions that involve CCR may include

- the use of collateral to mitigate risk;
- the use of legal netting or rights to offset contracts; and
- the use of re-margining agreements.

Measures reported: The set of metrics of CCR presented are

- **Expected Positive Exposure**,
- **Expected Exposure**, and
- **Potential Future Exposure**.

Banks typically compute **EPE**, **EE**, and **PFE** (see section 2.2.4 for definitions.) using a common stochastic model. The *EPE is generally viewed as the appropriate EAD measure to determine capital for CCR*. Additionally, the EAD for instruments with CCR must also be determined conservatively and conditionally on a 'bad state' of the economy with the IMM and SM scale EPE using multipliers, termed 'alpha'-α and 'beta' -β, respectively. Both alpha and beta are set at 1.4, but supervisors have the flexibility to raise either parameter in appropriate situations. This highlights to a certain extent the non-dynamical feature of those risk measures.

Netting Sets: A 'netting set' is a group of transactions with a single counterparty that are subject to a legally enforceable bilateral netting arrangement and permitted to be netted as identified in [BIS95].

Summary of Current Exposure Method:

For those banks that do not qualify for the use of the Internal Model Method, the **Current Exposure Method** takes the form of the following equation:

$$\text{CCR Charge} = [(\text{RC} + \text{Add-on}) - \text{VAC}] * \text{RW} * 8\%$$

with

- RC: Current Replacement Cost,
- Add-on: The estimated amount of potential future exposure (Page 37 to page 42 in [BIS06]),
- VAC: Volatility Adjusted Collateral,
- RW: Risk Weight - The risk weight of the counterparty.

So under the CEM, EAD is equal to

$$\text{EAD} = [(\text{RC} + \text{Add-on}) - \text{VAC}]$$

Summary of the Standardised Method

The **Standardised Method** is also available for banks that do not qualify for the IMM method but that would like to adopt a more risk-sensitive method than the CEM. The Standardised Method is designed to capture certain key features of the IMM for Counterparty Credit with a simpler implementation. The concept is that CCR exposures are expressed in risk positions that reference short-term changes in valuation parameters (e.g. modified duration for debt instruments, delta concept for options). It also assumes that the positions that are open under a short-term forecasting horizon of, for example one day, remain open and unchanged throughout the forecasting horizon. There is however no recognition of diversification effects.

For transaction i, collateral l, hedging set j the EAD is:

$$\text{EAD} = \beta. \left(\text{CMV} - \text{CMC}, \sum_j \left(\sum_i \text{RPT}_{i,j} - \sum_l \text{RPC}_{l,j} \right) .\text{CCF}_j \right)^+ \tag{A.1}$$

where:

- CMV: "Current Market Value" of the portfolio of transactions within the netting set with a counterparty gross of collateral, i.e. CMV=$\sum_i \text{CMV}_i$ is the current market value of transaction i,
- CMC: "Current Market value of the Collateral" assigned to the netting set, i.e. CMC=$\sum_l \text{CMC}_l$ is the current market value of collateral l,
- $\text{RPT}_{i,j}$: "Risk Position from Transaction" i with respect to hedging set j,
- $\text{RPC}_{l,j}$: "Risk Position from Collateral" l with respect to hedging set j,

- CCF_j: Supervisory "Credit Conversion Factor" with respect to the hedging set j,
- β: Supervisory scaling parameter.(typical value 1.4)

The exposure amount EAD represents the product of

- the larger of the net current market value or a supervisory EPE, times,
- a scaling factor, termed beta β.

The first factor captures two key features of the Internal Model Method:

- for netting sets that are deep in the money, the EPE is primarily determined by the current market value of the netting set,
- for netting sets that are at the money, the current market value is not relevant, and Counterparty Credit Risk is driven only by the potential change in the value of the transactions.

Neither of these features are applicable in the Current Exposure Method.

Summary of Internal Model Method

Banks should be allowed to use the output of their 'own estimates' developed through Internal Models in an advanced EAD approach. The Internal Model Method permits qualifying institutions to employ internal estimates of the EPE of defined netting sets of their counterparty exposures in computing the Exposure Amount (EAD). An illustration of the time profile of Expected Exposure (EE) and the Expected Positive Exposure (EPE) is provided in figure A.1 respectively as the solid line and the dotted line .

Internal models commonly used for CCR estimate a time profile of Expected Exposure (EE) over each point in the future, which equals the average exposure, over possible future values of relevant market risk factors, such as interest rates, foreign exchange rates, etc.

To address the concern that Expected Exposure (EE) and Expected Positive Exposure (EPE) may not capture rollover risk or may underestimate the exposures of OTC derivatives with short maturities, EAD for CCR is to be set equal to a netting sets 'Effective EPE' multiplied by a factor α. Effective EPE is calculated using the time profile of estimated 'Effective Expected Exposure' for a netting set with 'Effective EE' defined recursively as

$$\text{Effective } \mathrm{EE}_{t_k} = (\text{Effective } \mathrm{EE}_{t_{k-1}}, \mathrm{EE}_{t_k})^+$$

with EE_{t_0} equal to the current exposure. Thus, in figure 2.3, the time profile of Effective EE is represented by the dashed line while Effective EPE (defined as the average of Effective EE) is represented by the solid straight line.

The Internal Method Method is obtained typically by Monte Carlo simulations with parameters and variables approval set in accordance to the Revised Frameworks requirements. These include notably:

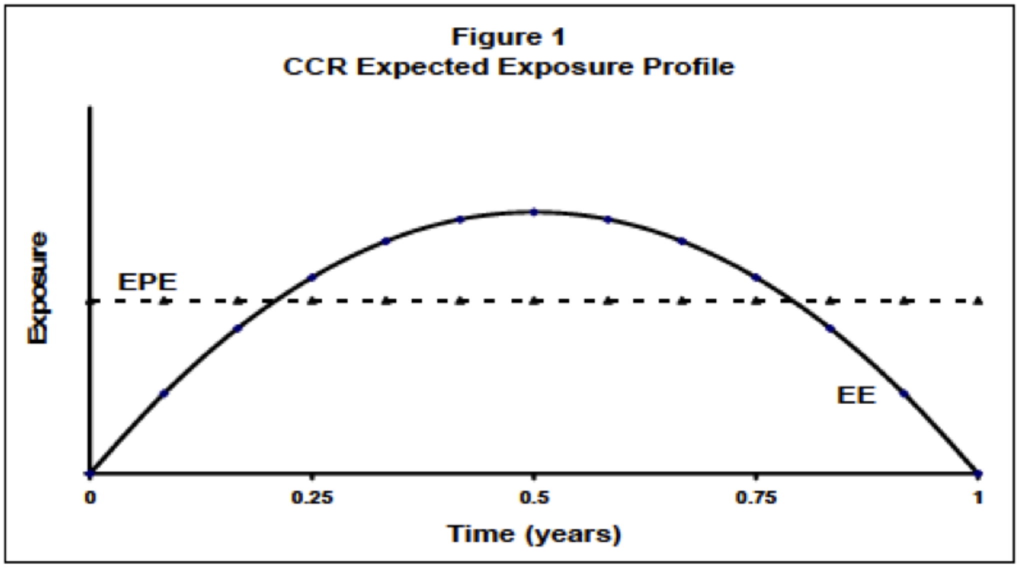


Figure A.1: Typical CCR Expected Exposure and Expected Positive Exposure illustration.

- **Time Horizon**: Effective EPE is to be measured as the average of Effective EE over one year. This is a key feature as most of the regulation is focused on a short-term horizon ($<$ 1year). We will try in this thesis to focus on longer term exposure subject to liquidity of hedging instruments.

- **Alpha Multiplier**: The α multiplier provides a means of conditioning internal estimates of EPE on a 'bad state' of the economy. In addition, it acts to adjust internal EPE estimates for both:

  - correlations of exposures across counterparties exposed to common risk factors, and,
  - the potential lack of granularity across a firms counterparty exposures.

  The α multiplier is also viewed as a method to offset model error or estimation error. Alphas may range from approximately 1.1 for large global dealer portfolios to more than 2.5 for new users of derivatives with concentrated exposures and little or no current exposure in their book.

- **Risk-neutral vs. Actual distributions**: Industry practice does not indicate that one single approach has gained favour. For this reason, supervisors are not requiring any particular distribution be used. We will se later that [**?**] uses both and additionally describe variable change of neutral-real measures of credit contagion models.

- **Maturity Adjustment**: Like corporate loan exposures with maturity greater than one year, counterparty exposure is susceptible to changes in economic value with the deterioration in the counterparty's creditworthiness. Supervisors believe that an **Effective Maturity parameter** (M) can reflect the impact of these changes on capital and that the existing maturity adjustment formula in the Revised Framework is appropriate for counterparty credit exposures (Page 81 and page 28 in [BIS05] and page 89 in [BIS06] - *Revised Framework Full Version*) with

$$\text{Effective maturity (M)} = \frac{\sum_t t.CF_t}{\sum_t CF_t} \tag{A.2}$$

where CF_t denotes the cash flows (principal, interest payments and fees) contractually payable by the borrower in period t. Conceptually, **M equals the effective credit duration** of the counterparty exposure.

Model approval:

The usage of the Internal Model Method is conditional to model approval by the regulation authorities. In the Full Framework, detailed in [BIS06], Risk Management approach is split in 3 Pillars,

- Pillar 1 which is focused on establishing the minimum capital requirement process.
- Pillar 2 which provides general background and specific guidance to cover counterparty credit risks that may not be fully covered by the Pillar 1.
- Pillar 3 which is focused on market discipline principles.

Thus, the 3 Pillars are complementary in establishing a risk system that captures the counterparty risk of a portfolio.

- the **Pillar 1 - Minimum Capital Requirement**, in document [BIS05] on page 25 and page 29, describes the approval method to adopt an Internal Model Method to estimate EAD. In essence, the process of validating the EPE model is close to validating a Value-At-Risk one with additional elements like
  - the forecasting of long-term horizon of market drivers (IR, FX, Equities),
  - capturing transaction specific information, and,
  - the pricing models to calculate counterparty exposure.

  Thus, **Expected Exposure** or **Peak Exposure** measures should be calculated based on a distribution of exposures that accounts for the possible 'non-normality' of the distribution of exposures, including the existence of leptokurtosis ('fat tails'). A specific mention is given in page 32 on the monitoring of **wrong-way risk** (see definition in section 2.2.4) where a bank must have procedures in place to identify, monitor and control cases of specific wrong-way risk, beginning at the inception of a trade and continuing through the life of the trade. The **Pillar 1 - Minimum Capital Requirement** also specifies in parallel to the EAD, the Risk Weighted Assets associated per each type of assets as reproduced in figure A.2.

Categories	External credit assessment	Specific risk capital charge
Government	AAA to AA-	0%
	A+ to BBB-	0.25% (residual term to final maturity 6 months or less)
		1.00% (residual term to final maturity greater than 6 and up to and including 24 months)
		1.60% (residual term to final maturity exceeding 24 months)
	BB+ to B-	8.00%
	Below B-	12.00%
	Unrated	8.00%
Qualifying		0.25% (residual term to final maturity 6 months or less)
		1.00% (residual term to final maturity greater than 6 and up to and including 24 months)
		1.60% (residual term to final maturity exceeding 24 months)
Other	Similar to credit risk charges under the standardised approach of the Revised Framework, e.g.:	
	BB+ to BB-	8.00%
	Below BB-	12.00%
	Unrated	8.00%

Figure A.2: Risk Weighted Assets table

- The **Pillar 2 -Framework** states that the bank must have counterparty credit risk management policies, processes and systems that are conceptually sound and implemented with integrity relative to the sophistication and complexity of a firms holdings of exposures that give rise to CCR. The details of the process can be found on page 42 in [BIS05], where Bank must review the value of the alpha factor α to capture the diversification of the portfolio, the correlation of default across counterparties, and the number and granularity of counterparty exposures.

- The **Pillar 3 - Market Disclosure** has established, due to the rapid growth of credit exposure from derivatives intermediation activities, a set of disclosures for CCR and OTC derivatives instruments like the one found in figure A.3.

An always moving concept

The **Internal Model Method**, detailed in [BIS01a] and [BIS01b], is the most dynamic model in addressing the CCR and is in constant evolution in order to capture counterparty credit risk. As an example, the recent paper [BIS10a] highlights the evolution of the validation process in quantitative aspects such as

- the frequency of backtesting,

- representativeness of portfolios,

- long risk horizons and real trade backtesting versus hypothetical trade backtesting.

General disclosure for CCR-related exposures

Qualitative Disclosures	(a)	The general qualitative disclosure requirement (paragraph 824 and 825) with respect to derivatives and CCR, including: • Discussion of methodology used to assign economic capital and credit limits for counterparty credit exposures; • Discussion of policies for securing collateral and establishing credit reserves; • Discussion of policies with respect to wrong-way risk exposures; • Discussion of the impact of the amount of collateral the bank would have to have to provide given a credit rating downgrade.
Quantitative Disclosures	(b)	Gross positive fair value of contracts, netting benefits, netted current credit exposure, collateral held (including type, e.g., cash, government securities, etc.), and net derivatives credit exposure.[33] Also report measures for exposure at default, or exposure amount, under the IMM, SM or CEM, whichever is applicable. The notional value of credit derivative hedges, and the distribution of current credit exposure by types of credit exposure.[34]
	(c)	Credit derivative transactions that create exposures to CCR (notional value), segregated between use for the institution's own credit portfolio, as well as in its intermediation activities, including the distribution of the credit derivatives products used[35], broken down further by protection bought and sold within each product group.
	(d)	The estimate of alpha if the bank has received supervisory approval to estimate alpha.

Figure A.3: CCR General Disclosure

The establishment of best practice in the industry is illustrated by [BIS98] which represents the results of market participants standard questionnaire about settlement procedures and counterparty risk management practices such as:

- Counterparty Credit limits,
- Master Agreements,
- Transaction processing and settlement,
- Close-Out Netting,
- Collateralisation,
- Clearing Houses.

All those practices and procedures will need to be addressed by a counterparty credit risk system and give rise to new risks and modelling issues, like:

- **Master Agreements and Trade Confirmations**: Backlogs of uncompleted agreements (between 5 and 20% of their counterparties) and trade confirmations discrepancies (in 5 to 10% of confirmations received) increase credit risk by jeopardising the enforceability of transactions.
- **Netting systems**: Incomplete systems integration make it difficult for dealers to calculate and administer net payments.

- **Close-out netting**: Legally enforceable netting provisions reportedly reduce aggregate counterparty credit exposure by 20 to 60%.
- **Collateral**: In recent years some dealers have rapidly expanded their use of collateral to mitigate counterparty credit risks. Those dealers with the most advanced programmes collateralise transactions with between 10 and 30% of their counterparties at the time of the paper [BIS10a].
- **Collateral Agreements and Custodian Risk**: Collateral Agreements may however give rise to **custody risk**, that is, the risk of loss of securities received from counterparties and held in custody because of insolvency, negligence or fraudulent action by the custodian.
- **Double Default concept**: The Part 2 of [BIS05] extends the concept of the treatment of **Double Default** detailed in [BIS06]-Full Framework, i.e, *the reduction in risk afforded by having credit protection place is recognised through a substitution approach.* This means that, for example in the Standardised Method approach, a bank may substitute the risk weight of the protection provider for that of the obligor. However in the current context no double default is being granted for:
  - Multi-name credit derivatives (other than n^{th}-default basket products),
  - Synthetic securitisations and other tranched products,
  - Funded credit derivatives (i.e Credit Linked Notes).

New types of risks arising? As the recent consultative paper [BIS10b] shows, proposed Basel III reforms require banks to more appropriately capitalise their exposures to **Central counterparties** (CCPs). It proposed that a qualifying CCP will receive a 2% risk weight while the Basel II Framework allows exposures to CCPs to be nil and, as such, provides significantly reduced capital charges for banks.

Appendix B

Numerical Results

B.1 Calibration Results

Results of the calibration to the spreads quoted in table B.1, page 158, with the corresponding reproduced CDS spread and intensities a_i in table B.2, page 158, with a multivariate phase-type distribution "semi-calibration" process

Table B.1: Single Name CDS Spread (bps) on 01/07/2010 - Source: Datastream -

Maturity (T in years)	5	7	10
Financials			
Axa	148.97	152.96	148.76
BNP Paribas	130.95	134.83	137.31
Crédit Agricole	159.01	161.68	163.67
Crédit Mutuel	125.40	129.80	136.10
Natixis	199.75	204.11	210.60
Société Génerale	145.04	150.12	148.76
Industrials			
Renault	346.03	345.12	339.36
Peugeot	353.67	370.03	388.12
Air Liquide	54.91	61.27	67.00
Sanofi	67.30	78.43	86.99
LVMH	64.45	67.24	72.71
Total	93.20	99.59	106.64
EDF	93.66	100.34	109.36

Table B.2: Calibration result to 10y Single Name CDS Spread using MPH

Name	Obtained Quote (bps)	base intensity a_i	Error (bps)
Financials			
Axa	148.76	0.0054	-0.04e^{-10}
BNP Paribas	137.31	0.0052	0.55e^{-10}
Crédit Agricole	163.67	0.0056	-0.01e^{-10}
Crédit Mutuel	136.10	0.0052	0.66e^{-10}
Natixis	210.60	0.0061	-0.06e^{-10}
Société Génerale	148.76	0.0054	0.04e^{-10}
Industrials			
Renault	339.36	0.0077	0.30e^{-10}
Peugeot	388.12	0.0079	-0.02e^{-10}
Air Liquide	67	0.0039	-0.04e^{-7}
Sanofi	86.99	0.0045	0.22e^{-7}
LVMH	72.71	0.0041	-0.004e^{-7}
Total	106.64	0.0049	-0.0016e^{-7}
EDF	109.36	0.0050	-0.0026e^{-7}

B.2 Probability distribution illustration under phase-type distribution

For $m = 2$ obligors, we provide an illustrative simulation of the marginal probability distribution $\mathbb{Q}(tau_1 > t)$ of obligor 1 as a function of the base intensity a_1 of obligor 1, the base intensity a_2 of obligor 2 and a interval of values of jump intensity $b_{2,1}$. The six scenarios represented in figure B.1, page 160, from left-upper right corner,

- obligor 1 and 2 are low risk obligors i.e. $a_1 = a_2 = 0.005$
- obligor 1 and 2 are normal risk obligors i.e. $a_1 = a_2 = 0.015$
- obligor 1 and 2 are high risk obligors i.e. $a_1 = a_2 = 0.03$
- obligor 1 is a low risk obligor and obligor 2 is a high risk obligors i.e. $a_1 = 0.005,\ a_2 = 0.03$
- obligor 1 is a high risk obligor and obligor 2 is a low risk obligors i.e. $a_1 = 0.03,\ a_2 = 0.005$
- obligor 1 is a very high risk obligor and obligor 2 is a low risk obligors i.e. $a_1 = 0.04,\ a_2 = 0.005$

Note that the value interval of values of jump intensity $b_{2,1}$ is set from low risk to high risk.

The figure B.1 highlight the fact the default of an obligor due to contagion requires under the multi-variate phase type distribution that another obligor has defaulted to kick-in contagion. This is why most of the spread of default lines is reduced in the first 10 years unless the one counterparty is really risky and the contagion paramerter is high.

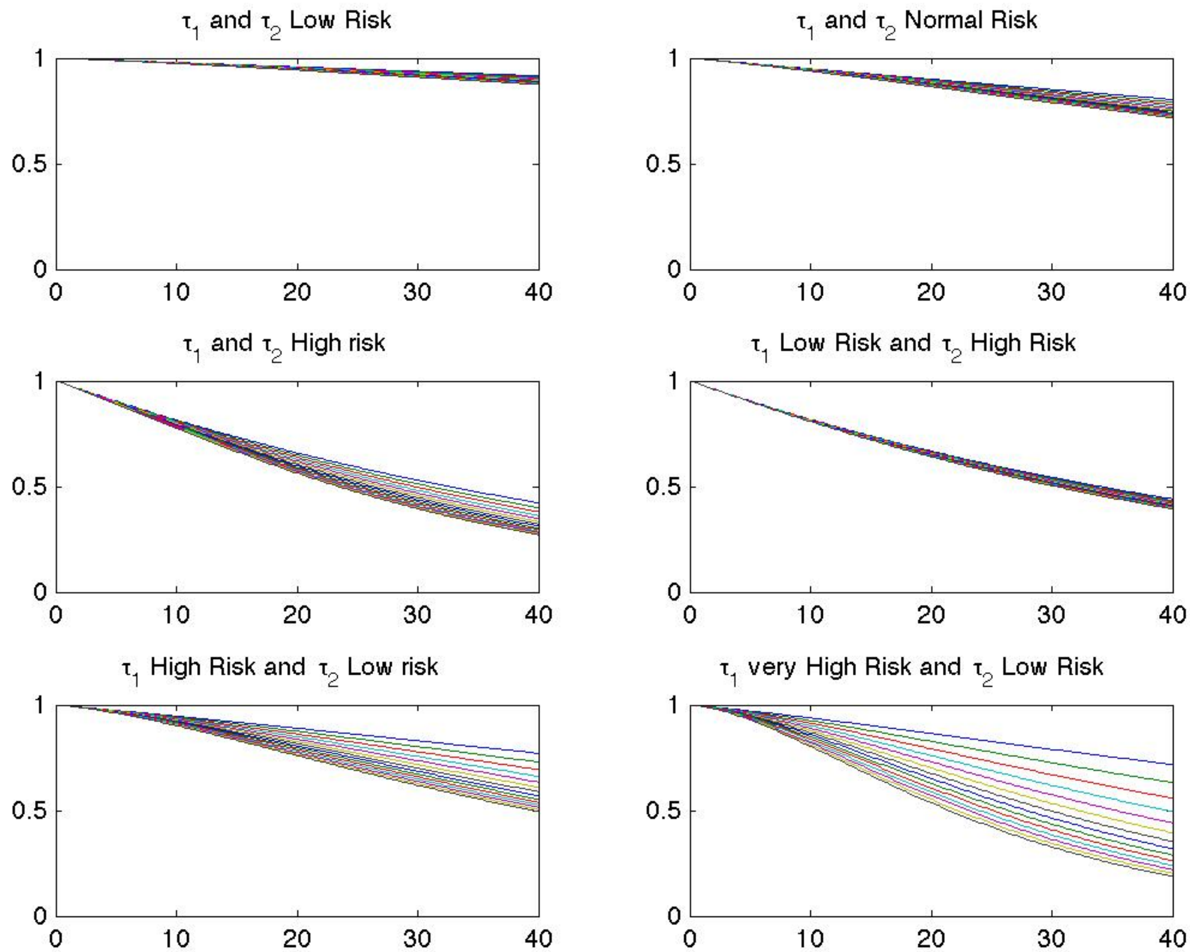


Figure B.1: Contagion Illustration: Impact of Jump Intensities on Survival Distributions of two counterparties - Time in years

B.3 Counterparty Credit Risk of Contingent Claims

Illustration of Counterparty Credit Risk for non-defaultable claims

We provide in this section implementation of typical financial claims, like interest-rate swaps (IRS) or cross-currency swaps (CCS), payer or receiver, annual or semi-annual payment frequency, to illustrate the implementation of a Counterparty Credit Risk model and to generate the distribution of products in the future using the relevant risk metrics as introduced in section 3.1.2. The Monte-Carlo simulation implementation using the C++-based quantitative free library Quantlib will generate the distribution of the price of the financial claims in the future and enable the computation of the Counterparty Credit Risk metrics such as EPE, ENE, PFE and ES. (See definitions in section 3.1.2)

The figure B.2, page 162, presents the results (from the upper left to the lower right) of the following products

- a vanilla interest-rate swap (IRS), 7-year maturity, annual frequency payment,
- a vanilla interest-rate swap (IRS), 15-year maturity, annual frequency payment,
- a vanilla interest-rate swap (IRS), 7-year maturity, semi-annual frequency payment, and,
- a vanilla interest-rate swap (IRS), 15-year maturity, semi-annual frequency payment.

Finally, figure B.3, page 163, presents the results of the price distribution and CCR metrics for a Cross Currency Swap.

As pointed by [Cesari09], [Gregory10], [Pykhtin05] and [Tang07], the Counterparty Credit Risk metrics in section 2.2.4 are a function of the structural features of the financial claim valued such as payer vs receiver, the payment frequency, the reference curves etc. Typically a cross-currency swap with the exchange of notional at maturity results in a exposure profile that is rising with time with the volatility of the underlying. In the case of an Interest-rate Swap, the exposure is limited to the interest payment which results in the profile rising under the effect of the volatility but decreasing after as a result of less interest payments due towards maturity.

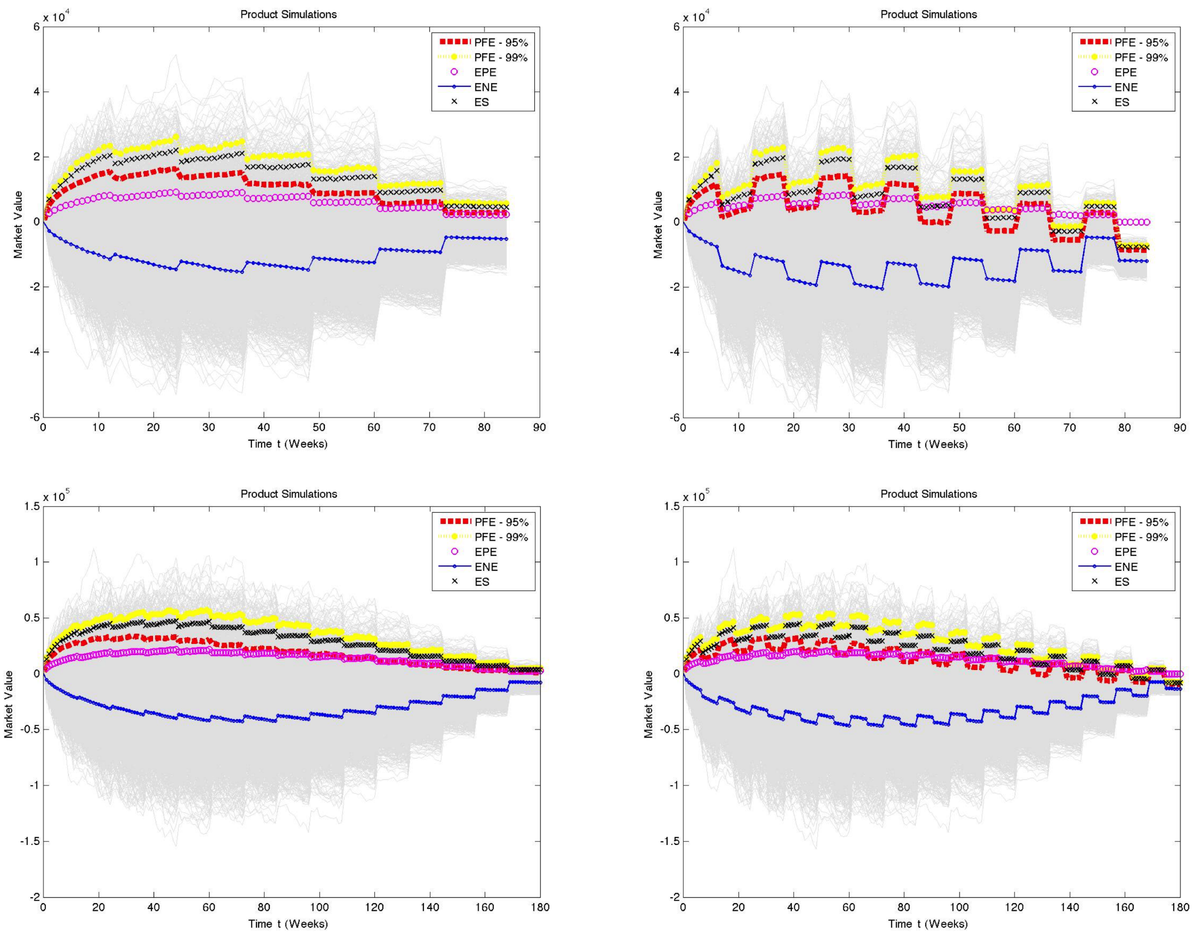


Figure B.2: Counterparty Credit Risk Metrics for Payer Swap. From upper-left to bottom-right: 1 - Maturity 7y, Annual - 2 - Maturity 7y, Semi-annual - 3 - Maturity 15y, Annual - 4 - Maturity 15y, Semi-annual.

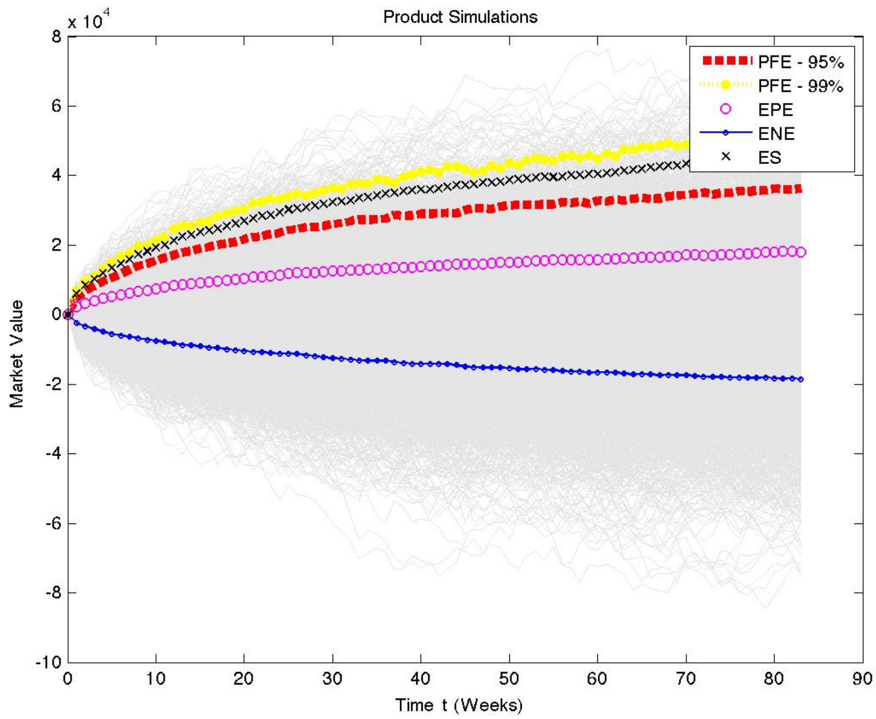


Figure B.3: Cross Currency Swap (CCS) Counterparty Credit Risk Exposure

Illustration of Counterparty Credit Risk for a portfolio of non-defaultable claims

Having identified and implemented, according to product-term sheets, the structural specificity of the financial claims, usually the Counterparty Credit Risk needs to be valued in a portfolio context as the risk measure for CCR are convex. Using the price distributions of the Monte-Carlo simulations for different products under the same underlying distributions and same time grid, one can aggregate the multi-product price distributions to realise netting in order to illustrate the benefit of netting and bilateral counterparty risk pricing. Thus, three illustrative simple portfolios are generated

- Portfolio 1 in figure B.4: 7-year Payer IRS with annual payment frequency and 15-year Payer IRS with annual payment frequency
- Portfolio 2 in figure B.5: 7-year Payer IRS with annual payment frequency and 15-year Receiver IRS with semi-annual payment frequency
- Portfolio 3 in figure B.6: 7-year Receiver IRS with semi-annual payment frequency and 15-year Payer IRS with annual payment frequency

As extensively covered in [Gregory10], in the case of a Credit Support Annex that allows netting, portfolio that have opposite sensitivities to underlying drivers will reduce risks like figure B.5 and B.6. In the case of figure B.4 there is an addition of the notional of the resulting exposure since sensitivities have the same sign.

Numerically, the results and portfolio simulated in appendix B.3, presents the impact of netting on the CVA and DVA in table B.3 for an investment-grade counterpart. This just highlights the typical results obtained in the case of a counterparty credit risk system in the case of an interest-rate swap with a notional of 1 million USD and a low volatility G8 interest rate environment. The credit contagion scenario is neutral with obligors having both low default risk with $a_1 = a_2 = b_{1,2} = 0.005$. The netting results are also illustrative of the benefit of a consistent methodology for portfolio CVA exposure and the directionality of the exposure (Payer vs Receiver).

Table B.3: CVA - DVA Netting effect for interest-rate swaps defined in section B.3. Prices are in bps followed within brackets of the MC error.

Products	CVA	DVA
Payer - Annual	5(0)	45(2)
Payer - Semi-Annual	3(0)	67(3)
Receiver - Semi-Annual	53(3)	4(0)
Portfolio1	21(1)	6(0)
Portfolio 2	7(1)	112(5)

Remark: In closing, most of the products that are presented in this section are products designed as "vanilla", i.e. simple with no optionality, like callability or putability, embedded.

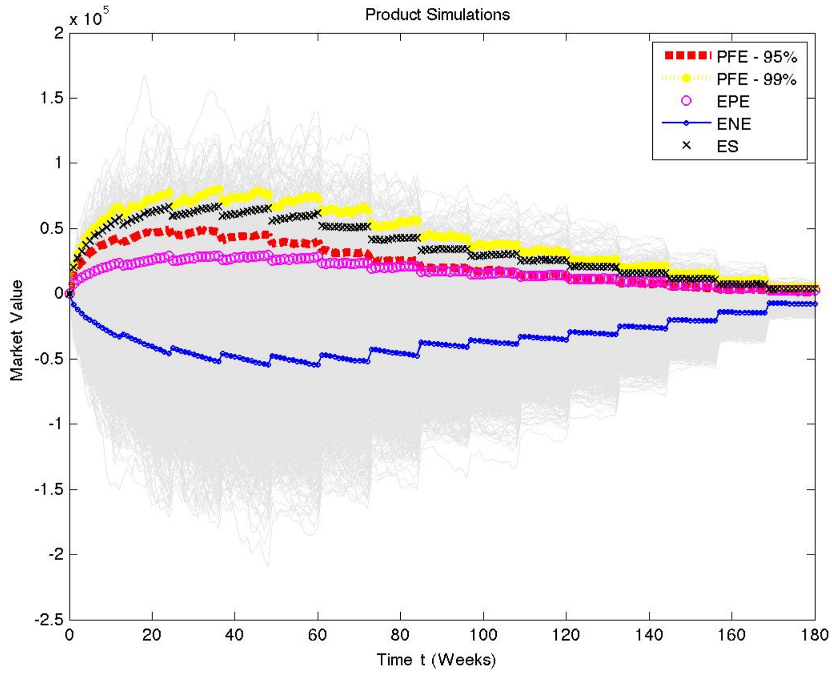


Figure B.4: Portfolio Counterparty Credit Risk: 7yr Payer IRS and 15 yr Payer IRS Portfolio

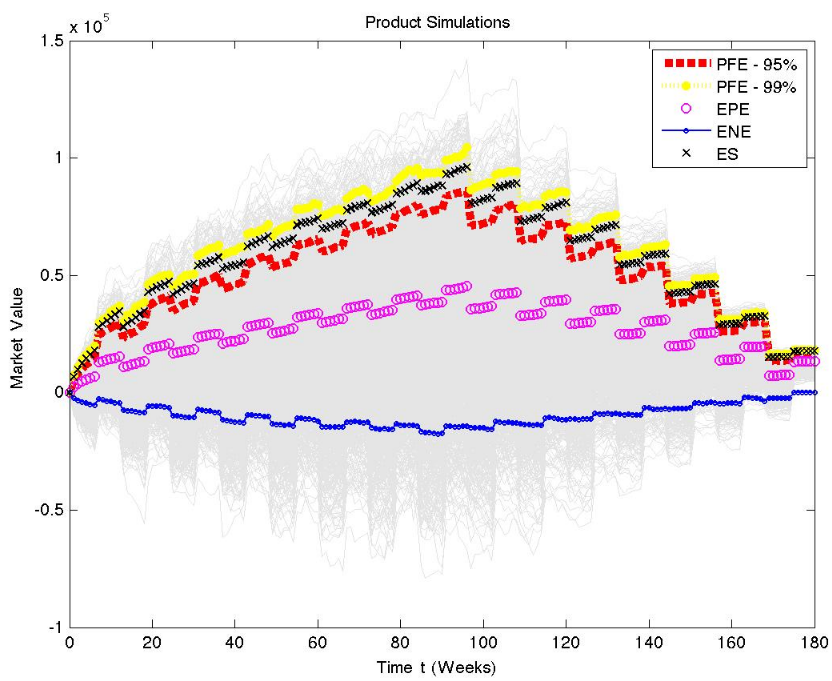


Figure B.5: Portfolio Counterparty Credit Risk: 7 yr Payer IRS and 15 yr Receiver IRS Portfolio

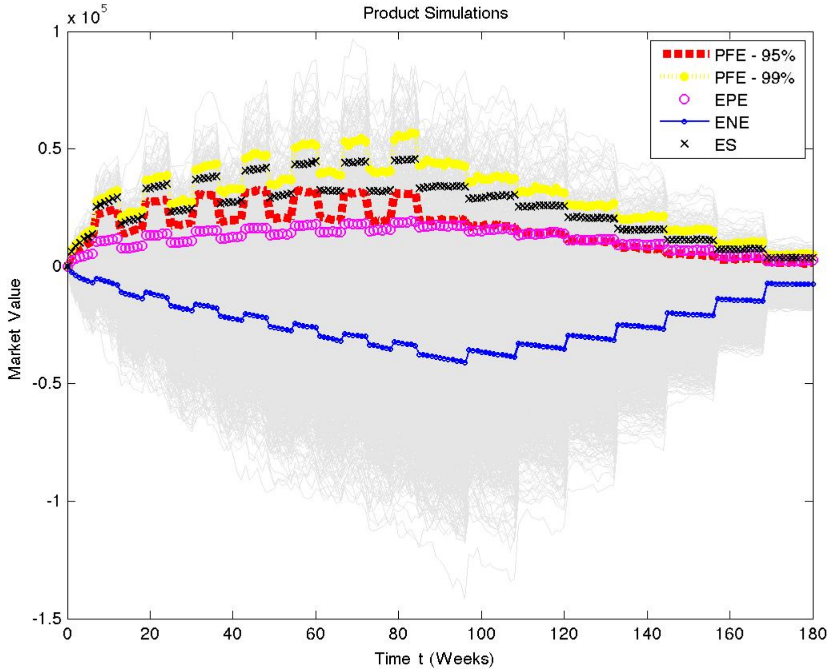


Figure B.6: Portfolio Counterparty Credit Risk: 7 yr Receiver IRS and 15 yr Payer IRS Portfolio

Since CCR metrics require the full distribution of a product of maturity T over the interval $(t, T]$, Monte-Carlo simulations are the main pricing method but do not account for early exercise feature. In case of early exercise, [Cesari09] and [Broadie04] describe a Monte-Carlo structure using the feature of early exercise through providing price and distribution of price at each point of a grid of discretised time points. This will permit to generate quantiles of credit exposures for all type of products. [Longstaff01] and [Tiley93] provide the algorithm mainly based on regression of state variables that can be included in a Monte-Carlo simulation.

B.4 Exponential Matrix computation

As listed in [Moler03], there are various ways to compute the exponential of a matrix:

- Series methods revolving around the Taylor series expansion,
- Ordinary Differential Equations methods by solving iteratively using numerical optimisation methods,
- Polynomial methods revolving around the characteristic polynomial and the usage of eigenvalues λ_i,
- Matrix decomposition methods using the properties of the matrix.

The preferred choice of the algorithm in terms of stability and efficiency will typically be a function of the distribution of the eigenvalues (confluent or not, situated in the left pane). Convergence conditions of the series will be done by introducing:

- the vector norm

$$\parallel x \parallel = \left[\sum_{i=1}^{n} |x_i|^2 \right]^{-1}$$

- the matrix norm

$$\parallel \boldsymbol{Q} \parallel = \max_{\parallel x \parallel = 1} \parallel \boldsymbol{Q} x \parallel$$

- the condition number of a matrix

$$cond(\boldsymbol{Q}) = \parallel \boldsymbol{Q} \parallel \parallel \boldsymbol{Q}^{-1} \parallel$$

  where the condition measures the asymptotically worst case of how much the matrix can change in proportion to small changes in the argument. Thus a low condition number is said to be well-conditioned and will provide that small iterative changes will not generate an important change in the matrix value thus enabling asymptotic convergence.

- the relative perturbation

$$\phi(t) = \frac{\parallel e^{(\boldsymbol{Q}+\boldsymbol{E})t} - e^{\boldsymbol{Q}t} \parallel}{\parallel e^{\boldsymbol{Q}t} \parallel}$$

The convergence of iterative methods will depend of the minimisation of the relative perturbation presented above with the theorem stated in [Moler03]:
If

$$\alpha(\boldsymbol{Q}) = \max\{Re(\lambda) | \lambda \text{ an eigenvalue of } \boldsymbol{Q}\}$$

and

$$\mu(\boldsymbol{Q}) = \max\{Re(\lambda)|\lambda \text{ an eigenvalue of } (\boldsymbol{Q} + \boldsymbol{Q}^*)/2\}$$

then

$$\phi(t) \leq t \parallel \boldsymbol{E} \parallel \exp[\mu(\boldsymbol{Q}) - \alpha(\boldsymbol{Q}) + \parallel \boldsymbol{E} \parallel]t \parallel$$

if $\boldsymbol{Q}$ is normal ($\boldsymbol{Q}\boldsymbol{Q}^* = \boldsymbol{Q}^*\boldsymbol{Q}$), then

$$\phi(t) \leq t \parallel \boldsymbol{E} \parallel \exp[\parallel \boldsymbol{E} \parallel]t \parallel$$

Calculation methods:

- The most classical series method is the Taylor Series of the exponential with

  $$\exp(\boldsymbol{Q}t) = \boldsymbol{I} + t\boldsymbol{Q} + \frac{\boldsymbol{Q}^2 t^2}{2!} + \frac{\boldsymbol{Q}^3 t^3}{3!} + ...$$

  Numerically, the implementation is a function of the precision of the computer where there exist a $k \in \mathbb{N}$ such that, the serie

  $$T_k(\boldsymbol{Q}) = \sum_{n=0}^{k} \frac{\boldsymbol{Q}^n t^n}{n!}$$

  has changes

  $$T_{k+1}(\boldsymbol{Q}) \simeq T_k(\boldsymbol{Q})$$

  that are not significant for the computer precision. The main problem with this approach is the error that is due when raising power of the elements of the matrix $\boldsymbol{Q}$ might produce rounding errors.

- The second Series method is the Pade Approximation based on the ratio of two rational functions of the matrix $\boldsymbol{Q}$. [Higham05] explains the method of Pade approximation which is based on a limited polynomial exponential to approximate the matrix exponential. The (p, q) Pade Approximation to $\exp(\boldsymbol{Q}t)$ is defined by

  $$R_{pq}[\boldsymbol{Q}] = [D_{pq}[\boldsymbol{Q}]]^{-1} N_{pq}[\boldsymbol{Q}]$$

  with

  $$N_{pq}[\boldsymbol{Q}] = \sum_{j=0}^{p} \frac{(p+q-j)!p!}{(p+q)!j!(p-j)!} \boldsymbol{Q}^j$$

  and

  $$D_{pq}[\boldsymbol{Q}] = \sum_{j=0}^{q} \frac{(p+q-j)!q!}{(p+q)!j!(q-j)!} \boldsymbol{Q}^j$$

The non-singularity of $D_{pq}[\boldsymbol{Q}]$ is assured if p and q are large enough or if the eigenvalues λ_i of $\boldsymbol{Q}$ are negative. In the case of the multivariate phase type distributions presented earlier, the construction of an upper triangular matrix $\boldsymbol{T}$ with negative elements on the diagonal will ensure that all the eigenvalues are thus negative. This will be the method used in the computation of the exponential matrix for the calibration to default probability embedded in credit derivatives claims. The functions is implemented in Matlab as *'expm()'*. Numerically, the choice of p and q will determine prescribed accuracy with

$$cond(D_{pq}[\boldsymbol{Q}]) \simeq cond(e^{\boldsymbol{Q}/2}) \geq e^{(\lambda_1 - \lambda_n)/2}$$

with $\lambda_1 \geq ... \geq \lambda_n$ are the real part of the eigenvalues $\boldsymbol{Q}$. Thus, the Pade Approximation can be used if $\| \boldsymbol{Q} \|$ is not too large which will be the case for the matrix $\boldsymbol{T}$.

- The Scaling and Squaring method is used when the spread of eigenvalues of $\boldsymbol{E}$ increases can be controlled by the property of the exponential functions

$$e^{\boldsymbol{Q}} = (e^{\boldsymbol{Q}/m})^m$$

The idea is to choose m to be a power of two for which $(e^{\boldsymbol{Q}/m})$ can be reliably and efficiently computed, and then to form the matrix $e^{\boldsymbol{Q}}$ by repeated squaring. This method cannot be applied in a calibration process of the exponential matrix with changing coefficients and still has the problem of convergence of the taylor series method.

- The classical Ordinary Differential Equation method refers to the optimisation presentation with the ODE solver and the function

$$f(x,t) = \frac{\partial x(t)}{\partial t} - \boldsymbol{Q}x(t)$$

with initial condition $x(0) = x_0$. The method involves an iterative sequence of values $t_0 = 0, t_1, ..., t_i = t$ with a fixed or variable step $h_i = t_{i+1} - t_i$. It will produce a sequence of vectors x_i that approximates $x(t_i)$. However, the backward and forward equation are of limited utility since the process is cumbersome with dimension greater than 2 with the possibility of eigenvalues that can turn complex in the iterative process.

- The Polynomial methods use the characteristic polynomial of $\boldsymbol{Q}$ defined as

$$c(z) = \det(z\boldsymbol{I} - \boldsymbol{Q}) = z^n - \sum_{k=0}^{n-1} c_k z^k$$

with the Caley-Hamilton theorem that states that every square matrix satisfies its own characteristic equation. Thus,

$$c(\boldsymbol{Q}) = 0$$

is a polynomial of degree n, and, hence

$$\boldsymbol{Q}^n = c_0\boldsymbol{I} + c_1\,\boldsymbol{Q} + ... + c_{n-1}\,\boldsymbol{Q}^{n-1}$$

Any power of $\boldsymbol{Q}$ can be expressed as coefficients of $\boldsymbol{I}$, $\boldsymbol{Q}, ..., \boldsymbol{Q}^{n-1}$ with

$$\boldsymbol{Q}^k = \sum_{j=0}^{n-1} \beta_{kj}\,\boldsymbol{Q}^j$$

The exponential matrix can be expressed as

$$\begin{aligned} e^{\boldsymbol{Q}t} &= \sum_{k=0}^{\infty} \frac{t^k\,\boldsymbol{Q}^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\left[\sum_{j=0}^{n-1} \beta_{kj}\,\boldsymbol{Q}^j\right] \\ &= \sum_{j=0}^{n-1}\left[\sum_{k=0}^{\infty} \beta_{kj}\frac{t^k}{k!}\right]\boldsymbol{Q}^j \\ &= \sum_{j=0}^{n-1} \alpha_j(t)\,\boldsymbol{Q}^j \end{aligned}$$

The other methods for determining the coefficients are:

– the Lagrange Interpolation with

$$e^{\boldsymbol{Q}t} = \sum_{j=0}^{n-1} e^{\lambda_j t} \prod_{k=1 k\neq j}^{n} \frac{(\boldsymbol{Q} - \lambda_k\boldsymbol{I})}{(\lambda_i - \lambda_k)}$$

– the Newton interpolation with

$$e^{\boldsymbol{Q}t} = e^{\lambda_1 t}\boldsymbol{I} + \sum_{j=2}^{n}[\lambda_1, ..., \lambda_j]\prod_{k=1}^{j-1}(\boldsymbol{Q} - \lambda_k\boldsymbol{I})$$

where the divided differences depend on t and are computed recursively by

$$[\lambda_1, \lambda_2] = (e^{\lambda_1 t} - e^{\lambda_2 t})/(\lambda_1 - \lambda_2)$$

$$[\lambda_1, ..., \lambda_{k+1}] = \frac{[\lambda_1, ..., \lambda_{k+1}] - [\lambda_2 - \lambda_{k+1}]}{\lambda_1 - \lambda_{k+1}} \tag{B.1}$$

– the Inverse Laplace Transform where $\mathcal{L}[e^{\boldsymbol{Q}t}]$ is the Laplace Transform of the matrix exponential

$$\mathcal{L}[e^{\boldsymbol{Q}t}] = (s\boldsymbol{I} - \boldsymbol{Q})^{-1}$$

where the entries of this matrix are rational functions of s. Thus, the Laplace transform

$$\mathcal{L}[e^{\boldsymbol{Q}t}] = (s\boldsymbol{I} - \boldsymbol{Q})^{-1} = \sum_{n=0}^{n-1} \frac{s^{n-k-1}}{c(s)} Q^k$$

will leading to

$$e^{\boldsymbol{Q}t} = \sum_{k=0}^{n-1} \mathcal{L}^{-1}[s^{n-k-1}/c(s)]\,\boldsymbol{Q}^k$$

Like in the case of the Taylor series method, those methods are seriously affected by roundoff errors.

- The Matrix Decomposition methods are likely to be most efficient for problems involving large matrices and repeated evaluation based on factorisations and decompositions of the matrix $\boldsymbol{Q}$.
  If $\boldsymbol{Q}$ happens to be symmetric, the matrix decomposition are based on the transformation of the form

$$\boldsymbol{Q} = \boldsymbol{S}\boldsymbol{R}\boldsymbol{S}^{-1}$$

thus,

$$e^{\boldsymbol{Q}t} = \boldsymbol{S}e^{\boldsymbol{R}t}\boldsymbol{S}^{-1}$$

The natural decomposition is to take $\boldsymbol{S}$ to be the matrix whose columns are eigenvectors of $\boldsymbol{Q}$ with

$$e^{\boldsymbol{Q}t} = \boldsymbol{V}e^{\boldsymbol{D}t}\boldsymbol{V}^{-1}$$

in case $\boldsymbol{V}$ is non singular and $\boldsymbol{D}$ is the diagonal matrix with the eigenvalues. The difficulty in that case is that $\boldsymbol{Q}$ may be close to singular which means that cond($\boldsymbol{Q}$) is large leading to non convergence. The multivariate phase-type distribution introduced earlier presents non confluent eigenvalues (Confluent eigenvalues occurs when the matrix $\boldsymbol{Q}$ does not have linearly independent vectors.) and are a good case for this methods. Another improvement can be obtained with the use of triangular estimation of the matrix $\boldsymbol{V}$ of eigenvectors using the QR algorithm.

From [Golub83], since the matrix $\boldsymbol{T}$ is upper-triangular, the product of upper-triangular matrices is an upper triangular and thus the power series, i.e the exponential is upper triangular. Furthermore, the eigenvalues of an upper triangular matrix are the values on the diagonal. Since none of the diagonal value is zero the matrix is also invertible. Additionally the exponential of a diagonal matrix is also a diagonal matrix with the value being the exponential of diagonal value.

$$\boldsymbol{D} = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

$$e^{\boldsymbol{D}} = \begin{pmatrix} e^{a_1} & 0 & 0 & 0 \\ 0 & e^{a_2} & 0 & 0 \\ 0 & 0 & e^{a_3} & 0 \\ 0 & 0 & 0 & e^{a_4} \end{pmatrix}$$

The exponential of an upper triangular matrix is a triangular matrix and the matrix $\boldsymbol{T}$ can be decomposed as

$$\boldsymbol{T} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}$$

where matrix $\boldsymbol{D}$ is a diagonal matrix with the eigenvalues of matrix $\boldsymbol{T}$ and matrix $\boldsymbol{U}$ is an upper matrix representing the eigenvectors of $\boldsymbol{T}$. Additionally, the m^{th} power of matrix $\boldsymbol{T}$ is

$$\begin{aligned} \boldsymbol{T}^n &= \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}\boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}...\boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}...\boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}\boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1} \\ &= \boldsymbol{U}\boldsymbol{D}^n\boldsymbol{U}^{-1} \end{aligned}$$

Thus the exponential of the matrix $\boldsymbol{T}$ can be expressed as

$$\begin{aligned} \exp^{\boldsymbol{T}t} &= \sum_{n=0}^{\infty} \frac{\boldsymbol{T}^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\boldsymbol{U}\boldsymbol{D}^n\boldsymbol{U}^{-1} t^n}{n!} \\ &= \boldsymbol{U}\left(\sum_{n=0}^{\infty} \frac{\boldsymbol{D}^n t^n}{n!}\right)\boldsymbol{U}^{-1} \\ &= \boldsymbol{U}\exp^{\boldsymbol{D}t}\boldsymbol{U}^{-1} \end{aligned}$$

Computation using backward substitution:
The computation of the matrix $\boldsymbol{U}$ is obtained with the iterative process called the backward substitution for upper triangular or forward substitution for lower triangular matrices.

Let $\boldsymbol{T}$ be upper triangular and let $\boldsymbol{U}$ be a matrix whose columns are the eigenvectors to $\boldsymbol{T}$. Furthermore, assume all the eigenvalues of $\boldsymbol{T}$ are distinct. Additionally, since $\boldsymbol{U}$ and $\boldsymbol{U}^{-1}$ are the matrices with the eigenvectors, an additional condition is $\boldsymbol{U}\boldsymbol{U}^{-1} = \boldsymbol{I}$ With those set of conditions we can establish the recursive equations for $n = 1, 2, ..m$:

$$u_{j,n} = \begin{cases} 0 & j > n \\ 1 & j = n \\ \frac{t_{n-1,n}}{t_{n,n}-t_{n-1,n-1}} & j = n-1 \\ \frac{t_{n-k,n}+\sum_{q=1}^{k-1} t_{n-k,n-k+q}u_{n-k+q,n}}{t_{n,n}-t_{n-k,n-k}} & j = n-k, 2 \leq k \leq n-1 \end{cases}$$

and

$$u^{-1}_{j,n} = \begin{cases} 0 & j < n \\ 1 & j = n \\ -\sum_{k=0}^{j-n-1} u^{-1}_{n,n+k} u_{n+k,j} & j > n \end{cases}$$

Thus, the exponential matrix is calculated recursively by using the symmetrical properties of the $\boldsymbol{T}$ matrix.

B.5 QuantLib Implementation

The overall process of the Quantlib implementation for the Counterparty Credit Risk treatment is presented in figure B.7 with the following steps

- Having generated the csv files of the default probability curves for the reference names in the obligor set, a vanilla swap and the default probability term structure classes are initiated using QuantLib-based C++ products.

- Based on the object-oriented generic Monte-Carlo simulation model, underlying stochastic processes are iterated across different paths and use a *ProductEngine* class pricer to generate the NPV of all the transactions and store the distribution. In order to do so, an occurrence of the product term sheet (in the most basic case a Vanilla Swap) need to be instantiated using the class *InstrumentDeclaration*. The Monte-Carlo execution will be controlled by the class *SimulationArchitecture* but calling the *CurveCalibration* class to reprice along the path and store the distribution across time for Counterparty Credit Risk valuation. In the most common case, a pricing at $t = 0$ for valuation is needed and doesn't require to store the value of the financial product along the path. However, in the case of CCR, the product distribution need to be stored to generate EPE profiles.

- Using the previous distribution of product prices along the discretised paths, the Counterparty Credit Risk (CCR) metrics like EPE (see formula (2.3)), PFE (see formula (2.1)) and CVA are generated.

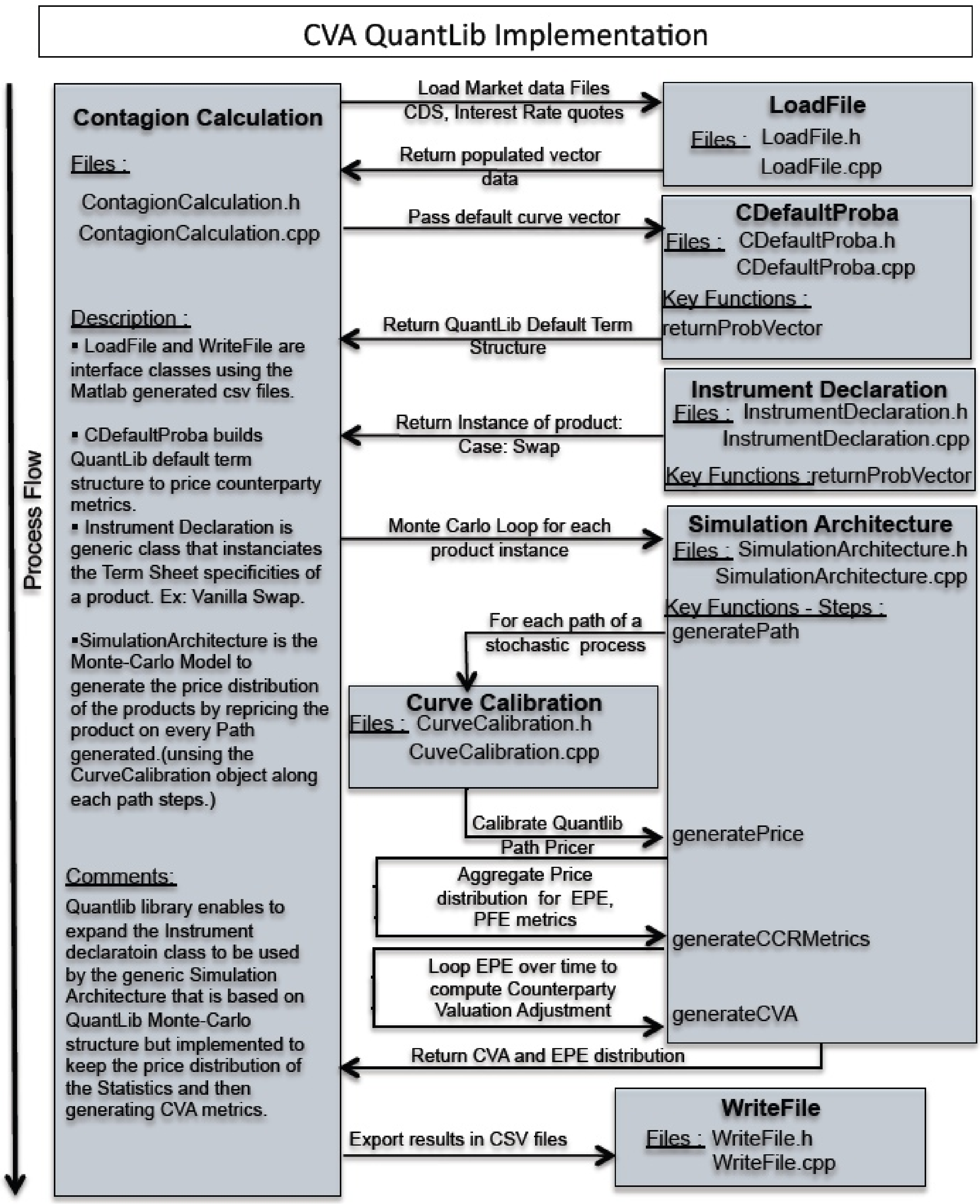


Figure B.7: CVA QuantLib Implementation.

Bibliography

[Artzner99] P. Artzner, F. Delbaen, J.-M. Eber and D. Heath. Coherent measures of risk, *Math, Finance, pp. 203-228, 1999.*

[Assaf82] D. Assaf, and B. Levikson. Closure of Phase-type distributions under operations arising in Reliability Theory, The Annals of Probability, Vol. 10, No. 1,pp 265-269, 1982.

[Assaf84] D. Assaf, N. Langberg, T. Savits, and M. Shaked. Multivariate phase-type distributions, Operations Research 32, 688-702. 1984.

[Assaf85] D. Assaf, and N. Langberg. Presentation of Phase-type distributions as proper mixtures, Journal of Applied Probability, Vol. 22, No. 1,pp 247-250, 1985.

[Asmu03] S. Asmussen. Applied Probability and Queues. Second Edition. Springer, London,2003.

[Baake] M. Baake and U. Schlagel. The Peano-Baker series. *Department of Mathematical and Statistical Sciences, University of Alberta, January 2012.*

[Ballabio1] L. Ballabio. Implementing QuantLib website. Chapter 2 - Financial instruments and pricing engines.

`http://sites.google.com/site/luigiballabio/qlbook`

[Ballabio2] L. Ballabio. Implementing QuantLib website. Chapter 3 - Term structures.

`http://sites.google.com/site/luigiballabio/qlbook`

[Ballabio3] L. Ballabio. Implementing QuantLib website. Chapter 4 - Cash-flow analysis.

`http://sites.google.com/site/luigiballabio/qlbook`

[Ballabio4] L. Ballabio. Implementing QuantLib website. Chapter 6 - The Monte-Carlo framework.

`http://sites.google.com/site/luigiballabio/qlbook`

[Batchvarov01] A. Batchavarov. ABS/MBS/CMBS/CDO 101. *Merrill Lynch Global Securities Research & Economics Group, 2001.*

[Bellman60] R. Bellman. Introduction to Matrix Analysis. *Society for Industrial and Applied Mathematics, 1960*

[Berndt74] E. K. Berndt, B. H. Hall, R. E. Hall and J. A. Hausman. Estimation and Inference in nonlinear structural models. *Annals of Economic and Social Measurement, 1974.*

[Bielecki01] T. R. Bielecki and M. Rutkowski. Credit Risk: Modeling, Valuation and Hedging.*Springer Verlag, 2001.*

[Bielecki06] T. R. Bielecki, A. Vidozzi, and L. Vidozzi, L. An efficient approach to valuation of credit basket products and ratings triggered step-up bonds. Working Paper, 2006.

[Bielecki06b] T. R. Bielecki, A. Vidozzi, and L. Vidozzi, L. Pricing and hedging of basket default swaps and related derivatives. Working Paper, 2006.

[Bielecki07] T. R. Bielecki, S. Crepey, M. Jeanblanc, and M. Rutkowski. Valuation of basket credit derivatives in the credit migrations environment. Handbook of Financial Engineering, J. Birge and V. Linetsky eds., Elsevier, 2007.

[Bielecki08] T. R. Bielecki, J. Jakubowski, A. Vidozzi, and L. Vidozzi. Study of Dependence for Some Classes of Stochastic Processes. Stoch. Anal. and Appl., vol. 26, no. 4, pp. 903-924, 2008.

[Bielecki08] T. R. Bielecki, A. Vidozzi, and L. Vidozzi. A markov Copulae Approach to Pricing and Hedging of Credit index Derivatives and Ratings Triggered Step-Up Bonds, *Journal of Credit Risk, Vol. 4, Num.1 2008*

[Bielecki09] T. R. Bielecki, S. Crepey, and A. Herberstsson. Markov Chain Models of Portfolio Credit Risk, *Oxford Handbook of Credit Derivatives, 2009.*

[Bielecki10] T. R. Bielecki, S. Crepey, M. Jeanblanc. Up and Down Credit Risk. *Quantitative Finance. 2010*

[Bielecki11] T. R. Bielecki, J. Jakubowski, and M. Niewegkowski. Intricacies of dependence between components of multivariate markov chains: weak Markov consistency and Markov Copulae. *Working paper, 2011*

[Bielecki13] T. R. Bielecki, S. Crepey, and A. Herberstsson. Dynamic Hedging of Portfolio Credit Risk in a Markov Copula Model. *Journal of Optimisation Theory and Applications, 2013*

[Bielecki15a] T. R. Bielecki, J. Jakubowski, and M. Niewegkowski. Conditional Markov Chains revisited. Part I: Construction and Properties. *Working paper, 2015*

[Bielecki15b] T. R. Bielecki, J. Jakubowski, and M. Niewegkowski. Conditional Markov Chains revisited. Part II: Consistency and Copulae. *Working paper, 2015*

[Bingham98] N. H. Bingham and R. Kiesel. Risk-Neutral Valuation. *Springer, 1998*

[BIS95] BIS: Bank for International Settlement. Treatment of Potential Exposure for off-balance sheet Items. *April 1995.*

[BIS98] BIS: Bank for International Settlement. OTC derivatives: Settlement procedures and Counterparty Risk Management. *September 1998.*

[BIS01a] BIS: Bank for International Settlement. The internal ratings-based approach: supporting document to the new Basel Capital Accord. *January 2001.*

[BIS01b] BIS: Bank for International Settlement. Consultative document. The internal ratings-based approach. *January 2001.*

[BIS05] BIS: Bank for International Settlement. The application of Basel II to trading activities and the treatment of Double Default effects. *July 2005.*

[BIS06] BIS: Bank for International Settlement. International convergence of capital measurement and capital standards. A Revised Framework comprehensive version. *June 2006.*

[BIS10a] BIS: Bank for International Settlement. Sound practices for backtesting counterparty credit risk models. *December 2010.*

[BIS10b] BIS: Bank for International Settlement. Consultative document. Capitalisation of bank exposures to central counterparties. *December 2010.*

[Black76] C. Black and J.C. Cox. Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance, 31, 351-367, 1976.*

[Braithwaite12] T Braithwaite, M Mackenzie and K Scannell. Deutsche Bank: Show of strength or a fiction? *Financial Times, 5 December 2012.*

[Bremaud81] P. Bremaud. Point Processes and Queues. Martingales Dynamics. Springer-Verlag, Berlin Heidelberg New-York. *1981.*

[Bremaud78] P. Bremaud and M. Yor. Changes of Filtrations and of Probability Measures. *Z. Wahr. Very. Gebiete, 45:265-295, 1978.*

[Brig04] D. Brigo. Credit default swap calibration and equity swap valuation under counterparty risk with a tractable structural model. *Presented at the FEA 2004 Conference at MIT, Cambridge, Massachusetts, 2004.*

[Brig05] D. Brigo, and M.Tarenghi. Credit default swap calibration and counterparty Risk valuation with a scenario-based First passage model. Credit Models - Banca IMI, 2005.

[Brig06] D. Brigo, and M. Masetti. Risk neutral pricing of counterparty risk. Counterparty Credit Risk modeling: Risk management, pricing and regulation, *Risk Books, 2006.*

[Brig07a] D. Brigo and A. Pallavicini. Counterparty Risk under correlation between default and interest rates. Numerical methods for finance. *Chapman Hall. 2007.*

[Brig07b] D. Brigo and A. Pallavicini. Implied Expected Tranched Loss surface from CDO data. *Credit Models, Banca IMI, 2007.*

[Brig07c] D. Brigo, A. Pallavicini, and R. Torresetti. CDO calibration with the dynamical Generalised Poisson Loss model. *Risk Magazine, June issue, 2007.*

[Brig07d] D. Brigo, A. Pallavicini, and R. Torresetti. Cluster-based extension of the generalised poisson loss dynamics and consistency with single names. *International Journal of Theoretical and Applied Finance, Vol 10,n. 4., 2007.*

[Brig08a] D. Brigo and A. Pallavicini. Counterparty risk and Contingent CDS valuation under correlation between interest-rates and default. *Risk magazine, 2008.*

[Brig08b] D. Brigo and A. Pallavicini. Counterparty risk and Contingent CDS with stochastic intensity hybrid models. *Risk Magazine, February issue, 2008.*

[Brig09] D. Brigo and K. Chourdakis. Counterparty risk for credit default swaps: impact of spread volatility and default correlation. International Journal of Theoretical and Applied Finance, 12(7): 1007-26

[Brig10a] D. Brigo, M. Morini, and A. Pallavicini. Counterparty credit risk, collateral and funding with pricing cases for all asset classes. Wiley Finance, 2010.

[Brig10b] D. Brigo, A. Pallavicini, and V. Papatheodorou. Bilateral counterparty risk valuation for interest-rate products: impact of volatilities and correlations. *2010.*

[Brig11a] D. Brigo, A. Pallavicini, and R. Torresetti. Credit Models and the Crisis: An overview. *Journal of Risk Management in Financial institutions, Volume 4, Number 3, pp. 243-253, 2011.*

[Brig11b] D. Brigo, A. Pallavicini, and R. Torresetti. Credit Models and the Crisis: Default cluster dynamics and the Generalised Poisson Loss model. *Journal of Credit Risk, Volume 6, Number 4, pp. 39-81, 2011.*

[Brig11c] D. Brigo, A. Pallavicini and R. Torresetti, Credit Models and the Crisis: A journey into CDOs, Copulas, Correlations and Dynamic Models. *Wiley, 2010.*

[Brig12] D. Brigo, C. Buescu, and C. Morini. Impact of the first to default time on Bilateral CVA. *International Journal of theoretical and Applied Finance, Vol. 15, 2012.*

[Brig14] D. Brigo, A. Capponi, and A. Pallavicini. Aribtrage-free bilateral counterparty risk valuation under collateralisation and application to credit default swaps. *Mathematical Finance, Volume 24, Issue 1, pages 125-146, 2014.*

[Broadie04] M. Broadie and P.Glasserman. A stochastic mesh method for pricing high-dimensional American options. *Journal of Computational Finance, Volume 7, No.4, pp. 35-72, 2004.*

[Canabarro03] E. Canabarro and D. Duffie. Measuring and marking counterparty risk. *Asset/Liability Management of Financial Institutions, Institutional Investor Books, 2003.*

[Canabarro10] E. Canabarro. Counterparty Credit Risk. *Risk Books, 2010.*

[Carmona09] R. Carmona and S. Crepey. Importance Sampling and Interacting Particle Systems for the Estimation of Markovian Credit Portfolios Loss Distribution, Forthcoming in International Journal of Theoretical and Applied Finance. 2009

[Carver11] L. Carver. Dealers predict CVA-CDS loop will create sovereign volatility. *Risk magazine, 2011*

[Cesari09] G. Cesari and al. Modelling, Pricing and Hedging Counterparty Credit Exposure. *Springer Verlag, 2009.*

[Charpentier07] A. Charpentier and J. Segers. Lower tail dependence for Archimedean copulas: characterisations and pitfalls. *Insurance Math. Econom., 40(3):525532, 2007.*

[Chapovsky06] A. Chapovsky, A. Rennie and A. Tavares. Stochastic intensity modelling for structured credit exotics. *Credit Correlation: life after copulas, World Scientific, 2006.*

[Chopra04] S. Chopra and M. Sodhi. Managing risk to avoid supply-chain breakdown. *MIT SLOAN Management Review, Volume 46, 2004.*

[Citigroup09] Citigroup Reports Q1 2009. Cititgroup website.

`http://icg.citi.com/transactionservices/home/about_us/press_room/archives/2009/2009_0417.jsp`

[Citigroup10] Citigroup Reports Full Year 2010. Cititgroup website.

`http://www.citigroup.com/citi/press/2011/110118a.htm`

[Cont08] R. Cont and A. Minca. Recovering Portfolio Default Intensities Implied by CDO Quotes. Working Paper, 2008.

[Cohen06] L. Cohen and A. Frazzini. Economic links and predictable returns. *Yale School of Management and University of Chicago, Graduate School of Business, 2006.*

[Cont10] R. Cont. Model calibration. Encyclopaedia of quantitative finance. *Columbia University, 2010.*

[Cox77] D.R. Cox and H.D. Miller. The theory of stochastic processes. *Chapman & Hall/CRC Press, 1977.*

[Crepey09] S. Crepey, M. Jeanblanc and B. Zargary. Counterpart Risk on a CDS in a Markov Chain Copula Model with Joint Defaults. *Université d'Évry Val d'Essonne, 2009.*

[Crosbie01] P. Crosbie and J. Bohn. Modeling default risk. *KMV Llc, 2001.*

[Dacunha05] J. J. Dacunha. Transition matrix and generalised matrix exponential via the Paneo-Baker series. *J. Diff. Eq. Appl., 11, 1245-1264, 2005.*

[Daley02] D.J. Daley, D. Vere-Jones. An Introduction to the theory of point processes. Springer, 2002.

[Das98a] S.Das. Credit Derivatives - Instruments. pp. 7-77. *Credit Derivatives: trading and management of Credit and Default risk, J. Wiley, 1998*

[Davis99] M. Davis and V. Lo. Infectious defaults, *Tokyo Mitsubishi International plc. 1999.*

[Davis09] M. Davis. Contagion models in credit risk, *Imperial College London, 2009.*

[Dellacherie92] C. Dellacherie, B. Maisonneuve and P-A. Meyer. Probabilites et Potentiels. Chapitre XVII-XXIV, Processus de Markov (fin). Complements de calcul stochastique. *Herman, Paris, 1992.*

[Duffie99] D. Duffie and K. Singleton. Simulating correlated defaults. Working paper, *May 1999.*

[Duffie01] D. Duffie and N. Garleanu. Risk and Valuation of Collateralised Debt Obligations. *Financial Analysts Journal, 57(1) 2001.*

[Duffie03] D. Duffie and K. Singleton. Credit Risk. Pricing, Measurement and Management. Princeton University Press, Princeton, *2003.*

[Economist12] The Economist. Economic epidemiology: Predicting financial contagion. *The Economist, 16 June 2012.*

[Elliott04] R. J. Elliott and P. E. Kopp. Measures of risk. Mathematics of financial markets. *Springer Finance, pp 303-328, 2004.*

[Fabozzi12] Fabozzi J. F. The Handbook of Fixed Income Securities, Eighth Edition. *The McGraw-Hill Companies, 2012.*

[Fletcher63] R. Fletcher and M. Powell. A rapidly convergent descent method for minimisation. *Computer Journal, Volume 6, 1963.*

[Gallo06] A. Gallo and al. Credit Derivatives Handbook. Volume 1 and Volume 2. *Merrill Lynch Global Securities Research & Economics Group, 2006.*

[Ginty04a] L. McGinty and R. Ahluwalia. Credit correlation: A Guide. *JP Morgan Technical Document, 2004.*

[Ginty04b] L. McGinty and R. Ahluwalia. Introducing base correlation. *JP Morgan Technical Document, 2004.*

[Glasserman04] P. Glasserman. Monte-Carlo methods in Financial Engineering. *Springer, Berlin, 2004.*

[Goldman10] Overview of Goldman Sachs interaction with AIG and Goldman Sachs approach to risk management. Goldman Sachs website.

`http://www2.goldmansachs.com/our-firm/on-the-issues/viewpoint/archive/aig-summary.html`

[Golub83] G. H. Golub and C. F. Van Loan. Matrix Computation. *Johns Hopkins University Press, p. 384, 1983.*

[Gregory10] J. Gregory. Counterparty credit risk. The new challenge for global financial markets. *Wiley editions, 2010.*

[Harrison81] J.M. Harisson and S.R. Pliska. Martingales and stochastic integrals in the theory of continuous trading. Stochastic Process. Applications. Vol 11, pp 250-260, *1981*

[Hartmann04] P. Hartmann, S. Straetmans and C. G. De Vries. Asset market linkages in crisis periods. *The Review of Economics and Statistics , The MIT Press, pp. 313-326, 2004.*

[Haworth06] H. Haworth. Structural models of credit with default contagion. Thesis. *University of Oxford, 2006.*

[Herbert05] A. Herbertsson. Dynamic dependence modelling in credit risk. Thesis. *Department of Mathematics, Göteborg University, 2005.*

[Herb07] A. Herbertsson. Pricing synthetic CDO tranches in a model with default contagion using matrix-analytic approach. *University of Gothenburg, 2007.*

[Herb08a] A. Herbertsson. Default Contagion in Large Homogeneous Portfolios. *The credit derivatives handbook. Global perspectives, Innovations and market drivers, Chapter 14, McGrawHill, 2008.*

[Herb08b] A. Herbertsson and A. Rootzen. Pricing k^{th}-to-default swaps under default contagion: The matrix-analytic approach. *The Journal of Computational Finance, pp. 49-78, 2008.*

[Herb10] A. Herbertsson. Modelling default contagion using multivariate phase-type distributions. *Springer Verlag, 2010.*

[Higham05] N. J. Higham. The Scaling and Squaring Method for the Matrix Exponential Revisited. *Society for Industrial and Applied Mathematics, Volume 26, pp. 1179-1193, 2005.*

[Hull11] J. Hull. Options, Futures and Other Derivatives. *Pearson, 2011.*

[Jarrow97] R. Jarrow, D. Lando and S. Turnbull. A Markov Model for the Term Structure of Credit Risk Spreads. *The Review of Financial Studies, vol. 10, no. 2, pp. 481-523, 1997*

[Jarrow01] R. A. Jarrow and F. Yu. Counterparty risk and the pricing of defaultable securities. Journal of Finance, 16:17651800, *2001.*

[Jeanblanc10] M. Jeanblanc, M. Yor and M. Chesney. Mathematical Methods for financial markets. *Springer Verlag, 2010.*

[Jeulin80] T. Jeulin. Semi-Martingales et Grossissement d'une filtration. Lecture Notes in Mathmatiques. Institut de Mathmatiques. Universit de Strasbourg. *Springer, 1980.*

[Kijima00] M. Kijima and Y. Muromachi. Credit events and the valuation of credit derivatives of basket type. *Rev. Derivatives Res. 4, 55-70, 2000.*

[Lamb03] D. Lamb and E. Picoult. Economic capital for counterparty credit risk. *Economic capital: A practitioner Guide, Risk Books, 2003.*

[Lando00] D.Lando. On correlated defaults in a rating-based model: common state variables versus simultaneous defaults. *University of Copenhagen, 2000.*

[Lando04] D.Lando. Credit Risk modelling. Theory and Applications. *Princeton University Press, Princeton, 2004.*

[Last95] G. Last, A. Brandt. Marked Point Processes on the Real Line: The Dynamical Approach. Springer, 1995.

[Laurent07] J.-P. Laurent, A. Cousin , J-D Fermanian. Hedging default risks of CDOs in Markovian contagion models. Working Paper, 2007.

[Leung05] S. Y. Leung and Y. K. Kwok. Credit default swap valuation with counterparty risk, *Kyoto Economics Review, vol. 74(1), pp 25-45, 2005.*

[Li00] D. X. Li. On default correlation: A copula function approach. The Journal of Fixed Income, 9:4354, *2000.*

[Lindskog] F. Lindskog, and A. McNeil. Common Poisson Shock Models: Applications to Insurance and Credit Risk Modelling. *Astin Bulletin, 33, pp. 209-238, 2003.*

[Lipton09] A. Lipton and A. Sepp. Credit value adjustment for credit default swaps via the structural default model. *Journal of Credit Risk, 5, pp 123-146, 2009.*

[London04] J. London. Modeling derivatives in C++, *Wiley Finance, 2004.*

[Longstaff01] F. Longstaff and E. Schwartz. Valuing american option by simulation: A simple least-squares approach. *Review of Financial Studies, pp 113-147, 2001.*

[Markowitz52] H. Markowitz. Portfolio Selection. *The Journal of Finance, Volume 7, No 1, pp. 77-91, 1952.*

[Marquardt63] D. Marquardt. An algorithm for least-squares estimation of nonlinear parameters. *Society for Industrial and Applied Mathematics, Volume 11, pp. 431-441, 1963.*

[Merton74] R. Merton. On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance, Volume 29, pp 449-470, 1974.*

[Moler03] C. Moler and C. Van Loan. Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later. *Society for Industrial and Applied Mathematics, 2003.*

[More77] J. J. Moré. The Levenberg-Marquardt algorithm: Implementation and theory. *Lecture notes in mathematics, Numerical analysis, Springer Verlag, pp. 105-116, 1977.*

[Musiela97] M. Musiela and M. Ruthkowski. Martingales methods in financial modelling. Springer-Verlag, *1997.*

[Nelsen99] R. B. Nelsen. An Introduction to Copulas. Springer, New York, *1999.*

[Nel12] R. Nelson: Probability, Stochastic processes, and Queueing Theory. Springer - Verlag.

[Nelson04a] B. Nelson and M. Taaffe. The $Ph_t/Ph_t/\infty$ Queueing System: Part I-The Single Node. INFORMS Journal on Computing, Vol. 16, No. 3, pp 266-274, 2004

[Nelson04b] B. Nelson and M. Taaffe. The $[Ph_t/Ph_t/\infty]^k$ Queueing System: Part II-The Multiclass Network, INFORMS Journal on Computing, Vol. 16, No. 3, pp 275-283, 2004

[Neuts78] M. Neuts: Matrix - Geometric Solutions in Stochastic Models. An algorithmic Approach. Johns Hopkins, 1978.

[Nocedal06] J. Nocedal and S. Wright. Numerical Optimization. *Springer Series in Operations Research and Financial Engineering, Springer Verlag, 2006.*

[Norros86] I. Norros. A compenastor representation of multivariate life length distributions, with applications. Scandinavian Journal of Statistics, 13:99112, *1986.*

[Picoult05] E. Picoult. Measuring and dynamically hedging counterparty credit exposure and risk. *Department of Industrial Engineering and Operations Research. Columbia University, 2005.*

[Powell11] P.D. Powell. Calculating determinants of block matrix. Department of Physics. University of Illinois. 2011

[Protter90] p. Protter. Stochastic integration and differential equations. *Springer Verlag, 1990*

[Pykhtin05] M. Pykhtin. Counterparty credit risk modeling. *Risk Books, 2005.*

[Shaked87] M. Shaked and J. G. Shanthikumar. The multivariate hazard construction. Stochastic Processes and Their Applications, 24:241258, *1987.*

[Reiswich10a] D. Reiswich. A short introduction to selected classes of the Boost C++ library. *QuantLib, 2010.*

[Reiswich10b] D. Reiswich. Introduction to selected Classes of the QuantLib Library I. *QuantLib, 2010.*

[Reiswich10c] D. Reiswich. Introduction to selected Classes of the QuantLib Library II. *QuantLib, 2010.*

[Rolski98] T. Rolski, H. Schmidli, V. Schmidt, J. Teuggels. Stochastic Processes for Insurance and Finance, J. Wiley, Chichester, 1998.

[Rogers00] L.C.G. Rogers, D. Williams. Diffusions, Markov Processes and Martingales. 2nd edition.Cambridge University Press, 2000.

[Schőnbucher03] P. Schonbucher. Credit derivatives pricing models: Models, pricing and implementation. *The Wiley Finance Series, 2003.*

[Senate10] The United States Senate Committee on Finance. *The United States Senate, Press release, 2010.*

`http://finance.senate.gov/newsroom/ranking/release/?id=cb2c54ae-fb8b-43e0-abeb-9d12a422810c`

[Sharpe64] W. F. Sharpe. Capital asset prices: A theory of market equilibrium under conditions of risk. *The Journal of Finance, Volume 19, No. 3, 1964.*

[Singh12] M. Singh and P. Stella. Money and collateral. *International Monetary Fund Working Paper, 2012.*

`http://www.imf.org/external/pubs/ft/wp/2012/wp1295.pdf`

[Sorensen94] E. H, Sorensen and T. F. Bollier. Pricing swap default risk, *Financial Analyst Journal, Volume 50, 1994.*

[Vidyamurthy04] G.Vidyamurthy. Pairs trading: Quantitative methods and analysis, *Wiley Finance, 2004.*

[Vitali11] S. Vitali, J. Glattfelder and S. Battiston. The network of global corporate control. *ETH Zurich, 2011.*

[Tang07] Y. Tang and B. Li. Quantitative analysis, derivatives modelling and trading strategies. In the presence of counterparty credit risk for the fixed-income market. *World Scientific Publishing, 2007.*

[Tiley93] J.Tiley. Valuing american options in a path simulation model. *Transactions, Society of Actuaries Schaumburg, Volume 45, pp 499-520, 1993.*

[Xydias11] A. Xydias. Synthetic ETF backers pull most money in two years in move to real assets. *Financial Times, 1 November 2011.*

[Yu07] F. Yu. Correlated defaults in intensity-based models. *Mathematical Finance, Volume 17, pp. 155-173, 2007*

[Zagst02] R. Zagst. Interest-Rate Management. *Springer Finance, pp 227-271, 2002.*

[Zee06] E. Zee. Pairs trading model. *London Business School, 2006.*

[Zheng10] H. Zheng. Contagion models a la carte: Which one to choose?. *Department of Mathematics, Imperial College London, 2010.*