Block matrix determinant calculation and its application to regime phase-type distributions.

Pierre Nowicki

Abstract: This small document extracted from my *"thesis"* intends to present a block matrix determinant calculation through the Schur complement decomposition when blocks matrices exhibit a structure composed of:

- regime matrices component on the block matrix diagonal, and,
- regime-transitions component matrices off-diagonal.

Such block matrix calculations are useful in ensuring the singularity of the overall matrix and thus the existence of a limiting distribution for the process represented by the matrix.

Contents

1 Block matrix determinant

1.1 In the general case

In [Powell11], we find a method for recursively identify the determinant of a matrix based on the determinant of its N^2 blocks using the **Schur complement structure**. The theorem is as follows:

Theorem 1.1. *Let S be an $(nN)\times(nN)$ complex matrix, which is partitioned into N^2 blocks each of size $n\times n$*

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} & ... & \boldsymbol{S}_{1N} \\ \boldsymbol{S}_{21} & \boldsymbol{S}_{22} & ... & \boldsymbol{S}_{2N} \\ ... & ... & ... & ... \\ \boldsymbol{S}_{N1} & \boldsymbol{S}_{N2} & ... & \boldsymbol{S}_{NN} \end{pmatrix}$$

Then the determinant $\det(S) = \prod_{k=1}^{N} \det(\alpha_{kk}^{(N-k)})$ *where* α^k *are defined by*

$$\alpha_{ij}^{(0)} = S_{ij}$$
$$\alpha_{ij}^{(k)} = S_{ij} - \sigma_{i,N-k+1}^{t}\tilde{S}_k^{-1} s_{N-k+1,j},\ k\geq 1$$

and the vectors σ_{ij}^t *and* s_{ij} *are*

$$s_{ij} = (S_{ij}\ S_{i+1,j}\ \dots\ S_{N,j})^t,\ \sigma_{ij}^t = (S_{ij}\ S_{i,j+1}\ \dots\ S_{i,N})$$

and

$$\tilde{\boldsymbol{S}}_k = \begin{pmatrix} \boldsymbol{S}_{N-k+1,N-k+1} & \boldsymbol{S}_{N-k+1,N-k+2} & ... & \boldsymbol{S}_{N-k+1,N} \\ \boldsymbol{S}_{N-k+2,N-k+1} & \boldsymbol{S}_{22} & ... & \boldsymbol{S}_{N-k+2,N} \\ ... & ... & ... & ... \\ \boldsymbol{S}_{N,N-k+1} & \boldsymbol{S}_{N,N-k+2} & ... & \boldsymbol{S}_{N,N} \end{pmatrix}$$

the paper [Powell11] also establishes this proof by induction instead of using the "**Baniachewic identity**"

Lemma 1.1. *Let $\boldsymbol{S}$ be a complex block matrix of the form*

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} & ... & \boldsymbol{S}_{1N} \\ \boldsymbol{S}_{21} & \boldsymbol{S}_{22} & ... & \boldsymbol{S}_{2N} \\ ... & ... & ... & ... \\ \boldsymbol{S}_{N1} & \boldsymbol{S}_{N2} & ... & \boldsymbol{S}_{NN} \end{pmatrix}$$

and let us define the set of block matrices $\{\alpha^{(0)},\alpha^{(1)},\dots,\alpha^{(N-1)}\}$ *where* $\alpha^{(k)}$ *is an* $(N-k)\times(N-k)$ *block matrix with blocks*

$$\alpha_{ij}^{(0)} = S_{ij}$$
$$\alpha_{ij}^{(k+1)} = \alpha_{ij}^{(k)} - \alpha_{i,N-k}^{(k)}\left(\alpha_{N-k,N-k}^{(k)}\right)^{-1}\alpha_{N-k,j}^{(k)},\ k\geq 1$$

Then the determinants of consecutive $\alpha^{(k)}$ *are related via*

$$\det(\alpha^{(k)}) = \det(\alpha^{(k+1)}) \det(\alpha^{(k)}_{N-k,N-k})$$

with the partition

$$\alpha^{(k)} = \left(\begin{array}{cccc} \alpha^{(k)}_{11} & \dots & \alpha^{(k)}_{1,N-k-1} & \alpha^{(k)}_{1,N-k} \\ \vdots & & \vdots & \vdots \\ \alpha^{(k)}_{N-k-1,1} & \dots & \alpha^{(k)}_{N-k-1,N-k-1} & \alpha^{(k)}_{N-k-1,N-k} \\ \alpha^{(k)}_{N-k,1} & \dots & \alpha^{(k)}_{N-k,N-k-1} & \alpha^{(k)}_{N-k,N-k} \end{array} \right)$$

Proof. From [Powell11], we can use the following theorem

Theorem 1.2. *Given a complex block matrix of the form and the matrices $\alpha^{(k)}_{11}$ the determinant of $\boldsymbol{S}$ is given by*

$$\det(S) = \prod_{k=1}^{N} \det(\alpha^{(N-k)}_{kk})$$

1.2 Case of regime transition of a Markov Jump process

1.2.1 Process description

We define a Markov processes X_t with $X_t \sim \mathbf{Q}(Y_t)$ for example as a phase-type distribution process with $Y_t \sim \mathbf{R}(t)$ a hidden Markov chain defining regime shift R_t. X_t defines the a Markov jump process and Y_t defines a hidden Markov chain that defines the n regimes of X_t. For example, in the case of $n = 2$ regimes, we could typically identify R_1 as a *normal regime* and R_2 as a *high regime*.

Conceptually, the Markov chain Y_t expressing the regime chain can be dependent or not of the transition intensities $\lambda_{i,j}$ of X_t within a set regime. For example, several specificities could define the relation between X_t and Y_t.

- In finance, increased defaulted obligors in X_t could increase the probability of shifting the chain in a higher regime in Y_t,
- Y_t can be defined as endogenously or exogenously to the states of X_t. This will result either to the regime transition rate of Y_t to be incorporated or not in the transition rate of X_t.
- possibilities off simultaneous phase transition in X_t and regime transition in Y_t.

1.2.2 Case of unique regime

Considering the two representation $(\alpha_1, \mathbf{T}_1)$ and $(\alpha_2, \mathbf{T}_2)$. The matrix representing the case of two regimes can be identified as the blocks:

$$\mathbf{T} = \left(\begin{array}{c|c} \mathbf{T}_1 & \mathbf{R}_{12} \\ -- & -- \\ \mathbf{R}_{21} & \mathbf{T}_2 \end{array} \right)$$

The matrix $\mathbf{T}_1$ and $\mathbf{T}_2$ represent respectively the jump matrix of X_t when Y_t is in state $Y_t = R_1$ and $Y_t = R_2$. The matrix $\mathbf{R}_{12}$ and $\mathbf{R}_{21}$ represent respectively the regime transition matrix respectively, from regime 1 to regime 2 and vice versa.

1.2.3 Case of multiple regimes

Thus for $m > 2$ states for X_t and $n > 2$ regime with representation $(\alpha_i, \mathbf{T}_i), i \in \{1, \ldots, n\}$: This matrix can be identified as the blocks:

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}_{R_1} & \mathbf{R}_{12} & \ldots & \mathbf{R}_{1n} \\ \mathbf{R}_{21} & \mathbf{T}_{R_2} & \ldots & \mathbf{R}_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{R}_{n1} & \mathbf{R}_{n2} & \ldots & \mathbf{T}_{R_n} \end{pmatrix}$$

1.2.4 Existence of limiting distribution

Consider a finite set E containing m states and a set R identifying n regimes.

For each regime $j = 1, \ldots, n$ the triplet $(\boldsymbol{E}, \alpha_j, \boldsymbol{T}_j)$ is a phase type distribution where F is the distribution of the time to absorption τ with the closed-form formula:

$$F(t) = \mathbb{P}[\tau < t] = 1 - \alpha e^{\boldsymbol{T}_j t \boldsymbol{1}}$$
$$f(t) = \alpha e^{\boldsymbol{T}_j t \boldsymbol{1}}$$

with

$$e^{\boldsymbol{T}_j t} = \sum_{n=0}^{\infty} \frac{1}{n!} \boldsymbol{T}_j{}^n t^n$$

Using the work of [Assaf82] and [Assaf85] about the closure properties of phase-type distributions mixtures and their properties, we can augment the dimension of time-homogeneous parameter. As mentioned in [Neuts78] and in the multivariate case in [Assaf84], phase-type distribution identify a unique absorption state Δ leading to an absorption vector $\mathbf{t}$ on the right-hand side of matrix $\mathbf{Q}$ in order to apply the phase-type distribution methodology and obtain closed-form formula.

Usually the closure properties of the representation $(\alpha_R, \mathbf{T}_R)$ as a function of all the representation $(\alpha_i, \mathbf{T}_i), \forall i \in \{1, \ldots, n\}$ is established though the Laplace Transform to express the transition matrix $\mathbf{T}_R$ and exit vector $\mathbf{t}_R$ as a function of the underlying phase. Theorem 2.2.2 and 2.2.4 in [Neuts78] establish closure properties of $\mathbf{T}_R$ for convolutions and finite mixtures as:

Theorem 1.3 (Closure of multivariate phase-type distributions)**.** *Let $T = (T_1, \ldots, T_n)$ and $S = (S_1, \ldots, S_m)$ be independent multi-variate phase-type random vectors. Then the conjunction $(T, S) = (T_1, \ldots, T_n, S_1, \ldots, S_m)$ is a multi-variate phase-type random vector.*

However most of the applications are concerned with successive phases like theorem 1.3 instead of alternating between phases. We will instead have to assess the singularity of the transition matrix $\mathbf{T}_R$ using Lemma 2.2.1 p45 in [Neuts78].

Lemma 1.2. *The states* $1, \dots, m$ *are transient if and only if the matrix* $\boldsymbol{T}$ *is singular.*

We are interested in constructing the different multivariate phase between regimes using the closure theorem 2.2.1 of absorption state Δ_i of multivariate phase-type distribution in [Assaf84] using communication between phases though regime shifts. Therefore we need to establish the singularity of matrix $\boldsymbol{T}$ in the case of incremental states towards the absorption state Δ.

Assumption on processes: As we intend to demonstrate the existence of the phase property for the general case of Markov jump process applied to credit modelling, the assumption is that process X_t starts in an initial state and transits to a final state (represented by the absorption state Δ) without transiting back to a state that has already been visited. Therefore, the $\mathbf{T}_{R_i} = [\lambda_{ij}]_{1 \leq i,j \leq K}$and is upper-triangular. We make no specific assumption of credit contagion in the intensities. Additionally, the condition $t_{i,i} << \lambda_{i,i}, \forall i \in K$ in our context is equivalent to the fact that the regime shifts occur less often than transitions between states which seems a sensible assumption. So by definition the matrices $R_{ij}, \forall i, j \in R, i \neq j$ are diagonal matrices with diagonal values representing the transition rate from regime i to regime j and $T_i, i \in R$ is upper-triangular with the transitions rate towards absorption when the Markov chain is in regime i. Thus, we can recursively establish the singularity of matrix $\boldsymbol{T}$ based on its sub-components which determinants are defined as its diagonal values.

In the multi-variate setting, we loose the whole upper triangular structure of $\mathbf{T}$ for the phase type process to gain easy existence and calculation for the exponential matrix through the eigenvalues on the diagonal axis. However each of the sub-matrix $\mathbf{T}_{R_i}$ has the upper triangular structure and the same of convergence criteria for the exponential matrix.

We need to proceed as in the document phase in [Neuts78] and show

- that $\boldsymbol{T}$ is singular so it is invertible and thus the absorption is certain.
- Using the Perron-Frobenius Theorem 6.5 p26 in [Asmu03] that the eigenvalues are negatives and not null to have an ergodic markov jump process and thus that there exists a distribution.

Proposition 1.1. 1.2.5 Demonstration

We consider the general matrix

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{T}_{R_1} & \boldsymbol{R}_{12} & \dots & \boldsymbol{R}_{1n} \\ \boldsymbol{R}_{21} & \boldsymbol{T}_{R_2} & \dots & \boldsymbol{R}_{2n} \\ \dots & \dots & \dots & \dots \\ \boldsymbol{R}_{n1} & \boldsymbol{R}_{n2} & \dots & \boldsymbol{T}_{R_n} \end{pmatrix}$$

where $\boldsymbol{T}_{R_i} = [\lambda_{i,j}]_{1\le i\le j\le K}$ *define upper-triangular matrices with negative eigenvalues with modulus lower than 1 and* $\boldsymbol{R}_{ij} == [t_{i,i}]_{1\le i\le K}$ *define diagonal matrices with positive values and suppose that* $t_{i,i} << \lambda_{i,i}, \forall i \in K$ *then the matrix* $\boldsymbol{T}$ *is singular.*

Recursive Demonstration.

Case $n = 1$. $\mathbf{T}^1 = \mathbf{T}_1$ with $\mathbf{T}_1$ upper-triangular with non null eigenvalues on the diagonal. Thus $\mathbf{T}^1$ non singular.

Case $n = 2$: Suppose

$$\mathbf{T}_2 = \begin{pmatrix} \mathbf{T}^1 & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{T}^2 \end{pmatrix}$$

With $\mathbf{T}^i, \forall i \in R$ upper-triangular non-null eigenvalue structure. All sub-matrices have same dimension. We recall the result

- Block identical
$$\det\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \det(\mathbf{AD} - \mathbf{BC})$$
- Block unidentical
$$\det\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \det(\mathbf{D})\det(\mathbf{A} - \mathbf{BD}^{-1}\mathbf{C})$$
with $\mathbf{A}$ a $m \times m$ matrix, $\mathbf{D}$ a $n \times n$ matrix, $\mathbf{B}$ a $n \times m$ matrix and $\mathbf{C}$ a $m \times n$ matrix. (This is the "Baniachewic identity").

Thus we have

$$\det(\mathbf{T}_2) = \det(\mathbf{T}^1\mathbf{T}^2 - \mathbf{R}_{12}\mathbf{R}_{21})$$

Which is a function of the eigenvalue produced by the product 2 upper-triangular matrices minus the product of 2 diagonal matrices. We thus obtain a condition on the eigenvalue of non singularity as

$$\lambda^1_{ii}\lambda^2_{ii} - t^{12}_{ii}t^{21}_{ii} \neq 0, \forall i \in 1, \dots, m \tag{1}$$

The matrix $\mathbf{T}_2$ is invertible.

Case $n = 3$: Suppose

$$\mathbf{T}_3 = \begin{pmatrix} \mathbf{T}^1 & \mathbf{R}_{12} & \mathbf{R}_{13} \\ \mathbf{R}_{21} & \mathbf{T}^2 & \mathbf{R}_{23} \\ \mathbf{R}_{31} & \mathbf{R}_{32} & \mathbf{T}^3 \end{pmatrix} = \begin{pmatrix} \mathbf{T}_2 & \mathbf{R}_{X3} \\ \mathbf{R}_{3X} & \mathbf{T}^3 \end{pmatrix}$$

with $\det(\mathbf{T}_2) \neq 0$ as previously shown and

$$\mathbf{T}_3^t = \begin{pmatrix} (\mathbf{T}^3)^t & \mathbf{R}^t_{3X} \\ \mathbf{R}^t_{X3} & \mathbf{T}^t_2 \end{pmatrix} \tag{2}$$

By application if the Baniachewic identity, we have

$$\det(\mathbf{T}_3) = \det(\mathbf{T}^3)\det(\mathbf{T}_2 - \mathbf{R}_{X3}(\mathbf{T}^3)^{-1}\mathbf{R}_{3X})$$

we have $\det(\mathbf{T}^3) \neq 0$ but we do not have any specific information about the eigenvalues of the matrix $\mathbf{T}_2$. However with

$$\det(\mathbf{T}_3) = \det(\mathbf{T}_3^t) = \det(\mathbf{T}_2^t)\det((\mathbf{T}^3)^t - \mathbf{R}_{3X}^t(\mathbf{T}_2^t)^{-1}\mathbf{R}_{X3}^t)$$

with $\det(\mathbf{T}_2) \neq 0$ as shown previously with (1) valid. The transpose matrices are either lower triangular or diagonal with eigenvalue being the inverse since with (1) valid none is null. Thus the second determinant is a derterminant of lower triangular structure with the condition

$$\lambda_{ii}^3 - t_{ii}^{31}t_{ii}^{32}(\lambda_{ii}^2)^{-1}t_{ii}^{13}t_{ii}^{23} \neq 0, \forall i \in 1,\ldots,m \tag{3}$$

Suppose now that the case is valid for n with

$$\det(\mathbf{T}_n) = \det(\mathbf{T}_n^t) \neq 0$$

and none of its associated eigenvalue $\theta_{ii}, \forall i \in E$ are null. We need to show that $\mathbf{T}_{n+1}$ is such that

$$\det(\mathbf{T}_{n+1}) = \det(\mathbf{T}_{n+1}^t) \neq 0$$

$$\mathbf{T}_{n+1} = \begin{pmatrix} (\mathbf{T}_n)^t & \mathbf{R}_{Xn} \\ \mathbf{R}_{nX} & \mathbf{T}^{n+1} \end{pmatrix} \tag{4}$$

with $\mathbf{T}^{n+1}$ being an upper triangular matrix representing regime $n+1$ and $\mathbf{R}_{Xn}, \mathbf{R}_{nX}$ transition diagonal matrices. Or

$$\det(\mathbf{T}_{n+1}^t) = \det(\mathbf{T}_n^t)\det((\mathbf{T}^{n+1})^t - \mathbf{R}_{nX}^t(\mathbf{T}_n^t)^{-1}\mathbf{R}_{Xn}^t)$$

Thus we need to find that none of the eigenvalues of $\det((\mathbf{T}^{n+1})^t - \mathbf{R}_{nX}^t(\mathbf{T}_n^t)^{-1}\mathbf{R}_{Xn}^t)$ is null. The matrix $(\mathbf{T}^{n+1})^t$ is lower-triangular and $\mathbf{R}_{nX}, \mathbf{R}_{Xn}$ are diagonal matrices. As for our assumptions, for all values in the matrix

$$(\mathbf{T}^{n+1})^t - \mathbf{R}_{nX}^t(\mathbf{T}_n^t)^{-1}\mathbf{R}_{Xn}^t \approx (\mathbf{T}^{n+1})^t$$

thus

$$\det((\mathbf{T}^{n+1})^t) \approx \det((\mathbf{T}^{n+1})^t - \mathbf{R}_{nX}^t(\mathbf{T}_n^t)^{-1}\mathbf{R}_{Xn}^t)$$

We have to condition the fact that

$$\forall \lambda_{ii}, i \in E, \forall t_{ii} i \in E,\ t_{ii}\lambda_{ii} << \lambda_{ii} \tag{5}$$

We have thus established the non-singularity of $\boldsymbol{T}$ and thus the fact that all its states are transient. □

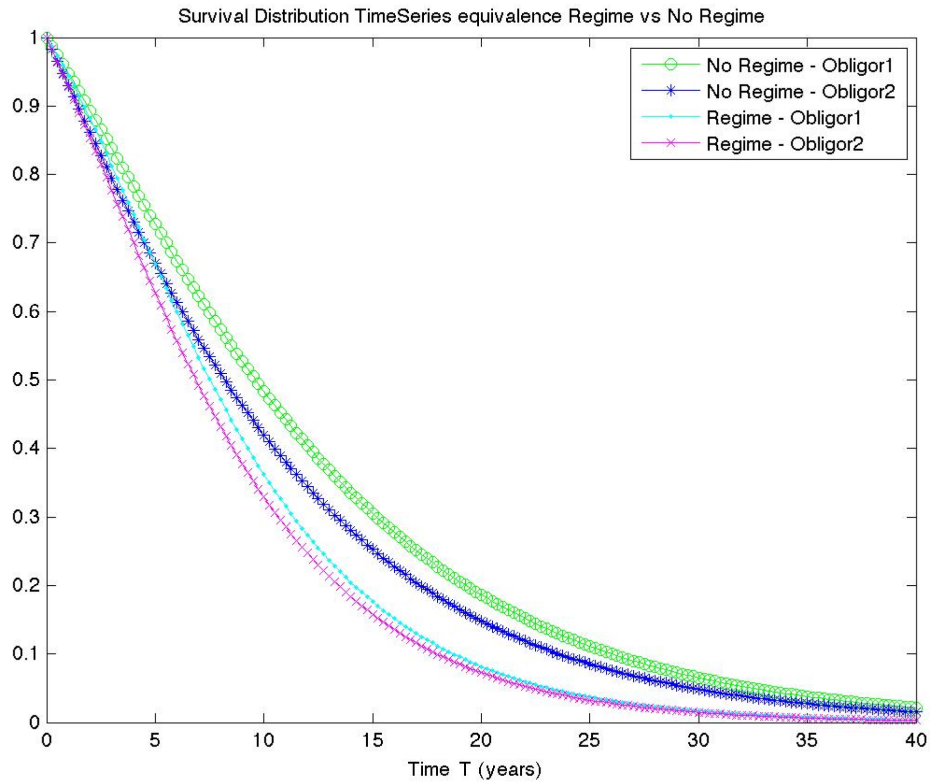

Fig. 1: Marginal survival distribution $\mathbb{Q}[\tau_i > t]$ for 2 obligors with or without regime shift.

1.3 Illustration to credit risk modelling

1.3.1 Example: $m = 2$ obligors and $n = 2$ regimes:

Taken for the thesis *"Counterparty credit risk under credit risk contagion using time-homogeneous phase-type distribution"* for the two-regime representation $(\alpha_1, \mathbf{T}_1)$ and $(\alpha_2, \mathbf{T}_2)$, we can highlight the flexibility of the approach with :

- 2 regimes converging towards an equivalent single regime marginal distribution (fig 1, p 9),
- 2 different regimes capturing high and low credit volatility situations (fig 2, p 10.).

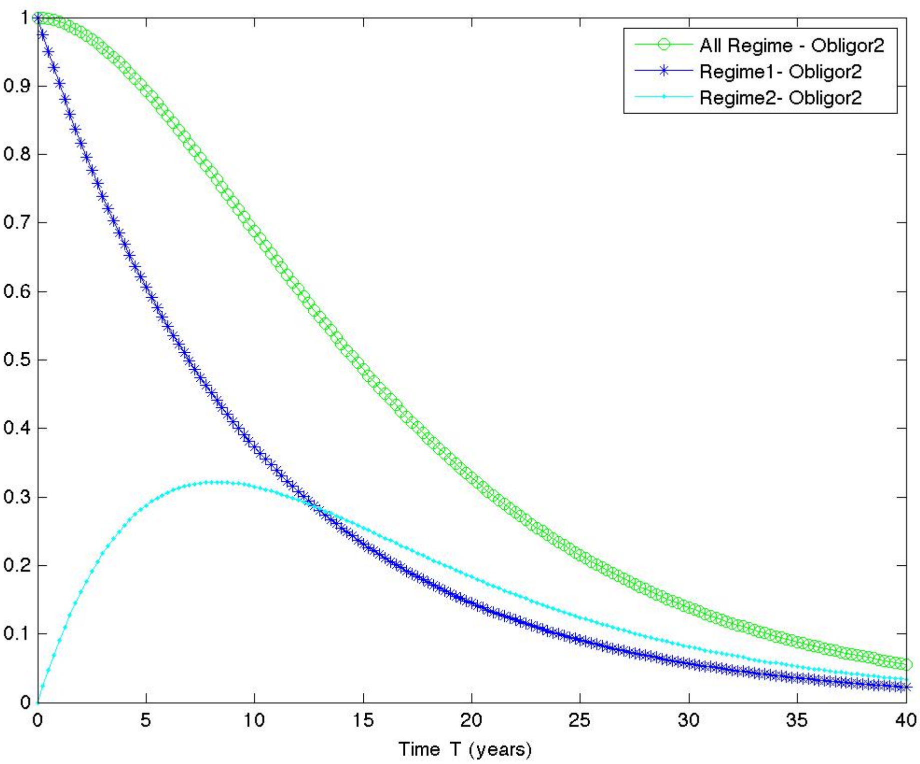

Fig. 2: Marginal survival distribution $\mathbb{Q}[\tau_1 > t, i \in R_j]$ per regime j for obligor 1 in case of $m = 2$ obligors and $n = 2$ regimes.

References

[Assaf82] D. Assaf, and B. Levikson. Closure of Phase-type distributions under operations arising in Reliability Theory, The Annals of Probability, Vol. 10, No. 1,pp 265-269, 1982.

[Assaf84] D. Assaf, N. Langberg, T. Savits, and M. Shaked. Multivariate phase-type distributions, Operations Research 32, 688-702. 1984.

[Assaf85] D. Assaf, and N. Langberg. Presentation of Phase-type distributions as proper mixtures, Journal of Applied Probability, Vol. 22, No. 1,pp 247-250, 1985.

[Asmu03] S. Asmussen. Applied Probability and Queues. Second Edition. Springer, London,2003.

[Brig05] D. Brigo, and M.Tarenghi. Credit default swap calibration and counterparty Risk valuation with a scenario-based First passage model. Credit Models - Banca IMI, 2005.

[Herb10] A. Herbertsson. Modelling default contagion using multivariate phase-type distributions. *Springer Verlag, 2010.*

[Neuts78] M. Neuts: Matrix - Geometric Solutions in Stochastic Models. An algorithmic Approach. Johns Hopkins, 1978.

[Powell11] P.D. Powell. Calculating determinants of block matrix. Department of Physics. University of Illinois. 2011

[Rolski98] T. Rolski, H. Schmidli, V. Schmidt, J. Teuggels. Stochastic Processes for Insurance and Finance, J. Wiley, Chichester, 1998.