Homework 5
Solution

March 6th

Exercises from the book

Exercise 2.1 The range of X is $\{1, 3\}$ (x coordinates of the jumps), with

$$\mathbb{P}(X = 1) = \frac{1}{3}, \qquad \mathbb{P}(X = 3) = \frac{2}{3}$$

(heights of the jumps). Everything follows easily from this:

$$F(2) = \frac{1}{3}, \qquad \mathbb{P}(X > 1) = \frac{2}{3}, \qquad \mathbb{P}(X = 2) = 0, \qquad \mathbb{P}(X = 3) = \frac{2}{3}.$$

Exercise 2.2 There are 36 possible outcomes when we roll a pair of dice. Six of them are doubles $(1, 1), \ldots, (6, 6)$, while the other thirty involve pairs of distinct numbers.

(a) We have

$$\begin{aligned}
\mathbb{P}(X = 1) &= \frac{1 + 2 \cdot 5}{36} = \frac{11}{36}, & \mathbb{P}(X = 2) &= \frac{1 + 2 \cdot 4}{36} = \frac{9}{36}, \\
\mathbb{P}(X = 3) &= \frac{1 + 2 \cdot 3}{36} = \frac{7}{36}, & \mathbb{P}(X = 4) &= \frac{1 + 2 \cdot 2}{36} = \frac{5}{36}, \\
\mathbb{P}(X = 5) &= \frac{1 + 2 \cdot 1}{36} = \frac{1}{12}, & \mathbb{P}(X = 6) &= \frac{1}{36}.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\mathbb{P}(X = 5) &= \frac{2}{36} = \frac{1}{18}, & \mathbb{P}(X = 4) &= \frac{2 \cdot 2}{36} = \frac{1}{9}, & \mathbb{P}(X = 3) &= \frac{2 \cdot 3}{36} = \frac{1}{6}, \\
\mathbb{P}(X = 2) &= \frac{2 \cdot 4}{36} = \frac{2}{9}, & \mathbb{P}(X = 1) &= \frac{2 \cdot 5}{36} = \frac{5}{18}, & \mathbb{P}(X = 0) &= \frac{6}{36} = \frac{1}{6}.
\end{aligned}$$

Exercise 2.3 (a) We must have

$$1 = \sum_{k=0}^{\infty} \frac{c}{2^k} = c \sum_{k=0}^{\infty} \frac{1}{2^k} = 2c \Rightarrow c = \frac{1}{2}.$$

(b) We have

$$\mathbb{P}(X > 0) = 1 - \mathbb{P}(X = 0) = \frac{1}{2}.$$

(c) The probability that X is even is

$$\begin{aligned} &\mathbb{P}(X=0) + \mathbb{P}(X=2) + \mathbb{P}(X=4) + \cdots \\ &= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2^2} + \frac{1}{2} \cdot \frac{1}{2^4} + \cdots = \frac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots\right) \\ &= \frac{1}{2}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right) = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}. \end{aligned}$$

Exercise 2.4 (a) With replacement $X \sim \mathcal{B}in(n = 5, p = 4/52)$. The range of X is $\{0, 1, 2, 3, 4, 5\}$, and

$$\mathbb{P}(X = k) = \binom{5}{k}\left(\frac{48}{52}\right)^{5-k}\left(\frac{4}{52}\right)^k.$$

(b) Without replacement we have $X \sim \mathcal{H}yper\mathcal{G}eom(w = 4, b = 48, n = 5)$. The range of X is $\{0, 1, 2, 3, 4\}$ and

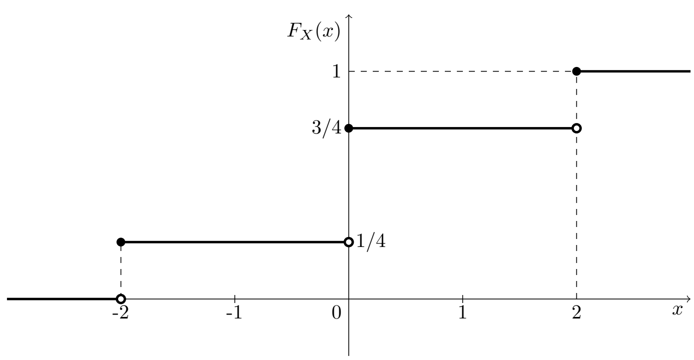
$$\mathbb{P}(X=0) = \frac{\binom{48}{5}}{\binom{52}{5}}, \qquad \mathbb{P}(X=1) = \frac{\binom{4}{1}\cdot\binom{48}{4}}{\binom{52}{5}}, \qquad \mathbb{P}(X=2) = \frac{\binom{4}{2}\cdot\binom{48}{3}}{\binom{52}{5}}$$

$$\mathbb{P}(X=3) = \frac{\binom{4}{3}\cdot\binom{48}{2}}{\binom{52}{5}}, \qquad \mathbb{P}(X=4) = \frac{\binom{4}{4}\cdot\binom{48}{1}}{\binom{52}{5}}.$$

Exercise 2.5 The range of X is $\{-2, 0, 2\}$. We have

$$\mathbb{P}(X=2) = \mathbb{P}(X=-2) = \frac{1}{4}, \qquad \mathbb{P}(X=0) = \frac{1}{2}, \qquad \mathbb{P}(X=x) = 0 \text{ otherwise.}$$

It means that the cumulative distribution function has 3 jumps at -2, 0 and 2, with sizes 1/4, 1/2 and 1/4:



Exercise 2.6 Write a for the number you pick. Denote by X the number of a-s when we toss three dice, and by W your win. The range of X is $\{0, 1, 2, 3\}$. Denote by p the probability mass function of X. We have

$$W = \begin{cases} -1 & \text{if } X = 0, \\ X & \text{if } X > 0, \end{cases}$$

so

$$\mathbb{E}[W] = -p(0) + p(1) + 2p(2) + 3p(3).$$

Note that $X \sim \mathcal{B}in(3, 1/6)$. We have

$$p(1) + 2p(2) + 3p(3) = \mathbb{E}[X] = 3 \cdot \frac{1}{6} = \frac{1}{2}, \qquad p(0) = \mathbb{P}(X = 0) = \left(\frac{5}{6}\right)^3.$$

Hence

$$\mathbb{E}[W] = \frac{1}{2} - \left(\frac{5}{6}\right)^3 \approx -0.079.$$

Exercise 2.8 The number B of different birthdays is a random variable with range $\{1, 2, 3, 4\}$. The pmf is computed as follows.

$$\mathbb{P}(B = 1) = \underbrace{\frac{1}{365^4}}_{\text{number of possible birthdays}} \cdot \overbrace{365}^{\text{common birthday}} = \frac{1}{365^3},$$

$$\mathbb{P}(B = 2) = \frac{1}{365^4} \overbrace{\binom{365}{2}}^{\text{two different birthdays}} \cdot \Bigg[\underbrace{\binom{4}{3} + \binom{4}{2} + \binom{4}{1}}_{\text{3, 2 or 1 people for the first birthday}} \Bigg] = \frac{364 \cdot 7}{365^3}$$

$$\mathbb{P}(B = 3) = \frac{1}{365^4} \binom{365}{3} \cdot \overbrace{3}^{\text{which birthday is shared}} \cdot \underbrace{(4)_2}_{\text{2 isolated people}} = \frac{(364)_2 \cdot 6}{365^3}$$

$$\mathbb{P}(B = 4) = \frac{(365)_4}{365^4} = \frac{(364)_3}{365^3}.$$

Using this data we compute the expectation to be

$$\mathbb{E}[B] = 1 \cdot \mathbb{P}(B = 1) + 2 \cdot \mathbb{P}(B = 2) + 3 \cdot \mathbb{P}(B = 3) + 4 \cdot \mathbb{P}(B = 4) \approx 3.98.$$

Exercise 2.9 Denote by H the number of heads Bob gets, and by G its gains. Then $H \in \{0, 1, 2, 3\}$ and

$$G = \begin{cases} 0.25 \cdot H, & H > 0, \\ -2, & H = 0. \end{cases}$$

Thus

$$\begin{aligned} \mathbb{E}[G] &= 0.25 \cdot \mathbb{P}(H = 1) + 0.5 \cdot \mathbb{P}(H = 2) + 0.75 \cdot \mathbb{P}(H = 3) - 2 \cdot \mathbb{P}(H = 0) \\ &= 0.25 \cdot \frac{\binom{3}{1} + 2\binom{3}{2} + 3\binom{3}{3}}{2^3} - \frac{2}{2^3} = 0.25 \cdot \frac{3}{2} - \frac{1}{4} = \frac{1}{8}. \end{aligned}$$

$$\mathbb{E}[G^2] = \frac{1}{16} \cdot \frac{\binom{3}{1} + 4 \cdot \binom{3}{2} + 9\binom{3}{3}}{8} + 4 \cdot \frac{1}{8}$$
$$= \frac{3 + 12 + 9}{128} + \frac{64}{128} = \frac{88}{128} = \frac{11}{16}.$$
$$\operatorname{var}(G) = \mathbb{E}[G^2] - \mathbb{E}[G]^2 = \frac{11}{16} - \frac{1}{64} = \frac{43}{64}.$$

Exercise 2.10 Note first that the order in which extract the balls is irrelevant when deciding which has the smallest label so we assume that we draw three balls simultaneously. There are $\binom{10}{3} = 120$ possibilities.

Next observe that

$$\mathbb{P}(X = 10) = \mathbb{P}(X = 9) = 0$$

while for $k \leq 8$ we there are $\binom{10-k}{2}$subsets of $\{1, \dots, 10\}$ that have k as the smallest element. Hence

$$\mathbb{P}(X = k) = \frac{\binom{10-k}{2}}{120}, \;\; 1 \leq k \leq 8.$$

The mean of X is

$$\mathbb{E}[X] = \frac{1}{120} \sum_{k=1}^{8} k \binom{10-k}{2} = 2.75.$$

Exercise 2.11 (a) In this case the number of tries $X \sim \mathcal{G}eom(1/5)$. Therefore $\mathbb{E}[X] = 5$ and $\operatorname{var}(X) = 20$.

(b) Now the number of tries X has range $\{1, \dots, 5\}$. For $k = 1, 2, \dots, 5$ we have

$$\mathbb{P}(X = k) = \frac{(4)_{k-1}}{(5)_k} = \frac{1}{5}.$$

Hence

$$\mathbb{E}[X] = \frac{1 + \cdots + 5}{5} = 3,$$
$$\mathbb{E}[X^2] = \frac{1^2 + \cdots + 5^2}{5} = 11, \;\; \operatorname{var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 2.$$

Exercise 1

We call $N_1, \dots, N_4$ the random variables for each question.

1. $N_1 \sim \mathcal{H}yper\mathcal{G}eom(w = 2, b = 998, n = 850)$, so
$$\mathbb{P}(N_1 = 0) = \frac{\binom{2}{0}\binom{998}{850}}{\binom{1000}{850}} = \frac{150 \cdot 149}{1000 \cdot 999}.$$
2. $N_2 \sim \mathcal{B}er(2/1000)$.
3. $N_3 \sim \mathcal{B}in(1000, 0.001)$, so
$$\mathbb{E}[N_3] = 1000 \cdot 0.001 = 1.$$
4. $N_4 \sim \mathcal{N}eg\mathcal{B}in(1000, 0.999)$, so
$$\mathbb{E}[\text{number of defective pieces}] = \mathbb{E}[N_4 - 1000] = \frac{1000}{0.999} - 1000 = \frac{1000}{999} \approx 1.001.$$

Bonus exercise

1. There are $\binom{n-1}{k+\ell-1}$ such subsets.

2. We can describe such a set by prescribing the first $(k-1)$ elements, and the last $(\ell-1)$: the kth is already given. The first $(k-1)$ have to be smaller than m, so there are $\binom{m-1}{k-1}$ possibilities. Similarly, there are $\binom{(n-1)-m}{\ell-1}$ possibilities for the last $(\ell-1)$, for a total of

$$\binom{m-1}{k-1}\binom{(n-m)-1}{\ell-1}$$

possibilities.

3. There is a mistake in the statement; it should read

$$\sum_{m=k}^{n-\ell}\binom{m-1}{k-1}\binom{(n-m)-1}{\ell-1}=\binom{n-1}{k+\ell-1}.$$

We know (question 1) that there are $\binom{n-1}{k+\ell-1}$ subsets of $\{1,2,\ldots,n-1\}$ with $k+\ell-1$ elements. But we also see that

$$\begin{aligned}
&\#\{\text{subsets of } \{1,2,\ldots,n-1\} \text{ of size } k+\ell-1\}\\
&\quad=\sum_{m=k}^{n-\ell}\#\{\text{subsets of } \{1,2,\ldots,n-1\} \text{ of size } k+\ell-1, \text{ with } m \text{ as the } k\text{th element}\}\\
&\quad=\sum_{m=k}^{n-\ell}\binom{m-1}{k-1}\binom{(n-1)-m}{\ell-1},
\end{aligned}$$

hence the equality of the question indeed holds.

4. Let N and M be such variables. For fixed $n \geq k+\ell$, we have

$$\begin{aligned}
\mathbb{P}(N+M=n) &= \sum_{m=k}^{n-\ell}\mathbb{P}(N=m \text{ and } M=n-m)\\
&=\sum_{m=k}^{n-\ell}\mathbb{P}(N=m)\cdot\mathbb{P}(M=n-m)\\
&=\sum_{m=k}^{n-\ell}\binom{m-1}{k-1}(1-p)^{m-k}p^k\cdot\binom{(n-m)-1}{\ell-1}(1-p)^{(n-m)-\ell}p^\ell\\
&=(1-p)^{n-(k+\ell)}p^{k+\ell}\sum_{m=k}^{n-\ell}\binom{m-1}{k-1}\binom{(n-m)-1}{\ell-1}\\
&=(1-p)^{n-(k+\ell)}p^{k+\ell}\binom{n-1}{k+\ell-1},
\end{aligned}$$

which is indeed the value expected for a distribution $\mathcal{N}eg\mathcal{B}in(k+\ell)$.