# Homework 4
# Solution

February 21st

## Exercises from the book

**Exercise 1.34** Denote by $N_k$ the event "the number $k$ does not appear during 3 successive rolls of the die". We are interested in the probability $\mathbb{P}(N_5|N_6)$. We have

$$\mathbb{P}(N_5|N_6) = \frac{\mathbb{P}(N_5 \cap N_6)}{\mathbb{P}(N_6)}.$$

We have

$$\mathbb{P}(N_6) = \frac{5^3}{6^3}, \ \ \mathbb{P}(N_5 \cap N_6) = \frac{4^3}{6^3},$$

so

$$\mathbb{P}(N_5|N_6) = \frac{4^3}{5^3} = \frac{64}{125} = 0.512.$$

**Exercise 1.36** Write $A_i$ (resp. $B_i$, $AB_i$, $O_i$) for the event that the individual individual $i$ has blood type A (resp. B, AB, O), and $\oplus_i$ (resp. $\ominus_i$) for the event that their Rh factor is positive (resp. negative). For the combined events, we will write for instance $A_i^-$ for $A_i \cap \ominus_i$, and similarly for all blood types.

(i)

$$\mathbb{P}(A_1^- \cap A_2^-) = \mathbb{P}(A_1 \cap \ominus_1)^2 = \big(\mathbb{P}(A_1) \cdot \mathbb{P}(\ominus_1)\big)^2 = (0.4 \times 0.16)^2 \approx 0.0041.$$

(ii)

$$\mathbb{P}(O_i^+) = \mathbb{P}(O_i \cap \oplus_i) = 0.45 \cdot 0.84 = 0.378,$$
$$\mathbb{P}\Big( (O_i^+)^\complement \Big) = 1 - 0.378 = 0.622.$$

The probability of the event described in the question is

$$\mathbb{P}\left( \left(O_1^+ \cap (O_2^+)^\complement\right) \cup \left((O_1^+)^\complement \cap O_2^+\right) \right).$$

Because this union is disjoint, the probability is in fact equal to

$$\mathbb{P}\left(O_1^+ \cap (O_2^+)^{\complement}\right) + \mathbb{P}\left((O_1^+)^{\complement} \cap O_2^+\right) = 2\,\mathbb{P}(O_1^+) \cdot \mathbb{P}\Big( (O_2^+)^{\complement}\Big)$$
$$= 2 \cdot 0.378 \cdot 0.622$$
$$\approx 0.470.$$

(iii)

$$\mathbb{P}(O_1^+ \cup O_2^+) = 1 - \mathbb{P}\left((O_1^+)^{\complement} \cap (O_2^+)^{\complement}\right) = 1 - 0.622^2 \approx 0.613.$$

(iv) Denote by $E$ the event of interest: "one person is Rh positive and the other is not AB". Then

$$E = \underbrace{\left( \oplus_1 \cap (AB_2)^{\complement} \right)}_{E_1} \cup \underbrace{\left( \oplus_2 \cap (AB_1)^{\complement} \right)}_{E_2}.$$

Let us transform 'or' statements into 'and' statements (note that the Rh factors and groups of the individuals are 4 independent variables):

$$\begin{aligned}
\mathbb{P}(E) &= 1 - \mathbb{P}\big( E_1^{\complement} \cap E^{\complement} \big) \\
&= 1 - \big(1 - \mathbb{P}(E_1)\big)(1 - \mathbb{P}(E_2)) \\
&= 1 - \Big(1 - \mathbb{P}(\oplus_1) \cdot \big(1 - \mathbb{P}(AB_2)\big)\Big)^2 \\
&= 2\,\mathbb{P}(\oplus_1)\big(1 - \mathbb{P}(AB_2)\big) - \Big(\mathbb{P}(\oplus_1)\big(1 - \mathbb{P}(AB_2)\big)\Big)^2 \\
&= 2(0.96 \cdot 0.84) - (0.96 \cdot 0.84)^2 \\
&\approx 0.9625.
\end{aligned}$$

(v) The probability is

$$\mathbb{P}(A_1)^2 + \mathbb{P}(B_1)^2 + \mathbb{P}(O_1)^2 + \mathbb{P}(AB_1)^2 = (0.4)^2 + (0.11)^2 + (0.45)^2 + (0.04)^2 = 0.3762.$$

(vi) The event that they have different Rh factors (written $R_{\neq}$) is independent of the event that they have equal ABO type (written $ABO_=$, and whose probability we computed above). The probability must then be

$$\begin{aligned}
\mathbb{P}(R_{\neq}) \cdot \mathbb{P}(ABO_=) &= \mathbb{P}\big((\oplus_1 \cap \ominus_2)) \cup (\ominus_1 \cap \ominus_2)\big) \cdot \mathbb{P}(ABO_=) \\
&= 2 \cdot \mathbb{P}(\oplus_1) \cdot \mathbb{P}(\ominus_2) \cdot \mathbb{P}(ABO_=) \\
&= 2 \cdot 0.84 \cdot 0.16 \cdot 0.3762 \\
&\approx 0.101.
\end{aligned}$$

**Exercise 1.39** Denote by $C \nrightarrow S$ the event "the luggage is missing in Sidney", by $C \nrightarrow LA$ the event "the luggage was lost between O'Hare and LAX" (mishandled in O'Hare) and by $LA \nrightarrow S$ the event “the luggage was lost between LAX and Sydney” (mishandled at LAX). We use $\to$ instead of $\nrightarrow$ for their complements. We know that for $p = 1\%$,

$$\mathbb{P}(C \nrightarrow LA) = p, \quad \mathbb{P}(LA \nrightarrow S|C \to LA) = p.$$

(i) We want to compute $\mathbb{P}(C \nrightarrow LA|M)$. We have

$$\mathbb{P}(C \nrightarrow LA|C \nrightarrow S) = \frac{\mathbb{P}(C \nrightarrow S|C \nrightarrow LA)\mathbb{P}(C \nrightarrow LA)}{\mathbb{P}(C \nrightarrow S)},$$

$$\mathbb{P}(C \nrightarrow S) = \mathbb{P}(C \nrightarrow S|C \nrightarrow LA)\mathbb{P}(C \nrightarrow LA) + \mathbb{P}(C \nrightarrow S|C \to LA)\mathbb{P}(C \to LA).$$

Note that $\mathbb{P}(C \nrightarrow S|C \nrightarrow LA) = 1$, $\mathbb{P}(C \nrightarrow S|C \to LA) = \mathbb{P}(LA \nrightarrow S|C \to LA) = p$. Hence

$$\mathbb{P}(C \nrightarrow S) = p + p(1-p),$$

$$\mathbb{P}(C \nrightarrow LA|C \nrightarrow S) = \frac{p}{p + p(1-p)} = \frac{1}{2-p} \approx 0.5025.$$

(ii) We have

$$1 = \mathbb{P}(LA \nrightarrow S|C \nrightarrow S) + \mathbb{P}(C \nrightarrow LA|C \nrightarrow S)$$

so that

$$\mathbb{P}(LA \nrightarrow S|C \nrightarrow S) = 1 - \frac{1}{2-p} \approx 0.4975.$$

**Exercise 1.40** Denote by $D_k$ the event “we get the number $k$ after one roll of the die”, $T_k$ the event “we get $k$ tails in a row” and by $T$ the event described in the exercise: “we get only tails when we first roll a die and then we flip a coin as many times as the number we get after the die roll".

Using the law of total probability we deduce

$$\mathbb{P}(T) = \sum_{k=1}^{6} \mathbb{P}(T|D_k)\mathbb{P}(D_k).$$

Observe that for any $k = 1, \ldots, 6$ we have

$$\mathbb{P}(T|D_k) = \mathbb{P}(T_k) = \frac{1}{2^k}, \ \ \mathbb{P}(D_k) = \frac{1}{6}.$$

Hence

$$\mathbb{P}(T) = \frac{1}{6}\sum_{k=1}^{6} 2^{-k} = \frac{1}{6}\left(1 - \frac{1}{2^6}\right) = \frac{21}{128}.$$

**Exercise 1.42** Consider the events $A$ “we get an ace when drawing a card” and $S$ “we get a spade that is not an ace when drawing a card”. Recall that the probability to see $A$ before $S$ is

$$\frac{\mathbb{P}(A)}{\mathbb{P}(A) + \mathbb{P}(S)}.$$

We see that the probability that we first get an ace is

$$\frac{\frac{4}{52}}{\frac{4}{52} + \frac{12}{52}} = \frac{4}{16} = \frac{1}{4}.$$

**Exercise 1.46** Let $E$ be the event "one ace was drawn from the thickened 2nd deck".

For $i = 0, 1, 2$ we denote by $A_i$ the event "there were $i$ aces among the two cards drawn from the original first deck". Observe that

$$\mathbb{P}(A_0) = \frac{(48)_2}{(52)_2} = \frac{\binom{48}{2}}{\binom{52}{2}} = \frac{48 \cdot 47}{52 \cdot 51} \approx 0.8506,$$

$$\mathbb{P}(A_1) = \frac{2 \cdot 4 \cdot 48}{(52)_2} \approx 0.1447,$$

$$\mathbb{P}(A_2) = \frac{(4)_2}{(52)_2} \approx 0.0045.$$

We have

$$\mathbb{P}(A_0|E) = \frac{\mathbb{P}(E|A_0)\mathbb{P}(A_0)}{\mathbb{P}(E|A_0)\mathbb{P}(A_0) + \mathbb{P}(E|A_1)\mathbb{P}(A_1) + \mathbb{P}(E|A_2)\mathbb{P}(A_2)}.$$

The thickened deck has 54 cards and, given $A_i$, it has $4 + i$ aces. Using this we deduce

$$\mathbb{P}(E|A_0) = \frac{4}{54}, \ \ \mathbb{P}(E|A_1) = \frac{5}{54}, \ \ \mathbb{P}(E|A_2) = \frac{6}{54}.$$

Hence

$$\begin{aligned}\mathbb{P}(A_0|E) &= \frac{\frac{4}{54}\mathbb{P}(A_0)}{\frac{4}{54}\mathbb{P}(A_0) + \frac{5}{54}\mathbb{P}(A_1) + \frac{6}{54}\mathbb{P}(A_2)} = \frac{4\mathbb{P}(A_0)}{4\mathbb{P}(A_0) + 5\mathbb{P}(A_1) + 6\mathbb{P}(A_2)} \\ &= \frac{4 \cdot 48 \cdot 47}{4 \cdot 48 \cdot 47 + 5 \cdot 2 \cdot 4 \cdot 48 + 6 \cdot 4 \cdot 3} = \frac{376}{459} \\ &\approx 0.8192.\end{aligned}$$

**Exercise 1.48** Denote by $U_i$ the event "the $i$-th urn was picked" and by $B_k$ the event "a ball labelled $k$ was drawn".

(a) In this case $\mathbb{P}(U_i) = 1/2$, in both cases $i = 1, 2$. We have to compute $\mathbb{P}(U_1|B_5)$. We have

$$\mathbb{P}(U_1|B_5) = \frac{\mathbb{P}(B_5|U_1)\mathbb{P}(U_1)}{\mathbb{P}(B_5)} = \frac{\mathbb{P}(B_5|U_1)\mathbb{P}(U_1)}{\mathbb{P}(B_5|U_1)\mathbb{P}(U_1) + \mathbb{P}(B_5|U_2)\mathbb{P}(U_2)}$$

$$= \frac{\frac{1}{10} \cdot 0.5}{\frac{1}{10} \cdot 0.5 + \frac{1}{100} \cdot 0.5} = \frac{0.1}{0.11} = \frac{1}{11} \approx 0.909.$$

(b) In this case we have

$$\mathbb{P}(U_1) = \frac{10}{110}, \qquad \mathbb{P}(U_2) = \frac{100}{110}, \qquad \mathbb{P}(B_5) = \frac{2}{110}.$$

We deduce

$$\mathbb{P}(U_1|B_5) = \frac{\mathbb{P}(B_5|U_1)\mathbb{P}(U_1)}{\mathbb{P}(B_5)} = \frac{\frac{1}{10} \cdot \frac{10}{110}}{\frac{2}{110}} = \frac{1}{2}.$$

**Exercise 1.52** (i) Denote by $A$ the event that there are no accidents during the next successive $n$ days after an accident. Then

$$\mathbb{P}(A) = (1-p_1)(1-p_2)\cdots(1-p_n).$$

If you are so inclined, you can use the notation

$$\mathbb{P}(A) = \prod_{k=1}^{n}(1-p_k).$$

(ii) Denote by $B$ the event that there is exactly one accident during the next successive $n$ days after an accident. We have

$$\begin{aligned}\mathbb{P}(B) &= p_1(1-p_1)\cdots(1-p_{n-1}) + (1-p_1)p_2(1-p_1)\cdots(1-p_{n-2})\\ &\quad + \cdots + (1-p_1)\cdots(1-p_{n-1})p_n.\end{aligned}$$

Again, if you like that sort of things, you can write it as

$$\mathbb{P}(B) = \prod_{k=1}^{n}(1-p_k)\cdot\sum_{k=1}^{n}\frac{p_k}{1-p_k}.$$

## Exercise 1

Write $O_1^S$ for the event "OK is sent for the first chamber", $E_2^R$ for "Error is received for the second chamber", and similarly for all other combinations.

1. Because the chambers and transmissions are independent, the probability is the square of the probability
$$\mathbb{P}\big(O_1^R \big| O_1^S\big).$$
The answer must then be
$$\mathbb{P}\big(O_1^R \big| O_1^S\big)^2 = 0.99^2 = 0.9801.$$

2. (a)
$$\mathbb{P}\big(O_1^S \cap E_2^S\big) = \mathbb{P}\big(O_1^S\big)\cdot\mathbb{P}\big(E_2^S\big) = \frac{1}{5}\cdot\frac{4}{5} = \frac{4}{25}.$$

 (b) This is a classic application of the Bayes law:
$$\begin{aligned}\mathbb{P}\big(E_1^S \big| O_1^R\big) &= \frac{\mathbb{P}\big(O_1^R \big| E_1^S\big)\cdot\mathbb{P}\big(E_1^S\big)}{\mathbb{P}\big(O_1^R\big)} = \frac{\mathbb{P}\big(O_1^R \big| E_1^S\big)\cdot\mathbb{P}\big(E_1^S\big)}{\mathbb{P}\big(O_1^R \big| E_1^S\big)\cdot\mathbb{P}\big(E_1^S\big) + \mathbb{P}\big(O_1^R \big| O_1^S\big)\cdot\mathbb{P}\big(O_1^S\big)}\\ &= \frac{0.01\cdot 0.2}{0.01\cdot 0.2 + 0.99\cdot 0.8} = \frac{1}{397} \approx 0.265\%.\end{aligned}$$

 (c) We make full use of the independence of the two chambers:
$$\begin{aligned}\mathbb{P}\big(E_1^S \cup E_2^S \big| O_1^R \cap O_2^R\big) &= 1 - \mathbb{P}\big(O_1^S \cap O_2^S \big| O_1^R \cap O_2^R\big)\\ &= 1 - \mathbb{P}\big(O_1^S \big| O_1^R\big)^2\\ &= 1 - \Big(1 - \mathbb{P}\big(E_1^S \big| O_1^R\big)\Big)^2\\ &= 1 - \Big(1 - \frac{1}{397}\Big)^2 \approx 0.503\%.\end{aligned}$$

## Exercise 2

1. It is the first time there is no edge between two points: $\mathcal{G}eom(1-p)$.

   It doesn't matter if this doesn't count as a 'success' to us: if you want, you can imagine that you are playing against someone, and they are successful if you are stopped.

2. The event "we can go to infinity", call it $E_\infty$, is included in the event "$N$ is at least $n$", that we call $E_n$. As see in class, $N$ is at least $n$ if the first $n$ experiments are 'failures', in our case if the first $n$ edges are present, and this event has probability $p^n$. Hence, we have

   $$\mathbb{P}(E_\infty) \leq \mathbb{P}(E_n) = p^n.$$

   We can take the limit as $n$ goes to infinity in the inequality, and deduce that $\mathbb{P}(E_\infty) \leq 0$, so this probability has to be zero.

3. The event $E_\infty$ "we can go to infinity" is included in the union $E_+ \cup E_-$, where $E_\pm$ is the event "all edges from zero to $\pm\infty$ are present". According to the previous question, $E_-$ and $E_+$ have probability zero.

   We can then either use Homework 1, and say that

   $$\mathbb{P}(E_\infty) \leq \mathbb{P}(E_+) + \mathbb{P}(E_-) = 0 + 0 = 0;$$

   alternatively, we can use the inclusion/exclusion formula:

   $$\mathbb{P}(E_\infty) = \mathbb{P}(E_+) + \mathbb{P}(E_-) - \mathbb{P}(E_+ \cap E_-) = -\mathbb{P}(E_+ \cap E_-) \leq 0.$$

   In any case, the probability cannot be positive, so it must be zero.