Cheat sheet

Introduction to Probability

Last updated: January 29, 2020

Events, Probabilities

January 15th: Probability spaces

Def. Sample space S: set of all possible outcomes (ex: $S = \mathbb{R}$ for the temperature tomorrow).
Def. Event A: subset of A (ex: $A = [50, +\infty)$ for the temperature being too hot).
Logic connectors $\leftrightarrow$ set operations (ex: "A and B" $\leftrightarrow$ $A \cap B$; "not A" $\leftrightarrow$ $S \setminus A = A^{\complement}$).
Def. A and B disjoint: $A \cap B = \emptyset$ ($\leftrightarrow$ "A and B cannot occur simultaneously").
$A_1, A_2, \ldots, A_n, \ldots$ pairwise disjoint: A_n and A_m disjoint for $n \neq m$.
Def. Probability space $(S, \mathbb{P})$: sample space S and probability $\mathbb{P}(A)$ for every event $A \subset S$, with

- $0 \leq \mathbb{P}(A) \leq 1$,
- $\mathbb{P}\big(\bigcup_{n \geq 0} A_n\big) = \sum_{n \geq 0} \mathbb{P}(A_n)$ for $A_1, A_2, \ldots$ pairwise disjoints.

January 17th: Operations on events

Rk. For a finite sample set S, it suffices to know $\mathbb{P}(\{x\})$ for all $x \in S$.
Rk. Motto: the sample space is less important than the events.
Def. A is included in B: $A \subset B$ ($\leftrightarrow$ "A implies B").
Prop. $\mathbb{P}(A^{\complement}) = 1 - \mathbb{P}(A)$.
Prop. $\mathbb{P}(A \cup B) = 1 - \mathbb{P}(A^{\complement} \cap B^{\complement})$.
Prop. $\mathbb{P}(A \cap B) = 1 - \mathbb{P}(A^{\complement} \cup B^{\complement})$.
Prop. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.
Prop. $\mathbb{P}(A) \leq \mathbb{P}(B)$ whenever $A \subset B$.

January 22nd: Operations on events / Conditional probability

Prop(Inclusion-Exclusion Principle)**.**

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$$

Def. $A_0, A_1, A_2, \ldots$ increasing: $A_0 \subset A_1 \subset A_2 \subset \cdots$.

$A_0, A_1, A_2, \dots$ decreasing: $A_0 \supset A_1 \supset A_2 \supset \cdots$.

Prop. $\mathbb{P}(\bigcup_{n\geq 0} A_n) = \lim_{n\to\infty} \mathbb{P}(A_n)$ for $(A_n)_{n\geq 0}$ increasing;
$\mathbb{P}(\bigcap_{n\geq 0} A_n) = \lim_{n\to\infty} \mathbb{P}(A_n)$ for $(A_n)_{n\geq 0}$ decreasing.

Counting

Rk. *Tuples* are ordered sequences, *sets* are unordered sequences.
They are denoted by $(a_1, \dots, a_k)$ and $\{a_1, \dots, a_k\}$ respectively.

Def. Conditional probability $\mathbb{P}(A|B)$ of A knowing B: probability of A happening considering only outcomes where B holds.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Def. A and B independent: $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

January 24th: Product and uniform probability spaces

Def. Product set: $S_1 \times S_2 = \{(a,b),\ a \in S_1,\ b \in S_2\}$ ($\leftrightarrow$ experience S_1, then experience S_2).

Prop. For finite sets, $\#(A \times B) = (\#A) \times (\#B)$.

Def. Product probability space (for finite spaces): $S_1 \times S_2$ with $\mathbb{P}(\{(a,b)\}) = \mathbb{P}_1(\{a\})\mathbb{P}_2(\{b\})$
($\leftrightarrow$ experience S_1, then experience S_2 independently).

Def. Uniform probability (finite set): all outcomes equally likely, i.e. $\mathbb{P}(\{a\}) = \mathbb{P}(\{b\})$.

Prop. For $(S, \mathbb{P})$ uniform, $\mathbb{P}(A) = \frac{\#A}{\#S}$ for every event $A \subset S$.

Thm. The product of two uniform probability spaces is again uniform.

Ex. Sampling with replacement: $U = \{1, \dots, n\}$ an urn with n elements,
$S = U^k \leftrightarrow$ draw k elements with replacement.

January 27th: Sampling without replacement

Def. kth falling factorial: $(n)_k = n^{\underline{k}} = n \times (n-1) \times \cdots \times (n-k+1)$ (k terms).

Prop. If U has n elements, $\#\{k\text{-tuples of distinct elements of } U\} = (n)_k$.

Rk. The notation $U^{\underline{k}} = \{k\text{-tuples of distinct elements of } U\}$ is sometimes used.

January 29th: Permutations, combinations

Def. Permutations: $\mathfrak{S}(U) = \{\text{Permutations of } U\} = \{\text{Bijections } f : U \to U\}$.

Def. $\mathfrak{S}_n = \mathfrak{S}(\{1, \dots, n\})$ ($\leftrightarrow$ orderings of $\{1, \dots, n\}$ via $f(\text{element}) = \text{position}$).

Def. Factorial: $n! = n \times (n-1) \times \cdots \times 1$.

Prop. If U has n elements, $\#\mathfrak{S}(U) = n!$; e.g. $\#\mathfrak{S}_n = n!$.

Def. Binomial coefficient "n choose k": $\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}$.

Prop. If U has n elements, $\#\{\text{subsets of } U \text{ with } k \text{ elements}\} = \binom{n}{k}$.

Rk. The notation $\binom{U}{k} = \{\text{subsets of } U \text{ with } k \text{ elements}\}$ is sometimes used.