Homework 7
Solution

April 3rd

Exercises from the book

2.37, and 2.45

Exercise 2.31 Note that setting $x = \sin(y)$, we have for $-1 \leq b \leq 1$

$$\int_{-1}^{b} \frac{c}{\sqrt{1-x^2}}\, dx = \int_{-\frac{\pi}{2}}^{\arcsin(b)} \frac{c}{\sqrt{1-\sin(y)^2}} \cos(y)\, dy = c\left(\arcsin(b) + \frac{\pi}{2}\right)$$

We compute

$$\int_{-1}^{1} \frac{c}{\sqrt{1-x^2}}\, dx = c\pi$$

so $c = 1/\pi$.

The distribution function is given by

$$F(t) = \begin{cases} \int_{-1}^{t} \frac{c}{\sqrt{1-x^2}}\, dx = \frac{1}{\pi}\arcsin(t) + \frac{1}{2} & -1 < t < 1, \\ 0 & t \leq -1, \\ 1 & t \geq 1. \end{cases}$$

Exercise 2.32 Since the density function $f(x)$ is odd we see that $\mathbb{E}[X] = 0$.

$$\mathrm{Var}(X) = \mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{0}^{\infty} x^2 e^{-x}\, dx$$

since $x^2 f(x)$ is even. Integrating by parts twice gives $\mathrm{Var}(X) = 2$. Alternatively, we can also notice that this last integral is the second moment of an exponential distribution of parameter 1; for $Y \sim \mathcal{E}xp(1)$,

$$\mathrm{Var}(X) = \int_{0}^{\infty} x^2 e^{-x}\, dx = \mathbb{E}[Y^2] = \mathrm{Var}(Y) + \mathbb{E}[Y]^2 = 1 + 1^2 = 2.$$

Exercise 2.34 Denote by X the random location where you break the stick, and by $L = L(X)$ the length of the longest of the two resulting segments. Then

$$L(X) = \begin{cases} 1 - X, & X \leq \frac{1}{2}, \\ X, & X > \frac{1}{2}. \end{cases}$$

We have

$$\mathbb{E}[L] = \int_0^{\frac{1}{2}} (1-x)dx + \int_{\frac{1}{2}}^1 x dx = -\frac{(1-x)^2}{2}\Bigg|_0^{1/2} + \frac{x^2}{2}\Bigg|_{1/2}^1 = \frac{3}{4}.$$

Exercise 2.35 (a) $T \sim \mathcal{E}xp(\lambda)$, $10 = \mu = \mathbb{E}[T] = \frac{1}{\lambda}$ so that $\lambda = 0.1$

$$\mathrm{Var}(T) = \mu^2 = 100.$$

(b)

$$\mathbb{P}(T \leq 5) = 10 \int_0^5 e^{-0.1t} dt = 1 - e^{-0.5}.$$

(c)

$$\mathbb{P}(T \leq 30 | T > 25) = 1 - \mathbb{P}(T > 30 | T > 25) = 1 - \mathbb{P}(T > 5) = \mathbb{P}(T \leq 5).$$

(d)

$$\mathbb{P}(T > \mathbb{E}[T]) = \mathbb{P}(T > 10) = e^{-10\lambda} = e^{-1}.$$

Exercise 2.36 Suppose that the lifetime T of a light bulb is exponentially distributed, so $T \sim \mathcal{E}xp(\lambda)$. The probability that that a bult will last more that a year is $\mathbb{P}(T > 1) = e^{-\lambda}$ so we must have

$$0.8 \approx e^{-\lambda}.$$

The probability that a light bulb will last more than two years must be $e^{-2\lambda}$ so we must have

$$0.3 \approx e^{-2\lambda} = (e^{-\lambda})^2 \approx (0.8)^2.$$

This is clearly not the case so the lifetime is not exponentially distributed.

Exercise 2.37 We have

$$\mathbb{P}(\lambda X \leq x) = \mathbb{P}(X \leq x/\lambda) = \lambda \int_0^{x/\lambda} e^{-\lambda x} dx = 1 - e^{-\lambda \frac{x}{\lambda}} = 1 - e^{-x}.$$

Thus, λX has the same cumulative distribution function as $\mathcal{E}xp(1)$.

Exercise 2.45 Let $Y = X^2$. Set $F_Y(y) := \mathbb{P}(Y \leq y)$. Note that $F_Y(y) = 0$ for $y < 0$. We have

$$F_Y(y) = \mathbb{P}(X^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$$

for Φ the cumulative distribution function of X. The probability density function of Y is $F'_Y(y)$ and, for $y > 0$, we have

$$\begin{aligned} F'_Y(y) &= \frac{1}{2\sqrt{y}}\Big(\Phi'(\sqrt{y}) + \Phi'(-\sqrt{y})\Big) \\ &= \frac{1}{2\sqrt{2\pi y}}\Big(\exp\big(-\sqrt{y}^2/2\big) + \exp\big(-(-\sqrt{y})^2/2\big)\Big) \\ &= \frac{1}{\sqrt{2\pi}} \cdot y^{1/2-1}e^{-y/2}. \end{aligned}$$

In particular, Y is in fact a gamma distribution with parameters $(\frac{1}{2}, \frac{1}{2})$.

Exercise 1

Write Y for the highest bid of the other contestants, so $Y \sim \mathcal{U}nif([70, 130])$.

1. If the bid x is at most 70, the probability is zero, and it is one if x is at least 130.
   If $70 < x < 130$, the probability of winning is

   $$\mathbb{P}(\text{winning}) = \mathbb{P}(Y < x) = \int_{-\infty}^{x} p_Y(y)\mathrm{d}y = \int_{70}^{x} \frac{1}{130-70}\mathrm{d}y = \frac{x-70}{60}.$$

   All in all, we get

   $$\mathbb{P}(\text{winning}) = \begin{cases} 0 & \text{for } x \leq 70, \\ \frac{x-70}{60} & \text{for } 70 < x < 130, \\ 1 & \text{for } x \geq 130. \end{cases}$$

2. Write X for the gain. For x fixed, X can take at most two values: 0 or $100 - x$. The expectation is then

   $$\mathbb{E}[X] = 0 \cdot \mathbb{P}(\text{losing}) + (100-x) \cdot \mathbb{P}(\text{winning}) = \begin{cases} 0 & \text{for } x \leq 70 \\ \frac{(x-70)(100-x)}{60} & \text{for } 70 < x < 130 \\ 100 - x & \text{for } x \geq 130. \end{cases}$$

3. For $x \geq 130$, the gain is negative. We have to study $(x-70)(100-x)$. For instance, we can see that

   $$(x-70)(100-x) = 225 - (x-85)^2,$$

   so that $(x-70)(100-x)$ is at most 225, and it the maximum is reached at $x = 85$. Since $100 - 225/60 = 385/4$ is positive, it is the maximal possible expectation, and the most profitable bid in average is 85.

Exercise 2

1. A function p is the probability density function of a continuous variable if and only if p is nonnegative and its integral is equal to one. So we need a and b to be nonnegative, and

$$1 = \int_{-\infty}^{+\infty} p_X(x)\mathrm{d}x = \int_{-1}^{1} a\mathrm{d}x + \int_{0}^{1} b\mathrm{d}x = a + b.$$

2. Let us compute the first two moments of X.

$$\mathbb{E}[X] = \int_{-1}^{0} ax \,\mathrm{d}x + \int_{0}^{1} bx \,\mathrm{d}x = \left.\frac{ax^2}{2}\right|_{-1}^{0} + \left.\frac{bx^2}{2}\right|_{0}^{1} = \frac{b-a}{2}$$

$$\mathbb{E}[X^2] = \int_{-1}^{0} ax^2 \,\mathrm{d}x + \int_{0}^{1} bx^2 \,\mathrm{d}x = \left.\frac{ax^3}{3}\right|_{-1}^{0} + \left.\frac{bx^3}{3}\right|_{0}^{1} = \frac{a+b}{3}$$

Hence, the variance of X is

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{4(a+b) - 3(b-a)^2}{12}.$$

Remember that $a + b = 1$, so the variance is minimal when $|b - a|$ is maximal, which means that (a, b) equals either $(0, 1)$ or $(1, 0)$.