# Homework 10
# Solution

April 30th

## Exercises from the book

### Exercise 2.35

(i) $T \sim \mathcal{E}xp(\lambda)$, $10 = \mu = \mathbb{E}[T] = \frac{1}{\lambda}$ so that $\lambda = 0.1$

$$\mathrm{Var}(T) = \mu^2 = 100.$$

(ii)

$$\mathbb{P}(T \leq 5) = 10 \int_0^5 e^{-0.1t} dt = 1 - e^{-0.5}.$$

(iii)

$$\mathbb{P}(T \leq 30 | T > 25) = 1 - \mathbb{P}(T > 30 | T > 25) = 1 - \mathbb{P}(T > 5) = \mathbb{P}(T \leq 5).$$

(iv)

$$\mathbb{P}(T > \mathbb{E}[T]) = \mathbb{P}(T > 10) = e^{-10\lambda} = e^{-1}.$$

**Exercise 2.42** We compute the cumulative distribution for $Y$, and deduce its density by taking the derivative. It will help us to keep in mind that the inverse of the logarithm is the exponential.

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\ln(X) \leq y) = \mathbb{P}(X \leq e^y) = F_X(e^y).$$

Since $F_X(x) = 1 - e^{-3x}$ for all $x > 0$, we get that

$$F_Y(y) = 1 - \exp(-3e^y)$$

and

$$p_Y(y) = 3e^y \cdot \exp(-3e^y) = 3\exp(-3e^y + y).$$

**Exercise 2.46** Again, we first compute the cumulative distribution for $Y$. Setting $f(x)$ such that $Y = f(X)$, we see that for $0 < y < 1$,

$$f(x) \le y \Leftrightarrow (x \le y \text{ and } x \le 1) \text{ or } (1/x \le y \text{ and } x > 1) \Leftrightarrow x \le y \text{ or } x \ge 1/y.$$

Then for $0 < y < 1$ we have

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(\{X \le y\} \cup \{X \ge 1/y\}) = \int_0^y e^{-x}\,dx + \int_{1/y}^{\infty} e^{-x}\,dx = 1 - e^{-y} + e^{-1/y}.$$

Differentiating, for $0 < y < 1$ the density function is $p_Y(y) = e^{-y} + \frac{1}{y^2}e^{-1/y}$. Since $f(x)$ has values in $[0,1]$ for $x \ge 0$, $Y$ has values in $[0,1]$ and $F_y(y) = 0$ out of $(0,1)$.

**Exercise 2.48**

(i)

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{c_n}^{\infty} \frac{c_n}{x^{n+1}}\,dx = \frac{1}{nc_n^{n-1}} = 1$$

since $f(x)$ is the density of a random variable. Therefore, if $n > 1$

$$c_n = \frac{1}{n^{1/(n-1)}}$$

while if $n = 1$, then $c_n$ could be anything.

(ii)

$$\mathbb{E}[X] = \int_{c_n}^{\infty} \frac{c_n}{x^n}\,dx.$$

This diverges if $n = 1$ but for $n > 1$ gives

$$\frac{1}{n-1} \cdot n^{(n-2)/(n-1)} = \frac{n}{(n-1)n^{1/(n-1)}}.$$

(iii) The random variable $Z_n$ has range $[\ln c_n, \infty)$. For $z$ in this range we have

$$F_{Z_n}(z) = \mathbb{P}(Z_n \le z) = \mathbb{P}(\ln(X_n) \le z) = \mathbb{P}(X_n \le e^z) = \int_{c_n}^{e^z} \frac{c_n}{x^{n+1}}\,dx = \frac{1}{nc_n^{n-1}} - \frac{c_n}{ne^{nz}}.$$

The density is then the derivative of the function above:

$$p_{Z_n}(z) = \begin{cases} c_n e^{-nz} & \text{for } z \ge \ln(c_n), \\ 0 & \text{else.} \end{cases}$$

(iv)

$$\mathbb{E}[X_n^{m+1}] = \int_{c_n}^{\infty} x^{m+1} \frac{c_n}{x^{n+1}}\,dx = \int_{c_n}^{\infty} \frac{c_n}{x^{n-m}}\,dx.$$

This is convergent only of $m < n - 1$.

**Exercise 3.5** The joint probability mass function of the random vector $(X, Y)$ is

$$\mathbb{P}(X = b, Y = 2b) = \mathbb{P}(X = 2b, X = b) = \frac{1}{2}, \qquad \mathbb{P}(X = b, Y = b) = \mathbb{P}(X = 2b, Y = 2b) = 0.$$

This shows that $X$ and $Y$ have identical distributions

$$p_X(b) = p_Y(b) = \frac{1}{2} = p_X(2b) = p_Y(2b).$$

(i) The conclusion follows from the above equalities. Indeed, $X, Y$ have identical distributions so

$$\mathbb{E}[X] = \mathbb{E}[Y] = \frac{1}{2}b + \frac{2b}{2} = \frac{3b}{2}.$$

(ii) Using the law of the subconscious statistician we deduce

$$\mathbb{E}\left[\frac{Y}{X}\right] = \sum_{x,y} \frac{y}{x}\mathbb{P}(X = x, Y = y) = \frac{1}{2}\frac{2b}{b} + \frac{1}{2}\frac{b}{2b} = 1 + \frac{1}{4} = \frac{5}{4}.$$

(iii) We have

$$\mathbb{E}[Z] = b\mathbb{P}(Z = b) + 2b\mathbb{P}(Z = 2b).$$

From the law of total probability we deduce

$$\mathbb{P}(Z = b) = \underbrace{\mathbb{P}(Z = b|X = b)}_{=1-p(b)}\mathbb{P}(X = b) + \underbrace{\mathbb{P}(Z = b|X = 2b)}_{=p(2b)}\mathbb{P}(X = 2b) = \frac{1}{2}\big(1 - p(b) + p(2b)\big).$$

Similarly

$$\mathbb{P}(Z = 2b) = \underbrace{\mathbb{P}(Z = 2b|X = b)}_{=p(b)}\mathbb{P}(X = b) + \underbrace{\mathbb{P}(Z = 2b|X = 2b)}_{=1-p(2b)}\mathbb{P}(X = 2b) = \frac{1}{2}\big(1 - p(2b) + p(b)\big).$$

Hence

$$\begin{aligned}\mathbb{E}[Z] &= \frac{b}{2}\Big(\big(1 - p(b) + p(2b)\big) + 2\big(1 - p(2b) + p(b)\big)\Big)\\ &= \frac{b}{2}\big(3 + p(b) - p(2b)\big) = \frac{3b}{2} + \frac{b}{2}\underbrace{\left(\frac{1}{1 + e^{2b}} - \frac{1}{1 + e^{4b}}\right)}_{>0} > \frac{3b}{2} = \mathbb{E}[X].\end{aligned}$$

**Exercise 3.6** As suggested, call $X_1, \dots, X_4$ the variables defined by $X_i = 1$ if the $i$the component works, $X_i = 0$ otherwise. This means that the $X_i$ are independent Bernoulli variables; more precisely, $X_1 \sim \mathcal{B}er(0.9)$, $X_2 \sim \mathcal{B}er(0.8)$, $X_3 \sim \mathcal{B}er(0.6)$, $X_4 \sim \mathcal{B}er(0.6)$.

(i)

$$\mathbb{E}[X] = \mathbb{E}[X_1 + \cdots + X_4] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_4] = 0.9 + 0.8 + 0.6 + 0.6 = 2.9$$

(ii) Because the variables are independent,

$$\mathrm{Var}(X) = \mathrm{Var}(X_1+\cdots+X_4) = \mathrm{Var}(X_1)+\cdots+\mathrm{Var}(X_4) = 0.9\cdot0.1+0.8\cdot0.2+0.6\cdot0.4+0.6\cdot0.4 = 0.73$$

(iii)

$$\mathbb{P}(X > 0) = 1-\mathbb{P}(X = 0) = 1-\mathbb{P}(X_1 = 0)\times\cdots\times\mathbb{P}(X_4 = 0) = 1-0.9\cdot0.8\cdot0.6\cdot0.6 = 0.7408.$$

(iv) As far as I know, there is no clever trick for this one as there was for the ones before.

$$\begin{aligned}\mathbb{P}(X = 1) &= \mathbb{P}(\text{The first component is the only one working}) \\ &\qquad + \cdots + \mathbb{P}(\text{The fourth component is the only one working}) \\ &= 0.9\cdot0.2\cdot0.4\cdot0.4+0.1\cdot0.8\cdot0.4\cdot0.4+0.1\cdot0.2\cdot0.6\cdot0.4+0.1\cdot0.2\cdot0.4\cdot0.6 \\ &= 0.0512\end{aligned}$$

## Exercise 1

Write $Y$ for the maximum of $X_1$ to $X_n$. It has to be between 0 and 1, so we already know that $p_Y(y) = 0$ for $y \le 0$ or $y \ge 1$. We choose $0 < y < 1$ and compute the cumulative distribution function.

$$\begin{aligned}F_Y(y) &= \mathbb{P}(Y \le y) = \mathbb{P}(\max(X_1,\ldots,X_n) \le y) \\ &= \mathbb{P}(X_1 \le y \text{ and } \cdots \text{ and } X_n \le y) = \mathbb{P}(X_1 \le y)\times\cdots\times\mathbb{P}(X_n \le y) = F_X(y)^n\end{aligned}$$

where $F_X(x)$ is the cumulative distribution function of one of the $X_i$ (they all have the same one). Since they are uniform between 0 and 1, $F_X(x) = x$ for $0 < x < 1$, so for $0 < y < 1$ we have

$$F_Y(y) = y^n.$$

This means

$$p_Y(y) = \begin{cases} F_Y'(y) = ny^{n-1} & \text{for } 0 < y < 1 \\ 0 & \text{else}\end{cases}$$

This is a Beta distribution $\mathcal{B}eta(n,1)$.

### Exercise 2

1. We must have $0 \le X < Y$.

2. Fix $x$ and $y$ such that $0 \le x < y$. Then

$$\begin{aligned}\mathbb{P}(X = x, Y = y) &= \mathbb{P}\big(\text{We do } y-1 \text{ throws giving 1,2,3,4 or 5, within which } x \text{ 1's, then a 6.}\big) \\ &= \frac{\#\{\text{possible positions for the 1's}\}\cdot\#\{\text{possible outcomes for the remaining throws}\}}{6^y} \\ &= \frac{\binom{y-1}{x}\cdot 4^{y-1-x}}{6^y}\end{aligned}$$