# Homework 3
# Solution

February 14th

## Exercises from the book

**Exercise 1.20** Because $A$, $B$ and $C$ are independent, we have

$$\mathbb{P}(A \cap (B \cap C)) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \mathbb{P}(A)\mathbb{P}(B \cap C),$$

which is the definition of $A$ and $B \cap C$ being independent.

Regarding $A$ and $B \cup C$, note that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C);$$

indeed,

$$\begin{aligned} x \in A \cap (B \cup C) &\Leftrightarrow x \text{ belongs to } A\text{, and also to either } B \text{ or } C \\ &\Leftrightarrow x \text{ belongs to either } A \text{ and } B\text{, or } A \text{ and } C \\ &\Leftrightarrow x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

(You may just write the equality without justification.)

Then, using the inclusion/exclusion principle, we get

$$\begin{aligned} \mathbb{P}\big(A \cap (B \cup C)\big) &= \mathbb{P}\big(\,(A \cap B) \cup (A \cap C)\,\big) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) - \mathbb{P}(A \cap B \cap C) \\ &= \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \mathbb{P}(A)\big(\,\mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B)\mathbb{P}(C)\,\big) \\ &= \mathbb{P}(A)\big(\,\mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C)\,\big) = \mathbb{P}(A)\mathbb{P}(B \cup C), \end{aligned}$$

as expected.

**Exercise 1.21** If $p = 1$ or $p = 0$, then $\mathbb{P}(B)$ is one, so $A$ and $B$ are independent.

Assume then that $p \in (0, 1)$. The probability of $A$ is the sum of (1) the probability of no tails, and (2) the probability of exactly one tail. We set $q = 1 - p$ and we deduce

$$\mathbb{P}(A) = p^3 + 3qp^2 = p^2(p + 3q) = p^2(3 - 2p).$$

Next

$$\mathbb{P}(B) = q^3 + p^3 = (1-p)^3 + p^3 = 1 - 3p + 3p^2, \qquad \mathbb{P}(A \cap B) = p^3.$$

The events $A$ and $B$ are independent if and only if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, i.e.

$$p^3 = (1 - 3p + 3p^2)p^2(3 - 2p),$$

$$p = (1 - 3p + 3p^2)(3 - 2p) = (3 - 9p + 9p^2) - (2p - 6p^2 + 6p^3) = -6p^3 + 15p^2 - 11p + 3.$$

We want to solve the equation

$$-6p^3 + 15p^2 - 12p + 3 = 0 \Leftrightarrow 2p^3 - 5p^2 + 4p - 1 = 0,$$

hence we look for the roots of the polynomial on the right hand side. Since it is a third degree polynomial, the general technique is tedious, and we want instead to find a simple root.

A good strategy in general is to remember that if $A$ or $B$ have probability zero or one, then they are automatically independent. In this case, $A$ has probability zero if $p = 1$, and indeed, $p = 1$ is a root of the polynomial:

$$2p^3 - 5p^2 + 4p - 1 = (p - 1)(2p^2 - 3p + 1).$$

There are many ways to finish the computations; I chose to use the fact that $p = 1$ is again a root of $2p^2 - 3p + 1$, so

$$2p^3 - 5p^2 + 4p - 1 = 0 \Leftrightarrow (p - 1)(2p^2 - 3p + 1) = 0 \Leftrightarrow (p - 1)^2(2p - 1) = 0.$$

The only solution $p$ of the equation $(2p - 1)(p - 1)^2 = 0$ in the open interval $(0, 1)$ is $p = \frac{1}{2}$. Thus, the events $A$ and $B$ are independent if and only if $p = 0.5$, i.e. the coin is fair.

**Exercise 1.22** For $i = 1, 2$ denote by $E_i$ the event "*the politician wins the* $i$*-th election*". We know that

$$\mathbb{P}(E_1) = 0.6, \;\; \mathbb{P}(E_2) = 0.5, \;\; \mathbb{P}(E_2|E_1) = 0.75.$$

(i)

$$\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_2|E_1)\mathbb{P}(E_1) = 0.75 \cdot 0.6 = 0.45.$$

(ii)

$$\mathbb{P}(E_1 \cap E_2^c) = \mathbb{P}(E_2^c|E_1)\mathbb{P}(E_1) = 0.25 \cdot 0.6 = 0.15.$$

(iii)

$$\mathbb{P}(E_1|E_2) = \frac{\mathbb{P}(E_1 \cap E_2)}{\mathbb{P}(E_2)} = \frac{0.45}{0.5} = 0.9.$$

(iv)

$$\mathbb{P}(E_2|E_1^c) = \frac{\mathbb{P}(E_1^c|E_2)\mathbb{P}(E_2)}{\mathbb{P}(E_1^c)} = \frac{\big(1 - \mathbb{P}(E_1|E_2)\big)\mathbb{P}(E_2)}{\mathbb{P}(E_1^c)} = \frac{0.1 \cdot 0.5}{0.4} = 0.125.$$

**Exercise 1.25** We have

$$\mathbb{P}(A)=\frac{1}{2},\ \ \mathbb{P}(B)=\frac{5}{6},\ \ \mathbb{P}(A\cap B)=\frac{1}{2}$$

$$\mathbb{P}(A|B)=\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}=\frac{\frac{1}{2}}{\frac{5}{6}}=\frac{6}{10},$$

$$\mathbb{P}(B|A)=\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)}=\frac{\frac{1}{2}}{\frac{1}{2}}=1.$$

**Exercise 1.26** We have

$$\mathbb{P}(G|T)\mathbb{P}(T)=\mathbb{P}(G\cap T)=\mathbb{P}(T|G)\mathbb{P}(G).$$

Hence

$$\frac{\mathbb{P}(G|T)}{\mathbb{P}(T|G)}=\frac{\mathbb{P}(G)}{\mathbb{P}(T)}.$$

Thus

$$\mathbb{P}(G|T)=\mathbb{P}(T|G)\ \Leftrightarrow\ \mathbb{P}(G)=\mathbb{P}(T).$$

**Exercise 1.27** Let $U_k$ denote the event "we pick the $k$th urn" and $G$ the event "the second ball we sample is green". Then

$$\mathbb{P}(G)=\mathbb{P}(G|U_1)\underbrace{\mathbb{P}(U_1)}_{1/20}+\mathbb{P}(G|U_2)\underbrace{\mathbb{P}(U_2)}_{1/20}+\cdots+\mathbb{P}(G|U_{20})\underbrace{\mathbb{P}(U_{20})}_{1/20}$$

$$=\frac{1}{20}\Big(\mathbb{P}(G|U_1)+\cdots+\mathbb{P}(G|U_{20})\Big).$$

For $k=1,2,\dots,n$, the $k$th urn contains $20-k$ green balls and $k-1$ red balls. Thus, when we take two samples without replacement from the $k$th urn the probability that the second ball is green is

$$\frac{(k-1)(20-k)+(20-k)(19-k)}{19\cdot 18}=\frac{(20-k)18}{19\cdot 18}=\frac{20-k}{19}.$$

Note that this is in accordance with the result we have seen in class, that sampling two balls and looking at the colour of the second, disregarding the first, will not change the probabilities when compared to just sampling one ball.

We deduce

$$\mathbb{P}(G)=\frac{1}{20}\cdot\frac{19+18+\cdots+1+0}{19}=\frac{\frac{19\cdot 20}{2}}{20\cdot 19}=\frac{1}{2}.$$

**Exercise 1.29** (a) The two rolls are independent events. If the first roll is 1, then the largest number will be equal to the number we get at the second roll. The probability that this number is 3 is 1 in 6.

(b) The probability that the largest number is 3 is equal to the probability that the second roll gives a number $\leq 3$. This probability is $\frac{1}{2}$.

**Exercise 1.30** Write $A'_k$ for the probability that the second die rolls on a $k$. We have $\mathbb{P}(A_k) = \frac{1}{6}$, hence

$$\mathbb{P}(B_n \cap A_k) = \mathbb{P}(A_k \cap A'_{n-k}) = (A'_{n-k}) \cdot \mathbb{P}(A_k) = \frac{1}{6}\mathbb{P}(A_{n-k}),$$

$$\mathbb{P}(B_n)\mathbb{P}(A_k) = \frac{1}{6}\mathbb{P}(B_n).$$

Thus, the events $A_k$ and $B_n$ are independent if

$$\mathbb{P}(A_{n-k}) = \mathbb{P}(B_n).$$

Observe that if $n \leq k$, then $\mathbb{P}(A_{n-k}) = 0$ so the events $B_n$ and $A_k$ are not independent. If $n > k$, then $B_n$ is independent of $A_k$ if

$$\mathbb{P}(B_n) = \mathbb{P}(A_{n-k}) = \frac{1}{6}.$$

This is possible only if $n = 7$. Thus $A_k$ is independent of $B_n$ if and only $n = 7$, for any $1 \leq k \leq 6$.

## Exercise 1

It is not possible for the dog to eat the couch right away. So the probability that your dog ate the couch is the sum of the probability that (1) it ate it as a second activity, or (2) it ate it as a third activity, but did not as a second activity (note that these events are disjoints). Moreover, if it ate the couch at some step, then it must have watched out the window just before; the total probability is in fact the sum of the probability that (1) it watched out the window right away then ate the couch directly after, and (2) it did some activity, then watched out the window, then ate the couch. There are only two activities that can both be a first activity and lead to watching out the window: having a nap, or playing. Hence (2) decomposes as (2a) have a nap then watch out the window the eat the couch, and (2b) play then watch out the window the eat the couch.

Let us write $A_1$, $A_2$ and $A_3$ for the first, second and third activities. What we just discussed rewrites as

$$\begin{aligned}\mathbb{P}(\text{eventually ate the couch}) &= \mathbb{P}(A_1 = \text{window} \cap A_2 = \text{couch})\\ &\quad + \mathbb{P}(A_1 = \text{nap} \cap A_2 = \text{window} \cap A_3 = \text{couch})\\ &\quad + \mathbb{P}(A_1 = \text{play} \cap A_2 = \text{window} \cap A_3 = \text{couch}).\end{aligned}$$

The first probability is

$$\mathbb{P}(A_1 = \text{window} \cap A_2 = \text{couch}) = \mathbb{P}(A_2 = \text{couch} \mid A_1 = \text{window})\mathbb{P}(A_1 = \text{window}) = \frac{1}{10} \cdot \frac{1}{3} = \frac{1}{30}.$$

Similarly,

$$\begin{aligned}&\mathbb{P}(A_1 = \text{nap} \cap A_2 = \text{window} \cap A_3 = \text{couch})\\ &\qquad = \mathbb{P}(A_3 = \text{couch} \mid A_1 = \text{nap} \cap A_2 = \text{window}) \cdot \mathbb{P}(A_1 = \text{nap} \cap A_2 = \text{window})\\ &\qquad = \mathbb{P}(A_3 = \text{couch} \mid A_2 = \text{window}) \cdot \mathbb{P}(A_2 = \text{window} \mid A_1 = \text{nap}) \cdot \mathbb{P}(A_1 = \text{nap})\\ &\qquad = \frac{1}{10} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{90}\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}(A_1 = \text{play} \cap A_2 = \text{window} \cap A_3 = \text{couch}) \\
&\qquad = \mathbb{P}(A_3 = \text{couch} \,|\, A_1 = \text{play} \cap A_2 = \text{window}) \cdot \mathbb{P}(A_1 = \text{nap} \cap A_2 = \text{window}) \\
&\qquad = \mathbb{P}(A_3 = \text{couch} \,|\, A_2 = \text{window}) \cdot \mathbb{P}(A_2 = \text{window} \,|\, A_1 = \text{play}) \cdot \mathbb{P}(A_1 = \text{play}) \\
&\qquad = \frac{1}{10} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{90}
\end{aligned}$$

We find

$$\mathbb{P}(\text{eventually ate the couch}) = \frac{1}{30} + \frac{1}{90} + \frac{1}{90} = \frac{1}{18}.$$