MATH 30350
Introduction to Probability

Name: Solution

February 4th

Quiz 1

The duration of this test is 15 minutes. Collaboration is prohibited, as are documents (textbook, personal notes) and calculators.

Partial credit will be given for incomplete yet relevant attempts.

1. Out of an urn with balls labelled from 1 to n, one draws k balls without replacement. Using only factorials, what is the number of possible outcomes, assuming they are unordered?

Choose k unordered elements out of n: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

2. A spaceship radio has broken down and is sending out a random sequence of 5 letters on loop. Assuming the random sequence is uniform, what is (a formula for) the probability that the sequence contains the letters "SOS" in order?

   (For instance, "SOSST" contains "SOS", but "VSOAS" and "SJJSO" do not.)

- 26^5 possible sequences of (ordered) letters
- A sequence containing "SOS" is described by:
  - the position of "SOS": S O S * * (3 possibilities)
    * S O S *
    * * S O S
  - the two remaining letters (ordered) (26^2 possibilities)
    ↪ they might be the same!
- Note that SOSOS has been counted twice!

$$\mathbb{P}(\text{"SOS" appears}) = \frac{3 \cdot 26^2 - 1}{26^5} \quad (\simeq 0.02\%)$$

3. In a class of 20 students, a group of 5 is selected uniformly at random to work on a gardening project.

(a) You don't mind participating, but only if you are with your buddy Mike. What is (a formula for) the probability that you will be disappointed?

Many approaches are available, some of you have had good original ideas! Here is my approach, with ordered and unordered versions.

$\mathbb{P}(\text{disappointed}) = \mathbb{P}(\text{you are selected but not Mike})$.

Ordered:

There are $(20)_5$ possible ordered groups of five people. A group containing you but not Mike is described by:
- your position in the group (5 possibilities)
- the other 4 students, who cannot be you nor Mike, and must be ordered ($(18)_4$ possibilities).

$\mathbb{P}(\text{disappointed}) = \frac{5 \cdot (18)_4}{(20)_5} = \frac{5 \cdot 15}{20 \cdot 19} (\approx 20\%)$

Unordered:

There are $\binom{20}{5}$ possible unordered groups of 5 people. A group containing you but not Mike is described by the remaining 4 students, who cannot be you nor Mike ($\binom{18}{4}$ possibilities).

$\mathbb{P}(\text{disappointed}) = \frac{\binom{18}{4}}{\binom{20}{5}} = \frac{5 \cdot 15}{20 \cdot 19}$

(b) Your pals Mike and Sarah have bad blood since that night at O'Malley's. Maybe working on something together will have them patch it up? **Knowing that you and Mike have been selected,** what is (a formula for) the probability that Sarah has been selected as well?

$= \frac{1}{6} \approx 17\%$, same for unordered

$\mathbb{P}(\text{Sarah is selected} \mid \text{you and Mike are selected})$

$= \frac{\mathbb{P}(\text{Sarah, you and Mike are selected})}{\mathbb{P}(\text{you and Mike are selected})} \overset{\text{(see below)}}{=} \frac{\frac{(5)_3 \cdot (17)_2}{(20)_5}}{\frac{(5)_2 \cdot (18)_3}{(20)_5}} = \frac{(5)_3 \cdot (17)_2}{(5)_2 \cdot (18)_3} = \frac{3}{18}$

Ordered:

$(20)_5$ possible groups of 5 people (ordered).

A group containing you and Mike is described by:
- your position in the group (5 possibilities)
- the position of Mike in the group (4 possibilities)

(same as the ordered positions of you and Mike together ($(5)_2$ possibilities)
- the other (ordered) 3 students, who cannot be you nor Mike ($(18)_3$ possibilities)

$\mathbb{P}(\text{you and Mike selected}) = \frac{5 \cdot 4 \cdot (18)_3}{(20)_5} = \frac{(5)_2 \cdot (18)_3}{(20)_5}$

For the same reason, $\mathbb{P}(\text{Sarah, you and Mike selected}) = \frac{5 \cdot 4 \cdot 3 \cdot (17)_2}{(20)_5} = \frac{(5)_3 \cdot (17)_2}{(20)_5}$

Unordered:

$\binom{20}{5}$ possible groups of 5 people (unordered).

A group containing you and Mike is described by the remaining 3 students, who cannot be you nor Mike ($\binom{18}{3}$ possibilities).

$\mathbb{P}(\text{you and Mike selected}) = \frac{\binom{18}{3}}{\binom{20}{5}}$.

For the same reason, $\mathbb{P}(\text{Sarah, you and Mike selected}) = \frac{\binom{17}{2}}{\binom{20}{5}}$