Homework 6
Solution

March 27th

Exercises from the book

Exercise 2.12 Let L denote the number of lids you need to lift until you find the hidden object. We have

$$\mathbb{P}(L=1)=\frac{1}{10},\ \ \mathbb{P}(L=2)=\frac{9}{10}\cdot\frac{1}{9}=\frac{1}{10},\ \ \mathbb{P}(L=3)=\frac{9}{10}\cdot\frac{8}{9}\cdot\frac{1}{8}=\frac{1}{10},$$

and similarly

$$\mathbb{P}(L=k)=\frac{1}{10},\ \ \forall k=1,2,\dots,10,$$

so $L\sim\mathcal{U}nif(1,\dots,10)$. Hence

$$\mathbb{E}[L]=\frac{1+10}{2}=5.5.$$

Exercise 2.14 If $X\sim\mathcal{B}in(n,p)$, then

$$\mathbb{E}[X]=np,\ \ \mathrm{Var}(X)=np(1-p)\Rightarrow 1-p=\frac{\mathrm{Var}(X)}{\mathbb{E}[X]}=\frac{0.9}{1}=0.9\Rightarrow p=0.1.$$

$$n=\frac{\mathbb{E}[X]}{p}=10.$$

$$\mathbb{P}(X>0)=1-\mathbb{P}(X=0)=1-(0.9)^{10}\approx 0.6513.$$

Exercise 2.16 We have $X\sim\mathcal{B}in(n=6,p=\frac{1}{6})$, $Y\sim\mathcal{B}in(n=12,p=\frac{1}{6})$. We have

$$\mathbb{E}[X]=1,\ \ \mathbb{E}[Y]=2$$

$$\mathbb{P}(X>1)=1-\mathbb{P}(X\le 1)\approx 0.2632.$$

$$\mathbb{P}(Y>2)=1-\mathbb{P}(Y\le 2)\approx 0.3225.$$

Exercise 1

By definition,

$$\mathbb{E}[X] = -1\cdot\mathbb{P}(X=-1)+0\cdot\mathbb{P}(X=0)+0.4\cdot\mathbb{P}(X=0.4)+1\cdot\mathbb{P}(X=1) = -\frac{1}{4}+0+\frac{2}{15}+\frac{1}{3} = \frac{13}{60} \approx 0.217,$$

$$\mathbb{E}[X^2] = (-1)^2\cdot\frac{1}{4}+0^2\cdot\frac{1}{12}+0.4^2\cdot\frac{1}{3}+1^2\cdot\frac{1}{3} = \frac{1}{4}+0+\frac{4}{75}+\frac{1}{3} = \frac{191}{300} \approx 0.637,$$

$$\mathbb{E}[X^3] = (-1)^3\cdot\frac{1}{4}+0^3\cdot\frac{1}{12}+0.4^3\cdot\frac{1}{3}+1^3\cdot\frac{1}{3} = -\frac{1}{4}+0+\frac{8}{375}+\frac{1}{3} = \frac{157}{1500} \approx 0.105.$$

In particular,

$$\mathrm{Var}(X) = \mathbb{E}[X^2]-\mathbb{E}[X]^2 = \frac{191}{300}-\frac{169}{3600} = \frac{2123}{3600} \approx 0.590.$$

Exercise 2

According to Markov's inequality, we have readily

$$\mathbb{P}(X \geq 4) \leq \frac{\mathbb{E}[X]}{4} = \frac{1}{2}.$$

Chebyshev's inequality gives

$$\mathbb{P}\big(\big|X-\mathbb{E}[X]\big| \geq 2\big) \leq \frac{\mathrm{Var}(X)}{2^2} = \frac{1}{4}.$$

Because the event $\{X \geq 4\}$ is included in $\{|X-2| \geq 2\}$, we have

$$\mathbb{P}(X \geq 4) \leq \mathbb{P}(|X-2| \geq 2) \leq \frac{1}{4}.$$

I'll add two remarks. First, if we knew higher moments of X, we could use Markov's inequality to get other inequalities. Most of the times, higher moments will give better estimates; for instance, if we know that $\mathbb{E}[X^3] \leq 14$ (weird value, but it actually is a possible moment), then

$$\mathbb{P}(X \geq 4) = \mathbb{P}(X^3 \geq 4^3) \leq \frac{\mathbb{E}[X^3]}{4^3} \leq \frac{14}{64} < \frac{1}{4}.$$

Actually, we can already do that with the second moment:

$$\mathbb{P}(X \geq 4) = \mathbb{P}(X^2 \geq 4^2) \leq \frac{\mathbb{E}[X^2]}{4^2} = \frac{\mathrm{Var}(X)+\mathbb{E}[X]^2}{16} = \frac{1+4}{16} = \frac{5}{16},$$

but in this case, it isn't a better bound.

Secondly, such estimates are subtle. I managed to find

$$\mathbb{P}(X \geq 4) \leq \mathbb{P}(|X-3/2| \geq 5/2) \leq \frac{\mathbb{E}[(X-3/2)^2]}{(5/2)^2} = \frac{\mathbb{E}[X^2]-3\mathbb{E}[X]+\frac{9}{4}}{\frac{25}{4}} = \frac{5-6+\frac{9}{4}}{\frac{25}{4}} = \frac{1}{5},$$

but I don't know if it is possible to get a better bound still (I suspect the best one can do should be 1/8). No general inequality is likely to be optimal, so the way to go is often to try both Markov (possibly with moments of different order) and Chebyshev, to see what works best in the case at hand, and to go with whatever they give you. Even if it might not be an excellent bound, you will be fine; they are reliable tools.