

# Introduction to Dynamical Systems

France-Kosovo Undergraduate Research School of Mathematics

March 2017

This introduction to dynamical systems was a course given at the march 2017 edition of the France Kosovo Undergraduate Research School of Mathematics. It was designed to be, and has in practice been eight hours long. I am happy to thank to thank Petru Mironescu and Agathe Bouchet for giving me this very nice teaching opportunity, Matthieu Dussaule and Oriane Blondel who have been wonderful partners in this adventure, and most of all the students of the school for their conviviality and enthusiasm through everything.

One will find in these notes an introduction to topological dynamical systems. Of most interest here are the definition of such systems, and some first results about recurrence and mixing. I have tried not to reduce the interested mathematician to a simple reader; in fact, they should take an active role in the construction of missing proofs. It may not be an easy work if one is not familiar with the objects at play; I hope that the many given examples will take care of this matter.

To this purpose, the document is divided into three dependent chapters. The first one consists of the notes themselves. They are mostly proofless, but illustrated with many examples. The second one is that of the proofs. Some may argue that exercises without correction cannot qualify as proper proofs. I believe that there is no peculiar difficulty other than the mistakes I have most probably put here and there, and that the reader, if compelled to, will be capable of writing proper proofs himself. In the third one, I have chosen to isolate three important examples. Namely, if one wants to apply the knowledge hardly extracted from these notes, they may find the detailed study of the translation on  $\mathbb{Z}$ , the rotations of the torus, and the Bernoulli shift.

Pierre Perruchaud

# Contents

<b>1</b>	<b>Notes</b>	<b>3</b>
1.1	Topological and Metric Spaces . . . . .	3
1.2	Compact Spaces . . . . .	5
1.3	Topological Dynamical Systems . . . . .	6
1.4	Recurrence . . . . .	8
1.5	Mixing . . . . .	10
<b>2</b>	<b>Proofs</b>	<b>11</b>
2.1	Topological and Metric Spaces . . . . .	11
2.2	Compact Spaces . . . . .	13
2.3	Topological Dynamical Systems . . . . .	15
2.4	Recurrence . . . . .	16
2.5	Mixing . . . . .	17
<b>3</b>	<b>Examples</b>	<b>18</b>
3.1	Translation of the Integers . . . . .	18
3.2	Rotations of the Torus . . . . .	18
3.3	Left-Shift in Product Spaces . . . . .	19

# Chapter 1

## Notes

### 1.1 Topological and Metric Spaces

**Definition 1.1.1.** Let  $X$  be a set. A function  $d : X \times X \rightarrow \mathbb{R}_+$  is a distance if for every points  $x, y$  and  $z$  in  $X$ ,

- $d(x, y) = d(y, x)$  ;
- $d(x, y) = 0$  if and only if  $x = y$  ;
- $d(x, y) \leq d(x, z) + d(z, y)$ .

We say that  $(X, d)$  is a metric space if  $d$  is a distance on  $X$ .

*Example 1.1.2.*

- The euclidean distance is a distance on  $\mathbb{R}^n$ .
- For every set  $X$ , the function  $d : (x, y) \mapsto \mathbb{1}_{x \neq y}$  is a distance.
- If  $\mathbb{T}^1 \subset \mathbb{C}$  is the unit circle of  $\mathbb{C}$ , then the function  $d : (e^{2i\pi s}, e^{2i\pi t}) \mapsto \min_{k \in \mathbb{Z}} |s - t + k|$  is well-defined and a distance on  $\mathbb{T}^1$ .
- For every prime  $p$  and  $n \in \mathbb{Z}$ , define  $|n|_p = p^{-k}$ , where  $k$  is the largest integer such that  $p^k | n$  (by convention,  $|0|_p = 0$ ). Then for every  $p$ ,  $d_p : n, m \mapsto |n - m|_p$  is a distance on  $\mathbb{Z}$ . It is called the  $p$ -adic distance.
- If  $(X, d)$  is a metric space, then  $(Y, d)$  is a metric space for every subset  $Y \subset X$ .
- If  $(X_n, d_n)$  is a metric spaces for every  $n \in \mathbb{Z}$ , then  $(X, d)$  is a metric space where  $X = \prod_{n \in \mathbb{Z}} X_n$  and

$$d(x, y) = \sum_{n \in \mathbb{Z}} \min \left( 2^{-|n|}, d(x_n, y_n) \right).$$

We will call it the product metric.

**Definition 1.1.3.** Let  $X$  be a set.

A subset  $\mathcal{T} \subset \mathcal{P}(X)$  is a topology on  $X$  if

- $\emptyset$  and  $X$  are elements of  $\mathcal{T}$  ;
- $\mathcal{T}$  is stable under finite intersection, i.e.  $\mathcal{U} \cap \mathcal{V} \in \mathcal{T}$  for every sets  $\mathcal{U}, \mathcal{V} \in \mathcal{T}$  ;
- $\mathcal{T}$  is stable under arbitrary union, i.e.  $\bigcup_{i \in I} \mathcal{U}_i \in \mathcal{T}$  for every family  $(\mathcal{U}_i)_{i \in I}$  of sets in  $\mathcal{T}$ .

In this case, we say that  $(X, \mathcal{T})$  is a topological space, and the elements of  $\mathcal{T}$  are called the open sets of  $X$ .

*Example 1.1.4.*

- The set  $\{\emptyset, X\}$  is always a topology on  $X$ . It is called the trivial topology.
- The set  $\mathcal{P}(X)$  is always a topology on  $X$ . It is called the discrete topology.
- If  $(X, \mathcal{T})$  is a topological space, then every subset  $Y \subset X$  is a topological space with respect to  $\mathcal{T}_Y = \{\mathcal{U} \cap Y, \mathcal{U} \in \mathcal{T}\}$
- Let  $\widehat{\mathbb{Z}} \cup \{\infty\}$ . We define the closed sets as these belonging to one of the following two types: the finite subsets of  $\mathbb{Z}$ , and the sets containing  $\infty$ . Then the complements of these sets form, indeed, a topology on  $\widehat{\mathbb{Z}}$ .

**Proposition 1.1.5.** *Every intersection of topologies over  $X$  is a topology over  $X$ .*

*In other words, if  $(\mathcal{T}_i)_{i \in I}$  are topologies on  $X$ , then  $\mathcal{T} = \{A \subset X \text{ such that } A \in \mathcal{T}_i \text{ for all } i \in I\}$  is a topology on  $X$ .*

**Definition 1.1.6.** Let  $\mathcal{E} \subset \mathcal{P}(X)$  be a set of sets in  $X$ .

The (non empty) intersection of every topology containing  $\mathcal{E}$  is called the topology generated by  $\mathcal{E}$ . It is the smallest topology containing  $\mathcal{E}$ .

*Example 1.1.7.*

- Let  $(X, d)$  be a metric space. The open ball  $B_x(r)$  of centre  $x \in X$  and radius  $r > 0$  is the set of  $y \in X$  such that  $d(x, y) < r$ . We call metric topology the topology generated by all the open balls.
- If  $X$  is a set, the metric topology associated to the distance  $d : x, y \mapsto \mathbb{1}_{x \neq y}$  is the discrete topology on  $X$ .
- Let  $(X_i, \mathcal{T}_i)$  be a topological space for every  $i \in I$ , and set  $X = \prod_{i \in I} X_i$ . For every open set  $\mathcal{U}_i$  of  $X_i$ , let  $\widetilde{\mathcal{U}}_i$  be the set of points  $x \in X$  such that  $x_i \in \mathcal{U}_i$ . We call product topology the topology on  $X$  generated by all the sets  $\widetilde{\mathcal{U}}_i$ .
- We can see  $\mathbb{R}^n$  as the product  $X = \prod_{1 \leq i \leq n} X_i$  where  $X_i = \mathbb{R}$  for every  $i$ . Then the product topology on  $\mathbb{R}^n$  is the same as the metric topology with respect to the euclidean distance.

**Definition 1.1.8.** Let  $(X, \mathcal{T})$  be a topological space. Let  $E \subset X$  be a subset of  $X$ .

The union  $\bigcup_{\mathcal{U} \subset E} \mathcal{U}$  over open sets included in  $E$  is open; it is the largest open set contained in  $E$ . We call it the *interior* of  $E$ , and denote it by  $\mathring{E}$ .

The intersection  $\bigcap_{F \supset E} F$  over closed sets containing  $E$  is closed; it is the smallest closed set containing  $E$ . We call it the *closure* of  $E$ , and denote it by  $\overline{E}$ .

**Definition 1.1.9.** Let  $(X, \mathcal{T})$  be a topological space. We say that a subset  $E$  is dense if it verifies one of the equivalent following properties.

- The closure of  $E$  is exactly  $X$ .
- Every non empty open set  $\mathcal{U}$  has non empty intersection with  $E$ .

**Definition 1.1.10.** Let  $(X, \mathcal{T})$  be a topological space.

A subset  $E \subset X$  is a neighbourhood of  $x \in X$  if there is an open set  $\mathcal{U} \in \mathcal{T}$  such that  $x \in \mathcal{U} \subset E$ . Note that a set is open if and only if it is a neighbourhood of each of its points.

**Definition 1.1.11.** Let  $(X, \mathcal{T})$  be a topological space. A base  $\mathcal{B} \subset \mathcal{T}$  of the topology is a family of open sets such that one of the following equivalent properties are satisfied.

- For all open neighbourhood  $\mathcal{U}$  of a point  $x$ , there is an element  $B \in \mathcal{B}$  such that  $x \in B \subset \mathcal{U}$ .
- Every open set  $\mathcal{U}$  is a union of elements of  $\mathcal{B}$ .

*Example 1.1.12.*

- The singletons in any space  $X$  form a base of the discrete topology on  $X$ .
- The bounded open intervals form a base of the topology of  $\mathbb{R}$ , as well as the rational ones, i.e. intervals of the form  $(\alpha, \beta)$  with  $\alpha$  and  $\beta$  rational. The same goes for the open rectangles in  $\mathbb{R}^n$ .
- The curved intervals  $(e^{2i\pi s}, e^{2i\pi t})$  form a base of the topology of  $\mathbb{T}^1$ .
- Let  $(X_i, \mathcal{T}_i)$  be topological spaces for every  $i \in I$ . If we define  $\tilde{\mathcal{U}}_i$  as previously for every open set  $\mathcal{U}_i$  of  $X_i$ , then the finite intersections of such open sets form a base of the product topology.

**Definition 1.1.13.** Let  $(X, \mathcal{T})$  and  $(X', \mathcal{T}')$  be topological spaces.

A sequence  $(x_n)_{n \in \mathbb{N}}$  of elements in  $X$  converges to  $x \in X$  if for every open neighbourhood  $\mathcal{U}$  of  $x$ ,  $x_n \in \mathcal{U}$  for every  $n$  large enough.

A function  $f : X \rightarrow X'$  is continuous if for every open set  $\mathcal{U}'$  of  $X'$ , its preimage  $f^{-1}\mathcal{U}'$  is open.

**Proposition 1.1.14.** Let  $(X, d)$  and  $(X', d')$  be metric spaces. Consider their metric topologies. Then we can restate the above definitions.

*Convergence.* Let  $(x_n)_{n \in \mathbb{N}}$  of elements in  $X$ . The sequence  $(x_n)_{n \in \mathbb{N}}$  converges to  $x \in X$  if and only if  $d(x_n, x) \rightarrow 0$ .

*Continuity.* Let  $f : X \rightarrow X'$  be any function. The following statements are equivalent.

- The function  $f$  is continuous.
- For every  $x \in X$  and  $\delta > 0$ , there is some  $\varepsilon > 0$  such that every point  $\varepsilon$ -close to  $x$  has image  $\delta$ -close to that of  $x$ .
- Every converging sequence  $x_n \rightarrow x$  in  $X$  has converging images, in the sense that  $f(x_n) \rightarrow f(x)$ .

## 1.2 Compact Spaces

**Definition 1.2.1.** Let  $(X, \mathcal{T})$  be a topological space.

We say that a topological space  $(X, \mathcal{T})$  is Hausdorff if  $\mathcal{T}$  separates the points of  $X$ , meaning that two distinct points are always contained in disjoint open neighbourhoods.

**Proposition 1.2.2.** If  $\mathcal{E} \subset \mathcal{P}(X)$  separates the points of  $X$ , then the topology generated by  $\mathcal{E}$  is Hausdorff.

*Example 1.2.3.*

- If  $(X, \mathcal{T})$  is a Hausdorff space, then its singletons  $\{x\}$  are closed.
- The discrete topology is always Hausdorff.
- The trivial topology on  $X$  is Hausdorff if and only if  $X$  contains at most one point.
- The metric topology is always Hausdorff.
- If  $(X_i, \mathcal{T}_i)$  is a Hausdorff topological space for every  $i \in I$ , then the product topology is Hausdorff on the product  $\prod_{i \in I} X_i$ .

**Definition 1.2.4.** A topological space  $(X, \mathcal{T})$  is compact if it is not empty and any of its finite open covers  $(\mathcal{U}_i)_{i \in I}$  of  $X$  admits a finite subcover  $(\mathcal{U}_{i_1}, \dots, \mathcal{U}_{i_k})$ .

Equivalently,  $(X, \mathcal{T})$  non empty is compact if for every empty intersection of closed sets  $\bigcap_{i \in I} F_i = \emptyset$ , there exists a finite subfamily  $(F_{i_1}, \dots, F_{i_n})$  of empty intersection.

*Example 1.2.5.*

- A non empty subset of a compact Hausdorff space is compact Hausdorff if and only if it is closed.
- If  $(X_i, \mathcal{T}_i)$  is a compact Hausdorff space for every  $1 \leq i \leq n$ , then  $X = \prod_{1 \leq i \leq n} X_i$  is a compact Hausdorff space with respect to the product topology. It is also true when the index set is infinite, but it is much harder to prove.
- A non empty bounded closed interval of  $\mathbb{R}$  is compact.
- A non empty subset of  $\mathbb{R}^n$  is a compact Hausdorff space if and only if it is bounded and closed.
- The circle  $\mathbb{T}^1 \subset \mathbb{C}$  is a compact Hausdorff space.
- The  $n$ -torus  $\mathbb{T}^n$  is a compact Hausdorff space.
- If  $(X_n, d_n)$  is a compact metric space for every  $n \in \mathbb{Z}$ , then the product  $(X, d)$  endowed with the product metric is a compact metric space.
- If  $A$  is a finite set endowed with its discrete topology, the product  $A^{\mathbb{Z}}$  can be seen as a compact metric space.

**Theorem 1.2.6.** Let  $(X, d)$  be a non empty metric space. Denote by  $\mathcal{T}$  the associated metric topology. The following statements are equivalent.

- The space  $(X, \mathcal{T})$  is compact.
- Every sequence  $(x_n)_{n \in \mathbb{N}}$  with values in  $X$  has a limit point.
- The metric verifies the following properties.
  - i. For every  $\varepsilon > 0$ , there exists a finite number of points  $x_1, \dots, x_n$  such that the open balls  $B_{x_1}(\varepsilon), \dots, B_{x_n}(\varepsilon)$  cover  $X$ .
  - ii. For every open cover  $(\mathcal{U}_i)_{i \in I}$  of  $X$ , there is some  $\varepsilon > 0$  such that every open ball  $B_x(\varepsilon)$  is contained on some open set  $\mathcal{U}_{i_x}$ .

**Proposition 1.2.7.** Every metric compact space  $(X, d)$  has a countable base.

## 1.3 Topological Dynamical Systems

**Definition 1.3.1.** A topological dynamical system  $(X, \mathcal{T}, T)$  is

- a Hausdorff compact space  $(X, \mathcal{T})$  ;
- a homeomorphism  $T : X \rightarrow X$ , meaning that  $T$  is invertible and  $T, T^{-1} : X \rightarrow X$  are continuous.

*Example 1.3.2.*

- The translation  $T : x \mapsto x + 1$  in the topological space  $\mathbb{Z}$  is a homeomorphism.
- In  $\widehat{\mathbb{Z}}$ , the transformation  $T$  which fixes  $\infty$  and translates  $\mathbb{Z}$  is a homeomorphism, meaning that  $(\widehat{\mathbb{Z}}, \mathcal{T}, T)$  is a topological dynamical system.

- The rotations  $T_\alpha$  of the torus are continuous for every  $\alpha \in \mathbb{T}^n$ , meaning that  $(\mathbb{T}^n, \mathcal{T}, T_\alpha)$  is a topological dynamical system.
- The left-shift  $\theta : x = (x_n)_{n \in \mathbb{Z}} \mapsto (x_{n+1})_{n \in \mathbb{Z}}$  is continuous in  $A^{\mathbb{Z}}$ , so that  $(A^{\mathbb{Z}}, \mathcal{T}, \theta)$  is a topological dynamical system.

**Definition 1.3.3.** Let  $(X, \mathcal{T}, T)$  be a topological dynamical system. If  $Y \subset X$  is a subset of  $X$ , the following properties are equivalent.

- $(Y, \{\mathcal{U} \cap Y, \mathcal{U} \in \mathcal{T}\}, T|_Y)$  is a topological dynamical system.
- $Y$  is non-empty, closed in  $X$  and stable by  $T$ , meaning that  $Tx \in Y$  and  $T^{-1}x \in Y$  for every  $x \in Y$ .

In any of these equivalent cases,  $Y$  is called a subsystem of  $(X, \mathcal{T}, T)$ . The subset  $X \subset X$  is always a subsystem; it is called the trivial subsystem.

*Example 1.3.4.*

- Any non empty intersection of subsystems is a subsystem.
- In  $\widehat{\mathbb{Z}}$ , there is only one non trivial subsystem: the singleton  $\{\infty\}$ .
- If  $T$  is a rational rotation of the torus  $\mathbb{T}^1$ , then every orbit is a subsystem.
- If  $T$  is an irrational rotation, then no orbit is closed; therefore, none of these can be a subsystem.
- If  $Y$  is a subsystem of  $X$  and  $x \in Y$  is a point of  $Y$ , then  $Y$  contains the orbit  $T^{\mathbb{Z}}x$  of  $x$ , as well as its closure  $\overline{T^{\mathbb{Z}}x}$ .

**Proposition 1.3.5.** For every point  $x$  of a topological dynamical system  $(X, \mathcal{T}, T)$ , the closure  $\overline{T^{\mathbb{Z}}x}$  of the orbit of  $x$  is a subsystem of  $X$ .

**Definition 1.3.6.** A topological dynamical system is called minimal if it has no non trivial subsystem.

*Example 1.3.7.*

- If  $x$  is a point of a topological dynamical system  $(X, \mathcal{T}, T)$  such that  $Tx = x$ , then  $\{x\}$  is a minimal subsystem.
- In particular,  $\{\infty\}$  is a minimal topological dynamical system, as well as the subset  $\{1\}$  of  $A^{\mathbb{Z}}$  if  $1 \in A$ .
- Define  $x = (x_n)_{n \in \mathbb{Z}}$  by  $x_{2k} = a$ ,  $x_{2k+1} = b$ , and  $y$  exchanging the roles of  $a$  and  $b$ . Then the pair  $\{x, y\}$  is a minimal topological dynamical system in  $\{a, b\}^{\mathbb{Z}}$ .
- Every orbit of a rational rotation is minimal.
- The whole circle  $\mathbb{T}^1$  is minimal if the considered rotation is irrational.

**Proposition 1.3.8.** A topological dynamical system  $(X, \mathcal{T}, T)$  is minimal if and only if for every point  $x \in X$ , the orbit of  $x$  is dense.

**Theorem 1.3.9.** Every topological dynamical system contains a subsystem which is minimal.

## 1.4 Recurrence

**Definition 1.4.1.** Let  $(X, \mathcal{T}, T)$  be a topological dynamical system.

We say that a point  $x \in X$  is

- invariant if  $Tx = x$  ;
- periodic if  $T^n x = x$  for some  $n \in \mathbb{Z} \setminus \{0\}$  ;
- recurrent if for every open neighbourhood  $\mathcal{U}$  of  $x$ , an infinite number of iterates of  $x$  falls in  $\mathcal{U}$  ;
- transient if it is not recurrent.

If  $x$  is periodic, then we say that the period  $P$  of  $x$  is the smallest  $n > 0$  such that  $T^n x = x$ . In other words,

$$\{n \in \mathbb{Z} \text{ such that } T^n x = x\} = P\mathbb{Z}.$$

Note that every invariant point is periodic, and every periodic point is recurrent.

*Example 1.4.2.*

- Let  $x$  be a point of a topological dynamical system  $(X, \mathcal{T}, T)$ . If  $T^{n_k} x \rightarrow x$  for some sequence  $(n_k)_{k \in \mathbb{N}} \in (\mathbb{Z} \setminus \{0\})^{\mathbb{N}}$ , then  $x$  is recurrent. If, in addition, the topology is metric, then the converse is also true.
- In  $\widehat{\mathbb{Z}}$ , every integer is transient, whereas  $\infty$  is invariant.
- In the circle  $\mathbb{T}^1$ , every point has the same behaviour. If  $T$  is a rational rotation, every point is periodic. One point is invariant if and only if every point is invariant, meaning that  $T$  is the identity.
- If  $T$  is an irrational rotation of  $\mathbb{T}^1$ , every point is recurrent.
- In  $A^{\mathbb{Z}}$ , the invariant points are the constant sequences; the periodic ones are the periodic sequences.
- If  $A$  contains at least two elements  $a \neq b$ ,  $A^{\mathbb{Z}}$  contains transient points. For example, the point  $x$  such that  $x_n = a$  for  $n \geq 0$  and  $x_n = b$  for  $n < 0$ .
- In the previous setting,  $A^{\mathbb{Z}}$  contains recurrent aperiodic points. For example, any point  $x$  containing every finite sequence of  $a$ 's and  $b$ 's is recurrent but not periodic.

Because systems can be very transients in some sense — see for example  $\widehat{\mathbb{Z}}$  — every open set cannot be recurrent. But in the general setting, one can always find arbitrarily small open sets which are recurrent.

**Theorem 1.4.3.** Let  $(X, \mathcal{T}, T)$  be a topological dynamical system, and  $(\mathcal{U}_i)_{i \in I}$  an open cover of  $X$ .

There exists an open set  $\mathcal{U}_{i_0}$  such that

$$\mathcal{U}_{i_0} \cap \bigcap_{n \in S} T^n \mathcal{U}_{i_0} \neq \emptyset$$

for some infinite set  $S \subset \mathbb{Z}$ .

There is a notion stronger than recurring but weaker than periodic: it is called almost periodicity.

Let  $(X, \mathcal{T}, T)$  be a topological dynamical system. For any point  $x \in X$  and any open neighbourhood  $\mathcal{U}$  of  $x$ , we call set of the return times the set

$$R_x(\mathcal{U}) = \{n \in \mathbb{Z} \text{ such that } T^n x \in \mathcal{U}\} \subset \mathbb{Z}.$$

**Property.** Let  $(X, \mathcal{T}, T)$  be a topological dynamical system.

A point  $x \in X$  is



- *invariant* if  $R_x(\mathcal{U}) = \mathbb{Z}$  for every open neighbourhood  $\mathcal{U}$  of  $x$ ;
- *periodic* if there is some  $n \in \mathbb{Z} \setminus \{0\}$  such that  $n\mathbb{Z} \subset R_x(\mathcal{U})$  for every open neighbourhood  $\mathcal{U}$  of  $x$ ;
- *recurring* if for every open neighbourhood  $\mathcal{U}$  of  $x$ ,  $R_x(\mathcal{U})$  is infinite;
- *transient* if  $R_x(\mathcal{U})$  is finite for some open neighbourhood  $\mathcal{U}$  of  $x$ .

**Definition 1.4.4.**

We say that a set  $S \subset \mathbb{R}$  is syndetic if it has bounded gaps, meaning that for some  $T \in \mathbb{N}$ , the set  $[t, t+T] \cap S$  must always contain at least a point.

*Example 1.4.5.*

- No finite subset of  $\mathbb{Z}$  is syndetic.
- The set  $n\mathbb{Z}$  is syndetic for  $n \neq 0$ . The set of leap years is syndetic.
- The set  $\{\pm 2^n, n \in \mathbb{N}, \pm \in \{-1, +1\}\}$  is not syndetic. The set  $\{n^3, n \in \mathbb{Z}\}$  is not syndetic.
- If  $\mathcal{P}$  is the set of primes, then  $\mathbb{Z}_- \cup \mathcal{P}$  is not syndetic.
- If one tosses a coin independently for every  $n \in \mathbb{Z}$ , the set of tails is not syndetic.

**Definition 1.4.6.**

A point  $x$  of a topological dynamical system  $(X, \mathcal{T}, T)$  is almost periodic if  $R_x(\mathcal{U})$  is syndetic for every open neighbourhood  $\mathcal{U}$  of  $x$ .

Note that a periodic point is almost periodic, and that an almost periodic point is recurring.

*Example 1.4.7.*

- Every point of  $\mathbb{T}^1$  is almost periodic; this proves, when  $T$  is irrational, that every almost periodic point needs not be periodic.
- There are points in  $A^{\mathbb{Z}}$  that are almost periodic but not periodic.
- If  $A = \{\text{heads}, \text{tails}\}$  and  $x \in A^{\mathbb{Z}}$  is a point given by independent tosses, then  $x$  has a 100% chance of being recurrent non almost periodic. In particular, a recurring point needs not be almost periodic.

**Theorem 1.4.8.** *Let  $(X, \mathcal{T}, T)$  be a minimal topological dynamical system. Then every point of  $X$  is almost periodic.*

**Corollary 1.4.9.** *Every topological dynamical system contains at least one almost periodic point.*

*Example 1.4.10.*

- In systems containing some invariant point, the corollary is not surprising:  $\infty$  is almost periodic in  $\hat{\mathbb{Z}}$ , every constant sequence is almost periodic in  $A^{\mathbb{Z}}$ .
- Since the irrational rotations are minimal, every point of  $\mathbb{T}^1$  is almost periodic in this case.

## 1.5 Mixing

**Definition 1.5.1.** Let  $(X, \mathcal{T}, T)$  be a topological dynamical system.

The system is transitive if for every open sets  $\mathcal{U}$  and  $\mathcal{V}$ , there is some  $n \in \mathbb{Z}$  such that  $T^n \mathcal{U} \cap \mathcal{V} \neq \emptyset$ .

*Example 1.5.2.*

- $\widehat{\mathbb{Z}}$  is transitive, because every open set contains at least one integer.
- The rational rotations in  $\mathbb{T}^1$  are not transitive.
- The irrational rotations in  $\mathbb{T}^1$  are transitive.
- The left shift in  $A^{\mathbb{Z}}$  is transitive.

**Theorem 1.5.3.** Let  $(X, \mathcal{T}, T)$  be a topological dynamical system.

If there exists some point  $x \in X$  such that  $X = \overline{T^{\mathbb{Z}}x}$ , then the system is transitive.

In addition, if the topology is metric, then the converse is true: a metric topological dynamical system contains a point whose orbit is dense.

*Example 1.5.4.*

- Every integer has dense orbit in  $\widehat{\mathbb{Z}}$ .
- Let  $A$  be a finite set. If  $x$  is a point in  $A$  that contains every finite sequence of element of  $A$ , then  $x$  has dense orbit. In fact, most points, in some sense, contain every finite sequence.

**Definition 1.5.5.** Let  $(X, \mathcal{T}, T)$  be a topological dynamical system.

The system is weakly mixing if for every open sets  $\mathcal{U}$ ,  $\mathcal{V}_1$  and  $\mathcal{V}_2$ , there is some  $n \in \mathbb{Z}$  such that  $T^n \mathcal{U} \cap \mathcal{V}_i \neq \emptyset$  for  $i \in \{1, 2\}$ .

It is strongly mixing if for every open sets  $\mathcal{U}$  and  $\mathcal{V}$ , there is some  $N \in \mathbb{N}$  such that  $T^n \mathcal{U} \cap \mathcal{V} \neq \emptyset$  for every  $n \in \mathbb{Z}$  such that  $|n| \geq N$ .

We can see easily that strongly mixing  $\Rightarrow$  weakly mixing, and that weakly mixing  $\Rightarrow$  transient. The fact that some systems are weakly mixing but not strongly mixing seems to be well-known; on the contrary, I have not been able to find one anywhere. If you know such an example, please write me an email, I will be happy to read it.

*Example 1.5.6.*

- The system  $\widehat{\mathbb{Z}}$  is not weakly mixing, showing that a transitive system is not always weakly mixing.
- The rotations of the circle  $\mathbb{T}^1$  are not mixing.
- The left shift is strongly mixing, therefore weakly mixing.

**Theorem 1.5.7.** Let  $(X, \mathcal{T}, T)$  be a strongly mixing topological dynamical system. Let  $(\mathcal{V}_i)_{i \in I}$  be an open cover of  $X$ , and  $\mathcal{U}$  an open set.

There exists a subcover indexed by  $J \subset I$  and an integer  $N \in \mathbb{N}$  such that

$$T^n \mathcal{U} \cap \mathcal{V}_j \neq \emptyset$$

for every  $|n| \geq N$ ,  $j \in J$ .

# Chapter 2

## Proofs

### 2.1 Topological and Metric Spaces

*Exercise 2.1.1. Some Metric Spaces.*

1. Prove that  $x \cdot y \leq \|x\| \|y\|$  for every vectors  $x, y \in \mathbb{R}^n$ . This is the Cauchy-Schwarz inequality.
2. Deduce that the euclidean distance is a distance on  $\mathbb{R}^n$ .
3. Show that for any space  $X$ , the function  $d : x, y \mapsto \mathbb{1}_{x \neq y}$  is a distance on  $X$ .

Fix  $p$  a prime number. Define  $|n|_p$  as  $p^{-k}$ , where  $k$  is the largest integer such that  $p^k | n$  (by convention,  $|0|_p = 0$ ).

4. Show that  $n, m \mapsto |n - m|_p$  is a distance on  $\mathbb{Z}$ . It is called the  $p$ -adic distance.

Let  $(X_n, d_n)$  be a metric space for every  $n \in \mathbb{Z}$ . Define  $X = \prod_{n \in \mathbb{Z}}$ , and on  $X$  the function

$$d : x, y \mapsto \sum_{n \in \mathbb{Z}} \min(2^{-|n|}, d_n(x_n, y_n)).$$

5. Show that the function is well-defined.
6. Prove that  $d$  is a distance on  $X$ .

*Exercise 2.1.2. Some Topological Spaces.*

1. Show that the trivial and discrete topologies are indeed topologies.
2. Let  $(X, \mathcal{T})$  be a topological space, and  $Y \subset X$  be any subset. Show that the topology restricted to  $Y$  is indeed a topology on  $Y$ .
3. Let  $X$  be a set, and  $\mathcal{T}_i$  be a topology on  $X$  for every  $i \in I$ . Show that the intersection  $\mathcal{T} = \bigcap_{i \in I} \mathcal{T}_i$  is a topology on  $X$ .

*Exercise 2.1.3. Interior, Closure.*

Let  $(X, \mathcal{T})$  be a topological space.

1. Prove that a subset of  $X$  is open if and only if it is a neighbourhood of each of its points.

2. Prove that a set  $E \subset X$  has full closure if and only if it intersects every non empty open set.

Suppose that the topology on  $X$  is the metric topology of some metric space  $(X, d)$ .

3. Show that  $x \in \overline{E}$  if and only if  $x_n \rightarrow x$  for some sequence  $(x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$ .
4. Show that  $E$  is dense in  $X$  if and only if every point of  $X$  is a limit of a sequence in  $E$ .

*Exercise 2.1.4. Equivalent Topologies.*

Let  $X$  be a set, together with two topologies  $\mathcal{T}$  and  $\mathcal{T}'$ .

Suppose that for every open set  $\mathcal{U}' \in \mathcal{T}'$  and every point  $x \in \mathcal{U}'$ , there exists an open set  $\mathcal{U} \in \mathcal{T}$  such that  $x \in \mathcal{U} \subset \mathcal{U}'$ ; in other words,  $\mathcal{U}'$  is a neighbourhood of  $x$  in  $(X, \mathcal{T})$ .

1. Show that  $\mathcal{T}' \subset \mathcal{T}$ .
2. Show that the result still holds if the statement is only true for  $\mathcal{U}' \in \mathcal{B}'$ , where  $\mathcal{B}'$  is a base of  $\mathcal{T}'$ .

Let  $X$  be a set, together with two distances  $d$  and  $d'$ . Denote by  $\mathcal{T}$  and  $\mathcal{T}'$  the metric topologies associated with  $d$  and  $d'$ .

Suppose that there is a constant  $C > 0$  such that  $d'(x, y) \leq Cd(x, y)$  for every  $x, y \in X$ .

3. Show that  $\mathcal{T}' \subset \mathcal{T}$ .

Let  $\mathcal{T}$  be the metric topology on  $\mathbb{R}$ ,  $\mathcal{T}^n$  the product topology on  $\mathbb{R}^n$  and  $\mathcal{T}'$  the metric topology associated to the euclidean distance on  $\mathbb{R}^n$ .

4. Show that the rectangles  $(a_1, b_1) \times \cdots \times (a_n, b_n)$  form a base of  $\mathcal{T}^n$ .
5. Show that  $\mathcal{T}' \subset \mathcal{T}^n$ .
6. Show that  $\mathcal{T}^n \subset \mathcal{T}'$ .

This show that the product topology on  $\mathbb{R}^n$  is the same as its metric topology, allowing us to consider the most fitted when dealing with a problem.

*Exercise 2.1.5. Metric Continuity.*

Let  $(X, d)$  and  $(X', d')$  be metric spaces.

1. Show that  $x_n \rightarrow x$  in  $X$  if and only if  $d(x_n, x) \rightarrow 0$ .
2. Show that if  $f : X \rightarrow X'$  is continuous, then every sequence  $x_n \rightarrow x$  in  $X$  verifies  $f(x_n) \rightarrow f(x)$ .
3. Show that if  $f : X \rightarrow X'$  verifies the previous convergence condition, then for every  $x \in X$  and  $\delta > 0$ , there exists some  $\varepsilon > 0$  so that every point  $\varepsilon$ -close to  $x$  has image  $\delta$ -close to that of  $x$ .
4. Show that if  $f : X \rightarrow X'$  verifies the previous  $\varepsilon$ - $\delta$  condition, then  $f$  is continuous.

## 2.2 Compact Spaces

*Exercise 2.2.1. Some Hausdorff Spaces.*

1. Let  $(X, \mathcal{T})$  be a Hausdorff topological space, and  $Y \subset X$  be any subset. Show that the restricted topology is a Hausdorff topology on  $Y$ .
2. Show that the discrete topology is always Hausdorff. Show that the trivial topology on  $X$  is not Hausdorff if  $X$  contains at least two elements.
3. Show that the metric topology is always Hausdorff.
4. Show that if  $(X_i, \mathcal{T}_i)$  is a Hausdorff topological space, then the product topology on the product  $X = \prod_{i \in I} X_i$  is Hausdorff.

Let  $x$  be a point of a Hausdorff space  $(X, \mathcal{T})$ . There exists, for every  $y \neq x$ , disjoint open sets  $\mathcal{U}_y$  and  $\mathcal{V}_y$  such that  $x \in \mathcal{U}_y$  and  $y \in \mathcal{V}_y$ .

5. Show that  $\{x\} = \bigcap_{y \neq x} \mathcal{V}_y$ .
6. Deduce that the singletons of a Hausdorff space are closed.

*Exercise 2.2.2. Some Compact Spaces.*

1. Let  $(X, \mathcal{T})$  be a Hausdorff compact space. Show that  $Y \subset X$  together with its restricted topology is compact if and only if  $Y$  is closed.

Let  $(X, \mathcal{T})$  and  $(X', \mathcal{T}')$  be Hausdorff compact spaces. We denote by  $\mathcal{T} \times \mathcal{T}'$  the product topology on  $X \times X'$ .

Let  $(\mathcal{U}_i)_{i \in I}$  be an open cover of  $(X \times X', \mathcal{T} \times \mathcal{T}')$ .

2. Let  $x$  be a point in  $X$ . Show that there is a finite subfamily  $(\mathcal{U}_{i_1^x}, \dots, \mathcal{U}_{i_{n_x}^x})$  that covers  $\{x\} \times X'$ .
3. Define  $\mathcal{U}_x = \bigcup_{1 \leq k \leq n_x} \mathcal{U}_{i_k^x}$ . Show that there is a finite subfamily  $(\mathcal{U}_{x_1}, \dots, \mathcal{U}_{x_n})$  that covers  $X \times X'$ .
4. Find a finite subcover of  $(\mathcal{U}_i)_{i \in I}$ . You have just proven that  $X \times X'$  is compact.
5. Convince yourself that a finite product of Hausdorff compact spaces is Hausdorff compact.

*Exercise 2.2.3. Metric Compactity.*

Let  $(X, d)$  be a non empty metric space, and denote by  $\mathcal{T}$  its associated metric topology. Let us show that the compactness properties are indeed equivalent.

Suppose that  $(X, \mathcal{T})$  is compact. Let  $(x_n)_{n \geq 0}$  be a sequence in  $X$ .

1. Set  $E = \bigcap_{N \in \mathbb{N}} \overline{\{x_n, n \geq N\}}$ . Prove that  $E$  is non empty.
2. Prove that  $x \in E$  if and only if there is an extraction  $(n_k)_{k \geq 0}$  such that  $x_{n_k} \xrightarrow[k \rightarrow \infty]{} x$ .
3. Show that  $(X, d)$  verifies the extraction property.

Suppose that  $(X, d)$  verifies the extraction property.

Fix  $\varepsilon > 0$ . Suppose that  $(X, d)$  does not verify  $i$ .

4. Show that there exists a sequence  $(x_n)_{n \geq 0}$  such that  $d(x_n, x_m) > \varepsilon$  for every indices  $n$  and  $m$ .

5. Find a contradiction. Deduce that  $(X, d)$  verifies *i*.

Let  $U$  be an open cover of  $X$ . Suppose that  $(X, d)$  does not verify *ii*.

6. Show that there exists a sequence  $(x_n)_{n \geq 0}$  such that  $B_x(\frac{1}{n})$  is not contained in any  $\mathcal{U} \in U$ .

7. Find a contradiction. Deduce that  $(X, d)$  verifies *ii*.

Suppose that  $(X, d)$  verifies *i*. and *ii*. Let  $U$  be an open cover of  $X$ .

8. Making good use of properties *i*. and *ii*., find a finite open subcover of  $U$ . Deduce that  $(X, d)$  verifies the Borel-Lebesgue property.

9. Step back, and convince yourself that the theorem is proven.

#### Exercise 2.2.4. The Euclidean Case.

Let us prove that  $[0, 1]$  is compact in  $\mathbb{R}$ . Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence with values in  $[0, 1]$ .

We will define some sets  $A_n$ ,  $A_n^+$  and  $A_n^-$  by induction. The set  $A_n$  will always contain an infinite number of elements of the sequence; it will always be the union  $A_n^+ \cup A_n^-$ ; it will always be a segment of length  $2^{-n}$ . The sets  $A_n^+$  and  $A_n^-$  will always be segments of length  $2^{-n-1}$ .

Let  $A_0$  be  $[0, 1]$ . Define  $A_0^+ = [\frac{1}{2}, 1]$  as its upper half, and  $A_0^- = [0, \frac{1}{2}]$  as its lower half. They verify the induction hypothesis. Now that we have initiated the induction, let us build the iteration. Suppose that  $A_n$ ,  $A_n^+$  and  $A_n^-$  have already been defined, and verify the induction hypothesis.

1. Prove that at least one of the sets  $A_n^+$  and  $A_n^-$  contain an infinite number of elements of the sequence.

Define  $A_{n+1}$  as one of them. Take  $A_{n+1}^+$  to be its upper half, and  $A_{n+1}^-$  its lower half.

2. Convince yourself that  $A_{n+1}$ ,  $A_{n+1}^+$  and  $A_{n+1}^-$  verify the induction hypothesis

3. Since  $A_n$  is a segment, we can find  $a_n$  and  $b_n$  such that  $A_n = [a_n, b_n]$ . Show that the sequences  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  converge to some real numbers  $a$  and  $b$ . Show that  $a = b$ .

4. Find a subsequence of  $(x_n)_{n \in \mathbb{N}}$  such that  $x_{n_k} \rightarrow a$ . You have just proven that the segment  $[0, 1]$  is compact.

5. Convince yourself that any closed bounded interval  $[\alpha, \beta]$  is compact in  $\mathbb{R}$ .

Let us consider the compact sets of  $\mathbb{R}^n$ .

6. Let  $K$  be a compact subset of  $\mathbb{R}^n$ . Prove that  $K$  is closed and bounded.

7. Let  $[a_1, b_1] \times \cdots \times [a_n, b_n]$  be a closed bounded rectangle in  $\mathbb{R}^n$ . Prove that it is compact.

8. Let  $E$  be a closed bounded subset of  $\mathbb{R}^n$ . Prove that it is a closed set in some closed bounded rectangle of  $\mathbb{R}^n$ .

9. Show that  $E \subset \mathbb{R}^n$  is compact if and only if it is closed and bounded.

#### Exercise 2.2.5. Countable Bases.

Let  $(X, d)$  be a metric compact space, and  $\mathcal{T}$  its metric topology. Let us find a countable base of  $(X, \mathcal{T})$ .

1. Fix  $\varepsilon > 0$ . Show that there is a finite number  $x_1^\varepsilon, \dots, x_{k_\varepsilon}^\varepsilon$  of points in  $X$  such that the open balls  $B_{x_1^\varepsilon}(\varepsilon), \dots, B_{x_{k_\varepsilon}^\varepsilon}(\varepsilon)$  form an open cover of  $X$ . We set

$$(\mathcal{U}_1, \mathcal{U}_2, \dots) = (B_{x_1^1}, \dots, B_{x_{k_1}^1}, B_{x_1^{1/2}}, \dots, B_{x_{k_{1/2}}^{1/2}}, B_{x_1^{1/4}}, \dots).$$

2. Let  $\mathcal{U}$  be an open set of  $X$ , and  $x \in \mathcal{U}$  be one of its points. Show that there is some  $i \in \mathbb{N}$  such that every open ball of radius  $2^{-i}$  containing  $x$  must be contained in  $\mathcal{U}$ .
3. Show that there is some  $1 \leq n \leq k_\varepsilon$  such that  $x \in B_{x_{k_\varepsilon}}^\varepsilon \subset \mathcal{U}$ , where  $\varepsilon = 2^{-1}$ .
4. Conclude: show that  $(\mathcal{U}_n)_{n \in \mathbb{N}}$  is a countable base of  $X$ .

## 2.3 Topological Dynamical Systems

*Exercise 2.3.1. Around Orbit Closures.*

Let  $x$  be a point of a topological dynamical system  $(X, \mathcal{T}, T)$ .

1. Knowing that  $T(T^\mathbb{Z}x) \subset \overline{T^\mathbb{Z}x}$ , show that  $T(\overline{T^\mathbb{Z}x}) \subset \overline{T^\mathbb{Z}x}$ .
2. Convince yourself that  $\overline{T^\mathbb{Z}x}$  is a subsystem.

Let  $Y$  be a subsystem of some topological dynamical system  $(X, \mathcal{T}, T)$ .

3. Show that if  $x \in Y$  is a point of  $Y$ , then  $T^\mathbb{Z}x \subset Y$  and  $\overline{T^\mathbb{Z}x} \subset Y$ .
4. Suppose that the subsystem  $Y$  is not minimal. Prove that there is some point  $x \in Y$  whose orbit is not dense in  $Y$ .
5. Suppose that  $Y$  is minimal, and let  $x \in Y$  be a point of  $Y$ . Show that  $\overline{T^\mathbb{Z}x} = Y$ .

You have just proven that a system is minimal if and only if every orbit is dense.

*Exercise 2.3.2. Existence of Minimal Subsystems.*

We will prove that every topological dynamical system  $(X, \mathcal{T}, T)$  that has a countable base admits a minimal subsystem. The proof follows the same idea when the space does not have a countable base, but uses some stronger version of the induction principle — one might use for example the Zorn lemma. The added technicality does not give a better idea of what is happening, and since the example that we consider are metric, I hope to have convinced you that this proof will be sufficient for now. The not convinced reader, if familiar with the Zorn lemma, will easily upgrade the proof to general topological dynamical systems.

Let  $(X, \mathcal{T}, T)$  be a topological dynamical system, and let  $(\mathcal{U}_1, \mathcal{U}_2, \dots)$  be a countable base of  $\mathcal{T}$ .

We define a decreasing sequence  $(X_n)_{n \in \mathbb{N}}$  of subsystems of  $X$  by induction; they will verify the following property.

$(\mathcal{P}_n)$  : For every  $1 \leq k \leq n$ , either  $X_n \cap \mathcal{U}_k = \emptyset$  or there exists no subsystem of  $X$  in  $X_n \setminus \mathcal{U}_k$ .

Set  $X_0 = X$ . If  $X_n$  is already constructed, define  $X_{n+1}$  as follows. If there is a point  $x \in X_n \setminus \mathcal{U}_{n+1}$  such that  $\overline{T^\mathbb{Z}x} \subset X_n \setminus \mathcal{U}_{n+1}$ , then set  $X_{n+1} = \overline{T^\mathbb{Z}x}$  for some point  $x$  that verifies this property. If there is no such  $x$ , then set  $X_{n+1}$  to be  $X_n$ .

1. Convince yourself that  $(X_n)_{n \in \mathbb{N}}$  is a decreasing sequence of subsystems in  $X$ . Proves that it verifies the following property.

Either  $X_n \cap \mathcal{U}_n = \emptyset$  or there exists no subsystem of  $X$  in  $X_n \setminus \mathcal{U}_n$ .

2. Deduce that  $(X_n)_{n \in \mathbb{N}}$  is a decreasing sequence of subsystems in  $X$  verifying the  $(\mathcal{P}_n)$  properties.
3. Define  $X_\infty = \bigcap_{n \in \mathbb{N}} X_n$ . Prove that it is a subsystem. Remember, it must be not empty!

4. Suppose that there is some point  $x \in X_\infty$  such that  $\overline{T^\mathbb{Z}x} \neq X_\infty$ . Prove that there is some  $n \in \mathbb{N}$  such that  $\overline{T^\mathbb{Z}x} \subset X_\infty \setminus \mathcal{U}_n \subsetneq X_\infty$ , where  $\subsetneq$  means ‘is a strict subset’.
5. Show that this contradicts  $(\mathcal{P}_n)$ . This means that  $\overline{T^\mathbb{Z}x}$  must be  $X_\infty$ .
6. Conclude: prove that  $X_\infty$  is minimal, therefore every topological dynamical system with a countable base admits a minimal subsystem.

## 2.4 Recurrence

*Exercise 2.4.1. Converging Iterates.*

Let  $(X, \mathcal{T}, T)$  be a topological dynamical system. Suppose that  $x$  verifies  $T^{n_k}x \rightarrow x$  for some sequence  $(n_k)_{k \in \mathbb{N}} \in (\mathbb{Z} \setminus \{0\})^\mathbb{N}$ .

1. Suppose that  $|n_k| \leq M$  for some  $M \in \mathbb{N}$ . Show that there is some  $-M \leq N \leq M$  such that  $n_k = N$  for an infinite number of  $N$ . Deduce that  $T^Nx = x$ , so  $x$  is periodic, hence recurrent.
2. Suppose that  $(n_k)_{k \in \mathbb{N}}$  is not bounded. Find an extraction  $(k_i)_{i \in \mathbb{N}}$  such that  $k_i \neq k_j$  for every  $i \neq j$ .
3. Prove that  $x$  is recurrent.

Suppose now that  $(X, \mathcal{T})$  is metric, and let  $x$  be a recurring point in  $X$ .

4. Show that there is a sequence  $(n_k)_{k \in \mathbb{N}}$  such that  $n_k \in \mathbb{Z} \setminus \{0\}$  for every  $k \in \mathbb{N}$  and  $T^{n_k}x \in B_x(2^{-k})$ .
5. Show that  $T^{n_k}x \rightarrow x$ .

*Exercise 2.4.2. Simple Recurrence Theorem.*

Let  $(X, \mathcal{T}, T)$  be a topological dynamical system, and  $(\mathcal{U}_i)_{i \in I}$  an open cover of  $X$ .

1. Let  $x \in X$  be any point. Show that there is an open set  $\mathcal{U}_i$  such that  $T^n x \in \mathcal{U}_i$  for infinitely many iterates of  $T$ .
2. Prove the theorem.

*Exercise 2.4.3. Return Times.*

Let  $x$  be a point in a topological dynamical system  $(X, \mathcal{T}, T)$ . Suppose that there is some  $n \in \mathbb{Z}$  such that  $n \in R_x(\mathcal{U})$  for every open neighbourhood  $\mathcal{U}$  of  $x$ .

1. Recall that  $X$  is Hausdorff. Show that  $T^n x = x$ .

Let  $(X, \mathcal{T}, T)$  be a topological dynamical system.

2. Show that  $x \in X$  is invariant if and only if  $R_x(\mathcal{U}) = \mathbb{Z}$  for every open neighbourhood  $\mathcal{U}$  of  $x$ .
3. Show that  $x \in X$  is invariant if and only if there exists some  $n \neq 0$  such that  $n\mathbb{Z} \subset R_x(\mathcal{U})$  for every open neighbourhood  $\mathcal{U}$  of  $x$ .

*Exercise 2.4.4. Some Syndetic and Not Syndetic Sets.*

We say that a set  $A \subset \mathbb{Z}$  is periodic if  $A + n = A$  for some  $n \neq 0$ .

1. Show that a finite set is not syndetic. Show that a non empty periodic set is syndetic.
2. Show that  $\{\pm 2^n, n \in \mathbb{N}\}$  is not syndetic. Show that  $\{n^3, n \in \mathbb{Z}\}$  is not syndetic.

Let  $\mathcal{P} \subset \mathbb{N}$  be the set of prime numbers.



3. Show that  $k|n| + k$  for every  $1 \leq k \leq n$ .
4. Deduce that the set  $\mathbb{Z}_- \cup \mathcal{P}$  is not syndetic.

*Exercise 2.4.5. Almost Periodicity Theorem*

Let  $(X, \mathcal{T}, T)$  be a minimal topological dynamical system. Suppose that some point  $x \in X$  is not almost periodic.

1. Find an open set  $\mathcal{U}$  and, for all  $k \in \mathbb{N}$ , a number  $n_k$  such that  $T^n x \notin \mathcal{U}$  for every  $n$  verifying  $|n - n_k| \leq k$ .
2. Show that, up to extraction,  $T^{n_k} x$  converges to a point  $y \in X$ .
3. Show that none of the iterates of  $y$  belong to  $\mathcal{U}$ .
4. Recall that a topological dynamical system is minimal if and only if every of its orbits are dense. Prove the Almost Periodicity Theorem.

## 2.5 Mixing

*Exercise 2.5.1. Dense Orbits in Transitive Systems.*

Let  $(X, \mathcal{T}, T)$  be a topological dynamical system. We will prove that if one of its orbit is dense, then the system is transitive. We will not prove the converse, as it involves more subtle topological tools, namely the Baire theorem for compact sets.

Suppose that  $X = \overline{T^{\mathbb{Z}}x}$  for some point  $x \in X$ . Let  $\mathcal{U}$  and  $\mathcal{V}$  be two open sets of  $X$ .

1. Show that there are some iterates  $T^n x$  and  $T^m x$  of  $x$  such that  $T^n x \in \mathcal{U}$  and  $T^m x \in \mathcal{V}$ .
2. Find a  $k \in \mathbb{Z}$  such that  $T^k \mathcal{U} \cap \mathcal{V} \neq \emptyset$ . Conclude.

*Exercise 2.5.2. Open Sets in Mixing Systems.*

Let  $(X, \mathcal{T}, T)$  be a strongly mixing topological dynamical system. Let  $\mathcal{U}$  be an open set.

1. Let  $\mathcal{V}_1, \dots, \mathcal{V}_n$  be some open sets of  $X$ . Show that there is some  $N \in \mathbb{N}$  such that  $T^n \mathcal{U}$  intersect every  $\mathcal{V}_i$  whenever  $|n| \geq N$ . In particular,  $X$  is weakly mixing.
2. Let  $(\mathcal{U}_i)_{i \in I}$  be an open cover of  $X$ . Show that there exists a subcover  $(\mathcal{U}_i)_{i \in J}$ ,  $J \subset I$  and some  $N \in \mathbb{N}$  such that  $T^n \mathcal{U}$  intersects every one of these sets whenever  $|n| \geq N$ .

# Chapter 3

## Examples

### 3.1 Translation of the Integers

We want to consider the translation  $n \mapsto n + 1$  on  $\mathbb{Z}$  as a topological dynamical system. Because, sadly,  $\mathbb{Z}$  is not compact, we need to modify this system a bit.

Define the set  $\widehat{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$ . The first step to make  $\widehat{\mathbb{Z}}$  into a topological dynamical system is to define its topology. We define closed sets of  $\widehat{\mathbb{Z}}$  as the sets of one of the following two types: finite sets of  $\mathbb{Z}$  and sets that contain  $\infty$ .

1. Show that this defines a Hausdorff topology on  $\widehat{\mathbb{Z}}$ .
2. Let  $U$  be an open cover of  $\widehat{\mathbb{Z}}$ . Considering first an open set containing  $\infty$ , find a finite subcover of  $U$ .

The set  $\widehat{\mathbb{Z}}$  is called the (Alexandroff) one point compactification of  $\mathbb{Z}$  (because we added one point to  $\mathbb{Z}$  to make it compact).

On  $\widehat{\mathbb{Z}}$ , we may define the translation  $T$  that maps  $n \in \mathbb{Z}$  to  $n + 1$  and fixes  $\infty$ .

3. Show that  $T$  is continuous. Convince yourself that  $T^{-1}$  is well-defined and continuous.

This system is the most important example of not interesting systems. In a sense, we could say that its points have the most uninteresting behaviour.

4. Show that the iterates of any point converge to  $\infty$ . It means that  $\widehat{\mathbb{Z}}$  has two types of points: the integers are transient, and the point  $\infty$  is invariant.

### 3.2 Rotations of the Torus

We consider the circle in  $\mathbb{C}$ , that we denote  $\mathbb{T}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ . As a subset of  $\mathbb{C}$ , it has a distance and an associated topology.

Define a new distance on  $\mathbb{T}^1$  by  $d(e^{2i\pi t}, e^{2i\pi s}) = \min_{k \in \mathbb{Z}} |t - s + k|$ .

5. Convince yourself that the distance is well-defined. Get an intuition of it.
6. Show that the topology induced by  $d$  is the same as the inherited topology.
7. With the pigeonhole principle in mind, show that  $(\mathbb{T}^1, d)$  is compact.

Define  $\mathbb{T}^n$  as the product  $\mathbb{T}^1 \times \cdots \times \mathbb{T}^1$ . This torus has an abelian group law, defined for  $x, y \in \mathbb{T}^n$  as  $x + y = (x_1 y_1, \dots, x_n y_n)$ .

8. Prove that  $\mathbb{T}^n$  is indeed an abelian group.
9. Show that the sum  $\mathbb{T}^n \times \mathbb{T}^n \rightarrow \mathbb{T}^n; x, y \mapsto x + y$  is continuous.

Groups  $G$  that have a compatible topology (meaning that  $x, y \mapsto xy$  and  $x \mapsto x^{-1}$ ) are called topological groups. Any element  $\alpha$  of a topological group defines a topological dynamical system, with the action  $T_\alpha : x \mapsto \alpha x$ . They are called translations.

10. Convince yourself that  $\mathbb{R}^n$  is a topological group, and that its translations are exactly the translations.

For now, we focus on the translations of the circle  $\mathbb{T}^1$ . We say that  $T_\alpha$  is rational if  $\alpha^n = 1$  for some  $n \in \mathbb{Z}$ ; if not, the translation is irrational.

11. Show that  $T_\alpha$  has a fixed point if and only if  $T_\alpha = \text{id}$  and  $\alpha = 1$ . Let's say that this system is not very profound, and suppose that  $T_\alpha \neq \text{id}$ .
12. Show that if  $T_\alpha$  is a rational translation, then every point of  $\mathbb{T}^1$  is periodic with the same period.
13. Fix  $\alpha$  such that  $T_\alpha$  is irrational. Show that no point is periodic.
14. Choose some  $\varepsilon > 0$ . Remembering the pigeonhole principle, show that there exists  $n \neq m \in \mathbb{Z}$  such that  $d(T^n 1, T^m 1) < \varepsilon$ . Deduce that every point of  $\mathbb{T}^1$  is recurrent.
15. Let  $\varepsilon > 0$  be fixed. Show that there exists a finite number of positive translates of 1 such that every point of  $\mathbb{T}^1$  lies at most at distance  $\varepsilon$  of some of the translates.
16. Deduce that any point of  $\mathbb{T}^1$  has dense orbit.
17. Deduce from question 11. that every point is almost periodic.

### 3.3 Left-Shift in Product Spaces

For any finite set  $A$ , we define  $X = A^{\mathbb{Z}}$  as the set of sequences  $(a_k)_{k \in \mathbb{Z}}$  with  $a_k \in A$ . The Bernoulli shift is the application  $\theta : X \rightarrow X$  that maps  $\underline{a}$  to the left-shifted sequence  $(\underline{a}_{n+1})_{n \in \mathbb{Z}}$ .

We will show that  $(X, \mathcal{T}, T)$  is a topological dynamical system for a well-defined topology  $\mathcal{T}$ .

1. Show that  $d(\underline{a}, \underline{b}) = \sum_{k \in \mathbb{Z}} 2^{-|k|} \mathbb{1}_{\underline{a}_k \neq \underline{b}_k}$  is a distance on  $X$ .
2. Prove that a sequence of elements  $\underline{a}^{(k)} \in X$  converges to  $\underline{a}$  if and only if for all  $n \in \mathbb{Z}$ ,  $\underline{a}_n^{(k)} = \underline{a}_n$  for  $k$  sufficiently large.

Let  $(\underline{a}^{(k)})_{k \geq 0}$  be a sequence in  $X$ , meaning that every  $\underline{a}^{(k)}$  is a collection of elements of  $A$ .

3. Show that for every fixed  $n \in \mathbb{N}$ , there exists an element  $a \in A$  such that  $\underline{a}_n^{(k)} = a$  for an infinite number of  $k$ . Show that the result holds even if one fixes a finite number of such  $a$ 's: for every  $n_1 < \dots < n_i$ , there exists  $a_1, \dots, a_i \in A$  such that  $\underline{a}_{n_1}^{(k)} = a_1, \dots, \underline{a}_{n_i}^{(k)} = a_i$  for an infinite number of  $k$ .
4. Extract a subsequence  $k_i$  such that for every  $i$ ,  $\underline{a}_{n_i}^{(k_i)} = a_{n_i}$  when  $k$  is large enough. Convince yourself that the sequence  $\underline{a}^{(k)}$  converges to  $\underline{a} = (a_n)_{n \in \mathbb{Z}}$ .

*Remark 3.3.1.* We can prove that if  $(K, d)$  is a compact metric space, then  $K^{\mathbb{Z}}$  with the distance  $d^{\mathbb{Z}}(x, y) = \sum_{k \in \mathbb{Z}} d(x_k, y_k)$  is a compact metric space.

We have just shown that  $X$  is compact. We know that  $X$  must be second countable; let's find an explicit countable basis.

5. Let  $\mathcal{U}_n^a$  be the set of all  $\underline{a} \in X$  such that  $\underline{a}_n = a$ . Show that  $\mathcal{U}_n^a$  is open and closed.
6. Prove that the sets  $\mathcal{U}_{n_1}^{a_1} \cap \cdots \cap \mathcal{U}_{n_i}^{a_i}$  form an open basis of  $X$ .
7. Prove that this basis is countable.

We must now show that the shift is continuous. Fix a converging sequence  $\underline{a}^{(k)} \rightarrow \underline{a}$  in  $X$ ; we must show that  $\theta(\underline{a}^{(k)}) \rightarrow \theta(\underline{a})$ .

8. Do it.

The dynamics in this space is incredibly rich. For example, there are points in  $\{0, 1\}^{\mathbb{Z}}$  that show that none of the implications invariant  $\Rightarrow$  periodic  $\Rightarrow$  almost periodic  $\Rightarrow$  recurrent are an equivalence. For the remainder of the exercise, we suppose that  $A$  contains at least two elements, so that we may find  $a \neq b \in A$ .

9. Find all invariant points of  $A^{\mathbb{Z}}$ . Find a periodic point in  $A^{\mathbb{Z}}$  that is not invariant.

Define the (rather complicated) point  $x$  such that

$$\{n \in \mathbb{Z} | x_n = a\} = \left\{ \sum_{k \in \mathbb{N}} 3^k e_k, e_k = 0 \text{ or } 1 \right\}$$

with  $b$ 's elsewhere. It looks like this (the underlined element is  $x_0$ ):

$\cdots bbbbbb\underline{a}baabbbbaabaabbbbbbbaabaabbbbaabaabbbb \cdots$

Let us represent only the places with an  $a$ , and start at the 0 index.

Aa aa	aa aa	aa aa	aa aa	aa...
1 3	9	27		81

10. Take a moment to understand the underlying structure of  $x$ . It might help you to know that some mathematicians call it (and similar-looking objects) the discrete Cantor set.
11. Show that  $T^{3^n}x \rightarrow x$ . This proves that  $x$  is recurrent.
12. If  $\mathcal{U}$  is the open set consisting of all  $\underline{a}$  such that  $\underline{a}_0 = a$ , then prove that  $\{n \in \mathbb{Z} | T^n x \in \mathcal{U}\}$  is not syndetic. This means that  $x$  is recurrent but not almost periodic.

This system is on the random side of the spectrum: in some sense, there is no underlying structure in this space.

13. Show that there is no invariant distance on  $A^{\mathbb{Z}}$ .
14. Using the base you find a few questions before, show that for every open set  $\mathcal{U}$  and every point  $x \in A^{\mathbb{Z}}$ , there is  $N \in \mathbb{N}$  such that  $x \in T^N \mathcal{U}$ .
15. Prove that  $A^{\mathbb{Z}}$  is (strongly) mixing.