Homework 2
Solution

February 7th

Exercises from the book

Exercise 1.10 (c) The probability we consider is the ratio of number of favourable outcomes by the number of possible outcomes. The outcomes are sequences of 3 distinct squares; if we consider them unordered, they represent $\binom{64}{3}$ possibilities.

Because they are disjoint events, the probability that the pieces are all in the same row is the sum of the probabilities that they are in each individual row. Fix such a row (say the first). There are $\binom{8}{3}$ unordered sets of 3 squares in this row, i.e. $\binom{8}{3}$ favourable outcomes for the event "all pieces are in that fixed column". Since this does not depend on the choice of the column, it means that the total probability is

$$\mathbb{P}(\text{All pieces are in the same row}) = 8 \cdot \mathbb{P}(\text{All pieces are in the first row}) = 8 \cdot \frac{\binom{8}{3}}{\binom{64}{3}} \approx 0.01562.$$

Maybe, if the pieces are different for instance, we want to consider ordered sequences. In this case, we could follow a different approach that would give us the same result:

$$\mathbb{P}(\text{All pieces are in the same row}) = 8 \cdot \mathbb{P}(\text{All pieces are in the first row}) = 8 \cdot \frac{(8)_3}{(64)_3} \approx 0.01562.$$

(d) This is more or less the same reasoning. The difference is that we have to isolate not only a row, but also a colour. The total probability will be 8×2 times the probability of all pieces being on a fixed row and a fixed colour (say the black squares on the first row), and the number of outcomes corresponding to that event is $\binom{4}{3}$. All in all, we have

$$\begin{aligned}
&\mathbb{P}(\text{All pieces are in the same row and on the same colour}) \\
&\qquad = 8 \cdot 2 \cdot \mathbb{P}(\text{All pieces are in the first row on black squares}) \\
&\qquad = 8 \cdot 2 \frac{\binom{4}{3}}{\binom{64}{3}} \approx 0.0014.
\end{aligned}$$

Again, an ordered version would yield the same result:

$$\mathbb{P}(\text{All pieces are in the same row and on the same colour}) = 8 \cdot 2 \frac{(4)_3}{(64)_3} \approx 0.0014.$$

Exercise 1.12 Denote by R the event "we do *not* get the colour red", by B the event "we do *not* get a black ball, and by W the event "we do *not* get a white ball". We are interested in the probability of the event $B \cup R \cup W$. We compute it using the inclusion-exclusion principle and we have

$$\mathbb{P}(B \cup R \cup W) = \mathbb{P}(B) + \mathbb{P}(R) + \mathbb{P}(W)$$
$$-\mathbb{P}(B \cap R) - \mathbb{P}(R \cap W) - \mathbb{P}(W \cap B) + \mathbb{P}(B \cap R \cap W).$$

Note first that $B \cap R \cap W = \emptyset$ because any ball we draw has one of the colours black, red or white.

Note also that $B \cap R$ is the event that all the balls we get are white. An outcome is a collection of k balls out of $3n$ possible, and the order in which we draw the balls is irrelevant. Thus the number of possible outcomes is $\binom{3n}{k}$. The number of outcomes favourable to $B \cap R$ is $\binom{n}{k}$ (k balls out of the n white ones). We have

$$\mathbb{P}(B \cap R) = \mathbb{P}(R \cap W) = \mathbb{P}(W \cap B) = \frac{\binom{n}{k}}{\binom{3n}{k}}.$$

Observe that $\mathbb{P}(B) = \mathbb{P}(R) = \mathbb{P}(W)$ because there are equal numbers of balls of different colours. To compute $\mathbb{P}(B)$ observe that B can be alternatively described as the event "we draw only red or white balls". The total number of red and white balls is $2n$. There are $\binom{2n}{k}$ ways of drawing k balls of one of these colours, so

$$\mathbb{P}(B) = \frac{\binom{2n}{k}}{\binom{3n}{k}},$$

and

$$\mathbb{P}(B \cup R \cup W) = 3\frac{\binom{2n}{k}}{\binom{3n}{k}} - 3\frac{\binom{n}{k}}{\binom{3n}{k}}.$$

Exercise 1

Let the four different amounts of money placed in the 4 distinct envelopes be

$$a_1 < a_2 < a_3 < a_4,$$

A possible way to represent the outcomes is to write down the amounts in the order in which they appear; for instance, (a_3, a_2, a_1, a_4) and (a_1, a_2, a_4, a_3) are possible outcomes. For the first one, the strategy 2 and 3 are winning strategies, whereas strategy 1 makes us loose (we don't keep the maximal amount). For the second one, only strategy 3 makes us win.

Outcomes are ordered sequences of 4 distinct elements of $\{a_1, \dots, a_4\}$; this means there are $(4)_4 = 4!$ of them. They are assumed to be uniform, so the probability of some event A is the cardinal of A divided by 4!

- Strategy 1 makes us win if the first element of the outcome is a_4. Such an outcome is described entirely by the remaining 3 elements (ordered), none of which can be a_4: this represents $(3)_3 = 3!$ possibilities.

$$\mathbb{P}(\text{Strategy 1 wins}) = \frac{3!}{4!} = \frac{1}{4}$$

- Strategy 2 makes us lose if a_4 is in first position, and win if a_4 is in second position (a_4 will always be higher than the first amount). If a_4 is in fourth position, then we win only if a_3 was in first position (otherwise we will pick a_3 or lower after seeing the first envelope). If a_4 is in third position, then the second element has to be lower than the first for us to win, and a bit of brainstorming gives us the winning combinations:

$$(a_2, a_1, a_4, a_3), \quad (a_3, a_1, a_4, a_2), \quad (a_3, a_2, a_4, a_1).$$

  Letting A be the event of one of the three above draws happening, this means that the probability must be

$$\begin{aligned}\mathbb{P}(\text{Strategy 2 wins}) = {} & \mathbb{P}(a_4 \text{ is in the second envelope}) \\ & + \mathbb{P}(a_4 \text{ is in the fourth envelope and } a_3 \text{ in the first}) + \mathbb{P}(A).\end{aligned}$$

  The probability of A is $3/4!$. The number of outcomes giving a_4 in second position is the number of ways the remaining 3 amounts can be ordered: $(3)_3 = 3!$. The number of outcomes with a_4 and a_3 in positions 4 and 1 is the number of ways one can place a_1 and a_2 in the remaining two slots: $(2)_2 = 2$. Coming back to the formula, we get

$$\mathbb{P}(\text{Strategy 2 wins}) = \frac{3!}{4!} + \frac{2}{4!} + \frac{3}{4!} = \frac{11}{24}.$$

- Strategy 3 makes us lose if a_4 was in one of the first two envelopes. If it was in the third envelope, we win (it will always be higher than the first two). If it was in the last envelope, we lose only if a_3 was in the third envelope (otherwise it means that a_3 was in one of the first two, and the only one that is highest is a_4). So the probability is

$$\begin{aligned}\mathbb{P}(\text{Strategy 3 wins}) = {} & \mathbb{P}(a_4 \text{ is in the third envelope}) + \mathbb{P}(a_4 \text{ is in the fourth envelope}) \\ & - \mathbb{P}(a_4 \text{ is in the fourth envelope and } a_3 \text{ in the third}).\end{aligned}$$

  Similarly to what was discussed above, there are $(3)_3 = 3!$ outcomes with a_4 in third (respectively in forth) position, because we have to choose the place for each remaining amount. The probability of a_4 and a_3 coming respectively third and fourth is $(2)_2 = 2$, corresponding to the possible ways to place a_1 and a_2. We deduce

$$\mathbb{P}(\text{Strategy 3 wins}) = \frac{3! + 3! - 2}{4!} = \frac{5}{12}.$$

Since

$$\frac{1}{4} < \frac{5}{12} < \frac{11}{24},$$

strategy 2 is the best, and strategy 1 the worst out of the proposed three.

Exercise 2

1. Let us try to understand what is needed to describe a two-pairs. We need to know where are the two pairs, which will be enough to know where the isolated card is. We need the rank of each pair, and of the isolated card. Finally, we have to identify the suits of each pair and of the isolated card.

There are $\binom{5}{2,2,1}$ ways to place the 5 cards of a hand into 3 categories of 2, 2 and 1 cards. However, there is a subtlety here; in doing so, there will be a first and a second pair; for instance, the partitions

$$(\{1,2\},\{3,4\},5), \qquad (\{3,4\},\{1,2\},5)$$

will be different.

There are many ways to deal with this issue. For instance, we could impose that the first pair will be the one with the highest rank. Depending on the approach, an other way that is likely to work is to simply divide the final product by two. I chose instead to count the number of unordered ways to divide a hand in two pairs and an isolated card: for instance, I consider

$$\{\{1,2\},\{3,4\},5\}, \qquad \{\{3,4\},\{1,2\},5\}$$

to be equal. Since I can group ordered such partitions by two according to what unordered partition they give me, there are exactly $\frac{1}{2}\binom{5}{2,2,1}$ ways to choose two pairs in an ordered hand of 5 cards, if we disregard the order of the pairs.

Once this is done, we can choose the ranks of the three groups involved, for instance from left to right in the hand. They must be distinct (otherwise we would have a four-of-a-kind or a full house), which represents $(13)_3$ possibilities. Lastly, we choose the suits for each group. We have $(4)_2$ possibilities for the first pair we encounter in the hand from the left, just as many for the other pair, and 4 possibilities for the isolated card.

In summary, the number of ordered two-pairs is

$$\underbrace{\frac{1}{2}\binom{5}{2,2,1}}_{\text{positions}} \cdot \overbrace{(13)_3}^{\text{ranks}} \cdot \underbrace{(4)_2 \cdot (4)_2 \cdot 4}_{\text{suits}}.$$

2. The probability that we are given a two-pairs is

$$\frac{\frac{1}{2}\binom{5}{2,2,1} \cdot (13)_3 \cdot (4)_2 \cdot (4)_2 \cdot 4}{(52)_5} = \frac{123,552 \cdot 120}{2,598,960 \cdot 120} = \frac{198}{4165} \simeq 0.048,$$

the same as the one in the book.

Exercise 3

There are $\binom{50}{14,8,28}$ ways to distribute 50 people in three categories of size 14, 8 and 28. The probability of an event A will be the cardinal $\#A$ divided by $\binom{50}{14,8,28}$.

1. A partition of the 50 people such that Alice and Bob are in the jury is described by the repartition of the other 48 people. There are $\binom{48}{12,8,28}$ such repartitions (two places in the jury are already taken), so the probability is

$$\frac{\binom{48}{12,8,28}}{\binom{50}{14,8,28}} = \frac{14 \cdot 13}{50 \cdot 49} \simeq 7.4\%$$

2. A first way to compute this probability is to use the same reasoning as above. The probability that Alice and Bob are both in the jury was computed above. The probability that Alice is in the jury and Bob an alternate is

$$\frac{\binom{48}{13,7,28}}{\binom{50}{14,8,28}},$$

because outcomes corresponding to this event are described by the repartition of the other 48 people within the 13 remaining seats in the jury and the 7 remaining alternates. The probability that Alice is an alternate and Bob in the jury is the same, and the probability that both Alice and Bob are alternates is

$$\frac{\binom{48}{14,6,28}}{\binom{50}{14,8,28}}.$$

The probability we were asked for must then be

$$\frac{1}{\binom{50}{14,8,28}}\left[\binom{48}{12,8,28} + 2\cdot\binom{48}{13,7,28} + \binom{48}{14,6,28}\right].$$

Another approach consists in realising that the distribution between "selected for either the jury or the alternates" and "sent home" is uniform as well. Indeed, for a given such distribution d, the probability that it occurs is

$$\begin{aligned}
&\mathbb{P}(\text{The distribution of people in two groups is } d)\\
&\qquad = \frac{\#\{\text{Decompositions of the selected people into jury and alternates}\}}{\binom{50}{14,8,28}}\\
&\qquad = \frac{\binom{14+8}{14,8}}{\binom{50}{14,8,28}}.
\end{aligned}$$

This does not depend on the distribution d, meaning that each of them is equally likely.

Based on this result, we know that the probability of Alice and Bob being selected is the number of ways to distribute the remaining 48 people into two groups of 20 (the selected ones) and 28 (the ones sent home), divided by the number of ways to distribute 50 people into two groups of 22 and 28 people:

$$\mathbb{P}(\text{Alice and Bob are selected}) = \frac{\binom{48}{20,28}}{\binom{50}{22,28}} = \frac{\binom{48}{20}}{\binom{50}{22}}$$