Homework 8
Solution

April 22nd

Exercises from the book

Exercise 2.42 We compute the cumulative distribution for Y, and deduce its density by taking the derivative. It will help us to keep in mind that the inverse of the logarithm is the exponential.

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(\ln(X) \le y) = \mathbb{P}(X \le e^y) = F_X(e^y).$$

Since $F_X(x) = 1 - e^{-3x}$ for all $x > 0$, we get that

$$F_Y(y) = 1 - \exp(-3e^y)$$

and

$$p_Y(y) = 3e^y \cdot \exp(-3e^y) = 3\exp(-e^y + y).$$

Exercise 2.46 Again, we first compute the cumulative distribution for Y. Setting $f(x)$ such that $Y = f(X)$, we see that for $0 < y < 1$,

$$f(x) \le y \Leftrightarrow (x \le y \text{ and } x \le 1) \text{ or } (1/x \le y \text{ and } x > 1) \Leftrightarrow x \le y \text{ or } x \ge 1/y.$$

Then for $0 < y < 1$ we have

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(\{X \le y\} \cup \{X \ge 1/y\}) = \int_0^y e^{-x}\,dx + \int_{1/y}^\infty e^{-x}\,dx = 1 - e^{-y} + e^{-1/y}.$$

Differentiating, for $0 < y < 1$ the density function is $p_Y(y) = e^{-y} + \frac{1}{y^2}e^{-1/y}$. Since $f(x)$ has values in $[0,1]$ for $x \ge 0$, Y has values in $[0,1]$ and $F_y(y) = 0$ out of $(0,1)$.

Exercise 2.48 (a)

$$\int_{-\infty}^\infty f(x)\,dx = \int_{c_n}^\infty \frac{c_n}{x^{n+1}}\,dx = \frac{1}{nc_n^{n-1}} = 1$$

since $f(x)$ is the density of a random variable. Therefore, if $n > 1$

$$c_n = \frac{1}{n^{1/(n-1)}}$$

while if $n = 1$, then c_n could be anything.
(b)

$$\mathbb{E}[X] = \int_{c_n}^{\infty} \frac{c_n}{x^n}\,dx.$$

This diverges if $n = 1$ but for $n > 1$ gives

$$\frac{1}{n-1} \cdot \frac{1}{n^{(n-2)/(n-1)}} = \frac{n^{1/(n-1)}}{n(n-1)}.$$

(c) The random variable Z_n has range $[\ln c_n, \infty)$. For z in this range we have

$$F_{Z_n}(z) = \mathbb{P}(Z_n \le z) = \mathbb{P}(\ln(X_n) \le z) = \mathbb{P}(X_n \le e^z) = \int_{c_n}^{e^z} \frac{c_n}{x^{n+1}}\,dx = \frac{1}{nc_n^{n-1}} - \frac{c_n}{ne^{nz}}.$$

The density is then the derivative of the function above:

$$p_{Z_n}(z) = \begin{cases} c_n e^{-nz} & \text{for } z \ge \ln(c_n), \\ 0 & \text{else.} \end{cases}$$

(d)

$$\mathbb{E}[X_n^{m+1}] = \int_{c_n}^{\infty} x^{m+1} \frac{c_n}{x^{n+1}}\,dx = \int_{c_n}^{\infty} \frac{c_n}{x^{n-m}}\,dx.$$

This is convergent only of $m < n - 1$.

Exercise 3.4 The number N of rolls of die until we get the first 6 is a geometric random variable with success probability $p = \frac{1}{6}$ and failure probability $q = \frac{5}{6}$. Denote by $p(i,j)$ the joint pmf of (X, Y).
(i) Note that

$$p(0,0) = \mathbb{P}(X = 0, Y = 0) = 0.$$

For $i + j > 0$ we have

$$\begin{aligned} p(i,j) &= \mathbb{P}\big(\, X = i, Y = j \,\big) = \mathbb{P}\big(\, X = i, Y = j | N = i + j \,\big)\mathbb{P}(N = i + j) \\ &= \binom{i+j}{i} \left(\frac{1}{2}\right)^{i+j} q^{i+j-1}p = \binom{i+j}{i} \left(\frac{q}{2}\right)^{i+j} \frac{p}{q}. \end{aligned}$$

(ii) For $i > 0$ we have

$$\begin{aligned} p_X(i) = \mathbb{P}(X = i) &= \mathbb{P}(X = i, Y = 0) + \mathbb{P}(X = i, Y = 1) + \mathbb{P}(X = i, Y = 2) + \cdots \\ &= \frac{p}{q}\binom{i+0}{i}\left(\frac{q}{2}\right)^{i+0} + \frac{p}{q}\binom{i+1}{i}\left(\frac{q}{2}\right)^{i+1} + \frac{p}{q}\binom{i+2}{i}\left(\frac{q}{2}\right)^{i+2} + \cdots \\ &= \frac{p}{q}\left(\binom{i+0}{i}\left(\frac{q}{2}\right)^{i+0} + \binom{i+1}{i}\left(\frac{q}{2}\right)^{i+1} + \binom{i+2}{i}\left(\frac{q}{2}\right)^{i+2} + \cdots\right) \end{aligned}$$

$(x = \frac{q}{2})$

$$= \frac{p}{q} x^i \left(\binom{i}{i} x^0 + \binom{i+1}{i} x + \binom{i+2}{i} x^2 + \cdots \right)$$

$$= \frac{p}{q} \frac{x^i}{(1-x)^{i+1}} = \frac{p}{q(1-x)} \left(\frac{x}{1-x} \right)^i$$

$$= \frac{2p}{q(1+p)} \left(\frac{q}{2-q} \right)^i = \frac{12}{35} \left(\frac{5}{7} \right)^i.$$

For $i = 0$

$$p_X(0) = \mathbb{P}(X=0) = \mathbb{P}(X=0, Y=0) + \mathbb{P}(X=0, Y=1) + \mathbb{P}(X=0, Y=2) + \cdots$$

$$= \frac{p}{q} \left(\left(\frac{q}{2} \right) + \left(\frac{q}{2} \right)^2 + \cdots \right)$$

$(x = \frac{q}{2})$

$$= \frac{p}{q} \big(x + x^2 + \cdots \big) = \frac{p}{q} \frac{x}{1-x} = \frac{1}{5} \frac{5}{7} = \frac{1}{7}.$$

A similar argument shows that

$$p_Y(j) = p_X(j) = \begin{cases} \frac{12}{35} \left(\frac{5}{7} \right)^j, & j > 0, \\ \\ \frac{1}{7}, & j = 0. \end{cases}$$

(iii) The variables X, Y are not independent since

$$p_X(0) p_Y(0) = \frac{1}{49} \neq 0 = \mathbb{P}(X=0, Y=0).$$

Exercise 3.5 The joint probability mass function of the random vector (X, Y) is

$$\mathbb{P}(X=b, Y=2b) = \mathbb{P}(X=2b, X=b) = \frac{1}{2}, \qquad \mathbb{P}(X=b, Y=b) = \mathbb{P}(X=2b, Y=2b) = 0.$$

This shows that X and Y have identical distributions

$$p_X(b) = p_Y(b) = \frac{1}{2} = p_X(2b) = p_Y(2b).$$

(i) The conclusion follows from the above equalities. Indeed, X, Y have identical distributions so

$$\mathbb{E}[X] = \mathbb{E}[Y] = \frac{1}{2} b + \frac{2b}{2} = \frac{3b}{2}.$$

(ii) Using the law of the subconscious statistician we deduce

$$\mathbb{E}\left[\frac{Y}{X} \right] = \sum_{x,y} \frac{y}{x} \mathbb{P}(X=x, Y=y) = \frac{1}{2} \frac{2b}{b} + \frac{1}{2} \frac{b}{2b} = 1 + \frac{1}{4} = \frac{5}{4}.$$

(iii) We have

$$\mathbb{E}[Z] = b\mathbb{P}(Z=b) + 2b\mathbb{P}(Z=2b).$$

From the law of total probability we deduce

$$\mathbb{P}(Z=b) = \underbrace{\mathbb{P}(Z=b|X=b)}_{=1-p(b)}\mathbb{P}(X=b) + \underbrace{\mathbb{P}(Z=b|X=2b)}_{=p(2b)}\mathbb{P}(X=2b) = \frac{1}{2}\big(\,1-p(b)+p(2b)\,\big).$$

Similarly

$$\mathbb{P}(Z=2b) = \underbrace{\mathbb{P}(Z=2b|X=b)}_{=p(b)}\mathbb{P}(X=b) + \underbrace{\mathbb{P}(Z=2b|X=2b)}_{=1-p(2b)}\mathbb{P}(X=2b) = \frac{1}{2}\big(\,1-p(2b)+p(b)\,\big).$$

Hence

$$\begin{aligned}\mathbb{E}[Z] &= \frac{b}{2}\Big(\,\big(\,1-p(b)+p(2b)\,\big) + 2\big(\,1-p(2b)+p(b)\,\big)\,\Big)\\ &= \frac{b}{2}\big(3+p(b)-p(2b)\,\big) = \frac{3b}{2} + \frac{b}{2}\underbrace{\left(\frac{1}{1+e^{2b}} - \frac{1}{1+e^{4b}}\right)}_{>0} > \frac{3b}{2} = \mathbb{E}[X].\end{aligned}$$

Exercise 3.14 Denote by p the joint probability mass function of (H,S);

$$p(i,j) = \mathbb{P}(H=i, S=j).$$

There are 13 hearts and 13 spades so $p(i,j)=p(j,i)$. Clearly $p(i,j)=0$ if $i+j>3$. We have

$$p(0,0) = \frac{\binom{26}{3}}{\binom{52}{3}} = \frac{2}{17},\;\; p(0,1) = \frac{\binom{26}{2}\binom{13}{1}}{\binom{52}{3}} = \frac{13}{68},$$

$$p(0,2) = \frac{\binom{26}{1}\binom{13}{2}}{\binom{52}{3}} = \frac{39}{425},\;\; p(0,3) = \frac{\binom{13}{3}}{\binom{52}{2}} = \frac{11}{850},$$

$$p(1,1) = \frac{13\cdot 13\cdot 26}{\binom{52}{3}} = \frac{169}{850},\;\; p(1,2) = \frac{13\cdot\binom{13}{2}}{\binom{52}{3}} = \frac{39}{850}.$$

Note that

$$\begin{aligned}\mathbb{E}[H] &= \mathbb{E}[S]\\ &= 0\cdot\mathbb{P}(H=0)\\ &\quad +1\cdot\mathbb{P}(H=1)\\ &\quad +2\cdot\mathbb{P}(H=2)\\ &\quad +3\cdot\mathbb{P}(H=3)\end{aligned}$$

$$= \frac{\binom{13}{1}\binom{39}{2}}{\binom{52}{3}} + 2\cdot\frac{\binom{3}{2}\binom{39}{1}}{\binom{52}{3}} + 3\cdot\frac{\binom{3}{3}}{\binom{52}{3}} \approx 0.1558,$$

$$\begin{aligned}\mathbb{E}[H^2] = \mathbb{E}[S^2] &= 0^2 \cdot \mathbb{P}(H=0) + 1^2 \cdot \mathbb{P}(H=1) + 2^2 \cdot \mathbb{P}(H=2) + 3^2 \cdot \mathbb{P}(H=3) \\ &= \frac{\binom{13}{1}\binom{39}{2}}{\binom{52}{3}} + 2^2 \cdot \frac{\binom{13}{2}\binom{39}{1}}{\binom{52}{3}} + 3^2 \cdot \frac{\binom{13}{3}}{\binom{52}{3}} \approx 0.75, \\ \mathrm{Var}(H) = \mathrm{Var}(S) &\approx 0.1780 - (0.1558)^2 \approx 1.1029.\end{aligned}$$

To calculate the covariance we first compute

$$\mathbb{E}[HS] = p(1,1) + 2p(1,2) + 2p(2,1) = \frac{13 \cdot 13 \cdot 26}{\binom{52}{3}} + \frac{4 \cdot 13\binom{13}{2}}{\binom{52}{3}} \approx 0.3823.$$

Hence

$$\mathrm{Cov}(H,S) \approx 0.3823 - (0.75)^2 \approx -0.1802.$$

Then

$$\rho[H,S] = \frac{\mathrm{Cov}(H,S)}{\sqrt{\mathrm{Var}(H)} \cdot \sqrt{\mathrm{Var}(S)}} \approx \frac{-0.1802}{1.1209} \approx -0.1633.$$

Exercise 4.1 (i) The constant c is determined from the equality

$$c \int_{|x| \leq y} (y^2 - x^2) e^{-y} dx dy = 1.$$

We have

$$\begin{aligned}\int_{|x| \leq y} (y^2 - x^2) e^{-y} dx dy &= \int_0^\infty dy \left(\int_{-y}^{y} (y^2 - x^2) e^{-y} \right) dx \\ &= \int_0^\infty \left(\int_{-y}^{y} y^2 e^{-y} dx - \int_{-y}^{y} x^2 e^{-y} dx \right) \\ &= \int_0^\infty 2y^3 e^{-y} dy - \frac{2}{3} \int_0^\infty y^3 e^{-y} dy = \frac{4}{3} \int_0^\infty y^3 e^{-y} dy.\end{aligned}$$

Iteratively integrative by parts, we get

$$\int_0^\infty y^3 e^{-y}\, dy = 3 \int_0^\infty y^2 e^{-y}\, dy = 3 \cdot 2 \int_0^\infty y e^{-y}\, dy = 3 \cdot 2 \cdot 1 \int_0^\infty e^{-y}\, dy = 6$$

so

$$c \cdot \frac{4}{3} \cdot 6 = 1.$$

This shows that $c = 1/8$.

(ii) Let us first compute the integrals

$$I_n(a) = \int_a^\infty y^n e^{-y} dy,$$

where $n = 0, 1, 2, \ldots$. We have

$$I_0(a) = \big(-e^{-y} \big) \Big|_{y=a}^{y=-\infty} = e^{-a}.$$

Next observe that

$$I_{n+1}(a) = -\int_a^\infty y^{n+1} d(e^{-y}) = \big(- y^{n+1} e^{-y} \big)\Big|_{y=a}^{y=\infty} + (n+1) \int_a^\infty y^n e^{-y} dy$$
$$= a^{n+1} e^{-a} + (n+1) I_n(a).$$

Hence

$$I_1(a) = ae^{-a} + I_0(a) = ae^{-a} + e^{-a} = (a+1)e^{-a},$$
$$I_2(a) = a^2 e^{-a} + 2I_1(a) = a^2 + 2(a+1)e^{-a} = (a^2 + 2a + 2)e^{-a}.$$

Denote by $f_X(x)$ and $f_Y(y)$ the marginal density of X and respectively Y. We have

$$f_X(x) = c \int_{|x|}^\infty (y^2 - x^2) e^{-y} dy = cI_2(|x|) - cx^2 I_0(|x|) = \frac{1}{8}(2|x| + 2)e^{-|x|}.$$

For $y < 0$ we have $f_Y(y) = 0$. For $y \geq 0$ we have

$$f_Y(y) = c \int_{-y}^{y} (y^2 - x^2) e^{-y} dx = 2cy^3 e^{-y} - \frac{2cy^3}{3} y^{-y} = \frac{4c}{3} y^3 e^{-y} = \frac{1}{6} y^3 e^{-y}.$$

(iii) $\mathbb{E}[X] = 0$ since $f_X(-x) = f_X(x)$.

Exercise 4.5 Let us write X and Y for the arrival times of Adam and Billy Bob. They are independent with respective distributions $\mathcal{U}nif([0.5, 1])$ and $\mathcal{U}nif([0.5, 1.25])$, so their joint density is $8/3$ over $[0.5, 1] \times [0.5, 1.25]$ and zero elsewhere.
(i) The probability that Billy Bob arrives first is

$$\mathbb{P}(Y \leq X) = \int_{x=-\infty}^{+\infty} \int_{y=-\infty}^{x} p(x,y)\, dy dx = \int_{0.5}^{1} \int_{0.5}^{x} \frac{8}{3}\, dy dx = \int_{1/2}^{1} \frac{8}{3} \left(x - \frac{1}{2} \right)\, dx = 1 - \frac{2}{3} = \frac{1}{3}.$$

(ii) The probability that Billy Bob has to wait at least 10 minutes is

$$\mathbb{P}(X \geq Y + 10/60) = \int_{-\infty}^{+\infty} \int_{-\infty}^{x-1/6} p(x,y)\, dy dx = \int_{1/2+1/6}^{1} \int_{1/2}^{x-1/6} \frac{8}{3}\, dy dx = \int_{2/3}^{1} \frac{8}{3} \left(x - \frac{2}{3} \right)\, dx$$
$$= \frac{20}{27} - \frac{16}{27} = \frac{4}{27}.$$

The probability that Adam has to wait at least 10 minutes is

$$\mathbb{P}(Y \geq X + 10/60) = \int_{-\infty}^{+\infty} \int_{x+1/6}^{+\infty} p(x,y)\, dy dx = \int_{0.5}^{1} \int_{x+1/6}^{1.25} \frac{8}{3}\, dy dx = \int_{1/2}^{1} \frac{8}{3} \left(\frac{13}{12} - x \right)\, dx$$
$$= \frac{13}{9} - 1 = \frac{4}{9}.$$

In total, the probability that one of them has to wait at least 10 minutes is

$$\frac{4}{27} + \frac{4}{9} = \frac{16}{27}.$$

Exercise 4.6 The equation $x^2 + Bx + C = 0$ has real roots if and only if $B^2 \geq 4C$. Thus, we need to compute $\mathbb{P}(4C \leq B^2)$. The density of (X, Y) is $1/(4n^2)$ over $[-n, n]^2$ and zero elsewhere, hence for $n \geq 1$, we have

$$\begin{aligned}
\mathbb{P}(4C \leq B^2) &= 1 - \mathbb{P}(4C > B^2) \\
&= 1 - \int_{b=-2\sqrt{n}}^{2\sqrt{n}} \int_{c=b^2/4}^{n} \left(\frac{1}{2n}\right)^2 dc\, db \\
&= 1 - \frac{1}{4n^2} \int_{b=-2\sqrt{n}}^{2\sqrt{n}} \left(n - \frac{b^2}{4}\right) db \\
&= 1 - \frac{1}{4n^2} \left(n \cdot 4\sqrt{n} - \frac{1}{12}(2\sqrt{n})^3 - \frac{1}{12}(2\sqrt{n})^3\right) \\
&= 1 - \frac{2}{3\sqrt{n}}.
\end{aligned}$$

We notice that the probability goes to 1 as $n \to \infty$.

Exercise 4.9 (a) We have

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = \left(\frac{1}{2}\right)^2.$$

(b) The variables X and $1/Y$ are independent and thus

$$\mathbb{E}[X/Y] = \mathbb{E}[X]\mathbb{E}[1/Y] = \frac{1}{2}\mathbb{E}[1/Y] = \frac{1}{2}\int_0^1 \frac{1}{y} dy = \infty.$$

(c) We have $\log(XY) = \log X + \log Y$ so that

$$\begin{aligned}
\mathbb{E}\left[\log(XY)\right] = \mathbb{E}\left[\log X\right] + \mathbb{E}\left[\log Y\right] = 2\mathbb{E}\left[\log X\right] = 2\int_0^1 \log x dx \\
= 2\left(x \log x - x\right)\Big|_0^1 = -2.
\end{aligned}$$

Exercise 1

If p is the joint probability mass function of some variables (U, V) with values in $\{0, 1, 2\} \times \{0, 1\}$ (for instance (X, Y) or (X', Y')), then

$$p_U(0) = p(0,0), \;\; p_U(1) = p(1,0) + p(1,1), \;\; p_U(2) = p(2,1),$$

$$p_V(0) = p(0,0) + p(1,0), \;\; p_V(1) = p(1,1) + p(2,1).$$

1. Since (X, Y) is jointly uniformly distributed,

$$p_X(0) = \frac{1}{4} = p_X(2), \;\; p_X(1) = \frac{1}{2},$$

$$p_Y(0) = \frac{1}{2}, \;\; p_Y(1) = \frac{1}{2}.$$

2. If X' and Y' are uniformly distributed on their ranges, then

$$p(0,0)+p(0,1)=p(1,0)+p(1,1)=p(2,0)+p(2,1)=\frac{1}{3},$$

$$p(0,0)+p(1,0)+p(2,0)=p(0,1)+p(1,1)+p(2,1)=\frac{1}{2}.$$

Setting $a=p(0,0)$, $b=p(1,0)$, we deduce

$$p(0,1)=\frac{1}{3}-a, \qquad p(1,1)=\frac{1}{3}-b, \qquad p(2,0)=\frac{1}{2}-(a+b), \qquad p(2,1)=(a+b)-\frac{1}{6}.$$

Any of these would do, provided

$$0\le a\le\frac{1}{3}, \qquad 0\le b\le\frac{1}{3}, \qquad \frac{1}{6}\le a+b\le\frac{1}{2}.$$