

Homework 1

Solution

February 19th

Exercises from the book

Exercise 1.1. Set $a = \mathbb{P}(\text{Heads})$, $b = \mathbb{P}(\text{Tails})$, $c = \mathbb{P}(\text{Edge})$. Then

$$a + b + c = 1, \quad c = 0.1, \quad a = 2b.$$

We deduce

$$0.9 = a + b = 3b \Rightarrow b = 0.3, \quad a = 0.6.$$

Exercise 1.5. Set A the event “rain on Saturday”, and B the event “rain on Sunday”. Then we are given the probabilities of, in order, A , B , $A \cap B$ and $A \cup B$.

(i) According to the inclusion-exclusion principle, we should have

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.7 + 0.6 - 0.4 = 0.9.$$

This does not correspond to the given probability, so the given numbers are inaccurate.

(ii) These numbers correspond to the probability space $S = \{RR, RS, SR, SS\}$ ($R \leftrightarrow$ rainy, $S \leftrightarrow$ sunny) with the probability function satisfying

$$\mathbb{P}(RR) = 0.4, \quad \mathbb{P}(RS) = 0.3, \quad \mathbb{P}(SR) = 0.2, \quad \mathbb{P}(SS) = 0.1.$$

The sum of these probabilities is one, so it is a valid probability space.

(iii) We have $A \cap B \subset A$, so we must have

$$0.8 = \mathbb{P}(A \cap B) \leq \mathbb{P}(A) = 0.7.$$

This is obviously not the case, hence the given probabilities are inconsistent. The same argument also works for $A \cap B \subset B$, $A \subset A \cup B$, $B \subset A \cup B$, and most notably $A \cap B \subset A \cup B$.

(iv) Using the inclusion-exclusion principle again, we must have

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.7 + 0.6 - 0.5 = 0.8.$$

This is again not satisfied, and these numbers do not correspond to any probability space.

Exercise 1.18 We have

$$\mathbb{P}(A|A \cup B) = \frac{\mathbb{P}(A \cap (A \cup B))}{\mathbb{P}(A \cup B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A) + \mathbb{P}(B)}.$$

We used $A \cap (A \cup B) = A$; this type of set equality is proved by showing that an element x belongs to the left hand side if and only if it belongs to the right hand side. In this case, it is not difficult.

Exercise 1.19 Because $B \cap C$ is not empty, C cannot be empty either. We have

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A \cap (B \cap C)) = \mathbb{P}(A|B \cap C)\mathbb{P}(B \cap C)$$

and

$$\mathbb{P}(B \cap C) = \mathbb{P}(B|C)\mathbb{P}(C),$$

and the first result follows directly.

As for the second identity,

$$\mathbb{P}(A|B \cap C) = \frac{\mathbb{P}(A \cap (B \cap C))}{\mathbb{P}(B \cap C)} = \frac{\mathbb{P}((A \cap B) \cap C)}{\mathbb{P}(B \cap C)} = \frac{\mathbb{P}(A \cap B|C)\mathbb{P}(C)}{\mathbb{P}(B|C)\mathbb{P}(C)} = \frac{\mathbb{P}(A \cap B|C)}{\mathbb{P}(B|C)}.$$

Exercise 1.20 Because A , B and C are independent, we have

$$\mathbb{P}(A \cap (B \cap C)) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \mathbb{P}(A)\mathbb{P}(B \cap C),$$

which is the definition of A and $B \cap C$ being independent.

Regarding A and $B \cup C$, note that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C);$$

indeed,

$$\begin{aligned} x \in A \cap (B \cup C) &\Leftrightarrow x \text{ belongs to } A, \text{ and also to either } B \text{ or } C \\ &\Leftrightarrow x \text{ belongs to either } A \text{ and } B, \text{ or } A \text{ and } C \\ &\Leftrightarrow x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

(You may just write the equality without justification.)

Then, using the inclusion/exclusion principle, we get

$$\begin{aligned} \mathbb{P}(A \cap (B \cup C)) &= \mathbb{P}((A \cap B) \cup (A \cap C)) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) - \mathbb{P}(A \cap B \cap C) \\ &= \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \mathbb{P}(A)(\mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B)\mathbb{P}(C)) \\ &= \mathbb{P}(A)(\mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C)) = \mathbb{P}(A)\mathbb{P}(B \cup C), \end{aligned}$$

as expected.

Exercise 1.22 For $i = 1, 2$ denote by E_i the event “the politician wins the i -th election”. We know that

$$\mathbb{P}(E_1) = 0.6, \quad \mathbb{P}(E_2) = 0.5, \quad \mathbb{P}(E_2|E_1) = 0.75.$$

(i)

$$\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_2|E_1)\mathbb{P}(E_1) = 0.75 \cdot 0.6 = 0.45.$$

(ii)

$$\mathbb{P}(E_1 \cap E_2^c) = \mathbb{P}(E_2^c|E_1)\mathbb{P}(E_1) = 0.25 \cdot 0.6 = 0.15.$$

(iii)

$$\mathbb{P}(E_1|E_2) = \frac{\mathbb{P}(E_1 \cap E_2)}{\mathbb{P}(E_2)} = \frac{0.45}{0.5} = 0.9.$$

(iv)

$$\mathbb{P}(E_2|E_1^c) = \frac{\mathbb{P}(E_1^c|E_2)\mathbb{P}(E_2)}{\mathbb{P}(E_1^c)} = \frac{(1 - \mathbb{P}(E_1|E_2))\mathbb{P}(E_2)}{\mathbb{P}(E_1^c)} = \frac{0.1 \cdot 0.5}{0.4} = 0.125.$$

Exercise 1

1, 2. A first possibility for S could be all the possible values for N : $\{2, 3, 4, \dots\} = \mathbb{N} \setminus \{0, 1\}$, where N is just the outcome. In this case, $A = \{2, 3\}$.

A second could be all the finite sequences of heads and tails with exactly two heads, and finishing with a heads:

$$S = \{(H, H), (H, T, H), (T, H, H), (H, T, T, H), (T, H, T, H), (T, T, H, H), \dots\}.$$

In this case, N is the length of the outcome, and $A = \{(H, H), (H, T, H), (T, H, H)\}$. The exercise did not ask for it, but the probability of a given outcome is $1/2^L = 2^{-L}$, where L is the length of the sequence.

A third one might be the set of couples (k, ℓ) , where k is the number of tails we have to go through before the first heads, and ℓ the number of tails between the first and the second heads. In this case, N is given by $k + 1 + \ell + 1 = k + \ell + 2$. It means that $S = \mathbb{N}^2$, and the outcomes corresponding to the ones in the previous example are

$$S = \{(0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0), \dots\}.$$

The probability of (k, ℓ) is $2^{-k-\ell-2}$, and the set A is $\{(0, 0), (1, 0), (0, 1)\}$ in this case.

I did cheat a bit in this third example: it is the second one in disguise, since there is a correspondence between their elements. An actual different example is the set of *unordered* pairs $\{k, \ell\}$, including $\{k, k\}$. Then $\{1, 3\}$ for instance may correspond to either (T, H, T, T, T, H) or (T, T, T, H, T, H) , but it does not change N : is still $k + \ell + 2$, independently of course of the ordering. The probability of $\{k, \ell\}$ is $2^{-k-\ell-2}$ if $k = \ell$, otherwise it is $2^{-k-\ell-1}$. In this instance, $A = \{\{0, 0\}, \{0, 1\}\}$.

A fifth and very natural possibility is the set of all sequences of heads and tails: $S = \{H, T\}^{\mathbb{N}}$. In this sample space, we have added useless tosses after the second heads, but of course we can recover N by finding the position of that second heads. The set A is the set of all sequences such that the first three elements are either (H, H, T) , (H, H, H) , (H, T, H) or (T, H, H) . In this sample

set, the probability of the event B consisting of all sequences (x_0, x_1, x_2, \dots) such that elements x_0, \dots, x_k (for instance, in our case we might want to consider $k = 3$) have fixed values y_0, \dots, y_k (in our case, maybe $y_1 = y_2 = H, y_3 = T$) is $\mathbb{P}(B) = 2^{-k}$.

Many more sample spaces are available, although it is likely that those given above are the most convenient.

3. We have

$$C = \{N \geq 5\} = \{N < 4\}^c = (\{N \leq 3\} \cup \{N = 4\})^c = (A \cup B)^c.$$

4. I will use the second probability space above.

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}(\{(H, H), (H, T, H), (T, H, H)\}) \\ &= \mathbb{P}((H, H)) + \mathbb{P}((H, T, H)) + \mathbb{P}((T, H, H)) \\ &= \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} \\ \mathbb{P}(B) &= \mathbb{P}(\{(H, T, T, H), (T, H, T, H), (T, T, H, H)\}) \\ &= \mathbb{P}((H, T, T, H)) + \mathbb{P}((T, H, T, H)) + \mathbb{P}((T, T, H, H)) \\ &= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{3}{16} \end{aligned}$$

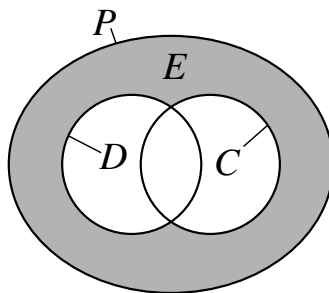
According to question 3, and noting that A and B are disjoint,

$$\mathbb{P}(C) = 1 - \mathbb{P}(A \cup B) = 1 - (\mathbb{P}(A) + \mathbb{P}(B)) = 1 - \left(\frac{1}{2} + \frac{3}{16}\right) = \frac{5}{16}.$$

Some of you may have used a variant of the fifth sample space above, and noted a sequence is in C if and only if its first 4 elements are either (T, T, T, T) , (H, T, T, T) , (T, H, T, T) , (T, T, H, T) or (T, T, T, H) , which gives the same result.

Exercise 2

We set P (resp. D, C) the probability that the randomly chosen household owns a pet (resp. a dog, a cat). Then the event we are interested in is $E = P \setminus (D \cup C)$.



Because $(D \cup C)$ is included in P , we have $P = (P \setminus (D \cup C)) \cup (D \cup C)$, and obviously the sets involved in this union are disjoint; hence

$$\mathbb{P}(P) = \mathbb{P}(E) + \mathbb{P}(D \cup C).$$

Using the inclusion-exclusion principle, we have

$$\mathbb{P}(D \cup C) = \mathbb{P}(D) + \mathbb{P}(C) - \mathbb{P}(D \cap C),$$

and finally

$$\begin{aligned} \mathbb{P}(E) &= \mathbb{P}(P) - \mathbb{P}(D \cup C) = \mathbb{P}(P) - (\mathbb{P}(D) + \mathbb{P}(C) - \mathbb{P}(D \cap C)) \\ &= 0.57 - 0.38 - 0.25 + 0.19 = 0.13. \end{aligned}$$

Exercise 3

The easiest way to prove that $\mathbb{P}(E) \leq \mathbb{P}(F)$ is to show that $E \subset F$. This exercise uses this trick over and over.

1. Of course $\mathbb{P}(B_0) \leq \mathbb{P}(A_0)$. Now by definition, B_{n+1} is A_{n+1} but with some elements taken away; it means that B_{n+1} is included in A_{n+1} , and $\mathbb{P}(B_{n+1}) \leq \mathbb{P}(A_{n+1})$. Of course it implies the second half immediately.

2. If $x \in A_n$, then in particular $x \in A_k$ for some k ($k = n$). Choose k to be the smallest possible. If $k = 0$, then $x \in A_0 = B_0$ and obviously $x \in \bigcup_{k=0}^n B_k$.

Suppose now that $k > 0$. Then $x \notin A_0, x \notin A_1, \dots, x \notin A_{k-1}, x \in A_k$. In other words, x is not in any $A_i, i < k$, hence not in $\bigcup_{i=0}^{k-1} A_i$. Since x however belongs to A_k , x must belong to B_k by definition of B_k , and $x \in \bigcup_{k=0}^n B_k$ using the fact that $k \leq n$.

To show the second part, take x in A . We will show that $x \in \bigcup_{n=0}^{\infty} B_n$, hence $A \subset \bigcup_{n=0}^{\infty} B_n$ and the conclusion will follow.

Since x is in A , it is in $\bigcup_{n=0}^{\infty} A_n$. In particular, it must belong to some fixed A_m . But according to the first part of the question, we see that it is in fact in $\bigcup_{k=0}^m B_k$, hence in $\bigcup_{n=0}^{\infty} B_n$. This is indeed what we expected.

3. Let n and m be two different integers. We want to show that B_n and B_m are disjoint, i.e. that no elements belongs to the two of them.

Suppose first that $n < m$. We have seen in question 1 that B_n is included in A_n . But by definition, B_m is A_m where we have taken away all the elements from the previous A_k , including A_n . It means that every element of B_n has been removed to get B_m , so that their intersection is indeed empty.

If $m < n$, then it is just the same argument the other way around.

4. Using, in order, question 2, 3 and 1, we get

$$\mathbb{P}(A) \leq \mathbb{P}\left(\bigcup_{n=0}^{\infty} B_n\right) = \sum_{n=0}^{\infty} \mathbb{P}(B_n) \leq \sum_{n=0}^{\infty} \mathbb{P}(A_n).$$