Homework 4
Solution

February 21st

Exercises from the book

1.34, 1.36, 1.39, 1.40, 1.42, 1.46, 1.48 and 1.52

Exercise 1.34 Denote by N_k the event "the number k does not appear during 3 successive rolls of the die". We are interested in the probability $\mathbb{P}(N_5|N_6)$. We have

$$\mathbb{P}(N_5|N_6) = \frac{\mathbb{P}(N_5 \cap N_6)}{\mathbb{P}(N_6)}.$$

We have

$$\mathbb{P}(N_6) = \frac{5^3}{6^3}, \quad \mathbb{P}(N_5 \cap N_6) = \frac{4^3}{6^3},$$

so

$$\mathbb{P}(N_5|N_6) = \frac{4^3}{5^3} = \frac{64}{125} = 0.512.$$

Exercise 1.36 First of all, I have to apologise; I misread the statement and thought it would be much easier to solve. But let us brace ourselves and get to work.

The model is as follows. First, choose the number C of children (for instance, 2 with probability 1/3). Then, choose the gender of the children, beginning with the oldest (for instance, a girl then a boy). Lastly, if there is a girl, choose one of them uniformly at random (in this case, there is no choice). We write G for the number of girls, and E for the event that we choose the oldest girl.[1] Using this approach, the probability we consider is $\mathbb{P}(E \,|\, G \geq 1)$.

[1] It may look strange to consider an event that is not well-defined if there is no girl. If you don't like abstract constructions, you have my word that this does not raise definitional difficulties, and you may leave this footnote. Otherwise, you can consider the experiment where you choose a number between 1 and 2, then a number between 1 and 3 (uniformly), then C, then the gender of the children. At this point, we say that E occurs if either (i) there is precisely one girl (ii) there are two girls and the first number we chose was a 1 (iii) there are three girls and the second number we chose was a 1.

By definition of the conditional probability, and then decomposing the probability according to the number of girls, we have

$$\mathbb{P}(E \,|\, G \geq 1) = \frac{\mathbb{P}(E \cap (G \geq 1))}{\mathbb{P}(G \geq 1)} = \frac{\mathbb{P}(E \cap (G = 1)) + \mathbb{P}(E \cap (G = 2)) + \mathbb{P}(E \cap (G = 3))}{\mathbb{P}(G = 1) + \mathbb{P}(G = 2) + \mathbb{P}(G = 3)}.$$

The probability $\mathbb{P}(E \cap (G = k))$ is $\mathbb{P}(E \,|\, G = k) \cdot \mathbb{P}(G = k) = \frac{1}{k}\mathbb{P}(G = k)$, so

$$\mathbb{P}(E \,|\, G \geq 1) = \frac{\frac{1}{1}\mathbb{P}(G = 1) + \frac{1}{2}\mathbb{P}(G = 2) + \frac{1}{3}\mathbb{P}(G = 3)}{\mathbb{P}(G = 1) + \mathbb{P}(G = 2) + \mathbb{P}(G = 3)}.$$

To find the probability mass function of the number of girls, we use the law of total probability, decomposing according the values of C. For instance,

$$\begin{aligned}\mathbb{P}(G = 2) &= \mathbb{P}(G = 2 \,|\, C = 1)\mathbb{P}(C = 1) + \mathbb{P}(G = 2 \,|\, C = 2)\mathbb{P}(C = 2) + \mathbb{P}(G = 2 \,|\, C = 3)\mathbb{P}(C = 3) \\ &= 0 + \frac{1}{3} \cdot \mathbb{P}(G = 2 \,|\, C = 2) + \frac{1}{3} \cdot \mathbb{P}(G = 2 \,|\, C = 3).\end{aligned}$$

There is only one way to have two girls out of two children, and a total of 4 possibilities (bb, bg, gb, gg), so the first probability is $1/4$. There is $\binom{3}{2} = 3$ ways to have two girls out of three children (we have to choose which of the three children are girls), and a total of $2^3 = 8$ possibilities so the second probability is $3/8$. Proceeding in this manner, we find for the probabilities $\mathbb{P}(G = k \,|\, C = n)$:

Children \ Girls	$G = 1$	$G = 2$	$G = 3$
$C = 1$	$\frac{1}{2}$		
$C = 2$	$\frac{2}{4} = \frac{1}{2}$	$\frac{1}{4}$	
$C = 3$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

This gives, using the formula above,

$$\mathbb{P}(G = 2) = 0 + \frac{1}{3} \cdot \mathbb{P}(G = 2 \,|\, C = 2) + \frac{1}{3} \cdot \mathbb{P}(G = 2 \,|\, C = 3) = \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{8} = \frac{5}{24}.$$

Similarly,

$$\mathbb{P}(G = 1) = \frac{1}{3} \cdot \mathbb{P}(G = 1 \,|\, C = 2) + \frac{1}{3} \cdot \mathbb{P}(G = 1 \,|\, C = 2) + \frac{1}{3} \cdot \mathbb{P}(G = 1 \,|\, C = 3) = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{3}{8} = \frac{11}{24}$$

and

$$\mathbb{P}(G = 3) = 0 + 0 + \frac{1}{3} \cdot \mathbb{P}(G = 3 \,|\, C = 3) = \frac{1}{3} \cdot \frac{1}{8} = \frac{1}{24}.$$

One can compute $\mathbb{P}(G = 0) = 7/24$, which checks out.

Recalling the formula for $\mathbb{P}(E \,|\, G \geq 1)$, and multiplying numerator and denominator by 24, we finally get

$$\mathbb{P}(E \,|\, G \geq 1) = \frac{\frac{1}{1} \cdot 11 + \frac{1}{2} \cdot 5 + \frac{1}{3} \cdot 1}{11 + 5 + 1} = \frac{\frac{83}{6}}{17} = \frac{83}{102}.$$

Exercise 1.39 Denote by $C \nrightarrow S$ the event "the luggage is missing in Sidney", by $C \nrightarrow LA$ the event "the luggage was lost between O'Hare and LAX" (mishandled in O'Hare) and by $LA \nrightarrow S$ the event "the luggage was lost between LAX and Sydney" (mishandled at LAX). We use $\rightarrow$ instead of $\nrightarrow$ for their complements. We know that for $p = 1\%$,

$$\mathbb{P}(C \nrightarrow LA) = p, \quad \mathbb{P}(LA \nrightarrow S|C \rightarrow LA) = p.$$

(i) We want to compute $\mathbb{P}(C \nrightarrow LA|M)$. We have

$$\mathbb{P}(C \nrightarrow LA|C \nrightarrow S) = \frac{\mathbb{P}(C \nrightarrow S|C \nrightarrow LA)\mathbb{P}(C \nrightarrow LA)}{\mathbb{P}(C \nrightarrow S)},$$

$$\mathbb{P}(C \nrightarrow S) = \mathbb{P}(C \nrightarrow S|C \nrightarrow LA)\mathbb{P}(C \nrightarrow LA) + \mathbb{P}(C \nrightarrow S|C \rightarrow LA)\mathbb{P}(C \rightarrow LA).$$

Note that $\mathbb{P}(C \nrightarrow S|C \nrightarrow LA) = 1$, $\mathbb{P}(C \nrightarrow S|C \rightarrow LA) = \mathbb{P}(LA \nrightarrow S|C \rightarrow LA) = p$. Hence

$$\mathbb{P}(C \nrightarrow S) = p + p(1-p),$$

$$\mathbb{P}(C \nrightarrow LA|C \nrightarrow S) = \frac{p}{p + p(1-p)} = \frac{1}{2-p} \approx 0.5025.$$

(ii) We have

$$1 = \mathbb{P}(LA \nrightarrow S|C \nrightarrow S) + \mathbb{P}(C \nrightarrow LA|C \nrightarrow S)$$

so that

$$\mathbb{P}(LA \nrightarrow S|C \nrightarrow S) = 1 - \frac{1}{2-p} \approx 0.4975.$$

Exercise 1.40 Denote by D_k the event "we get the number k after one roll of the die", T_k the event "we get k tails in a row" and by T the event described in the exercise: "we get only tails when we first roll a die and then we flip a coin as many times as the number we get after the die roll".

Using the law of total probability we deduce

$$\mathbb{P}(T) = \sum_{k=1}^{6} \mathbb{P}(T|D_k)\mathbb{P}(D_k).$$

Observe that for any $k = 1, \dots, 6$ we have

$$\mathbb{P}(T|D_k) = \mathbb{P}(T_k) = \frac{1}{2^k}, \;\; \mathbb{P}(D_k) = \frac{1}{6}.$$

Hence

$$\mathbb{P}(T) = \frac{1}{6}\sum_{k=1}^{6} 2^{-k} = \frac{1}{6}\left(1 - \frac{1}{2^6}\right) = \frac{21}{128}.$$

Exercise 1.42 Consider the events A "we get an ace when drawing a card" and S "we get a spade that is not an ace when drawing a card". Recall that the probability to see A before S is

$$\frac{\mathbb{P}(A)}{\mathbb{P}(A) + \mathbb{P}(S)}.$$

We see that the probability that we first get an ace is

$$\frac{\frac{4}{52}}{\frac{4}{52} + \frac{12}{52}} = \frac{4}{16} = \frac{1}{4}.$$

Exercise 1.46 Let E be the event "one ace was drawn from the thickened 2nd deck".

For $i = 0, 1, 2$ we denote by A_i the event "there were i aces among the two cards drawn from the original first deck". Observe that

$$\mathbb{P}(A_0) = \frac{(48)_2}{(52)_2} = \frac{\binom{48}{2}}{\binom{52}{2}} = \frac{48 \cdot 47}{52 \cdot 51} \approx 0.8506,$$

$$\mathbb{P}(A_1) = \frac{2 \cdot 4 \cdot 48}{(52)_2} \approx 0.1447,$$

$$\mathbb{P}(A_2) = \frac{(4)_2}{(52)_2} \approx 0.0045.$$

We have

$$\mathbb{P}(A_0|E) = \frac{\mathbb{P}(E|A_0)\mathbb{P}(A_0)}{\mathbb{P}(E|A_0)\mathbb{P}(A_0) + \mathbb{P}(E|A_1)\mathbb{P}(A_1) + \mathbb{P}(E|A_2)\mathbb{P}(A_2)}.$$

The thickened deck has 54 cards and, given A_i, it has $4 + i$ aces. Using this we deduce

$$\mathbb{P}(E|A_0) = \frac{4}{54}, \;\; \mathbb{P}(E|A_1) = \frac{5}{54}, \;\; \mathbb{P}(E|A_2) = \frac{6}{54}.$$

Hence

$$\begin{aligned}
\mathbb{P}(A_0|E) &= \frac{\frac{4}{54}\mathbb{P}(A_0)}{\frac{4}{54}\mathbb{P}(A_0) + \frac{5}{54}\mathbb{P}(A_1) + \frac{6}{54}\mathbb{P}(A_2)} = \frac{4\mathbb{P}(A_0)}{4\mathbb{P}(A_0) + 5\mathbb{P}(A_1) + 6\mathbb{P}(A_2)} \\
&= \frac{4 \cdot 48 \cdot 47}{4 \cdot 48 \cdot 47 + 5 \cdot 2 \cdot 4 \cdot 48 + 6 \cdot 4 \cdot 3} = \frac{376}{459} \\
&\approx 0.8192.
\end{aligned}$$

Exercise 1.48 Denote by U_i the event "the i-th urn was picked" and by B_k the event "a ball labelled k was drawn".

(a) In this case $\mathbb{P}(U_i) = 1/2$, in both cases $i = 1, 2$. We have to compute $\mathbb{P}(U_1|B_5)$. We have

$$\begin{aligned}
\mathbb{P}(U_1|B_5) = \frac{\mathbb{P}(B_5|U_1)\mathbb{P}(U_1)}{\mathbb{P}(B_5)} &= \frac{\mathbb{P}(B_5|U_1)\mathbb{P}(U_1)}{\mathbb{P}(B_5|U_1)\mathbb{P}(U_1) + \mathbb{P}(B_5|U_2)\mathbb{P}(U_2)} \\
&= \frac{\frac{1}{10} \cdot 0.5}{\frac{1}{10} \cdot 0.5 + \frac{1}{100} \cdot 0.5} = \frac{0.1}{0.11} = \frac{1}{11} \approx 0.909.
\end{aligned}$$

(b) In this case we have

$$\mathbb{P}(U_1) = \frac{10}{110}, \qquad \mathbb{P}(U_2) = \frac{100}{110}, \qquad \mathbb{P}(B_5) = \frac{2}{110}.$$

We deduce

$$\mathbb{P}(U_1|B_5) = \frac{\mathbb{P}(B_5|U_1)\mathbb{P}(U_1)}{\mathbb{P}(B_5)} = \frac{\frac{1}{10} \cdot \frac{10}{110}}{\frac{2}{110}} = \frac{1}{2}.$$

Exercise 1.52 (i) Denote by A the event that there are no accidents during the next successive n days after an accident. Then

$$\mathbb{P}(A) = (1 - p_1)(1 - p_2) \cdots (1 - p_n).$$

If you are so inclined, you can use the notation

$$\mathbb{P}(A) = \prod_{k=1}^{n} (1 - p_k).$$

(ii) Denote by B the event that there is exactly one accident during the next successive n days after an accident. We have

$$\begin{aligned}\mathbb{P}(B) = {} & p_1(1 - p_1) \cdots (1 - p_{n-1}) + (1 - p_1)p_2(1 - p_1) \cdots (1 - p_{n-2}) \\ & + \cdots + (1 - p_1) \cdots (1 - p_{n-1})p_n.\end{aligned}$$

Again, if you like that sort of things, you can write it as

$$\mathbb{P}(B) = \prod_{k=1}^{n} (1 - p_k) \cdot \sum_{k=1}^{n} \frac{p_k}{1 - p_k}.$$

Exercise 1

1. It is the first time there is no edge between two points: $\mathcal{G}eom(1 - p)$.

   It doesn't matter if this doesn't count as a 'success' to us: if you want, you can imagine that you are playing against someone, and they are successful if you are stopped.

2. The event "we can go to infinity", call it E_∞, is included in the event "N is at least n", that we call E_n. As see in class, N is at least n if the first n experiments are 'failures', in our case if the first n edges are present, and this event has probability p^n. Hence, we have

   $$\mathbb{P}(E_\infty) \leq \mathbb{P}(E_n) = p^n.$$

   We can take the limit as n goes to infinity in the inequality, and deduce that $\mathbb{P}(E_\infty) \leq 0$, so this probability has to be zero.

3. The event E_∞ "we can go to infinity" is included in the union $E_+ \cup E_-$, where $E_\pm$ is the event "all edges from zero to $\pm\infty$ are present". According to the previous question, E_- and E_+ have probability zero.

   We can then either use Homework 1, and say that

   $$\mathbb{P}(E_\infty) \leq \mathbb{P}(E_+) + \mathbb{P}(E_-) = 0 + 0 = 0;$$

   alternatively, we can use the inclusion/exclusion formula:

   $$\mathbb{P}(E_\infty) = \mathbb{P}(E_+) + \mathbb{P}(E_-) - \mathbb{P}(E_+ \cap E_-) = -\mathbb{P}(E_+ \cap E_-) \leq 0.$$

   In any case, the probability cannot be positive, so it must be zero.