Quiz 3

Exercise 1

Let X and Y by continuous random variables with the joint probability density function

$$p(x,y) = \begin{cases} \frac{x^2+\sin(\pi y)}{C} & \text{for } x,y \in [0,1], \\ 0 & \text{otherwise,} \end{cases}$$

for some constant $C > 0$.

1. What is the value of C?
2. What is the correlation of X and Y? Are X and Y independent?

1. The integral of the density over $\mathbb{R}^2$ should be 1.

$$1 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(x,y)\mathrm{d}x\mathrm{d}y = \frac{1}{C} \int_0^1 \int_0^1 \left(x^2 + \sin(\pi y)\right) \mathrm{d}x\mathrm{d}y,$$

so

$$\begin{aligned} C &= \int_0^1 \int_0^1 x^2 \mathrm{d}x\mathrm{d}y + \int_0^1 \int_0^1 \sin(\pi y)\mathrm{d}x\mathrm{d}y \\ &= \left(\int_0^1 x^2 \mathrm{d}x\right)\left(\int_0^1 \mathrm{d}y\right) + \left(\int_0^1 \mathrm{d}x\right)\left(\int_0^1 \sin(\pi y)\mathrm{d}y\right) \\ &= \frac{1}{3} \cdot 1 + 1 \cdot \frac{2}{\pi} = \frac{1}{3} + \frac{2}{\pi}. \end{aligned}$$

2. We must compute the expectations of XY, X^2, Y^2, X and Y. A small tip: it looks like we are going to compute the same integrals many times, so I keep a small table somewhere with the integrals I already computed. In the table I already have the (simple, not double) integrals of 1, x^2 and $\sin(\pi y)$ from the question before.

$$
\begin{aligned}
C\mathbb{E}[XY] &= \int_0^1 \int_0^1 xy\left(x^2+\sin(\pi y)\right)\mathrm{d}x\mathrm{d}y \\
&= \left(\int_0^1 x^3 \mathrm{d}x\right)\left(\int_0^1 y\mathrm{d}y\right) + \left(\int_0^1 x\mathrm{d}x\right)\left(\int_0^1 y\sin(\pi y)\mathrm{d}y\right) \\
&= \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{2}\cdot\left(\left[y\cdot\frac{-\cos(\pi y)}{\pi}\right]_0^1 - \int_0^1 1\cdot\frac{-\cos(\pi y)}{\pi}\mathrm{d}y\right) \\
&= \frac{1}{8} + \frac{1}{2}\left(\frac{1}{\pi} - 0\right) = \frac{1}{8} + \frac{1}{2\pi}.
\end{aligned}
$$

The integrals of x (or y), x^3, $y\sin(\pi y)$ and $\cos(\pi y)$ are added to my toolbox.

$$
\begin{aligned}
C\mathbb{E}[X^2] &= \int_0^1 \int_0^1 x^2\left(x^2+\sin(\pi y)\right)\mathrm{d}x\mathrm{d}y \\
&= \left(\int_0^1 x^4 \mathrm{d}x\right)\left(\int_0^1 \mathrm{d}y\right) + \left(\int_0^1 x^2\mathrm{d}x\right)\left(\int_0^1 \sin(\pi y)\mathrm{d}y\right) \\
&= \frac{1}{5}\cdot 1 + \frac{1}{3}\cdot\frac{2}{\pi} = \frac{1}{5} + \frac{2}{3\pi}.
\end{aligned}
$$

The integral of x^4 is added to my toolbox (I already knew all the others!).

$$
\begin{aligned}
C\mathbb{E}[Y^2] &= \int_0^1 \int_0^1 y^2\left(x^2+\sin(\pi y)\right)\mathrm{d}x\mathrm{d}y \\
&= \left(\int_0^1 x^2 \mathrm{d}x\right)\left(\int_0^1 y^2\mathrm{d}y\right) + \left(\int_0^1 \mathrm{d}x\right)\left(\int_0^1 y^2\sin(\pi y)\mathrm{d}y\right) \\
&= \frac{1}{3}\cdot\frac{1}{3} + 1\cdot\left(\left[y^2\cdot\frac{-\cos(\pi y)}{\pi}\right]_0^1 - \int_0^1 2y\cdot\frac{-\cos(\pi y)}{\pi}\mathrm{d}y\right) \\
&= \frac{1}{9} + \frac{1}{\pi} + \frac{2}{\pi}\int_0^1 y\cos(\pi y)\mathrm{d}y \\
&= \frac{1}{9} + \frac{1}{\pi} + \frac{2}{\pi}\cdot\left(\left[y\cdot\frac{\sin(\pi y)}{\pi}\right]_0^1 - \int_0^1 1\cdot\frac{\sin(\pi y)}{\pi}\mathrm{d}y\right) \\
&= \frac{1}{9} + \frac{1}{\pi} + \frac{2}{\pi}\cdot\left(0 - \frac{2}{\pi^2}\right) = \frac{1}{9} + \frac{1}{\pi} - \frac{4}{\pi^3}
\end{aligned}
$$

Integrals of $y^2\sin(\pi y)$ and $y\cos(\pi y)$ added to the toolbox.

$$\begin{aligned}
C\mathbb{E}[X] &= \int_0^1 \int_0^1 x\left(x^2+\sin(\pi y)\right) \mathrm{d}x\mathrm{d}y \\
&= \left(\int_0^1 x^3 \mathrm{d}x\right)\left(\int_0^1 \mathrm{d}y\right) + \left(\int_0^1 x\mathrm{d}x\right)\left(\int_0^1 \sin(\pi y)\mathrm{d}y\right) \\
&= \frac{1}{4}\cdot 1 + \frac{1}{2}\cdot\frac{2}{\pi} = \frac{1}{4}+\frac{1}{\pi}.
\end{aligned}$$

$$\begin{aligned}
C\mathbb{E}[Y] &= \int_0^1 \int_0^1 y\left(x^2+\sin(\pi y)\right) \mathrm{d}x\mathrm{d}y \\
&= \left(\int_0^1 x^2 \mathrm{d}x\right)\left(\int_0^1 y\mathrm{d}y\right) + \left(\int_0^1 \mathrm{d}x\right)\left(\int_0^1 y\sin(\pi y)\mathrm{d}y\right) \\
&= \frac{1}{3}\cdot\frac{1}{2} + 1\cdot\frac{1}{\pi} = \frac{1}{6}+\frac{1}{\pi}.
\end{aligned}$$

For these last two computations, we actually already encountered every integral.

From this we deduce the variances and covariance:

$$\begin{aligned}
C^2\mathrm{Cov}(X,Y) &= C^2\mathbb{E}[XY] - C\mathbb{E}[X]\cdot C\mathbb{E}[Y] \\
&= \left(\frac{1}{3}+\frac{2}{\pi}\right)\left(\frac{1}{8}+\frac{1}{2\pi}\right) - \left(\frac{1}{4}+\frac{1}{\pi}\right)\left(\frac{1}{6}+\frac{1}{\pi}\right) \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
C^2\mathrm{Var}(X) &= C^2\mathbb{E}[X^2] - \left(C\mathbb{E}[X]\right)^2 \\
&= \left(\frac{1}{3}+\frac{2}{\pi}\right)\left(\frac{1}{5}+\frac{2}{3\pi}\right) - \left(\frac{1}{4}+\frac{1}{\pi}\right)\left(\frac{1}{4}+\frac{1}{\pi}\right) \\
&= \frac{1}{240}+\frac{11}{90\pi}+\frac{1}{3\pi^2},
\end{aligned}$$

$$\begin{aligned}
C^2\mathrm{Var}(Y) &= C^2\mathbb{E}[Y^2] - \left(C\mathbb{E}[Y]\right)^2 \\
&= \left(\frac{1}{3}+\frac{2}{\pi}\right)\left(\frac{1}{9}+\frac{1}{\pi}-\frac{4}{\pi^3}\right) - \left(\frac{1}{6}+\frac{1}{\pi}\right)\left(\frac{1}{6}+\frac{1}{\pi}\right) \\
&= \frac{1}{108}+\frac{2}{9\pi}+\frac{1}{\pi^2}-\frac{4}{3\pi^3}+\frac{8}{\pi^4}.
\end{aligned}$$

Then the correlation is

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\cdot\mathrm{Var}(Y)}} = 0.$$

However, independent variables have zero correlation (we say that they are decorrelated), but having no correlation does not mean that the variables are independent. In fact, in our case X and Y are not independent.

Indeed, if X and Y were independent, we would have $p(x,y) = p_X(x)p_Y(y)$ so

$$f(x)g(y) = x^2 + \sin(y)$$

over $(0,1)^2$ for some functions f and g. There are many ways to see that this is not possible (my instinct is to take the logarithm and differentiate with respect to x), for instance[1]

$$\frac{f(2/3)}{f(1/3)} = \frac{f(2/3)\cdot g(1/6)}{f(1/3)\cdot g(1/6)} = \frac{(\frac{2}{3})^2 + \sin(\frac{\pi}{6})}{(\frac{1}{3})^2 + \sin(\frac{\pi}{6})} = \frac{\frac{4}{9}+\frac{1}{2}}{\frac{1}{9}+\frac{1}{2}} = \frac{17}{11}$$

but

$$\frac{f(2/3)}{f(1/3)} = \frac{f(2/3)\cdot g(1/2)}{f(1/3)\cdot g(1/2)} = \frac{(\frac{2}{3})^2 + \sin(\frac{\pi}{2})}{(\frac{1}{3})^2 + \sin(\frac{\pi}{2})} = \frac{\frac{4}{9}+1}{\frac{1}{9}+1} = \frac{13}{10},$$

which is impossible.

Exercise 2

We consider a button wired to a tally counter, but through a random device. Initially, the counter indicates zero. Then if at some point the counter shows k when the button is pressed, the counter is incremented only with probability 2^{-k}, otherwise nothing happens. For instance, the first time the button is pressed, the counter indicates zero so it increases with probability 1. The second time, the counter is one so it goes to either 1 or 2 with equal probability 1/2.

We write X_n for what the counter shows after the button is pressed n times (so X_n is a random variable and $X_0 = 0$). We denote their probability mass function by p_n; since the variables take non-negative integer values, we have for instance $p_n(-2) = 0$ for all n.

1. What is p_2?

2. Suppose p_n is known. Express the probability mass function of (X_n, X_{n+1}) using p_n but not p_{n+1}.

3. Let Y be a random variable with values in $\mathbb{N}$. What is the value of the following sum?

$$\sum_{k\geq 0} p_Y(k-1)$$

4. Show that

$$\mathbb{E}\left[2^{X_{n+1}}\right] = \mathbb{E}\left[2^{X_n}\right] + 1.$$

For instance, you can see X_{n+1} as a function of (X_n, X_{n+1})

[1] I am dividing by $f(1/3)$ and various values of g, which is often dangerous because we don't know much about f and g, and they might vanish for all we know. But in this case, $f(x)g(y) = x^2 + \sin(y) > 0$ for all $x, y \in (0,1)^2$, so neither f nor g can take the value zero.

5. Use induction to show that
$$\mathbb{E}\left[2^{X_n} - 1\right] = n.$$

1. X_1 must always be one, so X_2 can take at most two values, 1 or 2. In particular, $p_2(x) = 0$ for all $x \notin \{1, 2\}$.

   Since X_1 is always 1, we increase the counter with probability 2^{-1}, and leave it as it is with probability $1 - 2^{-1}$. So we have $p_2(1) = p_2(2) = 1/2$.

2. Since the X_n are the values of a counter, and we can only increase it by at most one from X_n to X_{n+1}, the probability mass function $p_{X_n,X_{n+1}}(k, \ell)$ is zero unless k is a non-negative integer and $\ell \in \{k, k+1\}$. In this case, we have
$$\begin{aligned}
p_{X_n,X_{n+1}}(k, k+1) &= \mathbb{P}(X_n = k, X_{n+1} = k+1) \\
&= \mathbb{P}(X_{n+1} = k+1 | X_n = k) \cdot \mathbb{P}(X_n = k) \\
&= 2^{-k} p_n(k),
\end{aligned}$$
$$\begin{aligned}
p_{X_n,X_{n+1}}(k, k) &= \mathbb{P}(X_n = k, X_{n+1} = k) \\
&= \mathbb{P}(X_{n+1} = k | X_n = k) \cdot \mathbb{P}(X_n = k) \\
&= (1 - 2^{-k}) p_n(k).
\end{aligned}$$

3. Because p_Y is a probability mass function that is zero out of $\mathbb{N}$,
$$\sum_{k \geq 0} p_Y(k-1) = \sum_{k \geq -1} p_Y(k) = \sum_{k \geq 0} p_Y(k) = 1.$$

4. We use the expression of $p_{X_n,X_{n+1}}$ that we found in question 2:
$$\begin{aligned}
\mathbb{E}[2^{X_{n+1}}] &= \sum_{k \geq 0} 2^k p_{n+1}(k) \\
&= \sum_{k \geq 0} 2^k \cdot \Big(\mathbb{P}(X_n = k, X_{n+1} = k) + \mathbb{P}(X_n = k-1, X_{n+1} = k)\Big) \\
&= \sum_{k \geq 0} 2^k \cdot \Big((1 - 2^{-k})\, p_n(k) + 2^{-k+1}\, p_n(k-1)\Big) \\
&= \sum_{k \geq 0} 2^k p_n(k) - \sum_{k \geq 0} p_n(k) + 2 \sum_{k \geq 0} p_n(k-1) \\
&= \mathbb{E}[2^{X_n}] - 1 + 2 = \mathbb{E}[2^{X_n}] + 1,
\end{aligned}$$
   where we used question 2 at the last line.

5. **Base case.** Since X_0 is constant equal to zero, obviously $\mathbb{E}[2^{X_0} - 1] = 2^0 - 1 = 0$.

   Induction. Assume that $(2^{X_n} - 1)$ has mean n. Then the previous question shows that
$$\mathbb{E}\left[2^{X_{n+1}} - 1\right] = \mathbb{E}\left[2^{X_n+1}\right] - 1 = \mathbb{E}\left[2^{X_n}\right] = \mathbb{E}\left[2^{X_n} - 1\right] + 1 = n + 1$$
   as expected.

Exercise 3

Let X and Y be two variables which admit a joint probability density function p given by

$$p(x,y) = \begin{cases} 1/6 & \text{if } (x,y) \in (0,2) \times (0,1), \\ 2/3 & \text{if } (x,y) \in (2,3) \times (0,1), \\ 0 & \text{else.} \end{cases}$$

We imagine that X and Y are the price in a few months' time of some raw product. An investment company wants to choose between two strategies A and B, which have been designed to give a respective (random) profit e^{-X} and Y^2.

1. What are the probability density functions of the marginals X and Y?
2. What are the expectations of e^{-X} and Y^2? Which strategy is better in average?
3. Let $\Omega \subset \mathbb{R}^2$ be the subset of all (x,y) such that the strategy A is better than the strategy B when $X = x$, $Y = y$. Keeping in mind that plotters are not allowed, draw Ω roughly on a graph. Indicate the values of p on the same graph.
4. What is the probability that the strategy A is better than the strategy B?

1. The density $p_X(x)$ of the marginal X is the integral of $p(x,y)$ over y, and similarly for Y:

$$\begin{aligned} p_X(x) = \int_{-\infty}^{+\infty} p(x,y)\mathrm{d}y &= 0 && \text{for } x \leq 0 \text{ or } x = 2 \text{ or } x \geq 3, \\ &= \int_0^1 \frac{1}{6}\mathrm{d}y = \frac{1}{6} && \text{for } x \in (0,2), \\ &= \int_0^1 \frac{2}{3}\mathrm{d}y = \frac{2}{3} && \text{for } x \in (2,3). \end{aligned}$$

$$\begin{aligned} p_Y(y) = \int_{-\infty}^{+\infty} p(x,y)\mathrm{d}x &= 0 && \text{for } y \leq 0 \text{ or } y \geq 1, \\ &= \int_0^2 \frac{1}{6}\mathrm{d}x + \int_2^3 \frac{2}{3}\mathrm{d}x = \frac{1}{3} + \frac{2}{3} = 1 && \text{for } y \in (0,1). \end{aligned}$$

In short,

$$p_X(x) = \begin{cases} \frac{1}{6} & \text{if } x \in (0,2), \\ \frac{2}{3} & \text{if } x \in (2,3), \\ 0 & \text{else,} \end{cases} \qquad p_Y(y) = \begin{cases} 1 & \text{if } y \in (0,1), \\ 0 & \text{else.} \end{cases}$$

2.

$$
\begin{aligned}
\mathbb{E}\left[e^{-X}\right] &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x}\cdot p(x,y)\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{6}\int_{x=0}^{2}\int_{y=0}^{1} e^{-x}\mathrm{d}x\mathrm{d}y + \frac{2}{3}\int_{x=2}^{3}\int_{y=0}^{1} e^{-x}\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{6}\left[-e^{-x}\right]_0^2\cdot 1 + \frac{2}{3}\left[-e^{-x}\right]_2^3\cdot 1 \\
&= \frac{1}{6} - \frac{1}{6e^2} + \frac{2}{3e^2} - \frac{2}{3e^3} \\
&= \frac{1}{6} + \frac{1}{2e^2} - \frac{2}{3e^3} \approx 0.201
\end{aligned}
$$

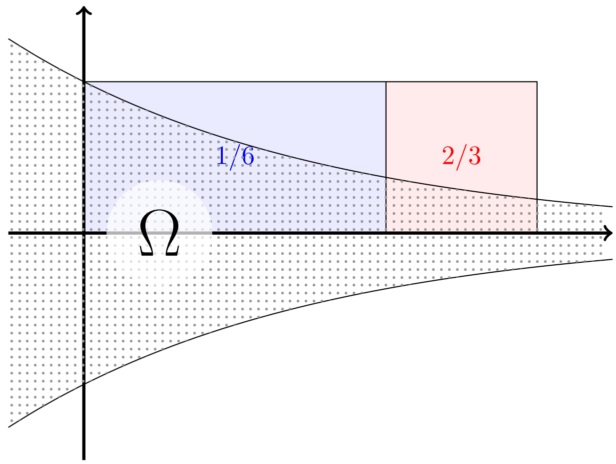
$$
\begin{aligned}
\mathbb{E}\left[Y^2\right] &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y^2\cdot p(x,y)\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{6}\int_{x=0}^{2}\int_{y=0}^{1} y^2\mathrm{d}x\mathrm{d}y + \frac{2}{3}\int_{x=2}^{3}\int_{y=0}^{1} y^2\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{6}\cdot 2\left[\frac{y^3}{3}\right]_0^1 + \frac{2}{3}\cdot 1\left[\frac{y^3}{3}\right]_0^1 \\
&= \frac{1}{9} - 0 + \frac{2}{9} - 0 \\
&= \frac{1}{3} \approx 0.333
\end{aligned}
$$

The strategy B is better on average.

3. The strategy A is better than B if $e^{-X} > Y^2$, so Ω is the set of points $(x,y)\in\mathbb{R}^2$ such that $e^{-x} > y^2$. But

$$e^{-x} > y^2 \Leftrightarrow |y| < \sqrt{e^{-x}} \Leftrightarrow -e^{-x/2} < y < e^{-x/2}$$

so Ω is the region between the curves $y = -\exp(-x/2)$ and $y = \exp(-x/2)$. On a graph, it looks like this:



4. The probability that A is better than B is the integral over Ω of $p(x, y)$:

$$
\begin{aligned}
\mathbb{P}(\text{A is better than B}) &= \iint_{\Omega} p(x, y)\mathrm{d}x\mathrm{d}y \\
&= \int_{x=0}^{2}\int_{y=0}^{\exp(-x/2)} \frac{1}{6}\,\mathrm{d}x\mathrm{d}y + \int_{x=2}^{3}\int_{y=0}^{\exp(-x/2)} \frac{2}{3}\,\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{6}\int_{0}^{2} e^{-x/2}\mathrm{d}x + \frac{2}{3}\int_{2}^{3} e^{-x/2}\mathrm{d}x \\
&= \frac{1}{6}\cdot\left[-2e^{-x/2}\right]_0^2 + \frac{2}{3}\cdot\left[-2e^{-x/2}\right]_2^3 \\
&= \frac{1}{3} - \frac{1}{3e} + \frac{4}{3e} - \frac{4}{3e^{3/2}} \\
&= \frac{1}{3} + \frac{1}{e} - \frac{4}{3e^{3/2}} \approx 0.404
\end{aligned}
$$

The strategy B is more likely to be the best than the strategy A.

Exercise 4

Let X be a variable with distribution $\mathcal{E}xp(2)$. Define

$$f(x) = \frac{2x}{1 + x^2}$$

and $Y = f(X)$.

1. Find the critical points of f and its limit at $+\infty$. Draw roughly the graph of f over $\mathbb{R}_+$.
2. Find all points $x \geq 0$ such that $f(x) \leq 1/\sqrt{2}$.
3. What is the density of Y?

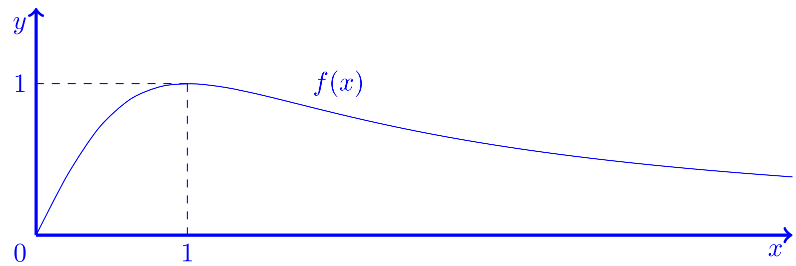
1. The critical points of $f(x)$ are the x such that $f'(x) = 0$:

$$f'(x) = 0 \Leftrightarrow \frac{2 \cdot (1 + x^2) - 2x \cdot 2x}{(1 + x^2)^2} = 0 \Leftrightarrow -2x^2 + 2 = 0 \Leftrightarrow x = -1 \text{ or } x = -1.$$

   Moreover,

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x} \cdot \frac{2}{1 + 1/x^2} = 0 \cdot \frac{2}{1 + 0} = 0.$$

   The graph of the curve looks something like this.



2. We have

$$f(x) \leq 1/\sqrt{2} \Leftrightarrow 2\sqrt{2}x \leq 1 + x^2 \Leftrightarrow x^2 - 2\sqrt{2}x + 1 \geq 0 \Leftrightarrow (x - \sqrt{2})^2 - 1 \geq 0 \Leftrightarrow |x - \sqrt{2}| \geq 1,$$

   so $x \geq 0$ satisfies $f(x) \leq 1/\sqrt{2}$ if and only if $x \in [0, \sqrt{2} - 1] \cup [\sqrt{2} + 1, +\infty)$.

3. We find first the cumulative distribution function of Y. Since the image of $\mathbb{R}_+$ by f is $[0, 1]$, Y has values in that same interval and $F_Y(y) = 0$ for $y \leq 0$, $F_Y(y) = 1$ for $y \geq 1$. Then for $y \in (0, 1)$,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(f(X) \leq y).$$

   Reasoning as in question 2,

$$\begin{aligned} f(x) \leq y &\Leftrightarrow yx^2 - 2x + y \geq 0 \\ &\Leftrightarrow y\left(x - \frac{1}{y}\right)^2 + y - \frac{1}{y} \geq 0 \\ &\Leftrightarrow \left|x - \frac{1}{y}\right| \geq \sqrt{\frac{1}{y^2} - 1} = \frac{\sqrt{1 - y^2}}{y} \end{aligned}$$

   so $f(X) \leq y$ if and only if X is in the set

$$\left[0, \frac{1 - \sqrt{1 - y^2}}{y}\right] \cup \left[\frac{1 + \sqrt{1 - y^2}}{y}, +\infty\right) = [0, b_y] \cup [a_y, +\infty) = \Omega_y.$$

Hence the probability we are interested in is

$$\begin{aligned}
F_Y(y) &= \mathbb{P}(f(X) \le y) \\
&= \mathbb{P}(X \in \Omega_y) \\
&= \mathbb{P}(X \le b_y) + \mathbb{P}(X \ge a_y) \\
&= F_X(b_y) + (1 - F_X(a_y)) \\
&= 1 - \exp\left(-2 \cdot \frac{1 - \sqrt{1-y^2}}{y}\right) + \exp\left(-2 \cdot \frac{1 + \sqrt{1-y^2}}{y}\right).
\end{aligned}$$

One can check that the limit of this expression is 0 (resp. 1) when y goes to 0 (resp. 1). The density of Y is then the derivative of $F_Y(y)$. It is zero out of $(0,1)$. For $y \in (0,1)$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1 \pm \sqrt{1-y^2}}{y}\right) &= \frac{\mp\frac{1}{2}\left(1-y^2\right)^{-1/2} \cdot 2y \cdot y - \left(1 \pm (1-y^2)^{1/2}\right) \cdot 1}{y^2} \\
&= \frac{\mp y^2 - \sqrt{1-y^2} \mp (1-y^2)}{y^2\sqrt{1-y^2}} \\
&= -\frac{1}{y^2}\left(1 \pm \frac{1}{\sqrt{1-y^2}}\right)
\end{aligned}$$

so we get

$$\begin{aligned}
p_Y(y) = \frac{2}{y^2} \exp\left(-2 \cdot \frac{1 - \sqrt{1-y^2}}{y}\right) \cdot \left(\frac{1}{\sqrt{1-y^2}} - 1\right) \\
+ \frac{2}{y^2} \exp\left(-2 \cdot \frac{1 + \sqrt{1-y^2}}{y}\right) \cdot \left(1 + \frac{1}{\sqrt{1-y^2}}\right)
\end{aligned}$$

for all $y \in (0,1)$.