Sommaire

Chapitre 1

Calcul du système à résoudre pour GD

1.1 Intro

$$
\begin{aligned}
\forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \int_{t_m}^{t_{m+1}} \mathbf{w_1} \left(\mathbf{M}\ \dot{\mathbf{v}} + \mathbf{C}\mathbf{v} + \mathbf{K}\mathbf{u} - \mathbf{f} \right)\ dt + \int_{t_m}^{t_{m+1}} \mathbf{w_2} \mathbf{K} \left(\dot{\mathbf{u}} - \mathbf{v} \right)\ dt \\
& + \mathbf{w_2}(t_m)\mathbf{K}[\mathbf{u}]_m + \mathbf{w_1}(t_m)\mathbf{M}[\mathbf{v}]_m \qquad = 0
\end{aligned}
\tag{1.1}
$$

$$
\begin{aligned}
\forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \int_{t_m}^{t_{m+1}} \mathbf{w_1}\mathbf{M}\dot{\mathbf{v}}\ dt + \int_{t_m}^{t_{m+1}} \mathbf{w_1}\mathbf{C}\mathbf{v}\ dt + \int_{t_m}^{t_{m+1}} \mathbf{w_1}\mathbf{K}\mathbf{u}\ dt \\
& - \int_{t_m}^{t_{m+1}} \mathbf{w_1}\mathbf{f}\ dt + \int_{t_m}^{t_{m+1}} \mathbf{w_2}\mathbf{K}\dot{\mathbf{u}}\ dt - \int_{t_m}^{t_{m+1}} \mathbf{w_2}\mathbf{K}\mathbf{v}\ dt \\
& + \mathbf{w_2}(t_m)\mathbf{K}[\mathbf{u}]_m + \mathbf{w_1}(t_m)\mathbf{M}[\mathbf{v}]_m \qquad = 0
\end{aligned}
\tag{1.2}
$$

$$
\begin{aligned}
\forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \int_{t_m}^{t_{m+1}} \mathbf{w_1}\mathbf{M}\dot{\mathbf{v}}\ dt + \int_{t_m}^{t_{m+1}} \mathbf{w_1}\mathbf{C}\mathbf{v}\ dt + \int_{t_m}^{t_{m+1}} \mathbf{w_1}\mathbf{K}\mathbf{u}\ dt \\
& - \int_{t_m}^{t_{m+1}} \mathbf{w_1}\mathbf{f}\ dt + \int_{t_m}^{t_{m+1}} \mathbf{w_2}\mathbf{K}\dot{\mathbf{u}}\ dt - \int_{t_m}^{t_{m+1}} \mathbf{w_2}\mathbf{K}\mathbf{v}\ dt \\
& + \mathbf{w_2}(t_m)\mathbf{K}(\mathbf{u}_m^+ - \mathbf{u}_m^-) + \mathbf{w_1}(t_m)\mathbf{M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \qquad = 0
\end{aligned}
\tag{1.3}
$$

1.2 Hypothèse de fonction de poids constantes

On choisit des fonctions de poids constantes sur l'intervalle :

$$\begin{aligned}\forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \mathbf{w_1} \int_{t_m}^{t_{m+1}} \mathbf{M}\dot{\mathbf{v}}\ dt + \mathbf{w_1} \int_{t_m}^{t_{m+1}} \mathbf{C}\mathbf{v}\ dt + \mathbf{w_1} \int_{t_m}^{t_{m+1}} \mathbf{K}\mathbf{u}\ dt \\ & -\mathbf{w_1} \int_{t_m}^{t_{m+1}} \mathbf{f}\ dt + \mathbf{w_2} \int_{t_m}^{t_{m+1}} \mathbf{K}\dot{\mathbf{u}}\ dt - \mathbf{w_2} \int_{t_m}^{t_{m+1}} \mathbf{K}\mathbf{v}\ dt \\ & +\mathbf{w_2}\mathbf{K}(\mathbf{u}_m^+ - \mathbf{u}_m^-) + \mathbf{w_1}\mathbf{M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \quad = 0\end{aligned} \tag{1.4}$$

Si on admet l'indépendance par rapport au temps des matrices M, C et K (cas linéaire) on a :

$$\begin{aligned}\forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \mathbf{w_1}\mathbf{M} \int_{t_m}^{t_{m+1}} \dot{\mathbf{v}}\ dt + \mathbf{w_1}\mathbf{C} \int_{t_m}^{t_{m+1}} \mathbf{v}\ dt + \mathbf{w_1}\mathbf{K} \int_{t_m}^{t_{m+1}} \mathbf{u}\ dt \\ & -\mathbf{w_1} \int_{t_m}^{t_{m+1}} \mathbf{f}\ dt + \mathbf{w_2}\mathbf{K} \int_{t_m}^{t_{m+1}} \dot{\mathbf{u}}\ dt - \mathbf{w_2}\mathbf{K} \int_{t_m}^{t_{m+1}} \mathbf{v}\ dt \\ & +\mathbf{w_2}\mathbf{K}(\mathbf{u}_m^+ - \mathbf{u}_m^-) + \mathbf{w_1}\mathbf{M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \quad = 0\end{aligned} \tag{1.5}$$

On admet la définition des fonctions sur chaque intervalles telle que :

$$\mathbf{q}(t) = \frac{t_{m+1} - t}{\Delta t}\mathbf{q}_m^+ + \frac{t - t_m}{\Delta t}\mathbf{q}_{m+1}^- \tag{1.6}$$

$$\begin{aligned}\forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \mathbf{w_1}\mathbf{M}\left[\mathbf{v}_{m+1}^- - \mathbf{v}_m^+\right] \\ & +\mathbf{w_1}\mathbf{C} \int_{t_m}^{t_{m+1}} \left[\frac{t_{m+1} - t}{\Delta t}\mathbf{v}_m^+ + \frac{t - t_m}{\Delta t}\mathbf{v}_{m+1}^-\right]\ dt \\ & +\mathbf{w_1}\mathbf{K} \int_{t_m}^{t_{m+1}} \left[\frac{t_{m+1} - t}{\Delta t}\mathbf{u}_m^+ + \frac{t - t_m}{\Delta t}\mathbf{u}_{m+1}^-\right]\ dt \\ & -\mathbf{w_1} \int_{t_m}^{t_{m+1}} \left[\frac{t_{m+1} - t}{\Delta t}\mathbf{f}_m + \frac{t - t_m}{\Delta t}\mathbf{f}_{m+1}\right]\ dt \\ & +\mathbf{w_2}\mathbf{K}\left[\mathbf{u}_{m+1}^- - \mathbf{u}_m^+\right] \\ & -\mathbf{w_2}\mathbf{K} \int_{t_m}^{t_{m+1}} \left[\frac{t_{m+1} - t}{\Delta t}\mathbf{v}_m^+ + \frac{t - t_m}{\Delta t}\mathbf{v}_{m+1}^-\right]\ dt \\ & +\mathbf{w_2}\mathbf{K}(\mathbf{u}_m^+ - \mathbf{u}_m^-) + \mathbf{w_1}\mathbf{M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \quad = 0\end{aligned} \tag{1.7}$$

On s'intéresse à la première intégrale :

$$\begin{aligned}\forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \mathbf{w_1 M}\left[\mathbf{v}_{m+1}^- - \mathbf{v}_m^+\right] \\ & + \frac{\mathbf{w_1 C}}{\Delta t}\int_{t_m}^{t_{m+1}} \left[(t_{m+1}-t)\mathbf{v}_m^+ + (t-t_m)\mathbf{v}_{m+1}^-\right]\ dt \\ & + \mathbf{w_1 K}\int_{t_m}^{t_{m+1}} \left[\frac{t_{m+1}-t}{\Delta t}\mathbf{u}_m^+ + \frac{t-t_m}{\Delta t}\mathbf{u}_{m+1}^-\right]\ dt \\ & - \mathbf{w_1}\int_{t_m}^{t_{m+1}} \left[\frac{t_{m+1}-t}{\Delta t}\mathbf{f}_m + \frac{t-t_m}{\Delta t}\mathbf{f}_{m+1}\right]\ dt \\ & + \mathbf{w_2 K}\left[\mathbf{u}_{m+1}^- - \mathbf{u}_m^+\right] \\ & - \mathbf{w_2 K}\int_{t_m}^{t_{m+1}} \left[\frac{t_{m+1}-t}{\Delta t}\mathbf{v}_m^+ + \frac{t-t_m}{\Delta t}\mathbf{v}_{m+1}^-\right]\ dt \\ & + \mathbf{w_2 K}(\mathbf{u}_m^+ - \mathbf{u}_m^-) + \mathbf{w_1 M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \qquad = 0\end{aligned} \tag{1.8}$$

On sépare :

$$\begin{aligned}\forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \mathbf{w_1 M}\left[\mathbf{v}_{m+1}^- - \mathbf{v}_m^+\right] \\ & + \frac{\mathbf{w_1 C}}{\Delta t}\left(\mathbf{v}_m^+\int_{t_m}^{t_{m+1}} (t_{m+1}-t)dt + \mathbf{v}_{m+1}^-\int_{t_m}^{t_{m+1}} (t-t_m)dt\right) \\ & + \mathbf{w_1 K}\int_{t_m}^{t_{m+1}} \left[\frac{t_{m+1}-t}{\Delta t}\mathbf{u}_m^+ + \frac{t-t_m}{\Delta t}\mathbf{u}_{m+1}^-\right]\ dt \\ & - \mathbf{w_1}\int_{t_m}^{t_{m+1}} \left[\frac{t_{m+1}-t}{\Delta t}\mathbf{f}_m + \frac{t-t_m}{\Delta t}\mathbf{f}_{m+1}\right]\ dt \\ & + \mathbf{w_2 K}\left[\mathbf{u}_{m+1}^- - \mathbf{u}_m^+\right] \\ & - \mathbf{w_2 K}\int_{t_m}^{t_{m+1}} \left[\frac{t_{m+1}-t}{\Delta t}\mathbf{v}_m^+ + \frac{t-t_m}{\Delta t}\mathbf{v}_{m+1}^-\right]\ dt \\ & + \mathbf{w_2 K}(\mathbf{u}_m^+ - \mathbf{u}_m^-) + \mathbf{w_1 M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \qquad = 0\end{aligned} \tag{1.9}$$

On calcule :

$$\begin{aligned}\forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \mathbf{w_1 M}\left[\mathbf{v}_{m+1}^- - \mathbf{v}_m^+\right] \\ & + \frac{\mathbf{w_1 C}}{\Delta t}\left(\mathbf{v}_m^+\left[\frac{(\Delta t)^2}{2}\right] + \mathbf{v}_{m+1}^-\left[\frac{(\Delta t)^2}{2}\right]\right) \\ & + \mathbf{w_1 K}\int_{t_m}^{t_{m+1}} \left[\frac{t_{m+1}-t}{\Delta t}\mathbf{u}_m^+ + \frac{t-t_m}{\Delta t}\mathbf{u}_{m+1}^-\right]\ dt \\ & - \mathbf{w_1}\int_{t_m}^{t_{m+1}} \left[\frac{t_{m+1}-t}{\Delta t}\mathbf{f}_m + \frac{t-t_m}{\Delta t}\mathbf{f}_{m+1}\right]\ dt \\ & + \mathbf{w_2 K}\left[\mathbf{u}_{m+1}^- - \mathbf{u}_m^+\right] \\ & - \mathbf{w_2 K}\int_{t_m}^{t_{m+1}} \left[\frac{t_{m+1}-t}{\Delta t}\mathbf{v}_m^+ + \frac{t-t_m}{\Delta t}\mathbf{v}_{m+1}^-\right]\ dt \\ & + \mathbf{w_2 K}(\mathbf{u}_m^+ - \mathbf{u}_m^-) + \mathbf{w_1 M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \qquad = 0\end{aligned} \tag{1.10}$$

En factorisant :

$$\begin{aligned}
\forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \mathbf{w_1 M}\left[\mathbf{v}_{m+1}^{-} - \mathbf{v}_m^{+}\right] \\
& + \mathbf{w_1 C}\frac{\Delta t}{2}\left(\mathbf{v}_m^{+} + \mathbf{v}_{m+1}^{-}\right) \\
& + \mathbf{w_1 K}\int_{t_m}^{t_{m+1}}\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{u}_m^{+} + \frac{t-t_m}{\Delta t}\mathbf{u}_{m+1}^{-}\right]\, dt \\
& - \mathbf{w_1}\int_{t_m}^{t_{m+1}}\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{f}_m + \frac{t-t_m}{\Delta t}\mathbf{f}_{m+1}\right]\, dt \\
& + \mathbf{w_2 K}\left[\mathbf{u}_{m+1}^{-} - \mathbf{u}_m^{+}\right] \\
& - \mathbf{w_2 K}\int_{t_m}^{t_{m+1}}\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{v}_m^{+} + \frac{t-t_m}{\Delta t}\mathbf{v}_{m+1}^{-}\right]\, dt \\
& + \mathbf{w_2 K}(\mathbf{u}_m^{+} - \mathbf{u}_m^{-}) + \mathbf{w_1 M}(\mathbf{v}_m^{+} - \mathbf{v}_m^{-}) \qquad = 0
\end{aligned} \tag{1.11}$$

en l'appliquant aux autres lignes :

$$\begin{aligned}
\forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \mathbf{w_1 M}\left[\mathbf{v}_{m+1}^{-} - \mathbf{v}_m^{+}\right] \\
& + \mathbf{w_1 C}\frac{\Delta t}{2}\left(\mathbf{v}_m^{+} + \mathbf{v}_{m+1}^{-}\right) \\
& + \mathbf{w_1 K}\frac{\Delta t}{2}\left(\mathbf{u}_m^{+} + \mathbf{u}_{m+1}^{-}\right) \\
& - \mathbf{w_1}\frac{\Delta t}{2}\left(\mathbf{f}_m + \mathbf{f}_{m+1}\right) \\
& + \mathbf{w_2 K}\left[\mathbf{u}_{m+1}^{-} - \mathbf{u}_m^{+}\right] \\
& - \mathbf{w_2 K}\frac{\Delta t}{2}\left(\mathbf{v}_m^{+} + \mathbf{v}_{m+1}^{-}\right) \\
& + \mathbf{w_2 K}(\mathbf{u}_m^{+} - \mathbf{u}_m^{-}) + \mathbf{w_1 M}(\mathbf{v}_m^{+} - \mathbf{v}_m^{-}) \qquad = 0
\end{aligned} \tag{1.12}$$

Matrice

$$\begin{aligned}
&\left(\mathbf{w_1}\left[\mathbf{K}\tfrac{\Delta t}{2} \quad \mathbf{K}\tfrac{\Delta t}{2} \quad \mathbf{C}\tfrac{\Delta t}{2} \quad \left(\mathbf{M} + \mathbf{C}\tfrac{\Delta t}{2}\right)\right] + \mathbf{w_2}\left[0 \quad \mathbf{K} \quad -\mathbf{K}\tfrac{\Delta t}{2} \quad -\mathbf{K}\tfrac{\Delta t}{2}\right]\right) \times \begin{bmatrix} \mathbf{u}_m^{+} \\ \mathbf{u}_{m+1}^{-} \\ \mathbf{v}_m^{+} \\ \mathbf{v}_{m+1}^{-} \end{bmatrix} \\
&\qquad = \mathbf{w_2 K u}_m^{-} + \mathbf{w_1 M v}_m^{-} + \mathbf{w_1}(\mathbf{f}_m + \mathbf{f}_{m+1})\tfrac{\Delta t}{2}
\end{aligned} \tag{1.13}$$

1.2.1 Hypothèse d'homogénéité : $\mathbf{w_1} = \dfrac{d\mathbf{w_2}}{dt}$

Matrice

Si on choisi $\mathbf{w_1} = \dfrac{d\mathbf{w_2}}{dt}$, on peut alors vérifier l'homogénéité des équations. La forme matricielle précédente est basée sur l'hypothèse que $\mathbf{w_2}$ est constante et donc que $\mathbf{w_1}$ est nulle. Alors :

$$\begin{aligned} &\mathbf{w_2}\begin{bmatrix}0 & \mathbf{K} & -\mathbf{K}\frac{\Delta t}{2} & -\mathbf{K}\frac{\Delta t}{2}\end{bmatrix}\begin{bmatrix}\mathbf{u}_m^+\\ \mathbf{u}_{m+1}^-\\ \mathbf{v}_m^+\\ \mathbf{v}_{m+1}^-\end{bmatrix} \\ &= \mathbf{w_2}\mathbf{K}\mathbf{u}_m^- \end{aligned} \tag{1.14}$$

1.3 Hypothèse de fonctions de poids affine

Si on veut plus d'équations il faut retourner à l'équation 1.3 juste avant de faire l'hypothèse de fonctions $\mathbf{w}$ constantes.

$$\begin{aligned} \forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \int_{t_m}^{t_{m+1}} \mathbf{w_1}\mathbf{M}\dot{\mathbf{v}}\ dt + \int_{t_m}^{t_{m+1}} \mathbf{w_1}\mathbf{C}\mathbf{v}\ dt + \int_{t_m}^{t_{m+1}} \mathbf{w_1}\mathbf{K}\mathbf{u}\ dt \\ & - \int_{t_m}^{t_{m+1}} \mathbf{w_1}\mathbf{f}\ dt + \int_{t_m}^{t_{m+1}} \mathbf{w_2}\mathbf{K}\dot{\mathbf{u}}\ dt - \int_{t_m}^{t_{m+1}} \mathbf{w_2}\mathbf{K}\mathbf{v}\ dt \\ & + \mathbf{w_2}(t_m)\mathbf{K}(\mathbf{u}_m^+ - \mathbf{u}_m^-) + \mathbf{w_1}(t_m)\mathbf{M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \quad = 0 \end{aligned} \tag{1.15}$$

1.3.1 Fonction de poids : $\mathbf{w_2} = \mathbf{w}.(t - t_m)$

On choisit des fonctions de poids sur l'intervalle $\mathbf{w_2} = \mathbf{w}.(t - t_m)$ alors $\mathbf{w_1} = \mathbf{w}$:

$$\begin{aligned} \forall \mathbf{w} \quad & \mathbf{w}\int_{t_m}^{t_{m+1}} \mathbf{M}\dot{\mathbf{v}}\ dt + \mathbf{w}\int_{t_m}^{t_{m+1}} \mathbf{C}\mathbf{v}\ dt + \mathbf{w}\int_{t_m}^{t_{m+1}} \mathbf{K}\mathbf{u}\ dt \\ & -\mathbf{w}\int_{t_m}^{t_{m+1}} \mathbf{f}\ dt + \mathbf{w}\int_{t_m}^{t_{m+1}} (t - t_m)\mathbf{K}\dot{\mathbf{u}}\ dt - \mathbf{w}\int_{t_m}^{t_{m+1}} (t - t_m)\mathbf{K}\mathbf{v}\ dt \\ & +\mathbf{w}(t_m - t_m)\mathbf{K}(\mathbf{u}_m^+ - \mathbf{u}_m^-) + \mathbf{w}\mathbf{M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \quad = 0 \end{aligned} \tag{1.16}$$

Si on admet l'indépendance par rapport au temps des matrices M, C et K (cas linéaire) on a :

$$\begin{aligned} \forall \mathbf{w} \quad & \mathbf{w}\mathbf{M}\int_{t_m}^{t_{m+1}} \dot{\mathbf{v}}\ dt + \mathbf{w}\mathbf{C}\int_{t_m}^{t_{m+1}} \mathbf{v}\ dt + \mathbf{w}\mathbf{K}\int_{t_m}^{t_{m+1}} \mathbf{u}\ dt \\ & -\mathbf{w}\int_{t_m}^{t_{m+1}} \mathbf{f}\ dt + \mathbf{w}\mathbf{K}\int_{t_m}^{t_{m+1}} (t - t_m)\dot{\mathbf{u}}\ dt - \mathbf{w}\mathbf{K}\int_{t_m}^{t_{m+1}} (t - t_m)\mathbf{v}\ dt \\ & +\mathbf{w}\mathbf{M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \quad = 0 \end{aligned} \tag{1.17}$$

On admet la définition des fonctions sur chaque intervalles telle que :

$$\mathbf{q}(t) = \frac{t_{m+1} - t}{\Delta t}\mathbf{q}_m^+ + \frac{t - t_m}{\Delta t}\mathbf{q}_{m+1}^- \qquad \text{avec} \qquad \Delta t = t_{m+1} - t_m \tag{1.18}$$

$$
\begin{aligned}
\forall \mathbf{w} \quad & \mathbf{wM}\left[\mathbf{v}_{m+1}^{-}-\mathbf{v}_{m}^{+}\right] \\
& +\mathbf{wC}\int_{t_m}^{t_{m+1}}\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{v}_{m}^{+}+\frac{t-t_m}{\Delta t}\mathbf{v}_{m+1}^{-}\right]\ dt \\
& +\mathbf{wK}\int_{t_m}^{t_{m+1}}\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{u}_{m}^{+}+\frac{t-t_m}{\Delta t}\mathbf{u}_{m+1}^{-}\right]\ dt \\
& -\mathbf{w}\int_{t_m}^{t_{m+1}}\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{f}_{m}+\frac{t-t_m}{\Delta t}\mathbf{f}_{m+1}\right]\ dt \\
& +\mathbf{wK}\frac{\left[\mathbf{u}_{m+1}^{-}-\mathbf{u}_{m}^{+}\right]}{\Delta t}\int_{t_m}^{t_{m+1}}(t-t_m) \\
& -\mathbf{wK}\int_{t_m}^{t_{m+1}}(t-t_m)\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{v}_{m}^{+}+\frac{t-t_m}{\Delta t}\mathbf{v}_{m+1}^{-}\right]\ dt \\
& +\mathbf{wM}(\mathbf{v}_{m}^{+}-\mathbf{v}_{m}^{-}) \quad =0
\end{aligned}
\tag{1.19}
$$

On retire $\mathbf{w}$:

$$
\begin{aligned}
& \mathbf{M}\left[\mathbf{v}_{m+1}^{-}-\mathbf{v}_{m}^{+}\right] \\
& +\mathbf{C}\int_{t_m}^{t_{m+1}}\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{v}_{m}^{+}+\frac{t-t_m}{\Delta t}\mathbf{v}_{m+1}^{-}\right]\ dt \\
& +\mathbf{K}\int_{t_m}^{t_{m+1}}\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{u}_{m}^{+}+\frac{t-t_m}{\Delta t}\mathbf{u}_{m+1}^{-}\right]\ dt \\
& -\int_{t_m}^{t_{m+1}}\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{f}_{m}+\frac{t-t_m}{\Delta t}\mathbf{f}_{m+1}\right]\ dt \\
& +\mathbf{K}\frac{\left[\mathbf{u}_{m+1}^{-}-\mathbf{u}_{m}^{+}\right]}{\Delta t}\int_{t_m}^{t_{m+1}}(t-t_m) \\
& -\mathbf{K}\int_{t_m}^{t_{m+1}}(t-t_m)\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{v}_{m}^{+}+\frac{t-t_m}{\Delta t}\mathbf{v}_{m+1}^{-}\right]\ dt \\
& +\mathbf{M}(\mathbf{v}_{m}^{+}-\mathbf{v}_{m}^{-}) \quad =0
\end{aligned}
\tag{1.20}
$$

On utilise le résultat obtenu précédemment :

$$
\int_{t_m}^{t_{m+1}}\left[\frac{t_{m+1}-t}{\Delta t}A+\frac{t-t_m}{\Delta t}B\right]\ dt=\frac{\Delta t}{2}(A+B) \tag{1.21}
$$

$$
\begin{aligned}
&\mathbf{M}\left[\mathbf{v}_{m+1}^{-}-\mathbf{v}_{m}^{+}\right] \\
&+\mathbf{C}\frac{\Delta t}{2}\left(\mathbf{v}_{m}^{+}+\mathbf{v}_{m+1}^{-}\right) \\
&+\mathbf{K}\frac{\Delta t}{2}\left(\mathbf{u}_{m}^{+}+\mathbf{u}_{m+1}^{-}\right) \\
&-\frac{\Delta t}{2}\left(\mathbf{f}_{m}+\mathbf{f}_{m+1}\right) \\
&+\mathbf{K}\frac{\left[\mathbf{u}_{m+1}^{-}-\mathbf{u}_{m}^{+}\right]}{\Delta t}\int_{t_m}^{t_{m+1}}(t-t_m) \\
&-\mathbf{K}\int_{t_m}^{t_{m+1}}(t-t_m)\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{v}_{m}^{+}+\frac{t-t_m}{\Delta t}\mathbf{v}_{m+1}^{-}\right]\,dt \\
&+\mathbf{M}(\mathbf{v}_{m}^{+}-\mathbf{v}_{m}^{-}) \quad = 0
\end{aligned}
\tag{1.22}
$$

Puis :

$$
\begin{aligned}
&\mathbf{M}\left[\mathbf{v}_{m+1}^{-}-\mathbf{v}_{m}^{+}\right] \\
&+\mathbf{C}\frac{\Delta t}{2}\left(\mathbf{v}_{m}^{+}+\mathbf{v}_{m+1}^{-}\right) \\
&+\mathbf{K}\frac{\Delta t}{2}\left(\mathbf{u}_{m}^{+}+\mathbf{u}_{m+1}^{-}\right) \\
&-\frac{\Delta t}{2}\left(\mathbf{f}_{m}+\mathbf{f}_{m+1}\right) \\
&+\mathbf{K}\frac{\left[\mathbf{u}_{m+1}^{-}-\mathbf{u}_{m}^{+}\right]}{\Delta t}\frac{(\Delta t)^2}{2} \\
&-\mathbf{K}\int_{t_m}^{t_{m+1}}(t-t_m)\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{v}_{m}^{+}+\frac{t-t_m}{\Delta t}\mathbf{v}_{m+1}^{-}\right]\,dt \\
&+\mathbf{M}(\mathbf{v}_{m}^{+}-\mathbf{v}_{m}^{-}) \quad = 0
\end{aligned}
\tag{1.23}
$$

L'intégrale restante :

$$
\begin{aligned}
& -\mathbf{K}\int_{t_m}^{t_{m+1}}(t-t_m)\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{v}_{m}^{+}+\frac{t-t_m}{\Delta t}\mathbf{v}_{m+1}^{-}\right]\,dt \\
= & -\mathbf{K}\int_{0}^{\Delta t} t\left[\frac{\Delta t-t}{\Delta t}\mathbf{v}_{m}^{+}+\frac{t}{\Delta t}\mathbf{v}_{m+1}^{-}\right]\,dt \\
= & -\mathbf{K}\int_{0}^{\Delta t}\left[\frac{t.\Delta t}{\Delta t}\mathbf{v}_{m}^{+}+\frac{-t^2}{\Delta t}\mathbf{v}_{m}^{+}+\frac{t^2}{\Delta t}\mathbf{v}_{m+1}^{-}\right]\,dt \\
= & -\mathbf{K}\left[\frac{t^2.\Delta t}{2\Delta t}\mathbf{v}_{m}^{+}+\frac{-t^3}{3\Delta t}\mathbf{v}_{m}^{+}+\frac{t^3}{3\Delta t}\mathbf{v}_{m+1}^{-}\right]_{0}^{\Delta t} \\
= & -\mathbf{K}\left[\frac{(\Delta t)^2}{2}\mathbf{v}_{m}^{+}-\frac{(\Delta t)^2}{3}\mathbf{v}_{m}^{+}+\frac{(\Delta t)^2}{3}\mathbf{v}_{m+1}^{-}\right]
\end{aligned}
\tag{1.24}
$$

Donc :

$$\begin{aligned}
&\mathbf{M}\left[\mathbf{v}_{m+1}^{-}-\mathbf{v}_{m}^{+}\right]\\
&+\mathbf{C}\frac{\Delta t}{2}\left(\mathbf{v}_{m}^{+}+\mathbf{v}_{m+1}^{-}\right)\\
&+\mathbf{K}\frac{\Delta t}{2}\left(\mathbf{u}_{m}^{+}+\mathbf{u}_{m+1}^{-}\right)\\
&-\frac{\Delta t}{2}\left(\mathbf{f}_{m}+\mathbf{f}_{m+1}\right)\\
&+\mathbf{K}\frac{\left[\mathbf{u}_{m+1}^{-}-\mathbf{u}_{m}^{+}\right]}{\Delta t}\frac{(\Delta t)^2}{2}\\
&-\mathbf{K}\left[\frac{(\Delta t)^2}{2}\mathbf{v}_{m}^{+}-\frac{(\Delta t)^2}{3}\mathbf{v}_{m}^{+}+\frac{(\Delta t)^2}{3}\mathbf{v}_{m+1}^{-}\right]\\
&+\mathbf{M}(\mathbf{v}_{m}^{+}-\mathbf{v}_{m}^{-})\qquad=0
\end{aligned}\tag{1.25}$$

et

$$\begin{aligned}
&\mathbf{M}\left[\mathbf{v}_{m+1}^{-}-\mathbf{v}_{m}^{+}\right]\\
&+\mathbf{C}\frac{\Delta t}{2}\left(\mathbf{v}_{m}^{+}+\mathbf{v}_{m+1}^{-}\right)\\
&+\mathbf{K}\frac{\Delta t}{2}\left(\mathbf{u}_{m}^{+}+\mathbf{u}_{m+1}^{-}\right)\\
&-\frac{\Delta t}{2}\left(\mathbf{f}_{m}+\mathbf{f}_{m+1}\right)\\
&+\mathbf{K}\left[\mathbf{u}_{m+1}^{-}-\mathbf{u}_{m}^{+}\right]\frac{\Delta t}{2}\\
&-\mathbf{K}\left[\frac{(\Delta t)^2}{2}\mathbf{v}_{m}^{+}-\frac{(\Delta t)^2}{3}\mathbf{v}_{m}^{+}+\frac{(\Delta t)^2}{3}\mathbf{v}_{m+1}^{-}\right]\\
&+\mathbf{M}(\mathbf{v}_{m}^{+}-\mathbf{v}_{m}^{-})\qquad=0
\end{aligned}\tag{1.26}$$

soit

$$\begin{aligned}
&\mathbf{M}\left[\mathbf{v}_{m+1}^{-}-\mathbf{v}_{m}^{+}\right]\\
&+\mathbf{C}\frac{\Delta t}{2}\left(\mathbf{v}_{m}^{+}+\mathbf{v}_{m+1}^{-}\right)\\
&+\mathbf{K}\frac{\Delta t}{2}\left(\mathbf{u}_{m}^{+}+\mathbf{u}_{m+1}^{-}\right)\\
&-\frac{\Delta t}{2}\left(\mathbf{f}_{m}+\mathbf{f}_{m+1}\right)\\
&+\mathbf{K}\left[\mathbf{u}_{m+1}^{-}-\mathbf{u}_{m}^{+}\right]\frac{\Delta t}{2}\\
&-\mathbf{K}\left[\frac{(\Delta t)^2}{6}\mathbf{v}_{m}^{+}+\frac{(\Delta t)^2}{3}\mathbf{v}_{m+1}^{-}\right]\\
&+\mathbf{M}(\mathbf{v}_{m}^{+}-\mathbf{v}_{m}^{-})\qquad=0
\end{aligned}\tag{1.27}$$

Matrice

$$\begin{bmatrix}0 & K\Delta t & \left(C\frac{\Delta t}{2} - K\frac{(\Delta t)^2}{6}\right) & \left(M + C\frac{\Delta t}{2} - K\frac{(\Delta t)^2}{3}\right)\end{bmatrix}\begin{bmatrix}\mathbf{u}_m^+ \\ \mathbf{u}_{m+1}^- \\ \mathbf{v}_m^+ \\ \mathbf{v}_{m+1}^-\end{bmatrix} = \mathbf{M}\mathbf{v}_m^- + (\mathbf{f}_m + \mathbf{f}_{m+1})\frac{\Delta t}{2} \tag{1.28}$$

1.3.2 Fonction de poids : $\mathbf{w_2} = \mathbf{w}.(t_{m+1} - t)$

On choisit des fonctions de poids sur l'intervalle : $\mathbf{w_2} = \mathbf{w}.(t_{m+1} - t)$ alors $\mathbf{w_1} = -\mathbf{w}$. Ce qui donne :

$$\begin{aligned}
&\mathbf{M}\left[\mathbf{v}_{m+1}^- - \mathbf{v}_m^+\right] \times (-1) \\
&+\mathbf{C}\frac{\Delta t}{2}\left(\mathbf{v}_m^+ + \mathbf{v}_{m+1}^-\right) \times (-1) \\
&+\mathbf{K}\frac{\Delta t}{2}\left(\mathbf{u}_m^+ + \mathbf{u}_{m+1}^-\right) \times (-1) \\
&-\frac{\Delta t}{2}\left(\mathbf{f}_m + \mathbf{f}_{m+1}\right) \times (-1) \\
&+\mathbf{K}\frac{\left[\mathbf{u}_{m+1}^- - \mathbf{u}_m^+\right]}{\Delta t}\int_{t_m}^{t_{m+1}}(t_{m+1} - t)\; dt \\
&-\mathbf{K}\int_{t_m}^{t_{m+1}}(t_{m+1} - t)\left[\frac{t_{m+1} - t}{\Delta t}\mathbf{v}_m^+ + \frac{t - t_m}{\Delta t}\mathbf{v}_{m+1}^-\right]\; dt \\
&+\mathbf{K}(\mathbf{u}_m^+ - \mathbf{u}_m^-)\Delta t + \mathbf{M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \times (-1) \qquad = 0
\end{aligned} \tag{1.29}$$

donc :

$$\begin{aligned}
&-\mathbf{M}\left[\mathbf{v}_{m+1}^- - \mathbf{v}_m^+\right] \\
&-\mathbf{C}\frac{\Delta t}{2}\left(\mathbf{v}_m^+ + \mathbf{v}_{m+1}^-\right) \\
&-\mathbf{K}\frac{\Delta t}{2}\left(\mathbf{u}_m^+ + \mathbf{u}_{m+1}^-\right) \\
&+\frac{\Delta t}{2}\left(\mathbf{f}_m + \mathbf{f}_{m+1}\right) \\
&+\mathbf{K}\left[\mathbf{u}_{m+1}^- - \mathbf{u}_m^+\right]\frac{\Delta t}{2} \\
&-\mathbf{K}\int_{t_m}^{t_{m+1}}(t_{m+1} - t)\left[\frac{t_{m+1} - t}{\Delta t}\mathbf{v}_m^+ + \frac{t - t_m}{\Delta t}\mathbf{v}_{m+1}^-\right]\; dt \\
&+\mathbf{K}(\mathbf{u}_m^+ - \mathbf{u}_m^-)\Delta t - \mathbf{M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \qquad = 0
\end{aligned} \tag{1.30}$$

L'intégrale restante :

$$
\begin{aligned}
& -\mathbf{K} \int_{t_m}^{t_{m+1}} (t_{m+1}-t) \left[\frac{t_{m+1}-t}{\Delta t}\mathbf{v}_m^+ + \frac{t-t_m}{\Delta t}\mathbf{v}_{m+1}^- \right] \, dt \\
= & -\mathbf{K} \int_0^{\Delta t} (\Delta t - t) \left[\frac{\Delta t - t}{\Delta t}\mathbf{v}_m^+ + \frac{t}{\Delta t}\mathbf{v}_{m+1}^- \right] \, dt \\
= & -\mathbf{K} \int_0^{\Delta t} \left[(\Delta t - t)\mathbf{v}_m^+ + t.\mathbf{v}_{m+1}^- \right] \, dt \\
& -\mathbf{K} \int_0^{\Delta t} - \left[\frac{t.\Delta t}{\Delta t}\mathbf{v}_m^+ + \frac{-t^2}{\Delta t}\mathbf{v}_m^+ + \frac{t^2}{\Delta t}\mathbf{v}_{m+1}^- \right] \, dt \\
= & -\mathbf{K}\frac{(\Delta t)^2}{2} \left[\mathbf{v}_m^+ + \mathbf{v}_{m+1}^- \right] \\
& +\mathbf{K} \left[\frac{t^2.\Delta t}{2\Delta t}\mathbf{v}_m^+ + \frac{-t^3}{3\Delta t}\mathbf{v}_m^+ + \frac{t^3}{3\Delta t}\mathbf{v}_{m+1}^- \right]_0^{\Delta t} \\
= & -\mathbf{K}\frac{(\Delta t)^2}{2} \left[\mathbf{v}_m^+ + \mathbf{v}_{m+1}^- \right] \\
& +\mathbf{K} \left[\frac{(\Delta t)^2}{2}\mathbf{v}_m^+ - \frac{(\Delta t)^2}{3}\mathbf{v}_m^+ + \frac{(\Delta t)^2}{3}\mathbf{v}_{m+1}^- \right] \\
= & -\mathbf{K}\frac{(\Delta t)^2}{2} \left[\mathbf{v}_{m+1}^- \right] \\
& +\mathbf{K} \left[-\frac{(\Delta t)^2}{3}\mathbf{v}_m^+ + \frac{(\Delta t)^2}{3}\mathbf{v}_{m+1}^- \right] \\
= & +\mathbf{K} \left[-\frac{(\Delta t)^2}{3}\mathbf{v}_m^+ + \frac{(\Delta t)^2}{3}\mathbf{v}_{m+1}^- - \frac{(\Delta t)^2}{2}\mathbf{v}_{m+1}^- \right] \\
= & +\mathbf{K} \left[-\frac{(\Delta t)^2}{3}\mathbf{v}_m^+ - \frac{(\Delta t)^2}{6}\mathbf{v}_{m+1}^- \right]
\end{aligned}
\tag{1.31}
$$

alors :

$$
\begin{aligned}
& -\mathbf{M} \left[\mathbf{v}_{m+1}^- - \mathbf{v}_m^+ \right] \\
& -\mathbf{C}\frac{\Delta t}{2} \left(\mathbf{v}_m^+ + \mathbf{v}_{m+1}^- \right) \\
& -\mathbf{K}\frac{\Delta t}{2} \left(\mathbf{u}_m^+ + \mathbf{u}_{m+1}^- \right) \\
& +\frac{\Delta t}{2} \left(\mathbf{f}_m + \mathbf{f}_{m+1} \right) \\
& +\mathbf{K} \left[\mathbf{u}_{m+1}^- - \mathbf{u}_m^+ \right] \frac{\Delta t}{2} \\
& +\mathbf{K} \left[-\frac{(\Delta t)^2}{3}\mathbf{v}_m^+ - \frac{(\Delta t)^2}{6}\mathbf{v}_{m+1}^- \right] \\
& +\mathbf{K}(\mathbf{u}_m^+ - \mathbf{u}_m^-)\Delta t - \mathbf{M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \qquad = 0
\end{aligned}
\tag{1.32}
$$

Matrice

$$\begin{bmatrix} 0 & 0 & \left(-C\frac{\Delta t}{2} - K\frac{(\Delta t)^2}{3}\right) & \left(-M - C\frac{\Delta t}{2} - K\frac{(\Delta t)^2}{6}\right) \end{bmatrix} \begin{bmatrix} \mathbf{u}_m^+ \\ \mathbf{u}_{m+1}^- \\ \mathbf{v}_m^+ \\ \mathbf{v}_{m+1}^- \end{bmatrix} = -\mathbf{M}\mathbf{v}_m^- + \mathbf{K}\mathbf{u}_m^-\Delta t - (\mathbf{f}_m + \mathbf{f}_{m+1})\frac{\Delta t}{2} \tag{1.33}$$

En changeant de signe

$$\begin{bmatrix} 0 & 0 & \left(C\frac{\Delta t}{2} + K\frac{(\Delta t)^2}{3}\right) & \left(M + C\frac{\Delta t}{2} + K\frac{(\Delta t)^2}{6}\right) \end{bmatrix} \begin{bmatrix} \mathbf{u}_m^+ \\ \mathbf{u}_{m+1}^- \\ \mathbf{v}_m^+ \\ \mathbf{v}_{m+1}^- \end{bmatrix} = \mathbf{M}\mathbf{v}_m^- - \mathbf{K}\mathbf{u}_m^-\Delta t + (\mathbf{f}_m + \mathbf{f}_{m+1})\frac{\Delta t}{2} \tag{1.34}$$

1.3.3 Fonction de poids : $\mathbf{w_2} = \mathbf{w}.(t - t_m - \Delta t/2)$

On choisit des fonctions de poids sur l'intervalle : $\mathbf{w_2} = \mathbf{w}.(t - t_m - \Delta t/2)$ alors $\mathbf{w_1} = \mathbf{w}$. Ce qui donne :

$$\begin{aligned}
&\mathbf{M}\left[\mathbf{v}_{m+1}^- - \mathbf{v}_m^+\right] \\
&+\mathbf{C}\frac{\Delta t}{2}\left(\mathbf{v}_m^+ + \mathbf{v}_{m+1}^-\right) \\
&+\mathbf{K}\frac{\Delta t}{2}\left(\mathbf{u}_m^+ + \mathbf{u}_{m+1}^-\right) \\
&-\frac{\Delta t}{2}\left(\mathbf{f}_m + \mathbf{f}_{m+1}\right) \\
&+\mathbf{K}\frac{\left[\mathbf{u}_{m+1}^- - \mathbf{u}_m^+\right]}{\Delta t}\int_{t_m}^{t_{m+1}}(t - t_m - \Delta t/2)\; dt \\
&-\mathbf{K}\int_{t_m}^{t_{m+1}}(t - t_m - \Delta t/2)\left[\frac{t_{m+1} - t}{\Delta t}\mathbf{v}_m^+ + \frac{t - t_m}{\Delta t}\mathbf{v}_{m+1}^-\right]\; dt \\
&+\mathbf{K}(\mathbf{u}_m^+ - \mathbf{u}_m^-)\frac{-\Delta t}{2} + \mathbf{M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \qquad = 0
\end{aligned} \tag{1.35}$$

Donc :

$$
\begin{aligned}
&\mathbf{M}\left[\mathbf{v}_{m+1}^{-}-\mathbf{v}_{m}^{+}\right] \\
&+\mathbf{C}\frac{\Delta t}{2}\left(\mathbf{v}_{m}^{+}+\mathbf{v}_{m+1}^{-}\right) \\
&+\mathbf{K}\frac{\Delta t}{2}\left(\mathbf{u}_{m}^{+}+\mathbf{u}_{m+1}^{-}\right) \\
&-\frac{\Delta t}{2}\left(\mathbf{f}_{m}+\mathbf{f}_{m+1}\right) \\
&-\mathbf{K}\int_{t_m}^{t_{m+1}}(t-t_m-\Delta t/2)\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{v}_{m}^{+}+\frac{t-t_m}{\Delta t}\mathbf{v}_{m+1}^{-}\right]\,dt \\
&-\mathbf{K}(\mathbf{u}_{m}^{+}-\mathbf{u}_{m}^{-})\tfrac{\Delta t}{2}+\mathbf{M}(\mathbf{v}_{m}^{+}-\mathbf{v}_{m}^{-}) \qquad =0
\end{aligned}
\tag{1.36}
$$

L'intégrale restante :

$$
\begin{aligned}
& -\mathbf{K}\int_{t_m}^{t_{m+1}}(t-t_m-\Delta t/2)\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{v}_{m}^{+}+\frac{t-t_m}{\Delta t}\mathbf{v}_{m+1}^{-}\right]\,dt \\
= & -\mathbf{K}\int_{0}^{\Delta t}(t-\Delta t/2)\left[\frac{\Delta t-t}{\Delta t}\mathbf{v}_{m}^{+}+\frac{t}{\Delta t}\mathbf{v}_{m+1}^{-}\right]\,dt \\
= & -\mathbf{K}\int_{0}^{\Delta t}t\left[\frac{\Delta t-t}{\Delta t}\mathbf{v}_{m}^{+}+\frac{t}{\Delta t}\mathbf{v}_{m+1}^{-}\right]\,dt \\
& -\mathbf{K}\int_{0}^{\Delta t}-\frac{\Delta t}{2}\left[\frac{\Delta t-t}{\Delta t}\mathbf{v}_{m}^{+}+\frac{t}{\Delta t}\mathbf{v}_{m+1}^{-}\right]\,dt \\
= & -\mathbf{K}\int_{0}^{\Delta t}\left[\frac{t.\Delta t}{\Delta t}\mathbf{v}_{m}^{+}+\frac{-t^2}{\Delta t}\mathbf{v}_{m}^{+}+\frac{t^2}{\Delta t}\mathbf{v}_{m+1}^{-}\right]\,dt \\
& -\mathbf{K}\int_{0}^{\Delta t}-\frac{1}{2}\left[(\Delta t-t)\mathbf{v}_{m}^{+}+t.\mathbf{v}_{m+1}^{-}\right]\,dt \\
= & -\mathbf{K}\left[\frac{t^2.\Delta t}{2\Delta t}\mathbf{v}_{m}^{+}+\frac{-t^3}{3\Delta t}\mathbf{v}_{m}^{+}+\frac{t^3}{3\Delta t}\mathbf{v}_{m+1}^{-}\right]_0^{\Delta t} \\
& +\frac{1}{2}\mathbf{K}\frac{(\Delta t)^2}{2}\left[\mathbf{v}_{m}^{+}+\mathbf{v}_{m+1}^{-}\right] \\
= & -\mathbf{K}\left[\frac{(\Delta t)^2}{2}\mathbf{v}_{m}^{+}-\frac{(\Delta t)^2}{3}\mathbf{v}_{m}^{+}+\frac{(\Delta t)^2}{3}\mathbf{v}_{m+1}^{-}\right] \\
& +\mathbf{K}\frac{(\Delta t)^2}{4}\left[\mathbf{v}_{m}^{+}+\mathbf{v}_{m+1}^{-}\right] \\
= & -\mathbf{K}\left[\frac{(\Delta t)^2}{6}\mathbf{v}_{m}^{+}+\frac{(\Delta t)^2}{3}\mathbf{v}_{m+1}^{-}\right] \\
& +\mathbf{K}\frac{(\Delta t)^2}{4}\left[\mathbf{v}_{m}^{+}+\mathbf{v}_{m+1}^{-}\right] \\
= & -\mathbf{K}\left[\frac{-(\Delta t)^2}{12}\mathbf{v}_{m}^{+}+\frac{(\Delta t)^2}{12}\mathbf{v}_{m+1}^{-}\right]
\end{aligned}
\tag{1.37}
$$

Alors :

$$\begin{aligned}
&\mathbf{M}\left[\mathbf{v}_{m+1}^{-}-\mathbf{v}_{m}^{+}\right] \\
&+\mathbf{C}\frac{\Delta t}{2}\left(\mathbf{v}_{m}^{+}+\mathbf{v}_{m+1}^{-}\right) \\
&+\mathbf{K}\frac{\Delta t}{2}\left(\mathbf{u}_{m}^{+}+\mathbf{u}_{m+1}^{-}\right) \\
&-\frac{\Delta t}{2}\left(\mathbf{f}_{m}+\mathbf{f}_{m+1}\right) \\
&-\mathbf{K}\left[\frac{-(\Delta t)^2}{12}\mathbf{v}_{m}^{+}+\frac{(\Delta t)^2}{12}\mathbf{v}_{m+1}^{-}\right] \\
&-\mathbf{K}(\mathbf{u}_{m}^{+}-\mathbf{u}_{m}^{-})\tfrac{\Delta t}{2}+\mathbf{M}(\mathbf{v}_{m}^{+}-\mathbf{v}_{m}^{-}) \qquad =0
\end{aligned} \tag{1.38}$$

Matrice

$$\begin{aligned}
&\begin{bmatrix}0 & K\frac{\Delta t}{2} & \left(C\frac{\Delta t}{2}+K\frac{(\Delta t)^2}{12}\right) & \left(M+C\frac{\Delta t}{2}-K\frac{(\Delta t)^2}{12}\right)\end{bmatrix}\begin{bmatrix}\mathbf{u}_{m}^{+}\\ \mathbf{u}_{m+1}^{-}\\ \mathbf{v}_{m}^{+}\\ \mathbf{v}_{m+1}^{-}\end{bmatrix} \\
&=+\mathbf{M}\mathbf{v}_{m}^{-}-\mathbf{K}\mathbf{u}_{m}^{-}\tfrac{\Delta t}{2}+(\mathbf{f}_{m}+\mathbf{f}_{m+1})\tfrac{\Delta t}{2}
\end{aligned} \tag{1.39}$$

1.4 Système des quatre équations

$$\begin{aligned}
&\begin{bmatrix}
0 & \mathbf{K} & -\mathbf{K}\frac{\Delta t}{2} & -\mathbf{K}\frac{\Delta t}{2}\\
0 & K\Delta t & \left(C\frac{\Delta t}{2}-K\frac{(\Delta t)^2}{6}\right) & \left(M+C\frac{\Delta t}{2}-K\frac{(\Delta t)^2}{3}\right)\\
0 & 0 & \left(C\frac{\Delta t}{2}+K\frac{(\Delta t)^2}{3}\right) & \left(M+C\frac{\Delta t}{2}+K\frac{(\Delta t)^2}{6}\right)\\
0 & K\frac{\Delta t}{2} & \left(C\frac{\Delta t}{2}+K\frac{(\Delta t)^2}{12}\right) & \left(M+C\frac{\Delta t}{2}-K\frac{(\Delta t)^2}{12}\right)
\end{bmatrix}\begin{bmatrix}\mathbf{u}_{m}^{+}\\ \mathbf{u}_{m+1}^{-}\\ \mathbf{v}_{m}^{+}\\ \mathbf{v}_{m+1}^{-}\end{bmatrix} \\
&=\begin{bmatrix}
\mathbf{K}\mathbf{u}_{m}^{-}\\
\mathbf{M}\mathbf{v}_{m}^{-}+(\mathbf{f}_{m}+\mathbf{f}_{m+1})\frac{\Delta t}{2}\\
\mathbf{M}\mathbf{v}_{m}^{-}-\mathbf{K}\mathbf{u}_{m}^{-}\Delta t+(\mathbf{f}_{m}+\mathbf{f}_{m+1})\frac{\Delta t}{2}\\
\mathbf{M}\mathbf{v}_{m}^{-}-\mathbf{K}\mathbf{u}_{m}^{-}\frac{\Delta t}{2}+(\mathbf{f}_{m}+\mathbf{f}_{m+1})\frac{\Delta t}{2}
\end{bmatrix}
\end{aligned} \tag{1.40}$$

1.4.1 Problème

Il y a un problème puisque la première colonne est nulle. D'autre part les Lignes 2 et 4 sont des combinaisons des lignes 1 et 3. Il n'y a donc que deux équation indépendantes.

1.5 Trouver un coefficient non nul pour $\mathbf{u}_m^+$

En retournant avant la moindre hypothèse, à l'équation 1.3 :

$$\begin{aligned} \forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \int_{t_m}^{t_{m+1}} \mathbf{w_1} \mathbf{M} \dot{\mathbf{v}} \; dt + \int_{t_m}^{t_{m+1}} \mathbf{w_1} \mathbf{C} \mathbf{v} \; dt + \int_{t_m}^{t_{m+1}} \mathbf{w_1} \mathbf{K} \mathbf{u} \; dt \\ & - \int_{t_m}^{t_{m+1}} \mathbf{w_1} \mathbf{f} \; dt + \int_{t_m}^{t_{m+1}} \mathbf{w_2} \mathbf{K} \dot{\mathbf{u}} \; dt - \int_{t_m}^{t_{m+1}} \mathbf{w_2} \mathbf{K} \mathbf{v} \; dt \\ & + \mathbf{w_2}(t_m) \mathbf{K} (\mathbf{u}_m^+ - \mathbf{u}_m^-) + \mathbf{w_1}(t_m) \mathbf{M} (\mathbf{v}_m^+ - \mathbf{v}_m^-) \qquad = 0 \end{aligned} \tag{1.41}$$

Je m'intéresse aux membres contenant $\mathbf{u}$.

$$\begin{aligned} \forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \int_{t_m}^{t_{m+1}} \mathbf{w_1} \mathbf{K} \mathbf{u} \; dt \\ & + \int_{t_m}^{t_{m+1}} \mathbf{w_2} \mathbf{K} \dot{\mathbf{u}} \; dt \\ & + \mathbf{w_2}(t_m) \mathbf{K} (\mathbf{u}_m^+ - \mathbf{u}_m^-) \end{aligned} \tag{1.42}$$

Par définition de $\mathbf{u}$, on obtient :

$$\begin{aligned} \forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \int_{t_m}^{t_{m+1}} \mathbf{w_1} \mathbf{K} \left[\frac{t_{m+1} - t}{\Delta t} \mathbf{u}_m^+ + \frac{t - t_m}{\Delta t} \mathbf{u}_{m+1}^- \right] \; dt \\ & + \int_{t_m}^{t_{m+1}} \mathbf{w_2} \mathbf{K} \frac{\left[\mathbf{u}_{m+1}^- - \mathbf{u}_m^+ \right]}{\Delta t} \; dt \\ & + \mathbf{w_2}(t_m) \mathbf{K} (\mathbf{u}_m^+ - \mathbf{u}_m^-) \end{aligned} \tag{1.43}$$

Je m'intéresse aux membres contenant $\mathbf{u}_m^+$.

$$\begin{aligned} \forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \int_{t_m}^{t_{m+1}} \mathbf{w_1} \mathbf{K} \frac{t_{m+1} - t}{\Delta t} \mathbf{u}_m^+ \; dt \\ & + \int_{t_m}^{t_{m+1}} \mathbf{w_2} \mathbf{K} \frac{-\mathbf{u}_m^+}{\Delta t} \; dt \\ & + \mathbf{w_2}(t_m) \mathbf{K} \mathbf{u}_m^+ \end{aligned} \tag{1.44}$$

donc :

$$\begin{aligned} \forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \mathbf{K} \frac{\mathbf{u}_m^+}{\Delta t} \int_{t_m}^{t_{m+1}} \mathbf{w_1} (t_{m+1} - t) \; dt \\ & - \mathbf{K} \frac{\mathbf{u}_m^+}{\Delta t} \int_{t_m}^{t_{m+1}} \mathbf{w_2} \; dt \\ & + \mathbf{w_2}(t_m) \mathbf{K} \mathbf{u}_m^+ \end{aligned} \tag{1.45}$$

1.5.1 Hypothèse d'homogénéité : $\mathbf{w_1} = \dfrac{d\mathbf{w_2}}{dt}$

Si on choisi $\mathbf{w_1} = \dfrac{d\mathbf{w_2}}{dt}$, on peut alors vérifier l'homogénéité des équations, et on obtient :

$$\begin{aligned} \forall \mathbf{w_2} \quad & \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\int_{t_m}^{t_{m+1}}\frac{d\mathbf{w_2}}{dt}(t_{m+1}-t)\ dt \\ & -\mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\int_{t_m}^{t_{m+1}}\mathbf{w_2}\ dt \\ & +\mathbf{K}\mathbf{u}_m^+\mathbf{w_2}(t_m) \end{aligned} \tag{1.46}$$

1.5.2 Fonction de poids : $\mathbf{w_2} = \mathbf{w}.(t - t_m)^2$

Si on choisi $\mathbf{w_2} = \mathbf{w}.(t - t_m)^2$:

$$\begin{aligned} \forall \mathbf{w} \quad & \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w}\int_{t_m}^{t_{m+1}}\frac{d(t-t_m)^2}{dt}(t_{m+1}-t)\ dt \\ & -\mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w}\int_{t_m}^{t_{m+1}}(t-t_m)^2\ dt \end{aligned} \tag{1.47}$$

donc :

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w}\left[\int_{t_m}^{t_{m+1}}2(t-t_m)(t_{m+1}-t)\ dt - \int_{t_m}^{t_{m+1}}(t-t_m)^2\ dt\right] \tag{1.48}$$

En faisant un changement de variable $(t - t_m) \to t$

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w}\left[\int_0^{\Delta t}2.t.(\Delta t - t)\ dt - \int_0^{\Delta t}t^2\ dt\right] \tag{1.49}$$

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w}\left[\int_0^{\Delta t}2.t.\Delta t\ dt - \int_0^{\Delta t}2.t^2\ dt - \int_0^{\Delta t}t^2\ dt\right] \tag{1.50}$$

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w}\left[2\Delta t\int_0^{\Delta t}t\ dt - 3\int_0^{\Delta t}t^2\ dt\right] \tag{1.51}$$

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w}\left[2\Delta t\frac{(\Delta t)^2}{2} - 3\frac{(\Delta t)^3}{3}\right] \quad = 0 \tag{1.52}$$

1.5.3 Fonction de poids : $\mathbf{w_2} = \mathbf{w}.(t - t_m)^n$

Si on choisi $\mathbf{w_2} = \mathbf{w}.(t - t_m)^n$:

$$\begin{aligned} \forall \mathbf{w} \quad & \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w} \int_{t_m}^{t_{m+1}} \frac{d(t-t_m)^n}{dt}(t_{m+1}-t)\ dt \\ & -\mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w} \int_{t_m}^{t_{m+1}} (t-t_m)^n\ dt \end{aligned} \tag{1.53}$$

donc :

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w} \left[\int_{t_m}^{t_{m+1}} n(t-t_m)^{(n-1)}(t_{m+1}-t)\ dt - \int_{t_m}^{t_{m+1}} (t-t_m)^n\ dt \right] \tag{1.54}$$

En faisant un changement de variable $(t - t_m) \rightarrow t$

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w} \left[\int_0^{\Delta t} n.t^{(n-1)}.(\Delta t - t)\ dt - \int_0^{\Delta t} t^n\ dt \right] \tag{1.55}$$

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w} \left[\int_0^{\Delta t} n.t^{(n-1)}.\Delta t\ dt - \int_0^{\Delta t} n.t^n\ dt - \int_0^{\Delta t} t^n\ dt \right] \tag{1.56}$$

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w} \left[n\Delta t \int_0^{\Delta t} t^{(n-1)}\ dt - (n+1) \int_0^{\Delta t} t^n\ dt \right] \tag{1.57}$$

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w} \left[n\Delta t \frac{(\Delta t)^n}{n} - (n+1)\frac{(\Delta t)^{(n+1)}}{n+1} \right] = 0 \tag{1.58}$$

1.5.4 Fonction de poids : $\mathbf{w_2} = \mathbf{w}.(t - t_m - \Delta t/2)^n$

Si on choisi $\mathbf{w_2} = \mathbf{w}.(t - t_m - \Delta t/2)^n$:

$$\begin{aligned} \forall \mathbf{w} \quad & \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w} \int_{t_m}^{t_{m+1}} \frac{d(t-t_m-\Delta t/2)^n}{dt}(t_{m+1}-t)\ dt \\ & -\mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w} \int_{t_m}^{t_{m+1}} (t-t_m-\Delta t/2)^n\ dt \\ & +\mathbf{K}\mathbf{u}_m^+\mathbf{w_2}(t_m) \end{aligned} \tag{1.59}$$

donc :

$$\begin{aligned} \forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w} \quad & \left[\int_{t_m}^{t_{m+1}} n(t-t_m-\Delta t/2)^{(n-1)}(t_{m+1}-t)\ dt \right. \\ & - \int_{t_m}^{t_{m+1}} (t-t_m-\Delta t/2)^n\ dt \\ & \left. +\Delta t.(-\Delta t/2)^n \right] \end{aligned} \tag{1.60}$$

En faisant un changement de variable $(t - t_m - \Delta t/2) \rightarrow t$

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w} \quad \begin{aligned}&\left[\int_{-\Delta t/2}^{\Delta t/2} n.t^{(n-1)}(\Delta t/2 - t)\ dt\right.\\ &-\int_{-\Delta t/2}^{\Delta t/2} t^n\ dt\\ &\left.+\Delta t.(-\Delta t/2)^n\right]\end{aligned} \tag{1.61}$$

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w} \quad \begin{aligned}&\left[n.\Delta t/2. \int_{-\Delta t/2}^{\Delta t/2} t^{(n-1)}\ dt\right.\\ &-n\int_{-\Delta t/2}^{\Delta t/2} t^n\ dt\\ &-\int_{-\Delta t/2}^{\Delta t/2} t^n\ dt\\ &\left.+\Delta t.(-\Delta t/2)^n\right]\end{aligned} \tag{1.62}$$

n est impaire

Si n est impaire alors la fonction t^n est impaire, donc son intégrale sur un domaine centré en zéro est nulle

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w} \quad \left[n.\Delta t/2. \int_{-\Delta t/2}^{\Delta t/2} t^{(n-1)}\ dt + \Delta t.(-\Delta t/2)^n\right] \tag{1.63}$$

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w} \left[n.\Delta t/2. \left(\frac{\left(\frac{\Delta t}{2}\right)^n}{n} - \frac{\left(-\frac{\Delta t}{2}\right)^n}{n}\right) + \Delta t.(-\Delta t/2)^n\right] \tag{1.64}$$

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w} \left[n.\Delta t/2. \left(\frac{\left(\frac{\Delta t}{2}\right)^n}{n} + \frac{\left(\frac{\Delta t}{2}\right)^n}{n}\right) - \Delta t.(\Delta t/2)^n\right] \tag{1.65}$$

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w} \left[\Delta t/2. \left(\left(\frac{\Delta t}{2}\right)^n + \left(\frac{\Delta t}{2}\right)^n\right) - \Delta t.(\Delta t/2)^n\right] \quad = 0 \tag{1.66}$$

n **est paire**

Si n est paire alors la fonction t^{n-1} est impaire, donc son intégrale sur un domaine centré en zéro est nulle

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w}\left[-n\int_{-\Delta t/2}^{\Delta t/2} t^n \; dt - \int_{-\Delta t/2}^{\Delta t/2} t^n \; dt + \Delta t.(-\Delta t/2)^n\right] \tag{1.67}$$

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w}\left[-(n+1)\int_{-\Delta t/2}^{\Delta t/2} t^n \; dt + \Delta t.(-\Delta t/2)^n\right] \tag{1.68}$$

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w}\left[-\frac{n+1}{n+1}\left(\left(\frac{\Delta t}{2}\right)^{n+1} - \left(\frac{-\Delta t}{2}\right)^{n+1}\right) + \Delta t.(-\Delta t/2)^n\right] \tag{1.69}$$

$$\forall \mathbf{w} \quad \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\mathbf{w}\left[-\left(\left(\frac{\Delta t}{2}\right)^{n+1} + \left(\frac{\Delta t}{2}\right)^{n+1}\right) + \Delta t.(\Delta t/2)^n\right] = 0 \tag{1.70}$$

1.5.5 Fonction de poids : $\mathbf{w_2} = e^t$

$$\begin{aligned} \forall \mathbf{w_2} \quad & \mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\int_{t_m}^{t_{m+1}} \frac{d\mathbf{w_2}}{dt}(t_{m+1} - t) \; dt \\ & -\mathbf{K}\frac{\mathbf{u}_m^+}{\Delta t}\int_{t_m}^{t_{m+1}} \mathbf{w_2} \; dt \\ & +\mathbf{K}\mathbf{u}_m^+\mathbf{w_2}(t_m) \end{aligned} \tag{1.71}$$

1.6 Abandon de l'hypothèse $\mathbf{w_1} = \dfrac{d\mathbf{w_2}}{dt}$

Retour en 1.3.

$$\begin{aligned} \forall \mathbf{w_1}, \forall \mathbf{w_2} \quad & \int_{t_m}^{t_{m+1}} \mathbf{w_1}\mathbf{M}\dot{\mathbf{v}} \; dt + \int_{t_m}^{t_{m+1}} \mathbf{w_1}\mathbf{C}\mathbf{v} \; dt + \int_{t_m}^{t_{m+1}} \mathbf{w_1}\mathbf{K}\mathbf{u} \; dt \\ & - \int_{t_m}^{t_{m+1}} \mathbf{w_1}\mathbf{f} \; dt + \int_{t_m}^{t_{m+1}} \mathbf{w_2}\mathbf{K}\dot{\mathbf{u}} \; dt - \int_{t_m}^{t_{m+1}} \mathbf{w_2}\mathbf{K}\mathbf{v} \; dt \\ & + \mathbf{w_2}(t_m)\mathbf{K}(\mathbf{u}_m^+ - \mathbf{u}_m^-) + \mathbf{w_1}(t_m)\mathbf{M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \qquad = 0 \end{aligned} \tag{1.72}$$

1.6.1 Hypothèse de fonctions de poids affine

$$
\begin{aligned}
\forall\ a,\ b,\ c \text{ et } d \quad & \int_{t_m}^{t_{m+1}} (a.t+b)\mathbf{M}\dot{\mathbf{v}}\ dt + \int_{t_m}^{t_{m+1}} (a.t+b)\mathbf{C}\mathbf{v}\ dt \\
& + \int_{t_m}^{t_{m+1}} (a.t+b)\mathbf{K}\mathbf{u}\ dt - \int_{t_m}^{t_{m+1}} (a.t+b)\mathbf{f}\ dt \\
& + \int_{t_m}^{t_{m+1}} (c.t+d)\mathbf{K}\dot{\mathbf{u}}\ dt - \int_{t_m}^{t_{m+1}} (c.t+d)\mathbf{K}\mathbf{v}\ dt \\
& +(c.t_m+d)\mathbf{K}(\mathbf{u}_m^+ - \mathbf{u}_m^-) + (a.t_m+b)\mathbf{M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \\
& = 0
\end{aligned}
\tag{1.73}
$$

$$
\begin{aligned}
\forall\ a, b, c \text{ et } d \quad & \int_{t_m}^{t_{m+1}} (a.t+b)\mathbf{M}\frac{\mathbf{v}_{m+1}^- - \mathbf{v}_m^+}{\Delta t}\ dt \\
& + \int_{t_m}^{t_{m+1}} (a.t+b)\mathbf{C}\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{v}_m^+ + \frac{t-t_m}{\Delta t}\mathbf{v}_{m+1}^-\right]\ dt \\
& + \int_{t_m}^{t_{m+1}} (a.t+b)\mathbf{K}\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{u}_m^+ + \frac{t-t_m}{\Delta t}\mathbf{u}_{m+1}^-\right]\ dt \\
& - \int_{t_m}^{t_{m+1}} (a.t+b)\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{f}_m + \frac{t-t_m}{\Delta t}\mathbf{f}_{m+1}\right]\ dt \\
& + \int_{t_m}^{t_{m+1}} (c.t+d)\mathbf{K}\frac{\mathbf{u}_{m+1}^- - \mathbf{u}_m^+}{\Delta t}\ dt \\
& - \int_{t_m}^{t_{m+1}} (c.t+d)\mathbf{K}\left[\frac{t_{m+1}-t}{\Delta t}\mathbf{v}_m^+ + \frac{t-t_m}{\Delta t}\mathbf{v}_{m+1}^-\right]\ dt \\
& +(c.t_m+d)\mathbf{K}(\mathbf{u}_m^+ - \mathbf{u}_m^-) + (a.t_m+b)\mathbf{M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \\
& = 0
\end{aligned}
\tag{1.74}
$$

On utilise le résultat obtenu précédemment :

$$
\int_{t_m}^{t_{m+1}} \left[\frac{t_{m+1}-t}{\Delta t}A + \frac{t-t_m}{\Delta t}B\right]\ dt = \frac{\Delta t}{2}(A+B)
\tag{1.75}
$$

Calculons :

$$
\begin{aligned}
& \int_{t_m}^{t_{m+1}} t.\left[\frac{t_{m+1}-t}{\Delta t}A+\frac{t-t_m}{\Delta t}B\right]\ dt \\
= & \int_0^{\Delta t}(t+t_m).\left[\frac{\Delta t-t}{\Delta t}A+\frac{t}{\Delta t}B\right]\ dt \\
= & \int_0^{\Delta t} t.\left[\frac{\Delta t-t}{\Delta t}A+\frac{t}{\Delta t}B\right]\ dt+t_m\int_0^{\Delta t}\left[\frac{\Delta t-t}{\Delta t}A+\frac{t}{\Delta t}B\right]\ dt \\
= & \int_0^{\Delta t} t.\left[\frac{\Delta t-t}{\Delta t}A+\frac{t}{\Delta t}B\right]\ dt+t_m\frac{\Delta t}{2}(A+B) \\
= & \int_0^{\Delta t}\left[\frac{t.\Delta t}{\Delta t}A-\frac{t^2}{\Delta t}A+\frac{t^2}{\Delta t}B\right]dt+t_m\frac{\Delta t}{2}(A+B) \\
= & \left[\frac{t^2.\Delta t}{2\Delta t}A-\frac{t^3}{3\Delta t}A+\frac{t^3}{3\Delta t}B\right]_0^{\Delta t}+t_m\frac{\Delta t}{2}(A+B) \\
= & \frac{(\Delta t)^2}{3}\left(\frac{1}{2}A+B\right)+t_m\frac{\Delta t}{2}(A+B)
\end{aligned}
\tag{1.76}
$$

Alors :

$$
\begin{aligned}
\forall\ a,b,c \text{ et } d \quad & \int_{t_m}^{t_{m+1}}(a.t+b)\mathbf{M}\frac{\mathbf{v}_{m+1}^- - \mathbf{v}_m^+}{\Delta t}\ dt \\
& +a\ \mathbf{C}\left[\frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{v}_m^+ + \mathbf{v}_{m+1}^-\right)+t_m\frac{\Delta t}{2}(\mathbf{v}_m^+ + \mathbf{v}_{m+1}^-)\right] \\
& +b\ \mathbf{C}\ \frac{\Delta t}{2}(\mathbf{v}_m^+ + \mathbf{v}_{m+1}^-) \\
& +a\ \mathbf{K}\left[\frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{u}_m^+ + \mathbf{u}_{m+1}^-\right)+t_m\frac{\Delta t}{2}(\mathbf{u}_m^+ + \mathbf{u}_{m+1}^-)\right] \\
& +b\ \mathbf{K}\ \frac{\Delta t}{2}(\mathbf{u}_m^+ + \mathbf{u}_{m+1}^-) \\
& -a\left[\frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{f}_m + \mathbf{f}_{m+1}\right)+t_m\frac{\Delta t}{2}(\mathbf{f}_m + \mathbf{f}_{m+1})\right] \\
& -b\ \frac{\Delta t}{2}(\mathbf{f}_m + \mathbf{f}_{m+1}) \\
& +\int_{t_m}^{t_{m+1}}(c.t+d)\mathbf{K}\frac{\mathbf{u}_{m+1}^- - \mathbf{u}_m^+}{\Delta t}\ dt \\
& -c\ \mathbf{K}\left[\frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{v}_m^+ + \mathbf{v}_{m+1}^-\right)+t_m\frac{\Delta t}{2}(\mathbf{v}_m^+ + \mathbf{v}_{m+1}^-)\right] \\
& -d\ \mathbf{K}\ \frac{\Delta t}{2}(\mathbf{v}_m^+ + \mathbf{v}_{m+1}^-) \\
& +(c.t_m+d)\mathbf{K}(\mathbf{u}_m^+ - \mathbf{u}_m^-)+(a.t_m+b)\mathbf{M}(\mathbf{v}_m^+ - \mathbf{v}_m^-) \\
& =0
\end{aligned}
\tag{1.77}
$$

Et si on calcule les intégrales restantes :

$$
\begin{aligned}
\forall\ a, b, c \text{ et } d \quad & \mathbf{M}\frac{\mathbf{v}_{m+1}^{-}-\mathbf{v}_{m}^{+}}{\Delta t}\left[a\frac{t_{m+1}^{2}-t_{m}^{2}}{2}+b\Delta t\right] \\
& +a\ \mathbf{C}\left[\frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{v}_{m}^{+}+\mathbf{v}_{m+1}^{-}\right)+t_m\frac{\Delta t}{2}(\mathbf{v}_{m}^{+}+\mathbf{v}_{m+1}^{-})\right] \\
& +b\ \mathbf{C}\ \frac{\Delta t}{2}(\mathbf{v}_{m}^{+}+\mathbf{v}_{m+1}^{-}) \\
& +a\ \mathbf{K}\left[\frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{u}_{m}^{+}+\mathbf{u}_{m+1}^{-}\right)+t_m\frac{\Delta t}{2}(\mathbf{u}_{m}^{+}+\mathbf{u}_{m+1}^{-})\right] \\
& +b\ \mathbf{K}\ \frac{\Delta t}{2}(\mathbf{u}_{m}^{+}+\mathbf{u}_{m+1}^{-}) \\
& -a\left[\frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{f}_{m}+\mathbf{f}_{m+1}\right)+t_m\frac{\Delta t}{2}(\mathbf{f}_{m}+\mathbf{f}_{m+1})\right] \\
& -b\ \frac{\Delta t}{2}(\mathbf{f}_{m}+\mathbf{f}_{m+1}) \\
& +\mathbf{K}\frac{\mathbf{u}_{m+1}^{-}-\mathbf{u}_{m}^{+}}{\Delta t}\left[c\frac{t_{m+1}^{2}-t_{m}^{2}}{2}+d\Delta t\right] \\
& -c\ \mathbf{K}\left[\frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{v}_{m}^{+}+\mathbf{v}_{m+1}^{-}\right)+t_m\frac{\Delta t}{2}(\mathbf{v}_{m}^{+}+\mathbf{v}_{m+1}^{-})\right] \\
& -d\ \mathbf{K}\ \frac{\Delta t}{2}(\mathbf{v}_{m}^{+}+\mathbf{v}_{m+1}^{-}) \\
& +(c.t_m+d)\mathbf{K}(\mathbf{u}_{m}^{+}-\mathbf{u}_{m}^{-})+(a.t_m+b)\mathbf{M}(\mathbf{v}_{m}^{+}-\mathbf{v}_{m}^{-}) \\
& =0
\end{aligned} \tag{1.78}
$$

1.7 Système complet

$$
\begin{bmatrix}
0 & 0 & -\frac{\mathbf{M}}{\Delta t}\left[a\frac{t_{m+1}^2-t_m^2}{2}+b\Delta t\right] & \frac{\mathbf{M}}{\Delta t}\left[a\frac{t_{m+1}^2-t_m^2}{2}+b\Delta t\right] \\
+0 & +0 & +a\ \mathbf{C}(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}) & +a\ \mathbf{C}(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}) \\
+0 & +0 & +b\ \mathbf{C}\frac{\Delta t}{2} & +b\ \mathbf{C}\frac{\Delta t}{2} \\
+a\ \mathbf{K}(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}) & +a\ \mathbf{K}(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}) & +0 & +0 \\
+b\ \mathbf{K}\frac{\Delta t}{2} & +b\ \mathbf{K}\frac{\Delta t}{2} & +0 & +0 \\
-\frac{\mathbf{K}}{\Delta t}\left[c\frac{t_{m+1}^2-t_m^2}{2}+d\Delta t\right] & +\frac{\mathbf{K}}{\Delta t}\left[c\frac{t_{m+1}^2-t_m^2}{2}+d\Delta t\right] & +0 & +0 \\
+0 & +0 & -c\ \mathbf{K}(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}) & -c\ \mathbf{K}(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}) \\
+0 & +0 & -d\ \mathbf{K}\frac{\Delta t}{2} & -d\ \mathbf{K}\frac{\Delta t}{2} \\
+\mathbf{K}(c.t_m+d) & +0 & +\mathbf{M}(a.t_m+b) & +0
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_m^+ \\ \mathbf{u}_{m+1}^- \\ \mathbf{v}_m^+ \\ \mathbf{v}_{m+1}^-
\end{bmatrix}
$$

$$
\begin{aligned}
&= (c.t_m+d)\mathbf{K}\mathbf{u}_m^- + (a.t_m+b)\mathbf{M}\mathbf{v}_m^- \\
&\quad + a\left[\frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{f}_m+\mathbf{f}_{m+1}\right)+t_m\frac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1})\right] \\
&\quad + b\ \frac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1})
\end{aligned}
\tag{1.79}
$$

$$
\begin{bmatrix}
\mathbf{K}\left[a\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right)+b\frac{\Delta t}{2}-\left(c\frac{t_{m+1}^2-t_m^2}{2\Delta t}+d\right)+(c.t_m+d)\right] \\
\mathbf{K}\left[a\left(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}\right)+b\frac{\Delta t}{2}+\left(c\frac{t_{m+1}^2-t_m^2}{2\Delta t}+d\right)\right] \\
-\mathbf{M}\left(a\frac{t_{m+1}^2-t_m^2}{2\Delta t}+b\right)+a\mathbf{C}(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2})+b\mathbf{C}\frac{\Delta t}{2}-c\mathbf{K}(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2})-d\mathbf{K}\frac{\Delta t}{2}+\mathbf{M}(a.t_m+b) \\
\mathbf{M}\left(a\frac{t_{m+1}^2-t_m^2}{2\Delta t}+b\right)+a\mathbf{C}(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2})+b\mathbf{C}\frac{\Delta t}{2}-c\mathbf{K}(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2})-d\mathbf{K}\frac{\Delta t}{2}
\end{bmatrix}
.
\begin{bmatrix}
\mathbf{u}_m^+ \\ \mathbf{u}_{m+1}^- \\ \mathbf{v}_m^+ \\ \mathbf{v}_{m+1}^-
\end{bmatrix}
$$

$$
\begin{aligned}
&= (c.t_m+d)\mathbf{K}\mathbf{u}_m^- + (a.t_m+b)\mathbf{M}\mathbf{v}_m^- \\
&\quad + a\left[\frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{f}_m+\mathbf{f}_{m+1}\right)+t_m\frac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1})\right] \\
&\quad + b\ \frac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1})
\end{aligned}
\tag{1.80}
$$

$$\begin{bmatrix} \mathbf{K}\left[a\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right)+b\frac{\Delta t}{2}-c\frac{t_{m+1}^2-t_m^2}{2\Delta t}+c.t_m\right] \\ \mathbf{K}\left[a\left(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}\right)+b\frac{\Delta t}{2}+\left(c\frac{t_{m+1}^2-t_m^2}{2\Delta t}+d\right)\right] \\ -\mathbf{M}a\frac{t_{m+1}^2-t_m^2}{2\Delta t}+a\mathbf{C}(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2})+b\mathbf{C}\frac{\Delta t}{2}-c\mathbf{K}(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2})-d\mathbf{K}\frac{\Delta t}{2}+\mathbf{M}a.t_m \\ \mathbf{M}\left(a\frac{t_{m+1}^2-t_m^2}{2\Delta t}+b\right)+a\mathbf{C}(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2})+b\mathbf{C}\frac{\Delta t}{2}-c\mathbf{K}(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2})-d\mathbf{K}\frac{\Delta t}{2} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_m^+ \\ \mathbf{u}_{m+1}^- \\ \mathbf{v}_m^+ \\ \mathbf{v}_{m+1}^- \end{bmatrix}$$

$$\begin{aligned} &= (c.t_m+d)\mathbf{K}\mathbf{u}_m^- + (a.t_m+b)\mathbf{M}\mathbf{v}_m^- \\ &\quad + a\left[\frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{f}_m+\mathbf{f}_{m+1}\right)+t_m\frac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1})\right] \\ &\quad + b\ \frac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1}) \end{aligned} \tag{1.81}$$

$$\frac{t_{m+1}^2-t_m^2}{2\Delta t}=\frac{(t_m+\Delta t)^2-t_m^2}{2\Delta t}=\frac{2\Delta t.t_m+(\Delta t)^2}{2\Delta t}=\frac{2t_m+\Delta t}{2} \tag{1.82}$$

1.7.1 $a=1,\ b=0,\ c=0$ et $d=0$

$$\begin{bmatrix} \mathbf{K}\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right) \\ \mathbf{K}\left(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}\right) \\ -\mathbf{M}\frac{\cancel{2t_m}+\Delta t}{2}+\mathbf{C}\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right)+\mathbf{M}.\cancel{t_m} \\ \mathbf{M}\frac{2t_m+\Delta t}{2}+\mathbf{C}\left(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}\right) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_m^+ \\ \mathbf{u}_{m+1}^- \\ \mathbf{v}_m^+ \\ \mathbf{v}_{m+1}^- \end{bmatrix} \tag{1.83}$$

$$= t_m\mathbf{M}\mathbf{v}_m^- + \left[\frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{f}_m+\mathbf{f}_{m+1}\right)+t_m\frac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1})\right]$$

1.7.2 $a=0,\ b=1,\ c=0$ et $d=0$

$$\begin{bmatrix} \mathbf{K}\frac{\Delta t}{2} \\ \mathbf{K}\frac{\Delta t}{2} \\ \mathbf{C}\frac{\Delta t}{2} \\ \mathbf{M}+\mathbf{C}\frac{\Delta t}{2} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_m^+ \\ \mathbf{u}_{m+1}^- \\ \mathbf{v}_m^+ \\ \mathbf{v}_{m+1}^- \end{bmatrix} \tag{1.84}$$

$$= \mathbf{M}\mathbf{v}_m^- + \tfrac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1})$$

1.7.3 $a = 0,\ b = 0,\ c = 1$ et $d = 0$

$$\begin{bmatrix} \mathbf{K}\left[-\frac{2t_m+\Delta t}{2}+t_m\right] \\ \mathbf{K}\frac{2t_m+\Delta t}{2} \\ -\mathbf{K}(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}) \\ -\mathbf{K}(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_m^+ \\ \mathbf{u}_{m+1}^- \\ \mathbf{v}_m^+ \\ \mathbf{v}_{m+1}^- \end{bmatrix} \tag{1.85}$$

$$= t_m \mathbf{K}\mathbf{u}_m^-$$

Soit :

$$\begin{bmatrix} \mathbf{K}\left[-\frac{\Delta t}{2}\right] \\ \mathbf{K}\frac{2t_m+\Delta t}{2} \\ -\mathbf{K}(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}) \\ -\mathbf{K}(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_m^+ \\ \mathbf{u}_{m+1}^- \\ \mathbf{v}_m^+ \\ \mathbf{v}_{m+1}^- \end{bmatrix} \tag{1.86}$$

$$= t_m \mathbf{K}\mathbf{u}_m^-$$

1.7.4 $a = 0,\ b = 0,\ c = 0$ et $d = 1$

$$\begin{bmatrix} 0 \\ \mathbf{K} \\ -\mathbf{K}\frac{\Delta t}{2} \\ -\mathbf{K}\frac{\Delta t}{2} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_m^+ \\ \mathbf{u}_{m+1}^- \\ \mathbf{v}_m^+ \\ \mathbf{v}_{m+1}^- \end{bmatrix} \tag{1.87}$$

$$= \mathbf{K}\mathbf{u}_m^-$$

1.7.5 Matrice complète

$$\begin{bmatrix} \mathbf{K}\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & -\mathbf{K}\frac{\Delta t}{2} & 0 \\ \mathbf{K}\left(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & \mathbf{K}\frac{2t_m+\Delta t}{2} & \mathbf{K} \\ -\mathbf{M}\frac{\Delta t}{2}+\mathbf{C}\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right) & \mathbf{C}\frac{\Delta t}{2} & -\mathbf{K}(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}) & -\mathbf{K}\frac{\Delta t}{2} \\ \mathbf{M}\frac{2t_m+\Delta t}{2}+\mathbf{C}\left(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}\right) & \mathbf{M}+\mathbf{C}\frac{\Delta t}{2} & -\mathbf{K}(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}) & -\mathbf{K}\frac{\Delta t}{2} \end{bmatrix}^T \begin{bmatrix} \mathbf{u}_m^+ \\ \mathbf{u}_{m+1}^- \\ \mathbf{v}_m^+ \\ \mathbf{v}_{m+1}^- \end{bmatrix}$$

$$= \begin{bmatrix} t_m\mathbf{M}\mathbf{v}_m^- + \left[\frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{f}_m+\mathbf{f}_{m+1}\right)+t_m\frac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1})\right] \\ \mathbf{M}\mathbf{v}_m^- + \frac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1}) \\ t_m\mathbf{K}\mathbf{u}_m^- \\ \mathbf{K}\mathbf{u}_m^- \end{bmatrix} \tag{1.88}$$

$$\begin{bmatrix} \mathbf{K}\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & -\mathbf{K}\frac{\Delta t}{2} & 0 \\ \mathbf{K}\left(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & \mathbf{K}\frac{2t_m+\Delta t}{2} & \mathbf{K} \\ -\mathbf{M}\frac{\Delta t}{2}+\mathbf{C}\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right) & \mathbf{C}\frac{\Delta t}{2} & -\mathbf{K}(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}) & -\mathbf{K}\frac{\Delta t}{2} \\ \mathbf{M}\frac{2t_m+\Delta t}{2}+\mathbf{C}\left(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}\right) & \mathbf{M}+\mathbf{C}\frac{\Delta t}{2} & -\mathbf{K}(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}) & -\mathbf{K}\frac{\Delta t}{2} \end{bmatrix}^T =$$

$$\begin{bmatrix} \mathbf{K}\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right) & \mathbf{K}\left(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}\right) & -\mathbf{M}\frac{\Delta t}{2}+\mathbf{C}\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right) & \mathbf{M}\frac{2t_m+\Delta t}{2}+\mathbf{C}\left(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}\right) \\ \mathbf{K}\frac{\Delta t}{2} & \mathbf{K}\frac{\Delta t}{2} & \mathbf{C}\frac{\Delta t}{2} & \mathbf{M}+\mathbf{C}\frac{\Delta t}{2} \\ -\mathbf{K}\frac{\Delta t}{2} & \mathbf{K}\frac{2t_m+\Delta t}{2} & -\mathbf{K}(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}) & -\mathbf{K}(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}) \\ 0 & \mathbf{K} & -\mathbf{K}\frac{\Delta t}{2} & -\mathbf{K}\frac{\Delta t}{2} \end{bmatrix} \tag{1.89}$$

1.8 Amélioration du système

1.8.1 système Initial

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{K}\left(\frac{(\Delta t)^2}{6} + \frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & -\mathbf{K}\frac{\Delta t}{2} & 0 \\ \mathbf{K}\left(\frac{(\Delta t)^2}{3} + \frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & \mathbf{K}\frac{2t_m+\Delta t}{2} & \mathbf{K} \\ -\mathbf{M}\frac{\Delta t}{2} + \mathbf{C}\left(\frac{(\Delta t)^2}{6} + \frac{t_m.\Delta t}{2}\right) & \mathbf{C}\frac{\Delta t}{2} & -\mathbf{K}(\frac{(\Delta t)^2}{6} + \frac{t_m.\Delta t}{2}) & -\mathbf{K}\frac{\Delta t}{2} \\ \mathbf{M}\frac{2t_m+\Delta t}{2} + \mathbf{C}\left(\frac{(\Delta t)^2}{3} + \frac{t_m.\Delta t}{2}\right) & \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} & -\mathbf{K}(\frac{(\Delta t)^2}{3} + \frac{t_m.\Delta t}{2}) & -\mathbf{K}\frac{\Delta t}{2} \end{bmatrix}^T \tag{1.90}$$

1.8.2 combinaison 1

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 + L_2 \\ L_4 \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{K}\left(\frac{(\Delta t)^2}{6} + \frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & 0 & 0 \\ \mathbf{K}\left(\frac{(\Delta t)^2}{3} + \frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & \mathbf{K}(t_m + \Delta t) & \mathbf{K} \\ -\mathbf{M}\frac{\Delta t}{2} + \mathbf{C}\left(\frac{(\Delta t)^2}{6} + \frac{t_m.\Delta t}{2}\right) & \mathbf{C}\frac{\Delta t}{2} & -\mathbf{K}(\frac{(\Delta t)^2}{6} + \frac{t_m.\Delta t}{2}) + \mathbf{C}\frac{\Delta t}{2} & -\mathbf{K}\frac{\Delta t}{2} \\ \mathbf{M}\frac{2t_m+\Delta t}{2} + \mathbf{C}\left(\frac{(\Delta t)^2}{3} + \frac{t_m.\Delta t}{2}\right) & \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} & -\mathbf{K}(\frac{(\Delta t)^2}{3} + \frac{t_m.\Delta t}{2}) + \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} & -\mathbf{K}\frac{\Delta t}{2} \end{bmatrix}^T \tag{1.91}$$

1.8.3 combinaison 2

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 + L_2 - (t_m + \Delta t).L_4 \\ L_4 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{K}\left(\frac{(\Delta t)^2}{6} + \frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & 0 & 0 \\ \mathbf{K}\left(\frac{(\Delta t)^2}{3} + \frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & 0 & \mathbf{K} \\ -\mathbf{M}\frac{\Delta t}{2} + \mathbf{C}\left(\frac{(\Delta t)^2}{6} + \frac{t_m.\Delta t}{2}\right) & \mathbf{C}\frac{\Delta t}{2} & -\mathbf{K}(\frac{(\Delta t)^2}{6} + \frac{t_m.\Delta t}{2} - \frac{\Delta t}{2}(t_m + \Delta t)) + \mathbf{C}\frac{\Delta t}{2} & -\mathbf{K}\frac{\Delta t}{2} \\ \mathbf{M}\frac{2t_m+\Delta t}{2} + \mathbf{C}\left(\frac{(\Delta t)^2}{3} + \frac{t_m.\Delta t}{2}\right) & \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} & -\mathbf{K}(\frac{(\Delta t)^2}{3} + \frac{t_m.\Delta t}{2} - \frac{\Delta t}{2}(t_m + \Delta t)) + \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} & -\mathbf{K}\frac{\Delta t}{2} \end{bmatrix}^T \tag{1.92}$$

Soit :

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 + L_2 - (t_m + \Delta t).L_4 \\ L_4 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{K}\left(\frac{(\Delta t)^2}{6} + \frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & 0 & 0 \\ \mathbf{K}\left(\frac{(\Delta t)^2}{3} + \frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & 0 & \mathbf{K} \\ -\mathbf{M}\frac{\Delta t}{2} + \mathbf{C}\left(\frac{(\Delta t)^2}{6} + \frac{t_m.\Delta t}{2}\right) & \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{3} + \mathbf{C}\frac{\Delta t}{2} & -\mathbf{K}\frac{\Delta t}{2} \\ \mathbf{M}\frac{2t_m+\Delta t}{2} + \mathbf{C}\left(\frac{(\Delta t)^2}{3} + \frac{t_m.\Delta t}{2}\right) & \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{6} + \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} & -\mathbf{K}\frac{\Delta t}{2} \end{bmatrix}^T \tag{1.93}$$

1.8.4 combinaison 3

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3+L_2-(t_m+\Delta t).L_4 \\ -\frac{(\Delta t)^2}{6}L_4+L_1 \end{bmatrix} \begin{bmatrix} \mathbf{K}\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & 0 & \mathbf{K}\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right) \\ \mathbf{K}\left(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & 0 & \mathbf{K}\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right) \\ -\mathbf{M}\frac{\Delta t}{2}+\mathbf{C}\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right) & \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{3}+\mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12}-\mathbf{M}\frac{\Delta t}{2}+\mathbf{C}\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right) \\ \mathbf{M}\frac{2t_m+\Delta t}{2}+\mathbf{C}\left(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}\right) & \mathbf{M}+\mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{6}+\mathbf{M}+\mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12}+\mathbf{M}\frac{2t_m+\Delta t}{2}+\mathbf{C}\left(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}\right) \end{bmatrix}^T \tag{1.94}$$

1.8.5 combinaison 4

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3+L_2-(t_m+\Delta t).L_4 \\ -\frac{(\Delta t)^2}{6}L_4+L_1-\frac{\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}}{\frac{\Delta t}{2}}.L_2 \end{bmatrix} \begin{bmatrix} \mathbf{K}\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & 0 & 0 \\ \mathbf{K}\left(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & 0 & 0 \\ -\mathbf{M}\frac{\Delta t}{2}+\mathbf{C}\left(\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}\right) & \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{3}+\mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12}-\mathbf{M}\frac{\Delta t}{2} \\ \mathbf{M}\frac{2t_m+\Delta t}{2}+\mathbf{C}\left(\frac{(\Delta t)^2}{3}+\frac{t_m.\Delta t}{2}\right) & \mathbf{M}+\mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{6}+\mathbf{M}+\mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12}+\mathbf{M}\left(\frac{2t_m+\Delta t}{2}-\frac{\frac{(\Delta t)^2}{6}+\frac{t_m.\Delta t}{2}}{\frac{\Delta t}{2}}\right)+\mathbf{C}\frac{(\Delta t)^2}{6} \end{bmatrix}^T \tag{1.95}$$

Soit :

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 + L_2 - (t_m + \Delta t).L_4 \\ -\frac{(\Delta t)^2}{6} L_4 + L_1 - \left(\frac{\Delta t}{3} + t_m\right).L_2 \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{K}\left(\frac{(\Delta t)^2}{6} + \frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & 0 & 0 \\ \mathbf{K}\left(\frac{(\Delta t)^2}{3} + \frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & 0 & 0 \\ -\mathbf{M}\frac{\Delta t}{2} + \mathbf{C}\left(\frac{(\Delta t)^2}{6} + \frac{t_m.\Delta t}{2}\right) & \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{3} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12} - \mathbf{M}\frac{\Delta t}{2} \\ \mathbf{M}\frac{2t_m+\Delta t}{2} + \mathbf{C}\left(\frac{(\Delta t)^2}{3} + \frac{t_m.\Delta t}{2}\right) & \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{6} + \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12} + \mathbf{M}\left(\frac{2t_m+\Delta t}{2} - \frac{\Delta t}{3} - t_m\right) + \mathbf{C}\frac{(\Delta t)^2}{6} \end{bmatrix}^T \tag{1.96}$$

Soit :

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 + L_2 - (t_m + \Delta t).L_4 \\ -\frac{(\Delta t)^2}{6} L_4 + L_1 - \left(\frac{\Delta t}{3} + t_m\right).L_2 \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{K}\left(\frac{(\Delta t)^2}{6} + \frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & 0 & 0 \\ \mathbf{K}\left(\frac{(\Delta t)^2}{3} + \frac{t_m.\Delta t}{2}\right) & \mathbf{K}\frac{\Delta t}{2} & 0 & 0 \\ -\mathbf{M}\frac{\Delta t}{2} + \mathbf{C}\left(\frac{(\Delta t)^2}{6} + \frac{t_m.\Delta t}{2}\right) & \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{3} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12} - \mathbf{M}\frac{\Delta t}{2} \\ \mathbf{M}\frac{2t_m+\Delta t}{2} + \mathbf{C}\left(\frac{(\Delta t)^2}{3} + \frac{t_m.\Delta t}{2}\right) & \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{6} + \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12} + \mathbf{M}\frac{\Delta t}{6} + \mathbf{C}\frac{(\Delta t)^2}{6} \end{bmatrix}^T \tag{1.97}$$

1.8.6 combinaison 5

$$\begin{bmatrix} L_1 - \frac{\frac{(\Delta t)^2}{6} + \frac{t_m.\Delta t}{2}}{\frac{\Delta t}{2}}.L_2 \\ L_2 \\ L_3 + L_2 - (t_m + \Delta t).L_4 \\ -\frac{(\Delta t)^2}{6} L_4 + L_1 - \left(\frac{\Delta t}{3} + t_m\right).L_2 \end{bmatrix}$$
$$\begin{bmatrix} 0 & \mathbf{K}\frac{\Delta t}{2} & 0 & 0 \\ \mathbf{K}\frac{(\Delta t)^2}{6} & \mathbf{K}\frac{\Delta t}{2} & 0 & 0 \\ -\mathbf{M}\frac{\Delta t}{2} & \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{3} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12} - \mathbf{M}\frac{\Delta t}{2} \\ \mathbf{M}\left(\frac{2t_m+\Delta t}{2} - \frac{\Delta t}{3} - t_m\right) + \mathbf{C}\frac{(\Delta t)^2}{6} & \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{6} + \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12} + \mathbf{M}\frac{\Delta t}{6} + \mathbf{C}\frac{(\Delta t)^2}{6} \end{bmatrix}^T \tag{1.98}$$

Soit :

$$\begin{bmatrix} L_1 - \left(\frac{\Delta t}{3} + t_m\right).L_2 \\ L_2 \\ L_3 + L_2 - (t_m + \Delta t).L_4 \\ -\frac{(\Delta t)^2}{6} L_4 + L_1 - \left(\frac{\Delta t}{3} + t_m\right).L_2 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{K}\frac{\Delta t}{2} & 0 & 0 \\ \mathbf{K}\frac{(\Delta t)^2}{6} & \mathbf{K}\frac{\Delta t}{2} & 0 & 0 \\ -\mathbf{M}\frac{\Delta t}{2} & \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{3} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12} - \mathbf{M}\frac{\Delta t}{2} \\ \mathbf{M}\frac{\Delta t}{6} + \mathbf{C}\frac{(\Delta t)^2}{6} & \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{6} + \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12} + \mathbf{M}\frac{\Delta t}{6} + \mathbf{C}\frac{(\Delta t)^2}{6} \end{bmatrix}^T \tag{1.99}$$

Soit :

$$\begin{bmatrix} L_1 - \left(\frac{\Delta t}{3} + t_m\right).L_2 \\ L_2 \\ L_3 + L_2 - (t_m + \Delta t).L_4 \\ -\frac{(\Delta t)^2}{6} L_4 + L_1 - \left(\frac{\Delta t}{3} + t_m\right).L_2 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{K}\frac{(\Delta t)^2}{6} & -\mathbf{M}\frac{\Delta t}{2} & \mathbf{M}\frac{\Delta t}{6} + \mathbf{C}\frac{(\Delta t)^2}{6} \\ \mathbf{K}\frac{\Delta t}{2} & \mathbf{K}\frac{\Delta t}{2} & \mathbf{C}\frac{\Delta t}{2} & \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} \\ 0 & 0 & \mathbf{K}\frac{(\Delta t)^2}{3} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{6} + \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} \\ 0 & 0 & \mathbf{K}\frac{(\Delta t)^3}{12} - \mathbf{M}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12} + \mathbf{M}\frac{\Delta t}{6} + \mathbf{C}\frac{(\Delta t)^2}{6} \end{bmatrix} \tag{1.100}$$

Ou en permutant :

$$\begin{bmatrix} L_2 \\ L_1 - \left(\frac{\Delta t}{3} + t_m\right).L_2 \\ L_3 + L_2 - (t_m + \Delta t).L_4 \\ -\frac{(\Delta t)^2}{6} L_4 + L_1 - \left(\frac{\Delta t}{3} + t_m\right).L_2 \end{bmatrix} \begin{bmatrix} \mathbf{K}\frac{\Delta t}{2} & \mathbf{K}\frac{\Delta t}{2} & \mathbf{C}\frac{\Delta t}{2} & \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} \\ 0 & \mathbf{K}\frac{(\Delta t)^2}{6} & -\mathbf{M}\frac{\Delta t}{2} & \mathbf{M}\frac{\Delta t}{6} + \mathbf{C}\frac{(\Delta t)^2}{6} \\ 0 & 0 & \mathbf{K}\frac{(\Delta t)^2}{3} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{6} + \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} \\ 0 & 0 & \mathbf{K}\frac{(\Delta t)^3}{12} - \mathbf{M}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12} + \mathbf{M}\frac{\Delta t}{6} + \mathbf{C}\frac{(\Delta t)^2}{6} \end{bmatrix} \tag{1.101}$$

1.8.7 combinaison 6

$$\begin{bmatrix} L_2-(L_3+L_2-(t_m+\Delta t).L_4) \\ L_1-\left(\frac{\Delta t}{3}+t_m\right).L_2 \\ L_3+L_2-(t_m+\Delta t).L_4 \\ -\frac{(\Delta t)^2}{6}L_4+L_1-\left(\frac{\Delta t}{3}+t_m\right).L_2 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{K}\frac{\Delta t}{2} & \mathbf{K}\frac{\Delta t}{2} & -\mathbf{K}\frac{(\Delta t)^2}{3} & -\mathbf{K}\frac{(\Delta t)^2}{6} \\ 0 & \mathbf{K}\frac{(\Delta t)^2}{6} & -\mathbf{M}\frac{\Delta t}{2} & \mathbf{M}\frac{\Delta t}{6}+\mathbf{C}\frac{(\Delta t)^2}{6} \\ 0 & 0 & \mathbf{K}\frac{(\Delta t)^2}{3}+\mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{6}+\mathbf{M}+\mathbf{C}\frac{\Delta t}{2} \\ 0 & 0 & \mathbf{K}\frac{(\Delta t)^3}{12}-\mathbf{M}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12}+\mathbf{M}\frac{\Delta t}{6}+\mathbf{C}\frac{(\Delta t)^2}{6} \end{bmatrix} \tag{1.102}$$

1.8.8 combinaison 7

$$\begin{bmatrix} L_2-(L_3+L_2-(t_m+\Delta t).L_4) \\ L_1-\left(\frac{\Delta t}{3}+t_m\right).L_2+\frac{(\Delta t)^2}{6}L_4-L_1+\left(\frac{\Delta t}{3}+t_m\right).L_2 \\ L_3+L_2-(t_m+\Delta t).L_4 \\ -\frac{(\Delta t)^2}{6}L_4+L_1-\left(\frac{\Delta t}{3}+t_m\right).L_2 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{K}\frac{\Delta t}{2} & \mathbf{K}\frac{\Delta t}{2} & -\mathbf{K}\frac{(\Delta t)^2}{3} & -\mathbf{K}\frac{(\Delta t)^2}{6} \\ 0 & \mathbf{K}\frac{(\Delta t)^2}{6} & -\mathbf{K}\frac{(\Delta t)^3}{12} & -\mathbf{K}\frac{(\Delta t)^3}{12} \\ 0 & 0 & \mathbf{K}\frac{(\Delta t)^2}{3}+\mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{6}+\mathbf{M}+\mathbf{C}\frac{\Delta t}{2} \\ 0 & 0 & \mathbf{K}\frac{(\Delta t)^3}{12}-\mathbf{M}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12}+\mathbf{M}\frac{\Delta t}{6}+\mathbf{C}\frac{(\Delta t)^2}{6} \end{bmatrix} \tag{1.103}$$

1.8.9 combinaison 8

$$\begin{bmatrix} (L_2-(L_3+L_2-(t_m+\Delta t).L_4)).\frac{2}{\Delta t} \\ \left(L_1-\left(\frac{\Delta t}{3}+t_m\right).L_2+\frac{(\Delta t)^2}{6}L_4-L_1+\left(\frac{\Delta t}{3}+t_m\right).L_2\right).\frac{6}{(\Delta t)^2}=L_4 \\ L_3+L_2-(t_m+\Delta t).L_4 \\ -\frac{(\Delta t)^2}{6}L_4+L_1-\left(\frac{\Delta t}{3}+t_m\right).L_2 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{K} & \mathbf{K} & -\mathbf{K}\frac{2\Delta t}{3} & -\mathbf{K}\frac{\Delta t}{3} \\ 0 & \mathbf{K} & -\mathbf{K}\frac{\Delta t}{2} & -\mathbf{K}\frac{\Delta t}{2} \\ 0 & 0 & \mathbf{K}\frac{(\Delta t)^2}{3}+\mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{6}+\mathbf{M}+\mathbf{C}\frac{\Delta t}{2} \\ 0 & 0 & \mathbf{K}\frac{(\Delta t)^3}{12}-\mathbf{M}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12}+\mathbf{M}\frac{\Delta t}{6}+\mathbf{C}\frac{(\Delta t)^2}{6} \end{bmatrix} \tag{1.104}$$

Soit :

$$\begin{bmatrix} \left(-L_3+(t_m+\Delta t).L_4\right).\frac{2}{\Delta t} \\ L_4 \\ L_3+L_2-(t_m+\Delta t).L_4 \\ -\frac{(\Delta t)^2}{6}L_4+L_1-\left(\frac{\Delta t}{3}+t_m\right).L_2 \end{bmatrix} \\ \begin{bmatrix} \mathbf{K} & \mathbf{K} & -\mathbf{K}\frac{2\Delta t}{3} & -\mathbf{K}\frac{\Delta t}{3} \\ 0 & \mathbf{K} & -\mathbf{K}\frac{\Delta t}{2} & -\mathbf{K}\frac{\Delta t}{2} \\ 0 & 0 & \mathbf{K}\frac{(\Delta t)^2}{3}+\mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{6}+\mathbf{M}+\mathbf{C}\frac{\Delta t}{2} \\ 0 & 0 & \mathbf{K}\frac{(\Delta t)^3}{12}-\mathbf{M}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12}+\mathbf{M}\frac{\Delta t}{6}+\mathbf{C}\frac{(\Delta t)^2}{6} \end{bmatrix} \tag{1.105}$$

1.9 Calcul du second membre

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix} = \\ \begin{bmatrix} t_m\mathbf{M}\mathbf{v}_m^- + \left[\frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{f}_m+\mathbf{f}_{m+1}\right)+t_m\frac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1})\right] \\ \mathbf{M}\mathbf{v}_m^- + \frac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1}) \\ t_m\mathbf{K}\mathbf{u}_m^- \\ \mathbf{K}\mathbf{u}_m^- \end{bmatrix} \tag{1.106}$$

$$\begin{bmatrix} \left(-L_3+(t_m+\Delta t).L_4\right).\frac{2}{\Delta t} \\ L_4 \\ L_3+L_2-(t_m+\Delta t).L_4 \\ -\frac{(\Delta t)^2}{6}L_4+L_1-\left(\frac{\Delta t}{3}+t_m\right).L_2 \end{bmatrix} = \\ \begin{bmatrix} \left[-\left(t_m\mathbf{K}\mathbf{u}_m^-\right)+(t_m+\Delta t).\left(\mathbf{K}\mathbf{u}_m^-\right)\right].\frac{2}{\Delta t} \\ \mathbf{K}\mathbf{u}_m^- \\ \left(t_m\mathbf{K}\mathbf{u}_m^-\right)+\left(\mathbf{M}\mathbf{v}_m^-+\frac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1})\right)-(t_m+\Delta t).\left(\mathbf{K}\mathbf{u}_m^-\right) \\ -\frac{(\Delta t)^2}{6}\left(\mathbf{K}\mathbf{u}_m^-\right)+t_m\mathbf{M}\mathbf{v}_m^-+\left[\frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{f}_m+\mathbf{f}_{m+1}\right)+t_m\frac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1})\right]-\left(\frac{\Delta t}{3}+t_m\right).\left(\mathbf{M}\mathbf{v}_m^-+\frac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1})\right) \end{bmatrix} \tag{1.107}$$

Soit :

$$\begin{bmatrix} \left[\Delta t\mathbf{K}\mathbf{u}_m^-\right].\frac{2}{\Delta t} \\ \mathbf{K}\mathbf{u}_m^- \\ \left(\mathbf{M}\mathbf{v}_m^-+\frac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1})\right)-\Delta t.\left(\mathbf{K}\mathbf{u}_m^-\right) \\ -\frac{(\Delta t)^2}{6}\left(\mathbf{K}\mathbf{u}_m^-\right)+\left[\frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{f}_m+\mathbf{f}_{m+1}\right)\right]-\frac{\Delta t}{3}.\left(\mathbf{M}\mathbf{v}_m^-+\frac{\Delta t}{2}(\mathbf{f}_m+\mathbf{f}_{m+1})\right) \end{bmatrix} \tag{1.108}$$

Soit :

$$\begin{bmatrix} 2\mathbf{K}\mathbf{u}_m^- \\ \mathbf{K}\mathbf{u}_m^- \\ \left(\mathbf{M}\mathbf{v}_m^- + \frac{\Delta t}{2}(\mathbf{f}_m + \mathbf{f}_{m+1})\right) - \Delta t.\left(\mathbf{K}\mathbf{u}_m^-\right) \\ -\frac{(\Delta t)^2}{6}\mathbf{K}\mathbf{u}_m^- + \frac{(\Delta t)^2}{3}\left(\frac{1}{2}\mathbf{f}_m + \frac{1}{2}\mathbf{f}_{m+1}\right) + \frac{(\Delta t)^2}{3}\frac{1}{2}\mathbf{f}_{m+1} - \frac{\Delta t}{3}.\left(\mathbf{M}\mathbf{v}_m^- + \frac{\Delta t}{2}(\mathbf{f}_m + \mathbf{f}_{m+1})\right) \end{bmatrix} \tag{1.109}$$

Soit :

$$\begin{bmatrix} 2\mathbf{K}\mathbf{u}_m^- \\ \mathbf{K}\mathbf{u}_m^- \\ \left(\mathbf{M}\mathbf{v}_m^- + \frac{\Delta t}{2}(\mathbf{f}_m + \mathbf{f}_{m+1})\right) - \Delta t.\left(\mathbf{K}\mathbf{u}_m^-\right) \\ -\frac{(\Delta t)^2}{6}\mathbf{K}\mathbf{u}_m^- + \frac{(\Delta t)^2}{6}\mathbf{f}_{m+1} - \frac{\Delta t}{3}.\mathbf{M}\mathbf{v}_m^- \end{bmatrix} \tag{1.110}$$

Soit :

$$\begin{bmatrix} 2\mathbf{K}\mathbf{u}_m^- \\ \mathbf{K}\mathbf{u}_m^- \\ \left(\mathbf{M}\mathbf{v}_m^- + \frac{\Delta t}{2}(\mathbf{f}_m + \mathbf{f}_{m+1})\right) - \Delta t.\left(\mathbf{K}\mathbf{u}_m^-\right) \\ -\frac{(\Delta t)^2}{6}\left(\mathbf{K}\mathbf{u}_m^- - \mathbf{f}_{m+1}\right) - \frac{\Delta t}{3}.\mathbf{M}\mathbf{v}_m^- \end{bmatrix} \tag{1.111}$$

1.10 Système complet final

$$\begin{bmatrix} \mathbf{K} & \mathbf{K} & -\mathbf{K}\frac{2\Delta t}{3} & -\mathbf{K}\frac{\Delta t}{3} \\ 0 & \mathbf{K} & -\mathbf{K}\frac{\Delta t}{2} & -\mathbf{K}\frac{\Delta t}{2} \\ 0 & 0 & \mathbf{K}\frac{(\Delta t)^2}{3} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{6} + \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} \\ 0 & 0 & \mathbf{K}\frac{(\Delta t)^3}{12} - \mathbf{M}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^3}{12} + \mathbf{M}\frac{\Delta t}{6} + \mathbf{C}\frac{(\Delta t)^2}{6} \end{bmatrix} \begin{bmatrix} \mathbf{u}_m^+ \\ \mathbf{u}_{m+1}^- \\ \mathbf{v}_m^+ \\ \mathbf{v}_{m+1}^- \end{bmatrix}$$
$$\begin{bmatrix} 2\mathbf{K}\mathbf{u}_m^- \\ \mathbf{K}\mathbf{u}_m^- \\ \left(\mathbf{M}\mathbf{v}_m^- + \frac{\Delta t}{2}(\mathbf{f}_m + \mathbf{f}_{m+1})\right) - \Delta t.\left(\mathbf{K}\mathbf{u}_m^-\right) \\ -\frac{(\Delta t)^2}{6}\left(\mathbf{K}\mathbf{u}_m^- - \mathbf{f}_{m+1}\right) - \frac{\Delta t}{3}.\mathbf{M}\mathbf{v}_m^- \end{bmatrix} \tag{1.112}$$

Ou :

$$\begin{bmatrix} \mathbf{K} & 0 & -\mathbf{K}\frac{\Delta t}{6} & \mathbf{K}\frac{\Delta t}{6} \\ 0 & \mathbf{K} & -\mathbf{K}\frac{\Delta t}{2} & -\mathbf{K}\frac{\Delta t}{2} \\ 0 & 0 & \mathbf{K}\frac{(\Delta t)^2}{3} + \mathbf{C}\frac{\Delta t}{2} & \mathbf{K}\frac{(\Delta t)^2}{6} + \mathbf{M} + \mathbf{C}\frac{\Delta t}{2} \\ 0 & 0 & \mathbf{K}\frac{(\Delta t)^2}{12} - \mathbf{M}\frac{1}{2} & \mathbf{K}\frac{(\Delta t)^2}{12} + \mathbf{M}\frac{1}{6} + \mathbf{C}\frac{\Delta t}{6} \end{bmatrix} \begin{bmatrix} \mathbf{u}_m^+ \\ \mathbf{u}_{m+1}^- \\ \mathbf{v}_m^+ \\ \mathbf{v}_{m+1}^- \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{K}\mathbf{u}_m^- \\ \mathbf{K}\mathbf{u}_m^- \\ \left(\mathbf{M}\mathbf{v}_m^- + \frac{\Delta t}{2}(\mathbf{f}_m + \mathbf{f}_{m+1})\right) - \Delta t.\left(\mathbf{K}\mathbf{u}_m^-\right) \\ -\frac{\Delta t}{6}\left(\mathbf{K}\mathbf{u}_m^- - \mathbf{f}_{m+1}\right) - \frac{1}{3}.\mathbf{M}\mathbf{v}_m^- \end{bmatrix} \tag{1.113}$$