Recursive Methods

Lecture 3: Optimal Growth Model

Julien Prat

CNRS, CREST

2015

Motivation

We have shown in the previous two lectures that "standard" dynamic programs can be solved recursively. Thus sequential problems have already been solved in a numerical sense.

But we would like to gather insights that go beyond specific numerical outcomes.

Analyzing the recursive problems, we have been able to establish some properties of the value and policy functions, such as uniqueness, concavity and differentiability.

However, besides these general properties, we still know very little about the behavior of endogenous variables.

Motivation

Although there is no general method that can be used to characterize the behavior of any solution, some techniques turn out to be widely applicable.

Instead of directly discussing them, we start with an analysis of the canonical growth model which illustrates how a simple problem can generate a variety of solutions, some very simple, others rather complex.

The planner seeks to solve

$$\max \sum_{t=0}^{\infty} \beta^t U[f(k_t) - k_{t+1}], \tag{1}$$

$$s.t.\ 0 \leq k_{t+1} \leq f(k_t),\ \ t = 0, 1, \ldots$$

given $k_0 \geq 0$.

Assumptions

A1. The discount factor $\beta \in (0,1)$.

A2. The utility function $U : \mathbb{R}_+ \to \mathbb{R}$ is continuous, strictly increasing, concave and continuously differentiable. Furthermore, $\lim_{c\to\infty} U'(c) = 0$.

A3. The production function $f : \mathbb{R}_+ \to \mathbb{R}$ is continuous, strictly increasing, (weakly) concave and continuously differentiable. Furthermore, $f(0) = 0$, $\lim_{k\to 0} f'(k) = +\infty$ and $\lim_{k\to\infty} f'(k) = 1 - \delta \in (0,1)$.

Validity of Recursive Formulation

First notice that assumption $(A3)$ ensures that there exists a unique k_{max} such that $k_{max} = f(k_{max})$. Hence k_{max} is the maximum maintainable capital stock.

Thus we can restrict or attention to the compact interval $[0, k_{max}]$ and, since both f and U are continuous, returns are bounded over the relevant range.

Exercise 3.1: Show that under (A1)-(A3): (i) the hypotheses of Theorems 1.1-1.4 are satisfied, (ii) the hypotheses of Theorems 2.1-2.5 are satisfied.

From part (i), we conclude that solutions to (1) coincide with those to

$$v(k) = \max_{0 \leq y \leq f(k)} \left\{ U[f(k) - y] + \beta v(y) \right\}.$$

From part (ii), we conclude that the value function v is continuous and differentiable, whereas the policy correspondence g is single-valued and continuous.

Euler Equation

A classical approach, dating back to Euler's works in the XVIIIth century, characterizes the optimality conditions by building on the observation that, $\{x_t^*\}_{t=0}^{\infty}$ solves the sequential problem, then

$$x_{t+1}^* = \arg\max_y \{F(x_t^*, y) + \beta F(y, x_{t+2}^*)\},$$
$$s.t.\ y \in \Gamma(x_t^*) \text{ and } x_{t+2}^* \in \Gamma(y).$$

This approach is called variational because it rules out the possibility that (one-shot) feasible variations could improve the optimal policy.

Since $F(\cdot)$ is assumed to be continuously differentiable and strictly concave, if $x_{t+1}^* \in \mathit{int}\,\Gamma(x_t^*)$ for all t, we can differentiate the condition above to obtain the Euler Equation:

$$F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*) = 0,\ \ t = 0, 1, \ldots.$$

Euler Equation and Envelope Condition

The Euler Equation can also be derived directly from the Bellman equation

$$v(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}.$$

Its FOC reads

$$F_2(x, g(x)) + \beta v'(g(x)) = 0,$$

while the envelope condition is

$$v'(x) = F_1(x, g(x)).$$

Replacing the envelope condition evaluated at $x = x_{t+1}$ into the FOC evaluated at $x = x_t$, we recover the Euler equation

$$F_2(x_t, x_{t+1}) + \beta F_1(x_{t+1}, x_{t+2}) = 0.$$

Optimal Growth

The FOC and envelope condition of the growth models are

$$\begin{aligned}(FOC) &: \quad U'[f(k) - g(k)] = \beta v'(g(k)), \\ (EC) &: \quad v'(k) = U'[f(k) - g(k)]f'(k).\end{aligned}$$

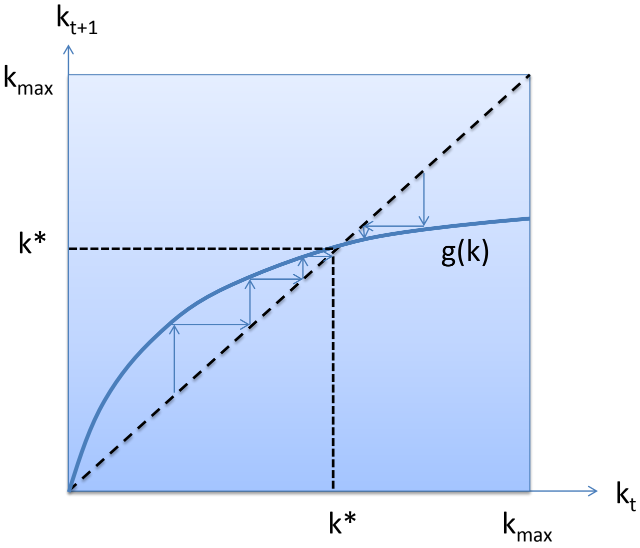
Replacing the steady-state condition $g(k^*) = k^*$ into (FOC) and (EC), we find that $1 = \beta f'(k^*)$ which, under assumption (A3) about the production function, admits a unique positive solution.[1]

Exercise 3.2: Show that, for any $k_0 \in (0, k_{max}]$, the optimal sequence $\{k_t\}_{t=0}^{\infty}$ defined by $k_{t+1} = g(k_t)$ converges monotonically to $k^* = f'^{-1}(1/\beta)$.

Exercise 3.3: Write a code in Matlab that uses an iterative procedure to solve for the value function.

[1] k=0 is also a trivial steady-state.

Phase Portrait



Turnpike Theorem

The optimal growth model has a unique attractive steady-state.

The turnpike theorem generalizes this finding to a wider class of models:

Theorem 3.1. (Turnpike Theorem)

Let $\{x_t\}_{t=0}^{\infty}$ maximize $\sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1})$ with $U(\cdot)$ strictly concave and $\beta \in (0,1)$. Then there exists a $\bar{\beta} \in (0,1)$ such that, for all $\beta \in [\bar{\beta}, 1)$, the policy function has a unique and globally stable fixed point.

PROOF: See Scheinkman, 1976, *Journal of Economic Theory*, Vol. 12(1), 11-30.

Intuition: Concave preferences imply that agents would like to stabilize the state variable x. When they are sufficiently patient, the long-run benefits of stabilization always outweigh the short-run costs of adjustment along the transition path.

Possibility Theorem

Can we conclude from the Turnpike Theorem that dynamic optimality guarantees global stability?

To the contrary, the Possibility Theorem shows that "(almost) everything goes":

Theorem 3.2. (Possibility Theorem)

Let X be a compact set in $\mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be any twice continuously differentiable function. Then there exists a return function F and a discount factor β satisfying assumptions 2.1-2.4 such that g is the optimal solution of the corresponding dynamic program.

PROOF: See constructive proof in Boldrin and Montrucchio, 1986, *Journal of Economic Theory*, Vol. 40(1), 26-39.

They show how to compute a fictitious economy for any desired dynamics.

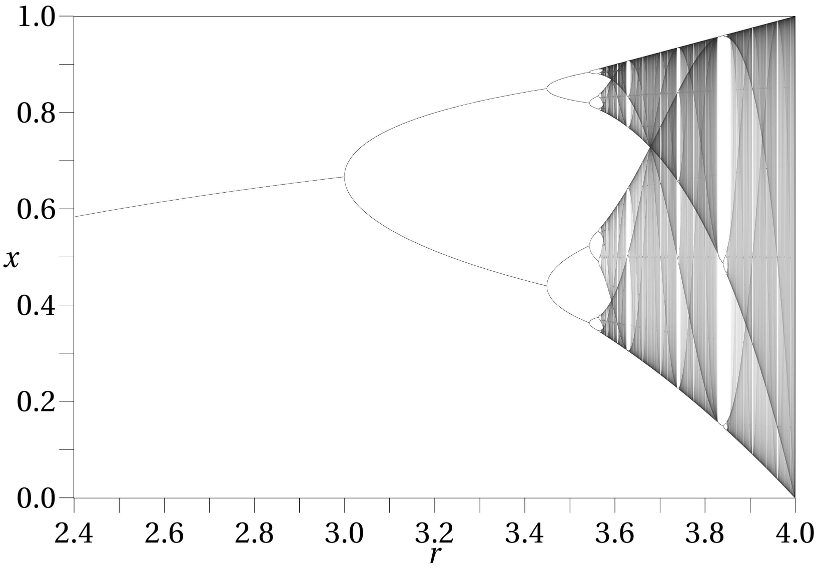
Logistic Map

The Possibility Theorem states that any smooth difference equation can be thought of as describing optimal behavior through time.

In particular, the Possibility Theorem implies that, setting $X = [0, 1]$ along with $\Gamma(X) = X$, we can select a return function and discount factor such that the policy function is a logistic map, i.e. $g(x) = rx(1 - x)$.

Then it is well known that, as the value of the coefficient r is increased from 0 to 4, the system will go from globally stable to exhibiting oscillations between a finite number of values, until it eventually displays chaotic behavior.

Bifurcation Diagram of Logistic Map



Endogenous fluctuations

The Possibility Theorem implies that deterministic programs can generate endogenous cycles or even erratic oscillations.

But are these economically meaningful possibilities? In other words, can one write "realistic" models whose dynamics are not globally stable?

The growth model suggests that the answer is no since, for that model, the Turnpike Theorem holds independently of the discount factor.

However, simple generalizations yield dramatically different results. For example, Benhabib and Nishimura (1985) shows that a two-sector growth model exhibits a flip bifurcation so that cycles of period 2 arise. (See Exercise 6.7 in SLP)

A vast literature studies whether such models can fit the data, with most of the debate centered on parametric restrictions. See Boldrin and Woodford, 1990, *Journal of Monetary Economics*, for a survey.

Conclusion

We have shown that, although the one-sector growth model is globally stable, optimal behavior in more sophisticated models can be consistent with endogenous fluctuations.

To study the local stability of the system, one has to characterize the slope of the policy function at the steady-state.

It is explained in Section 6.3-6.4 of SLP how this can be done by (i) taking a linear approximation of the Euler equation around the steady-state, and (ii) verifying that the absolute values of its characteristic roots are below one.