RECURSIVE CONTRACTS

Many economic relationships are repeated over time and regulated by contractual agreements. Contracting dramatically increases the set of feasible transfers. In general, contracts can specify payments that depend on the *whole history* of past realizations. This added flexibility makes it possible to optimally design payoffs so as to insure and motivate the agent. Thus it should not be surprising that contracts are so widespread in reality.

From a theoretical standpoint, the analysis of contracts in dynamic settings raises some serious challenges. A naïve approach quickly becomes intractable because it involves keeping track of all past states. We explain in this note how most contracting problems can be rewritten in recursive form through the introduction of the agent's promised value. We begin with a motivating example based on Holmstrom (1983).

1 Long term labor contracts

1.1 Implicit contracts

As in Holmstrom (1983), we first consider a two period setting so that $t = 0, 1$. The state s is known in period 0. By contrast, the state in period 1 is randomly assigned to one of S mutually exclusive states $s = 1, .., S$ with probability $\phi(s)$. Holmstrom distinguishes both periods and sates by the same index, so $s = 0$ will denote the state in period 0.

The model has two types of agents. Risk-neutral principals (firms) whose preferences are given by

$$\Pi(c_0, c_1, ..., c_S) = c_0 + \sum_{s=1}^{S} c_s \phi(s).$$

A principal's consumption is derived from the employment of an agent (worker) whose productivity f_s is a function of the realized state. Denoting the worker's wage w_s, we have $c_s = f_s - w_s$. Workers are risk-averse with preferences

$$\mathcal{U}(w_0, w_1, ..., w_S) = U(w_0) + \sum_{s=1}^{S} U(w_s)\phi(s).$$

Firms have the ability to commit to any wage schedule. Their problem therefore consists in finding the wage contract that maximizes their profits provided that workers are willing to participate. Formally, for any outside option $\underline{V}$, the wage contract solves[1]

$$\begin{aligned} \max_w \Pi(w) &= f_0 - w_0 + \sum_{s=1}^{S}(f_s - w_s)\phi(s), \\ s.t.\ \mathcal{U}(w) &\geq \underline{V}. \end{aligned}$$

Attaching the Lagrangian multiplier λ to the participation constraint, the contracting problem is equivalent to

$$\mathcal{L} = \max_w \left\{ f_0 - w_0 + \lambda U(w_0) + \sum_{s=1}^{S}[f_s - w_s + \lambda U(w_s)]\phi(s) - \lambda \underline{V} \right\}.$$

Its FOCs read

$$U'(w_s) = \frac{1}{\lambda}, \quad s = 0, 1, ..., S.$$

Hence wages are constant across states and periods. In other words, it is optimal to *fully insure* workers. Since they are risk averse, they are willing to trade income stabilization against lower wages, which raises the employer's profit. Wages are simply set equal to the lowest level such that the participation constraint is satisfied, i.e. $w = U^{-1}(\underline{V}/2)$.

1.2 Self-enforcing contracts

Implicit contract theory dates back to the work of Azariadis (1975). Holmstrom (1983) enriches Azariadis' model by allowing workers to receive new wage offers in the the second period. Let w_s^+ denote the best alternative wage the worker can earn in state s and $w^+ \equiv (w_1^+, ..., w_S^+)$. Adding this outside option restricts the set of feasible contracts since now the firm has to ensure that workers do not find it optimal to walk away from the relationship in $t = 1$. Since the agents' horizon ends at $t = 1$, they will quit whenever $w_s < w_s^+$.[2] Adding these self enforcing constraints allows us to define the principal's problem as

$$\begin{aligned} \max_w \Pi(w) &= f_0 - w_0 + \sum_{s=1}^{S}(f_s - w_s)\phi(s), \\ s.t.\ (i)\, \mathcal{U}(w) &\geq \underline{V}, \\ (ii)\, w_s &\geq w_s^+. \end{aligned}$$

[1] Note that the technology coefficient f_s are irrelevant since we could as well have minimized the wage bill, i.e., $\min_w w_0 + \sum_{s=1}^{S} w_s\phi(s)$.

[2] We can assume without loss of generality that workers do not quit when $w_s = w_s^+$.

Attaching the Lagrangian multipliers γ_s to the self-enforcing constraints, the contracting problem is equivalent to

$$\mathcal{L} = \max_w \left\{ f_0 - w_0 + \lambda U(w_0) + \sum_{s=1}^{S} [f_s - w_s + \lambda U(w_s) + \gamma_s(w_s - w_s^+)]\phi(s) - \lambda \underline{V} \right\},$$

and its FOCs read

$$\begin{aligned} U'(w_0) &= \frac{1}{\lambda}, \\ U'(w_s) &= \frac{1-\gamma_s}{\lambda}. \end{aligned}$$

There are two possibilities:

1. $w_s > w_s^+$: Then the self-enforcing constraint is slack and $\gamma_s = 0$. It follows that $w_s = w_0$ as in the implicit contract studied above.

2. $w_s = w_s^+$: Then the self-enforcing constraint binds and $\gamma_s > 0$. It follows that $w_s > w_0$ and so workers are not anymore perfectly insured.

Putting the two points above together, we see that $w_s = max\{w_0, w_s^+\}$. Wages are *downward rigid*: They never decrease but they can go up when the worker's outside option in the second period is high enough. Whether some of the self-enforcing constraints bind depends on the level of the initial wage whose value solves

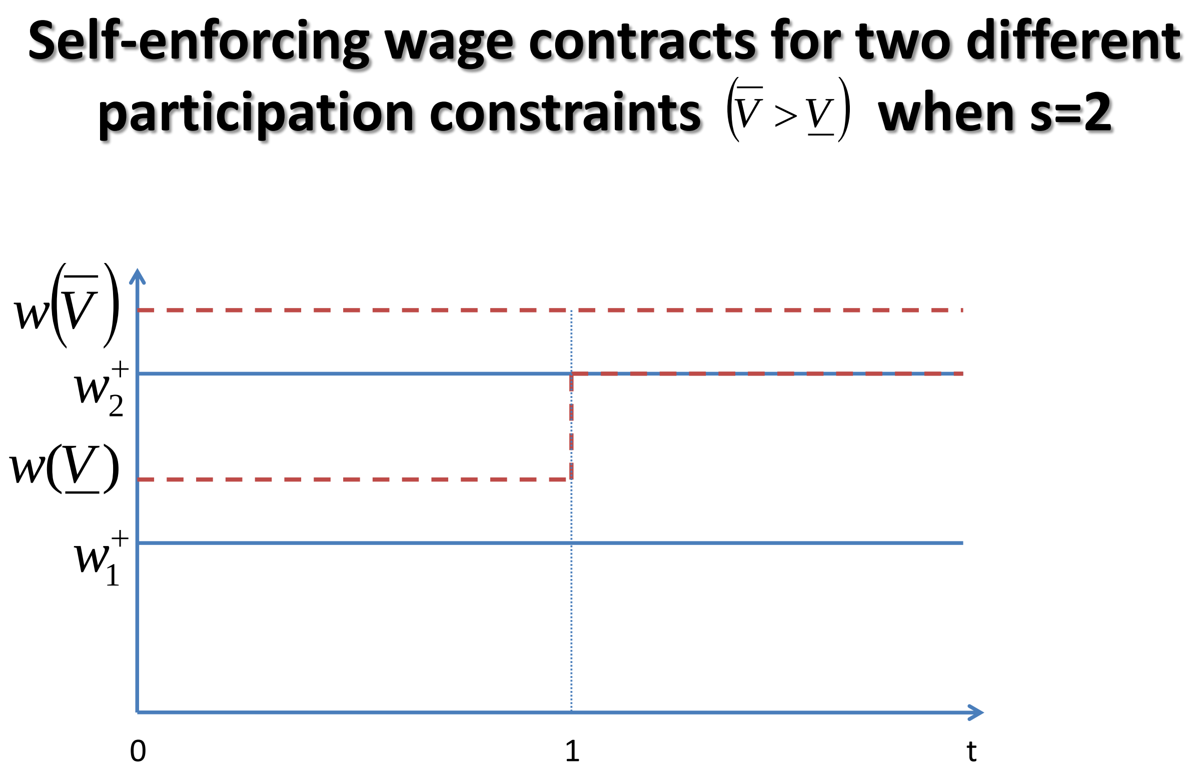
$$U(w_0) + \sum_{s=1}^{S} U(max\{w_0, w_s^+\})\phi(s) = \underline{V}.$$

Thus there exists a level of the outside option in the first period $\underline{V}$ below which some of the self-enforcing constraints start to bind. This also implies that wages in the second period will not only depend on the current state, but also on the worker's outside option when the contract was signed. Since the latter is correlated with wages in the first period, wages are path dependent.

In other words, optimal wages are not *Markovian*. This is most easily seen considering that the state in the first period can take one of two values. If the state is low, workers have a low participation constraint $\underline{V}$. Conversely, if the state is high, the participation constraint is higher $\bar{V} > \underline{V}$. Provided that the wedge separating the two outside options is wide enough, wages will be higher in some states of period 1 when the state was high in period 0, i.e., there will be some $s \in \{1, .., S\} : w(s; \bar{V}) > w(s; \underline{V})$.

Such a configuration is illustrated in Figure 1.2. It shows that wages are not fully determined by today's state. Instead they are a function of the *history of states*, formally: $w =$

$(w_0(s_0), w_1(s_0, s_1))$ where the subscripts now refer to *time indices*. While Keeping track of all past states is manageable in a 2-period setting, the problem quickly becomes untractable when we add further periods. We now explain how to write the contracting problem in recursive form so as to bypass the non-Markovian structure of transfers.



2 Recursive self-enforcing contracts

As in Thomas and Worrall (1988), we keep the assumptions of the model described above but we extend its horizon from 2 to an infinite number of periods. The state is *independently and identically distributed* according the probability distribution $\phi(s)$ with $s \in \{1, 2, ..., S\}$.

The contract is signed in the first period. While firms are fully committed, workers can walk away in any period. For simplicity, we assume that a worker who reneges on a contract must from then on trade on the spot market forever.[3] The worker's outside option is therefore equal to

$$\underline{V} = E\left[\sum_{t=0}^{\infty} \beta^t u(w^+(s_t))\right],$$

where β is the worker's discount factor. Note that the expectation and thus $\underline{V}$ do not depend on the current realization of s because states are i.i.d. Hence the worker's participation constraint after every *history* of states $s^t \equiv (s_0, s_1, ..., s_t)$ reads

$$u(w_t(s^t)) + \beta E_t\left[\sum_{j=1}^{\infty} \beta^{j-1} u(w_{t+j}(s^{t+j}))\right] \geq u(w^+(s_t)) + \beta\underline{V}.$$

The sequential problem therefore consists in solving

$$(SP): \Gamma^*(V_0) = \max_w \gamma(w; V_0) \equiv \max_w \left\{-E_0\left[\sum_{j=0}^{\infty} \beta^j w_j(s^j)\right]\right\}$$

$$s.t.\ E_0\left[\sum_{j=0}^{\infty} \beta^j u(w_j(s^j))\right] \geq V_0,$$

$$u(w_t(s^t)) + \beta E_t\left[\sum_{j=1}^{\infty} \beta^{j-1} u(w_{t+j}(s^{t+j}))\right] \geq u(w^+(s_t)) + \beta\underline{V}, \text{for all t>0 and } s^t \in \mathcal{S}^t, \quad (1)$$

where $\mathcal{S}^t$ is the tst Cartesian product of the set $\mathcal{S} \equiv \{1, 2, ..., S\}$. The difficulty with such a formulation is that wages are a function of history s^t whose dimension is so large and grows so rapidly with time. Fortunately we can bypass the history-dependance of contracts by assuming that wages are a function of the *promised discounted future value*

$$V_t(s^{t-1}) \quad \equiv \quad \sum_{\tau=t}^{\infty} \sum_{s^\tau \in \mathcal{S}^\tau} \beta^\tau u(w_\tau(s^\tau))\Phi(s^\tau|s^{t-1}) = E_t\left[\sum_{j=0}^{\infty} \beta^j u(w_{t+j}(s^{t+j}))\right],$$

where $\Phi(s^\tau|s^t) = \phi(s_{t+1})\phi(s_{t+2})...\phi(s_\tau)$ is the joint probability of history s^τ conditional on s^t.

Treating V_t as a state variable, the firm's problem consists in delivering the promised value at the lowest cost under the self-enforcing constraint. Its control variable are the state-dependent wages $w(s_t)$ and the *continuation values* $W(s_t) = V_{t+1}$.

Claim 1 *A contract w solving the sequential problem (SP) is also a solution of the following recursive problem under (i) the* Promise Keeping *constraint and (ii) the* Self-Enforcing *con-*

[3] See Assumption 1 in Thomas and Worrall (1988).

straints:

$$\begin{aligned}
\Gamma(V_t) \quad &= \quad \max_{w(s_t),W(s_t)} \sum_{s_t=1}^{S} \left[-w(s_t) + \beta\Gamma(W(s_t))\right]\phi(s_t), \\
s.t. \quad (PK): \quad & \sum_{s=1}^{S} \left[u(w(s_t)) + \beta W(s_t)\right]\phi(s_t) \geq V_t, \\
(SEC): \quad & u(w(s_t)) + \beta W(s_t) \geq u(w_{s_t}^{+}) + \beta \underline{V}, \ \ s_t = 1, ..., S.
\end{aligned} \tag{2}$$

Proof. See Appendix. ■

2.1 Analyzing the recursive solution

Γ solves a functional equation similar to the Bellman equations studied in previous lectures. Since the constraint set is convex, and provided that the objective is concave, we can form a Lagrangian

$$\begin{aligned}
\mathcal{L} \quad = \quad & \sum_{s_t=1}^{S} \left[-w(s_t) + \beta\Gamma(W(s_t))\right]\phi(s_t) + \mu\left\{\sum_{s=1}^{S}\left[u(w(s_t)) + \beta W(s_t)\right]\phi(s_t) - V_t\right\} \\
& + \sum_{s_t=1}^{S} \lambda_{s_t}\left\{u(w(s_t)) + \beta W(s_t) - \left[u(w_{s_t}^{+}) + \beta\underline{V}\right]\right\}.
\end{aligned}$$

For all V and each $s = 1, ..., S$; the FOCs are

$$\begin{aligned}
[\lambda_{s_t} + \mu\phi(s_t)]u'(w(s_t)) \quad &= \quad \phi(s_t), \\
\lambda_{s_t} + \mu\phi(s_t) \quad &= \quad -\phi(s_t)\Gamma'(W(s_t)).
\end{aligned}$$

Combining these two conditions, we find that

$$u'(w(s_t)) = -\Gamma'(W(s_t))^{-1}. \tag{3}$$

Intuitively, the worker's marginal utility should be equal to the employer's marginal rate of transformation.[4] The characterization of the contract will therefore be complete if we can determine how the continuation values $W(s_t)$ depend on the promised value V and the current state s. As in the 2-period model, the relationship depends on whether the self-enforcing constraint binds:

[4]By the envelope theorem, $\Gamma'(V) = -\mu$.

1. $\gamma_s > 0$: Combining the second FOC with the envelope condition, we find that $\Gamma'(W(s_t)) = \Gamma'(V_t) - \lambda_{s_t}/\phi(s_t)$. Hence, $\Gamma'(W(s_t)) < \Gamma'(V_t)$ when the self-enforcing constraint binds. $\Gamma(\cdot)$ being concave, it follows that $V_{t+1} = W(s_t) > V_t$ so that the firm raises the worker's continuation value. To determine $w(s_t)$, we notice that it solves

$$\begin{aligned} u(w(s_t)) + \beta W(s_t) &= u(w^+_{s_t}) + \beta \underline{V}, \\ u'(w(s_t)) &= -\Gamma'(W(s_t))^{-1}. \end{aligned}$$

   These two equations are independent of the promised value V. Thus we can define $w(s) = g_1(s)$ and $W(s) = l_1(s)$.

2. $\gamma_s = 0$: Then $\Gamma'(W(s_t)) = \Gamma'(V_t)$ which implies in turn that the promised value remains constant, i.e., $V_{t+1} = W(s_t) = V_t$. The optimality condition (3) is therefore equivalent to $u'(w(s_t)) = -\Gamma'(V_t)^{-1}$ and the wage depends only on the promised utility but not on the state. The wage and continuation utility when the self-enforcing constraint does not bind are given by $w(s) = g_2(V)$ and $W(s) = V$.

Combining the two points above, we find that

$$\begin{aligned} w(s) &= max\{g_1(s), g_2(V)\}, \\ W(s) &= max\{l_1(s), V\}. \end{aligned}$$

To interpret these definitions, notice that there exists a state $\bar{s}(V)$ such that the self-enforcing constraint binds solely if $s \geq \bar{s}(V)$. To find $\bar{s}(V)$, substitute $u'(g_2(V)) = -\Gamma'(V)^{-1}$ into the self-enforcing constraint evaluated at the threshold state

$$u(w^+_{\bar{s}(V)}) = u(g_2(V)) + \beta[V - \underline{V}].$$

By concavity of $\Gamma(\cdot)$, $\bar{s}(v)$ is increasing in V. For any promised utility, the optimal contract takes the following form:

1. If $s \leq \bar{s}(V)$: Both wages and continuation utility remain constant, i.e., $w = g_2(V)$ and $V_{t+1} = V_t$.

2. $s > \bar{s}(V)$: Both consumption and continuation utility increase until the value of remaining employed equals that of the outside option.

The optimal design of the infinite horizon contract is qualitatively similar to that of the 2-period contract. Wages and promised values never decrease. Workers benefit from partial insurance as their wages are raised solely when the threat to walk away from the relationship is credible. In particular, the highest wage is permanently awarded to the worker after any realization of the highest possible state S.

3 Moral hazard

The recursive approach described in the previous section can be used to analyze a much wider class of contracting problems. As an example we will study a dynamic version of Rogerson (1985). We consider a dynamic moral hazard problem where the agent's *unobservable effort* $e \in \mathcal{E}$ affects the output distribution, so that $\phi(s|e)$ now varies with e. We assume that raising e shifts the probability distribution in such a way that it First Order Stochastically Dominates the original one.

Then the principal's problem is to choose the effort level, wage and continuation utilities that maximize her profit under (i) the *Promise Keeping* constraint and (ii) the *Incentive* constraints:

$$\begin{aligned}
\Gamma(V_t) \quad &= \quad \max_{e_t, w(s_t), W(s_t)} \sum_{s_t=1}^{S} \left[y(s_t) - w(s_t) + \beta\Gamma(W(s_t))\right]\phi(s_t|e_t), \\
s.t. \quad (PK): &\quad \sum_{s_t=1}^{S} \left[u(w(s_t)) - e_t + \beta W(s_t)\right]\phi(s_t|e_t) \geq V_t, \\
(IC): &\quad \sum_{s_t=1}^{S} \left[u(w(s_t)) - e_t + \beta W(s_t)\right]\phi(s_t|e_t) \geq \sum_{s_t=1}^{S} \left[u(w(s_t)) - \tilde{e} + \beta W(s_t)\right]\phi(s_t|\tilde{e}), \;\; \forall \tilde{e} \in \mathcal{E}.
\end{aligned}$$

To simplify the analysis, let's say that e can take only two values, so that $\mathcal{E} = \{0, 1\}$, and that the principal always wants to implement high-effort $e_t = 1$ for all t. Then the Lagrangian simplifies to

$$\begin{aligned}
\mathcal{L} \;\; = \;\; & \sum_{s_t=1}^{S} \left[y(s_t) - w(s_t) + \beta\Gamma(W(s_t))\right]\phi(s_t|1) + \mu\left\{\sum_{s_t=1}^{S} \left[u(w(s_t)) - 1 + \beta W(s_t)\right]\phi(s_t|1) - V_t\right\} \\
& + \lambda\left\{\sum_{s_t=1}^{S} \left[u(w(s_t)) + \beta W(s_t)\right]\left(\phi(s_t|1) - \phi(s_t|0)\right) - \phi(s_t|1)\right\}.
\end{aligned}$$

The FOCs read

$$\begin{aligned} 0 &= -\phi(s_t|1) + \mu\phi(s_t|1)u'(w(s_t)) + \lambda[\phi(s_t|1) - \phi(s_t|0)]u'(w(s_t)), \\ 0 &= \phi(s_t|1)\beta\Gamma'(W(s_t)) + \mu\phi(s_t|1)\beta + \lambda\beta[\phi(s_t|1) - \phi(s_t|0)]. \end{aligned}$$

They are equivalent to

$$\frac{1}{u'(w(s_t))} = \mu + \lambda\left[1 - \frac{\phi(s_t|0)}{\phi(s_t|1)}\right], \tag{4}$$

$$\Gamma'(W(s_t)) = -\left(\mu + \lambda\left[1 - \frac{\phi(s_t|0)}{\phi(s_t|1)}\right]\right). \tag{5}$$

As before, the envelope condition is $\Gamma'(V_t) = -\mu$. Hence the FOC above implies that

$$\Gamma'(W(s_t)) = \Gamma'(V_t) - \lambda\left[1 - \frac{\phi(s_t|0)}{\phi(s_t|1)}\right].$$

Computing the expectation over all states

$$E\left[\Gamma'(W(s_t))\right] = \Gamma'(V_t) - \lambda\underbrace{E\left[1 - \frac{\phi(s_t|0)}{\phi(s_t|1)}\right]}_{=0} = \Gamma'(V_t).$$

The marginal returns of the promised value $\Gamma'(V)$ follows a martingale. But since (4) and (5) imply that $\Gamma'(W(s_t)) = -1/u'(w(s_t))$, the inverse of the marginal utility also has to follow a martingale, i.e.

$$E\left[\frac{1}{u'(w_t)}\right] = \frac{1}{u'(w_{t-1})}.$$

This equation is known as the **Inverse Euler Equation**. It equates the expected marginal cost of a unit of utility over time. Thus it differs from the standard Euler equation which equates the expected marginal utility of a Euro over time. The two conditions are mutually exclusive.[5] In particular, it is easy to check that if the Inverse Euler equation holds, $E\left[u'(w_t)\right] > u'(w_{t-1})$,[6] so that the agent would like to save when the rate of interest $1+r = 1/\beta$. In other words, the agent is *saving constrained*, which means that the optimal allocation can be implemented solely if the principal is able to observe the agent's savings.

[5]Show that the Euler equation holds when the Incentive Constraints do not bind.

[6]The inequality follows from Jensen inequality applied to the function f(x)=1/x.

APPENDIX

Proof. Claim 1 Suppose that a sequence w satisfies the sequential self-enforcing constraint (1) but that, for some t and associated history $\tilde{s}^t = (\tilde{s}_0, ..., \tilde{s}_t)$, it violates the recursive self-enforcing constraint (2). If we repeatedly reinsert the definition of the continuation values W_τ, for all $\tau > t$, and evaluate (2) at $\tilde{s}^t$, we find that

$$\begin{aligned} u(w^+_{\tilde{s}_t}) + \beta \underline{V} &> u(w(\tilde{s}_t)) + \beta W(\tilde{s}_t) \\ &= u(w_t(\tilde{s}^t)) + \sum_{\tau=t+1}^{\infty} \sum_{s^\tau \in S^\tau} \beta^{\tau - t} u(w_\tau(s^\tau)) \Phi(s^\tau | \tilde{s}^t) \\ &= u(w_t(\tilde{s}^t)) + \beta E_t \left[\sum_{j=1}^{\infty} \beta^{j-1} u(w_{t+j}(\tilde{s}^t, s^{t+1,t+j})) \right], \end{aligned}$$

where $s^{t+1,t+j} = (s_{t+1}, ..., s_{t+j})$ is the history from date $t+1$ to $t+j$. Since all histories occur with positive probability, the inequality above implies that the sequential self-enforcing constraint (1) evaluated at date 0 is also violated which contradicts our premise. Hence any contract that satisfies the sequential self-enforcing constraint must also satisfy the recursive ones.

Then let w^* denote an optimal contract for the sequential problem and $V^*_t(s^t)$ its associated promised values. By the argument above, w^* has to satisfy the recursive incentive constraints at every nodes of the history. Hence, for any history s^t, the contract satisfies the constraints of the recursive problem, and since $\Gamma(V)$ returns the minimum costs, we must have $\Gamma^*(s^t, V^*_t(s^t)) \leq \Gamma(V^*_t(s^t))$. If we can show that the inequality binds, we will have proven that the sequential and recursive solutions coincide.

Thus, let's assume for the sake of contradiction that $\Gamma^*(s^t, V^*_t(s^t)) < \Gamma(V^*_t(s^t))$. This means that, for some history $\tilde{s}^{t+1}$, the expected costs of the recursive contract are smaller than those of the sequential contract. Thus we can construct a new contract $\tilde{w}$ that keeps all the transfers as in the original contract w^* except at the history node $\tilde{s}^{t+1}$ where they are replaced by the solutions of the recursive problem. By construction, this new contract delivers the initial promised value V_0, but its expected discounted cost evaluated from date 0 is lower than that of w^* since

$$\begin{aligned} & \gamma(\tilde{w}; V_0) - \gamma(w^*; V_0) \\ = & -\left(\tilde{w}(\tilde{s}^{t+1}) - w^*(\tilde{s}^{t+1}) + \beta E_{t+1} \left[\sum_{j=1}^{\infty} \beta^{j-1} [\tilde{w}_{t+1+j}(\tilde{s}^{t+1+j}) - w^*_{t+1+j}(\tilde{s}^{t+1+j})] \right] \right) \Phi(\tilde{s}^{t+1} | s_0) \\ = & -[\tilde{w}(\tilde{s}^{t+1}) + \Gamma(\tilde{W}(\tilde{s}^{t+1})) - (w^*(\tilde{s}^{t+1}) + \Gamma^*(\tilde{s}^{t+1}, V^*_{t+1}(\tilde{s}^{t+1})))] \Phi(\tilde{s}^{t+1} | s_0) > 0, \end{aligned}$$

where the inequality follows from the definition of $\tilde{s}^{t+1}$ and our conjecture that $\Gamma^*(s^t, V_t^*(s^t)) < \Gamma(V_t^*(s^t))$. Since it contradicts the optimality of w^*, we must have $\Gamma^*(s^t, V_t^*(s^t)) = \Gamma(V_t^*(s^t))$ for all s^t, and the recursive solution coincides with that of the sequential problem. ■

References

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