Recursive Methods

Lecture 5: Job Search

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Motivation

In Walras' seminal model markets are **frictionless**.

Drawing analogy from centralized markets for commodities, it is assumed that an auctioneer adjust prices until the supply from seller equals the demand from buyers.

However, many transactions are too specific to be set in such a standardized way. Then the seller may have to search for a while before finding a suitable buyer.

This is particularly true in the labor market because workers all have different characteristics and aspirations.

From a technical standpoint, finding the optimal search process is equivalent to solving an optimal stopping problem.

The basic model

The model describes the behavior of a worker searching for a job in a world with imperfect information.

The basic model is based on the following hypotheses:

1. Workers are risk neutral.
2. The environment is stationary.
3. The intensity of search cannot be adjusted by the worker.
4. Workers cannot search for alternative job while working.

The basic model

We assume that:

- Time is discrete.
- Worker discount the future at rate $1/(1{+}r)$.
- When a firm meets a job seeker, it offers him a constant wage over the duration of the job.
- Jobs last forever.
- At the end of each period:
  1. **Employed workers** receive their wages.
  2. **Unemployed workers** receive benefits z and sample a wage offer from the exogenous distribution $H(w)$.

Discounted utilities

A job seeker can reject the wage offer and continue to search. Accordingly, the **Bellman equations** for employed and unemployed workers read

Discounted Utilities

Let E denote the discounted utility of an employed worker

$$E(w) = \frac{1}{1+r}(w + E(w)),$$

and U the discounted utility of a job seeker

$$U = \frac{1}{1+r}\left(z + \int_0^\infty max\{U, E(w)\} dH(w)\right). \tag{1}$$

Optimal search strategy

Job seekers have to solve an **optimal stopping problem**. Since $E(w) = w/r$ is strictly increasing, there exists a unique reservation wage w_r such that job seekers turn down every job offers if $w < w_r$ and accept every job offers if $w \geq w_r$.

At the reservation wage, workers are indifferent between working and being unemployed so that: $E(w_r) = U$. Replacing $U = w_r/r$ and $E(w) = w/r$ into (1) yields

Reservation Wage

$$\begin{aligned} w_r &= \frac{1}{1+r}\left(rz + \int_0^{\infty} max\{w, w_r\} dH(w)\right) \\ &= z + \frac{1}{r}\left(\int_{w_r}^{\infty}(w - w_r) dH(w)\right) \\ &= z + \frac{1}{r}\left(\int_{w_r}^{\infty}[1 - H(w)] dw\right). \end{aligned}$$

Exercise 5.1: Prove that the last equality holds true.

Continuous time

As shown in the Appendix, similar solutions can be derived considering the worker's asset equations in continuous time

Discounted Utilities in Continuous time

$$rE(w) = w,$$

$$rU = z + \lambda \left(\int_0^\infty max\{0, E(w) - U\} dH(w) \right),$$

where job seekers contact firms at rate λ.

In the remainder of the lecture notes, we favor this interpretation.

Job destruction

The basic model cannot account for the fact that job separations occur quite frequently (e.g. average job duration in US is close to 3 years). Thus we extend the model by considering that jobs are destroyed at the rate δ. Then the asset equations are

$$rE(w) = w + \delta(U - E(w))$$

$$rU = z + \lambda \left(\int_0^\infty max\{0, E(w) - U\} dH(w) \right)$$

The asset equation for E adds the flow income, w, to the capital loss, U-E, times the rate of job separation, δ. Notice that the gain derived from employment is

$$E(w) - U = \frac{w - rU}{r + \delta} \tag{2}$$

that is the discounted difference between the wage and the opportunity cost of employment.

Optimal reservation wage

Replacing the worker surplus in (8) and the definition of the reservation wage $rU = W_r$ yields

Optimal reservation wage

$$\begin{aligned} w_r &= z + \frac{\lambda}{r+\delta} \int_{w_r}^{\infty} (w - w_r) dH(w) \\ &= z + \frac{\lambda}{r+\delta} \int_{w_r}^{\infty} [1 - H(w)] dw \end{aligned}$$

It is easy to verify that the reservation wage

- is unique;
- maximizes the intertemporal utility of a job-seeker.

Interpretation of optimal reservation wage

The optimal reservation wage is higher than the flow income of job seekers ($w_r > z$) because it also includes the option to search

$$w_r = \underbrace{z}_{Flow\ \ income} + \underbrace{\frac{\lambda}{r+\delta}\int_{w_r}^{\infty}(w-w_r)dH(w)}_{Option\ \ to\ \ search}$$

Intuitively, by accepting a given job offer, the job seeker renounces to the possibility of finding a better job. Thus, he "kills" his option to search.

The job seeker takes into account this cost and so ask for a premium over his current flow income.

Notice that this effect is less important if workers can also search while being on-the-job.

Comparative statics on the optimal reservation wage

$w_r \uparrow$ when

- $z \uparrow$ because the opportunity cost of employment increases.
- $\lambda \uparrow$ because the option to search increases.

$w_r \downarrow$ when

- $r \uparrow$ because workers value less the future which decreases the option to search.
- $\delta \uparrow$ because the average job spell decreases which in turn decreases the option to search.

Comparative statics on unemployment duration

The average unemployment duration is

$$T_u = \frac{1}{\lambda[1 - H(w_r)]}$$

Accordingly T_u is an increasing function of z. So:

- higher unemployment benefits raise the average unemployment duration.
- λ has an ambiguous effect because it increases the rate of arrival of job offers but also augments the number of rejected job offers ($w_r \uparrow$). In practice, the former effect dominates so that $\partial T_u / \partial \lambda < 0$.

Unemployment rate

The rate of unemployment evolves as follows

$$\dot{u} = \underbrace{\delta(1-u)}_{\textit{Flows into Unemployment}} - \underbrace{\lambda[1-H(w_r)]u}_{\textit{Flows out of Unemployment}}$$

Equilibrium rate of unemployment

$$u = \frac{\delta}{\delta + \lambda[1-H(w_r)]}$$

The comparative statics for unemployment are therefore similar but with opposite signs to the ones for unemployment duration.

Diamond's critique

Diamond (1971) raised the following critique: since workers accept all wage offers above the reservation wage, firms have no incentive to offer a wage $w > w_r$. Hence the wage distribution should be concentrated at w_r.

But then the option to search has no value, and so the equilibrium wage is equal to z. This in turn raises the question of why workers are searching in the first place?

Diamond's critique highlights that the partial model cannot be extended to an equilibrium setting without further assumptions.

On Fishing and Islands

On way to answer this critique is to vary the model's interpretation by thinking of job seekers

- as fishermen looking for lakes. Then H would be the distribution of fish across lakes and thus can be taken as exogenous.
- as searching across islands. On each island there are many firms with a constant returns to scale technology using only labor. The productivity of a randomly selected island is distributed according to F. The labor market on each island is competitive, so that workers are paid their productivity. Again the wage distribution is exogenous.

From the partial model to the equilibrium model

There are other possible answers (with arguably more economic content) to Diamond's critique:

- **Worker heterogeneity:** under certain conditions, the wage distribution will correspond to the reservation wages of the different categories. This explanation, however, does not explain wage dispersion among similar workers.
- **On the job search:** then firms have an incentive to increase their wage offers above the reservation wage in order to retain their workers and also to attract workers which are already employed.
- **Match uncertainty:** if the productivity of the job is due to the quality of the firm-worker match and if workers have some bargaining power, then the wage distribution will reflect the productivity distribution of jobs.

Learning

We have assumed that the worker knows the rate at which she will receive job offers.

In practice the worker may be uncertain about her job market prospects.

Then she will *learn* over time what is her actual situation. More precisely, the longer she waits for job offers to arrive, the more pessimistic she will become about her prospects.

We now build a model that captures this learning channel.

Learning

We assume that the rate of arrival of job offer can take one of two values:

$$\lambda = \begin{cases} \lambda_h, \text{with } Pr = p_{h,0}, \\ \lambda_l, \text{with } Pr = p_{l,0} = 1 - p_{h,0}, \end{cases}$$

with $\lambda_h > \lambda_l$.

The probability $p_{h,0}$ is the prior of the worker that will be updated over time based on observations.

To simplify the algebra, we devise the model in discrete time so that $\lambda_h, \lambda_h \in [0, 1]$.

Rate at which he gets offers is E[lambda] = ph.L(h) + (1-ph).L(l)

Bayes' rule

Let ω_t denote the event in period t, so that $\omega_t = 1$ when the worker has received a job offer at date t and 0 otherwise.

Bayes' rule reads

$$
\begin{aligned}
p_{i,t}(\omega_{t-1}) &= Pr(\lambda = \lambda_i | \omega_{t-1}, p_{i,t-1}) && (3) \\
&= \frac{Pr(\omega_{t-1}|\lambda_i) p_{i,t-1}}{\sum_{i=l,h} Pr(\omega_{t-1}|\lambda_i) p_{i,t-1}}. && (4)
\end{aligned}
$$

In our model, this yields

$$
p_{i,t}(0) = \frac{(1-\lambda_i)p_{i,t-1}}{(\lambda_h - \lambda_l)p_{l,t-1} + 1 - \lambda_h},
$$

hence $p_{h,t}(0)/p_{l,t}(0) < p_{h,t-1}/p_{l,t-1}$, showing that the worker puts more weight on the low arrival rate λ_l when she has not received an offer.

Exercise 5.2: Show that the normal law is a conjugate prior for the normal likelihood function with know variance.

Model

We assume that jobs last forever and that unemployed workers leave the labor force by the end of the second period.

In period 2, if a worker does not get an offer or refuses the offer she gets, her outside option is to remain unemployed forever. In other words, $O = z/r$ is the worker's outside option in period 2.

$U_2(\omega_1) = \frac{1}{1+r}\left(z + \bar{\lambda}_2(\omega_1)\int_0^\infty max\{O, E(w)\}dH(w) + (1 - \bar{\lambda}_2(\omega_1))O\right),$

where $\bar{\lambda}_2(\omega_1) \equiv E[\lambda_2|\omega_1] = \sum_{i=l,h} p_{i,2}(\omega_1)\lambda_i$.
Reinserting $w_{r,2} = z$ in the Bellman equation yields

$$rU_2(\omega_1) = z + \frac{\bar{\lambda}_2(\omega_1)}{1+r}\int_z^\infty [1 - H(w)]dw,$$

so that $\partial U_2/\partial\bar{\lambda}_2(\omega_1) > 0$.

Model

Having characterized the model in period 2, we now proceed by backward induction and analyze the first period.

$$U_1 = \frac{1}{1+r}\left(z + \bar{\lambda}_1 \int_0^\infty max\{U_2(1), E(w)\}dH(w) + (1-\bar{\lambda}_1)U_2(0)\right).$$

Note that the value of being unemployed is not anymore stationary because it depends on whether or not the worker has received an offer in period 1.

For the same reason, the reservation wage is not stationary.

Moreover, the decision to accept a job is conditional on having received an offer, so that

$$E(w_{r,1}) = U_2(1) \Rightarrow w_{r,1} = rU_2(1).$$

Learning therefore increases the reservation wage in period 1 since $\bar{\lambda}_2(1) > \bar{\lambda}_1$ because workers are more optimistic about their prospect in period 2 after having received an offer. Hence $w_{r,1}$ is higher than its value in the model without learning.

Poisson process

The first step for deriving asset equations consists in defining the stochastic process that changes the state of the agents. For example, consider an unemployed worker. Suppose that on average she receives 3 job offers per year. Given this expected value, what is the probability that she will receive 5 offers in one year? Can we also compute the probability that she will receive a job offer in one month?

It is shown in this section how these two statistics can be easily computed using the Poisson distribution. In order to rigorously define Poisson processes, we need to introduce first Binomial distributions.

Binomial distribution

The binomial distribution is a discrete probability distribution which describes the number of successes in a sequence of n independent yes/no experiments, each of which yielding success with probability p.
If the random variable X follows the binomial distribution with parameters n and p, we write $X \sim B(n, p)$. The probability of getting exactly k successes is given by

$$\Pr\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k = 1, 2, ..., n$ and where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is the binomial coefficient.

Pascal's triangle

The Binomial distribution can be described visually using Pascal's triangle

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
\ldots
\end{array}
$$

Binomial and Poisson distribution

If $X \sim B(n, p)$, then the expected value of X is

$$E[X] = np$$

and the variance is

$$var(X) = np(1 - p)$$

Poisson distribution

The Poisson distribution arises when n is much larger than the expected number of successful outcomes. In other terms, Poisson processes are limits of binomial processes where $n \to \infty$ and $p \to 0$ while np remains constant.

Poisson random variable

Let's go back to the example considered at the beginning of this section. We know that on average an unemployed worker receives $\lambda = 3$ job offers per year. Given this expected value, how can we model the actual distribution of offer arrivals in a second? One possible model is to break up each year into tiny intervals of size $\delta > 0$ years, so there are a very large number, $n = 1/\delta$, of intervals in one year.
Then we declare that in each interval, a job offer arrives with probability $p = \lambda\delta$ (this gives the right expected number of arrivals). So it is as if the worker were drawing a sequence of n independent yes/no experiments, each of which yielding success with probability $\lambda\delta$. Note that this is not quite the same as counting the number of arrivals, since **more than one job offer may arrive in a given interval**. But if the interval is small enough, this is so unlikely that we can ignore the possibility. Hence the approximation holds in the limit where: $\delta \to 0 \Rightarrow n = 1/\delta \to \infty$.

Poisson random variable

Under this model, the number X of job offers which actually arrive in a year has a binomial distribution:

$$\Pr\{X = k\} = \binom{1/\delta}{k} (\lambda\delta)^k (1-\lambda\delta)^{1/\delta - k} \tag{5}$$

Now, let δ become infinitesimally small (while holding k fixed) and make use of three approximations:

$$\begin{aligned} \binom{1/\delta}{k} &\approx \frac{(1/\delta)^k}{k!} \\ (1-\lambda\delta)^{1/\delta} &\approx e^{-\lambda} \\ 1-\delta k &\approx 1 \end{aligned}$$

Poisson random variable

Plugging the last equations in (5) yields

$$
\begin{aligned}
\Pr\{X = k\} &= \binom{1/\delta}{k} (\lambda\delta)^k (1-\lambda\delta)^{1/\delta - k} \\
&= \binom{1/\delta}{k} (\lambda\delta)^k (1-\lambda\delta)^{(1-\delta k)/\delta} \\
&\approx \frac{(1/\delta)^k}{k!} (\lambda\delta)^k (1-\lambda\delta)^{1/\delta} \\
&= \frac{\lambda^k}{k!} (1-\lambda\delta)^{1/\delta} \\
&\approx \frac{\lambda^k}{k!} e^{-\lambda} \qquad (6)
\end{aligned}
$$

Poisson random variable

What happens if we want to answer a question that involves an interval of $t > 0$ units of time rather than 1 unit of time? Then we have a new random variable X_t that is Poisson distributed with mean λt. The probability of seeing exactly k counts in t units of time is

$$\Pr\{X_t = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

The probability distribution (6) is known as the Poisson distribution. When system events appear according to a Poisson density, the system is called a Poisson process.

Properties of the Poisson distribution

As a sanity check on our distribution, the probability values (6) had better sum to 1. Using the Taylor expansion for e^{λ} , we can verify that they do:

$$\sum_{k \in N} Pr\{X = k\} = \sum_{k \in N} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k \in N} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$$

A further sanity check is that the expected number of arrivals in a year is indeed λ, namely

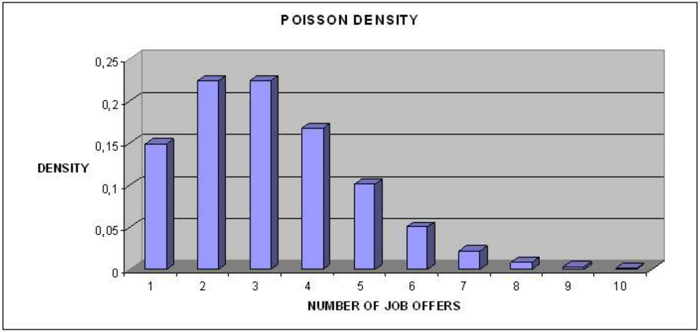
$$E[X] = \lambda$$

Similarly, the binomial distribution $B(n, p)$ has variance $np(1 - p)$. Since the Poisson distribution is the limit of $B(1/\delta, \lambda\delta)$ as δ vanishes, it ought to have variance

$$Var[X] = (1/\delta)(\lambda\delta)(1 - \lambda\delta) = \lambda(1 - \lambda\delta) \approx \lambda$$

The final approximation holds since $1 - \lambda\delta \approx 1$ for vanishing δ.

$\Pr\{X = k\}$ for $k = 1, 2, ..., 10$, when $E[X] = 3$



Waiting time and exponential distribution

Now we consider the waiting time W, which is **the time one must wait to see the next count**. The following two events are equivalent: $\{W > t\} = \{X_t = 0\}$. Both conditions say that no counts arrive in the first t units of time. Hence

$$\Pr\{W > t\} = \Pr\{X_t = 0\} = e^{-\lambda t} \tag{7}$$

Furthermore, since $\Pr\{W > t\} = 1 - \Pr\{W \leq t\}$, the cumulative distribution of W is $F(t) = \Pr\{W \leq t\} = 1 - e^{-\lambda t}$.
Finally, the density function $f(t)$ of W can be found by differentiating $F(t)$ with respect to t. We obtain $f(t) = F'(t) = \lambda e^{-\lambda t}$, for $t > 0$. This is the density function of the so-called exponential distribution with rate λ. It has mean $E[W] = 1/\lambda$. Intuitively, it means that the worker waits $1/3 = 4$ months before receiving an offer.

Waiting time and exponential distribution

Notice that a Taylor approximation of $\Pr\{W \leq t\}$ around $t = 0$ yields $\Pr\{W \leq t\} \underset{t \to 0}{\to} \lambda t$. This is why people often refer to λ as the "instantaneous rate of arrival".

An important property of the exponential distribution is that it is "memoryless." This means that, for $0 < s < t$, $\Pr\{W > t | W > s\} = \Pr\{W > t - s\}$. Given that there have been no counts up to time s, the conditional probability that there are no counts up to the later time t is just the probability of seeing no counts in a period of length $t - s$.

For example, if a machine breaks down according to a Poisson process, it doesn't matter if it is 10 years old or just new.

Surprisingly, the exponential law has remarkable explanatory power for the breakdown probability of many devices, one example being light bulbs.

Asset Equation

We want to derive the expected income of an unemployed worker. We begin by setting up the Bellman equation as though we were solving a discrete time problem. Let ε denote the length of a "time period".

While unemployed, the worker gets unemployment benefits equal to b at every instant. He consumes all his income. This payoff must be discounted back to the beginning of the "period," which is now, and that is the role played by the "instantaneous" discount factor

$$\frac{1}{1+r\varepsilon}$$

Therefore

$$\frac{u(b)\varepsilon}{1+r\varepsilon}$$

is the discounted value of the current period return over the short interval ε.

Asset Equation

Now, we assume that the worker randomly meet firms and then changes state from unemployed to employed. This event is modelled using a Poisson process: so in any small interval of time ε, the worker meets at most one potential employer. From (7), we know that the likelihood of of meeting an employer during the time interval ε is equal to $e^{-\lambda\varepsilon}$.

Let U and E denote the expected lifetime utility of an unemployed and employed worker

$$U(\varepsilon) = \left(\frac{1}{1+r\varepsilon}\right)\left(u(b)\varepsilon + \left(1 - e^{-\lambda\varepsilon}\right)E(\varepsilon) + e^{-\lambda\varepsilon}U(\varepsilon)\right)$$

$$\Rightarrow r\varepsilon U(\varepsilon) = u(b)\varepsilon + \left(1 - e^{-\lambda\varepsilon}\right)(E(\varepsilon) - U(\varepsilon))$$

$$\Rightarrow rU(\varepsilon) = u(b) + \left(\frac{1 - e^{-\lambda\varepsilon}}{\varepsilon}\right)(E(\varepsilon) - U(\varepsilon)) \qquad (8)$$

Asset Equation

Now consider the limit of (8) when $\varepsilon \to 0$. The numerator and denominator of $\left(\frac{1-e^{-\lambda\varepsilon}}{\varepsilon}\right)$ tend to zero. Hence we can use l'Hospital's rule to obtain the limit

$$\lim_{\varepsilon\to 0}\frac{1-e^{-\lambda\varepsilon}}{\varepsilon} = \frac{\lim\limits_{\varepsilon\to 0}\left(\frac{\partial\left(1-e^{-\lambda\varepsilon}\right)}{\partial\varepsilon}\right)}{\lim\limits_{\varepsilon\to 0}\left(\frac{\partial\varepsilon}{\partial\varepsilon}\right)} = \lim_{\varepsilon\to 0}\lambda e^{-\lambda\varepsilon} = \lambda \qquad (9)$$

Substituting (9) into (8) yields

$$\underbrace{rU}_{\textit{flow return}} = \underbrace{u(b)}_{\textit{flow income}} + \underbrace{\lambda\left(E-U\right)}_{\textit{capital gain}} \qquad (10)$$

where $U \equiv \lim\limits_{\varepsilon\to 0} U(\varepsilon)$ and $E \equiv \lim\limits_{\varepsilon\to 0} E(\varepsilon)$.

Asset Equation

There is a clear analogy with the pricing of financial assets (which yield periodic dividends and whose value may change over time), if we interpret the left-hand side of (10) as the flow return (opportunity cost) that a risk neutral investor demands if she invests an amount in a risk free asset with return r. The right-hand side of the equation contains the two components of the flow return on the alternative activity "unemployment": the expected dividend derived from consumption, and the expected change in the asset value of the activity or capital gain due to the switch from unemployment to employment.