Dynamic Models with Latent Variables

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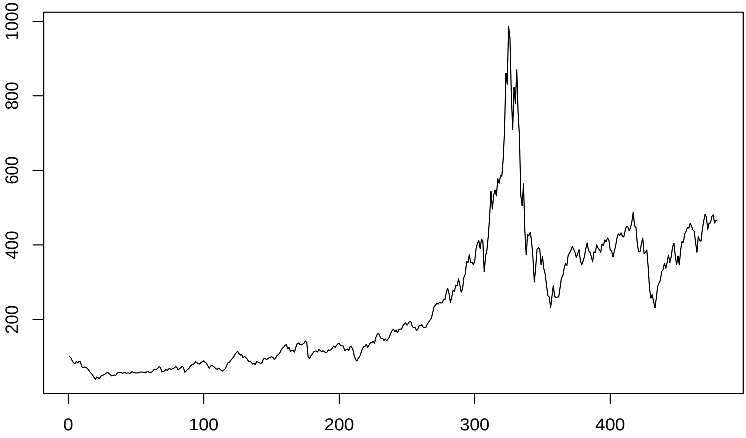
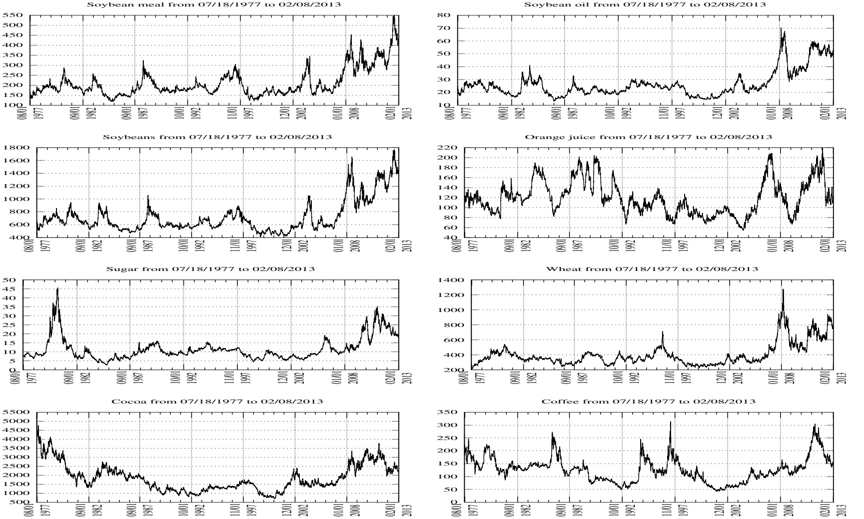
Bubbles modelling and non causal processes

Outline

Nasdaq composite price index : monthly, 1973 - 2012



Asymmetric bubbles



Bubbles in the econometric literature

Because of the presence of local explosive trends, depicted as bubbles, such series cannot be modelled by any traditional ARIMA models.

Bubbles are generally considered, in the time series literature, as nonstationary phenomena and treated similarly to the explosive stochastic trends due to unit roots.

Bubble phenomena in the economic literature

The price evolutions with bubble phenomena are usually derived in linear rational expectation models.
The starting point is the standard no arbitrage condition

$$P_t = E_t\left(\frac{P_{t+1} + D_{t+1}}{1+R}\right),$$

where E_t is the expectation conditional on the information at time t, P_t is the real stock price, D_t is the real dividend received from the asset for ownership between $t-1$ and t.

The information at time t thus includes the current and lagged values of P_t as well as the dividends.

The discount rate R is assumed to be positive and constant over time.

Bubble phenomena in the economic literature

By the law of iterated expectations ($E_t\{E_{t+1}[X]\} = E_t[X]$), solving forward K periods we have

$$P_t = E_t\left[\sum_{i=1}^{K}\left(\frac{1}{1+R}\right)^i D_{t+i}\right] + E_t\left[\left(\frac{1}{1+R}\right)^K P_{t+K}\right].$$

If we assume that

$$\lim_{K\to\infty} E_t\left[\left(\frac{1}{1+R}\right)^K P_{t+K}\right] = 0, \quad a.s. \tag{1}$$

there is a unique solution given by

$$P_t^0 = E_t\left[\sum_{i=1}^{\infty}\left(\frac{1}{1+R}\right)^i D_{t+i}\right],$$

provided that the right-hand side is well defined.

Bubble phenomena in the economic literature

If Assumption (1) is relaxed, the solutions are given by

$$P_t = P_t^0 + B_t, \quad \text{where} \quad B_t = E_t\left(\frac{B_{t+1}}{1+R}\right).$$

P_t^0 is called the **fundamental value** of the price, while B_t is an unobserved component often called a **rational bubble**.

By definition, both components belong to the information set at t.

The equation of B_t can be written under the AR(1) form

$$B_{t+1} = (1+R)B_t + u_{t+1}, \quad \text{where} \quad E_t(u_{t+1}) = 0.$$

For this reason B_t is often interpreted as a non stationary component although the latter equation **may admit stationary solutions**. When $R = 0$, B_t is a "generalized" (in the sense that $E(B_t)$ may not exist) martingale.

Economic remarks

- Equation $B_t = E_t\left(\frac{B_{t+1}}{1+R}\right)$ requires that any rational investor who is willing to buy that stock must expect the bubble to grow at rate R. If $B_t > 0$, a rational investor is willing to buy an "overpriced" stock because he believes that, through the increase of price, he will be sufficiently compensated for the extra payment of B_t.
- Extensions to other prices (not necessarily stocks) : P_t may be a house price and D_t a rent, or P_t may be the price of a grocery store and D_t the value of the benefits generated.

Explicit solutions

Additional assumptions have to be made to characterize the time series properties of the fundamental and bubble components.
A convenient assumption is to assume that

$$D_{t+1} = \mu + \phi D_t + \epsilon_{t+1}, \quad |\phi| \leq 1, \quad (\epsilon_t) \overset{i.i.d.}{\sim} (0, 1)$$

with ϵ_{t+1} independent from $\{P_s, D_s; s \leq t\}$. From an economic point of view it is not restrictive to assume $\phi \geq 0$.

$$\Rightarrow \quad E_t(D_{t+i}) = \mu(1 + \phi + \cdots + \phi^{i-1}) + \phi^i D_t$$

$$\Rightarrow \quad P_t^0 = \frac{\phi}{1 + R - \phi} D_t + \frac{\mu(1+R)}{R(1+R-\phi)}.$$

If $\phi < 1$, (D_t) and (P_t^0) are stationary.
If $\phi = 1$, both processes follow a random walk with drift. Even in the nonstationary case, P_t^0 can thus be distinguished from B_t which, if it is initialized, follows an **explosive AR process**.

Bubble models

The crucial condition for rational bubbles is given by

$$B_{t+1} = (1+R)B_t + u_{t+1}, \quad \text{where} \quad E_t(u_{t+1}) = 0$$

but this condition is compatible with a variety of processes.

Simplest example : deterministic bubble,

$$B_t = B_0(1+R)^t, \quad \text{for } t \geq 0.$$

In this case, the excess price is justified by the higher capital gain and the deviations grow exponentially.

To be rational, such an increase in the price must continue forever (very implausible).

Stochastic examples of bubbles models, in which the bubble does not necessarily grow forever, were considered in the economic literature.

Stochastic bubble models : Blanchard (1979)

The bubble dynamics is defined by :

$$B_{t+1} = \begin{cases} \frac{1}{\pi}(1+R)B_t, & \text{with probability } \pi, \\ 0 & \text{with probability } 1-\pi. \end{cases}$$

The model can equivalently be written

$$B_{t+1} = \frac{1}{\pi}(1+R)B_t \mathbf{1}_{\eta_{t+1} < \Phi^{-1}(\pi)}, \quad (\eta_t) \overset{iid}{\sim} \mathcal{N}(0,1),$$

where η_{t+1} is independent from the current and past values of the stock price.

This process corresponds to a **single bubble**, with 0 as an absorbing state. The rate of explosion of the bubble is fixed and equal to $(1+R)/\pi$. It is strictly larger than the average rate of explosion, which is equal to $1+R$. The average duration of the bubble is $(1-\pi)^{-1}$.

Stochastic bubble models : Blanchard and Watson (1982)

This model eliminates zero as an absorbing state. The dynamics of the bubble is

$$B_{t+1} = \begin{cases} \frac{1}{\pi}(1+R)B_t + \epsilon_{t+1}, & \text{with probability } \pi, \\ \epsilon_{t+1}, & \text{with probability } 1-\pi, \end{cases}$$

where $E_t(\epsilon_{t+1}) = 0$. This model allows for erratic changes in Y_t during the explosion spell. It also allows for **multiple bubbles**.

The model can be written as a random coefficient AR(1), as

$$B_{t+1} = a_{t+1}B_t + \epsilon_{t+1},$$

where (a_t) is an iid process

$$a_{t+1} = \begin{cases} \frac{1}{\pi}(1+R), & \text{with probability } \pi, \\ 0, & \text{with probability } 1-\pi. \end{cases}$$

Stochastic bubble models : Blanchard and Watson (1982)

Thus $E(a_{t+1}) = 1 + R$ which means that the bubble process is explosive in average. However, if

- (ϵ_t) is a strictly stationary and ergodic sequence such that $E \log |\epsilon_t^+| < \infty$,
- (ϵ_t) and (a_t) are independent sequences,

the model nevertheless admits a **strictly stationary solution** (with $t \in \mathbb{Z}$) given by

$$B_{t+1} = \epsilon_{t+1} + \sum_{k=0}^{\infty} a_{t+1} \ldots a_{t-k+1} \epsilon_{t-k}$$

where the condition $E \log(a_{t+1}) < 0$ (in fact equal to $-\infty$) ensures the absolute a.s. convergence of the infinite sum.

The model could be extended by allowing dependence in the sequence (a_t), for instance by assuming that it is a **Markov chain** instead of an iid process.

Stochastic bubble models : Evans (1991)

This bubble model has a stochastic rate of explosion :

$$B_{t+1} = \begin{cases} u_{t+1}(1+R)B_t, & \text{if } B_t \leq \alpha, \\ \left(\delta + \frac{1}{\pi}(1+R)\theta_{t+1}\{B_t - \delta(1+R)^{-1}\}\right) u_{t+1}, & \text{if } B_t > \alpha, \end{cases}$$

where $0 < \delta < \alpha$, (u_t) is i.i.d. with $u_t \geq 0$, $E_t(u_{t+1}) = 1$, (θ_t) is an i.i.d. Bernoulli process $\mathcal{B}(1, \pi)$ which is independent of (u_t).

- $B_t = E_t\left(\frac{B_{t+1}}{1+R}\right)$.
- $B_t > 0$ implies $B_s > 0$ for all $s \geq t$.
- If $B_0 < \alpha$, the bubble increases, in average, until it exceeds the value α. Thereafter, it can either collapse to δu_{t+1} (if $\theta_{t+1} = 0$), or increase with a faster rate ($(1+R)/\pi$ in average). Therefore, the bubble is periodically collapsing but never crashes to zero.

Stochastic bubble models : Evans (1991)

- α, δ and π impact frequency of the bubbles, the average time before collapse and the scale of the bubbles.
- The dynamics can be equivalently written under the form of a (seemingly) AR(1) model :

$$B_{t+1} = a_{t+1} B_t + \xi_{t+1},$$

where
$a_{t+1} = u_{t+1}(1+R)\{1-k_{t+1}\mathbf{1}_{B_t>\alpha}\}, \quad \xi_{t+1} = u_{t+1}\delta k_{t+1}\mathbf{1}_{B_t>\alpha},$
and $k_{t+1} = 1 - \frac{\theta_{t+1}}{\pi}$. We have

$$E(a_{t+1}) = 1 + R \quad \text{and} \quad E(\xi_{t+1}) = 0,$$

showing that the bubble process is **explosive in average**. However, contrary to what is done in Yoon (2012), this AR(1) representation cannot be used to analyze the strict stationarity of (B_t) (because the random coefficient a_{t+1} is function of B_t).

Noncausal linear heavy-tailed AR process

An alternative multiple bubble modelling :

- Stationary framework
- Easy to extend to multiple lags
- Easy to extend to the multivariate framework
- Amenable to statistical inference
- Bubbles are short-lived explosive patterns caused by extreme valued shocks.

Baseline paths

- **Noncausal AR(1) :** $Y_t = \rho Y_{t+1} + \varepsilon_t, \qquad |\rho| < 1,$

$$Y_t = \sum_{k \geq 0} \rho^k \varepsilon_{t+k} = \sum_{\tau \in \mathbb{Z}} \rho^{\tau - t} \mathbb{1}_{\{\tau \geq t\}} \varepsilon_\tau$$

$\Rightarrow$ a sample path of Y is a linear combination of deterministic baseline paths

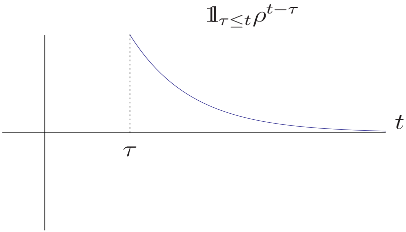
$$t \longmapsto \rho^{\tau - t} \mathbb{1}_{\{\tau \geq t\}}, \qquad \tau \in \mathbb{Z}.$$

An explosive growth followed by a vertical downturn at $t = \tau$.

- Causal AR(1) : A jump at $t = \tau$ followed by an exponential decrease

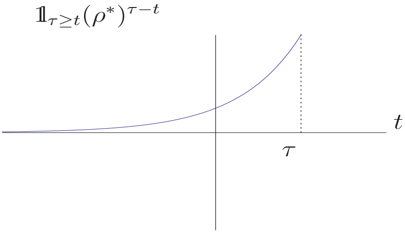
Baseline paths for causal and non causal AR(1) processes

$$Y_t = \rho Y_{t-1} + \varepsilon_t, \quad 0 < \rho < 1,$$



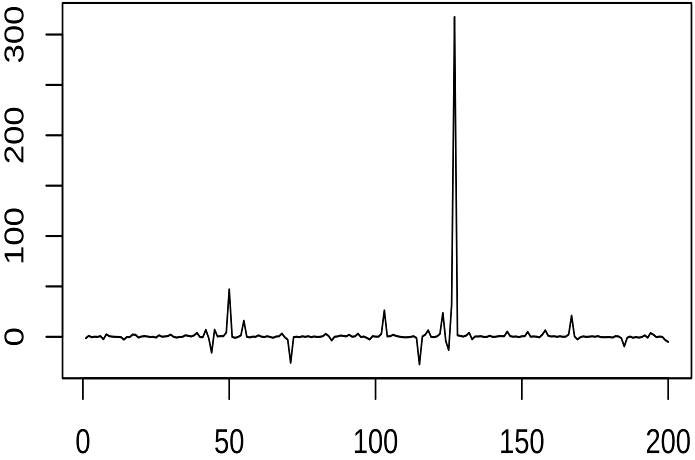
Causal

$$Y_t = \rho^* Y_{t+1} + \varepsilon_t, \quad 0 < \rho < 1$$

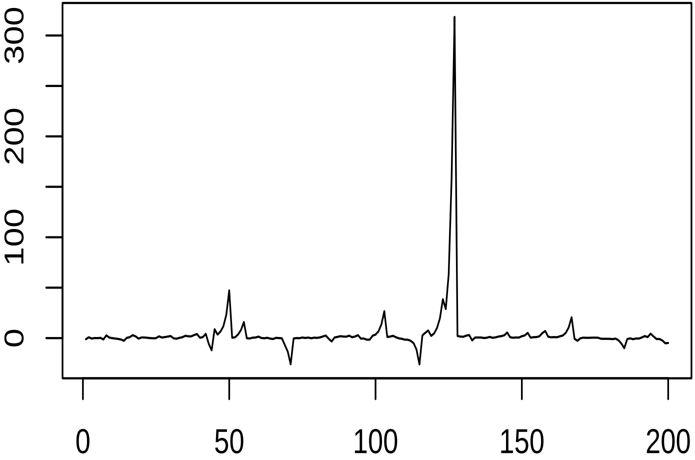


Noncausal

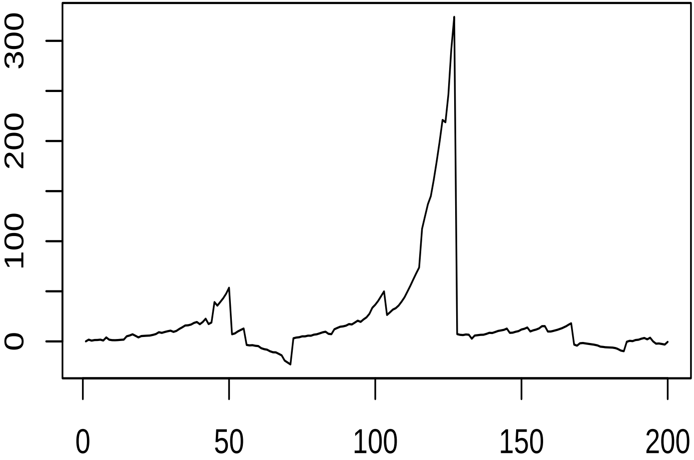
Simulations : $Y_t = 0.1Y_{t+1} + \varepsilon_t, \qquad (\varepsilon_t) i.i.d. \sim \mathcal{C}(0,1)$



Simulations : $Y_t = 0.5Y_{t+1} + \varepsilon_t, \qquad (\varepsilon_t) i.i.d. \sim \mathcal{C}(0,1)$



Simulations : $Y_t = 0.9Y_{t+1} + \varepsilon_t, \qquad (\varepsilon_t)$ i.i.d. $\sim \mathcal{C}(0,1)$



Some basics on stable distributions

- A probability measure μ on $\mathbb{R}$ is said to be **stable** if for every $a > 0$ and $b > 0$, there exists $c > 0$ and a real number e such that

$$aX_1 + bX_2 \stackrel{d}{=} cX + e,$$

where $\mathcal{L}(X_1) = \mathcal{L}(X_2) = \mathcal{L}(X) = \mu$, and X_1 and X_2 are independent.

- Equivalently : μ is stable iff for each $k \in \mathbb{N}$, there are $b_k > 0$ and e_k such that

$$X_1 + \cdots + X_k \stackrel{d}{=} b_k X + e_k,$$

where $\mathcal{L}(X_1) = \ldots = \mathcal{L}(X_k) = \mathcal{L}(X) = \mu$, and $X_1, \ldots, X_k$ are independent.
Moreover, there exists $\alpha \in [0, 2]$ such that $b_k = k^{1/\alpha}$ and we say that **μ is α-stable.**

Some basics on stable distributions

Univariate α-stable distributions are characterized by four parameters :

- $\alpha \in [0, 2]$ is called the **index of stability**, or tail exponent ;
- an asymmetry parameter $\beta \in [-1, 1]$,
- a scale parameter $\sigma \in (0, \infty)$,
- a location parameter $m \in \mathbb{R}$.

Thus we can write $X \sim \mathcal{S}(\alpha, \beta, \sigma, m)$ for a variable following a stable distribution. If $m = 0$ the distribution is called **strictly stable**.

The location and scale parameters are such that, if $X \sim \mathcal{S}(\alpha, \beta, 1, 0)$, then $Y = \sigma X + m \sim \mathcal{S}(\alpha, \beta, \sigma, m)$, where $\sigma > 0$.

Some basics on stable distributions

In general, the probability density function (pdf) of a stable distribution is not known explicitly, but its characteristic function $\psi(s) = E(e^{isX})$ has the closed form :

$$\log \psi(s) = -\sigma^{\alpha}|s|^{\alpha} \left\{ 1 - i\beta \left(\text{sign } s\right) \tan\left(\frac{\pi\alpha}{2}\right) \right\} + i\mu s,$$

if $\alpha \neq 1$, and

$$\log \psi(s) = -\sigma|s| \left\{ 1 + i\beta \left(\text{sign } s\right) \frac{2}{\pi} \log|s| \right\} + i\mu s,$$

if $\alpha = 1$.

Some basics on stable distributions

There exist other parameterizations for ψ, but this one presents the advantage that

$$f_\theta(x) := (2\pi)^{-1} \int_{\mathbb{R}} \exp\{-is(x-\mu)\}\, \psi_{\alpha,\beta}(\sigma s) ds$$

is differentiable with respect to both $x \in \mathbb{R}$ and $\theta \in \Lambda := (0,2) \times (-1,1) \times (0,\infty) \times \mathbb{R}$ (see Nolan, 2003). Let $f_{\alpha,\beta}$ be the stable density of parameter $\theta = (\alpha, \beta, 1, 0)$. We have for $\alpha \neq 1$,

$$f_{\alpha,\beta}(x) = \frac{1}{\pi} \int_0^\infty e^{-s^\alpha} \cos\left\{sx + \beta \tan\left(\frac{\pi\alpha}{2}\right)(s - s^\alpha)\right\} ds$$

and for $\alpha = 1$,

$$f_{\alpha,\beta}(x) = \frac{1}{\pi} \int_0^\infty e^{-s} \cos\left(sx + s\beta \frac{2}{\pi} \log s\right) ds$$

From the expansion $(1 - s^{\alpha-1}) \tan\left(\frac{\pi\alpha}{2}\right) = \frac{2}{\pi} \log s + o(\alpha - 1)$, the

Some basics on stable distributions

Note that $f_\theta(x) = \sigma^{-1} f_{\alpha,\beta}\left\{\sigma^{-1}(x-\mu)\right\}$ can be numerically evaluated from the previous expressions, or alternatively using the function `dstable()` of the `R` package `fBasics`.

Particular cases :

- Gaussian distribution $\mathrm{N}(\mu, \sigma^2)$: obtained for $\alpha = 2$ (β non identifiable).
- Cauchy distribution $\mathcal{C}(\mu, \sigma)$: obtained for $\alpha = 1$ and $\beta = 0$.
- Lévy distribution : obtained for $\alpha = 1/2$ and $\beta = 1$. Density (when $\sigma = 1$ and $\mu = 0$) :

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x^{3/2}} \exp\left(\frac{-1}{2x}\right) \mathbb{1}_{x>0}.$$

Some basics on stable distributions

The coefficient α determines the tail of the distribution of $X \sim S(\alpha, \beta, \sigma, \mu)$ in the sense that, when $\alpha < 2$, as $x \to \infty$,

$P(X < -x) \sim c_\alpha (1-\beta) x^{-\alpha}$ **and** $P(X > x) \sim c_\alpha (1+\beta) x^{-\alpha}$,

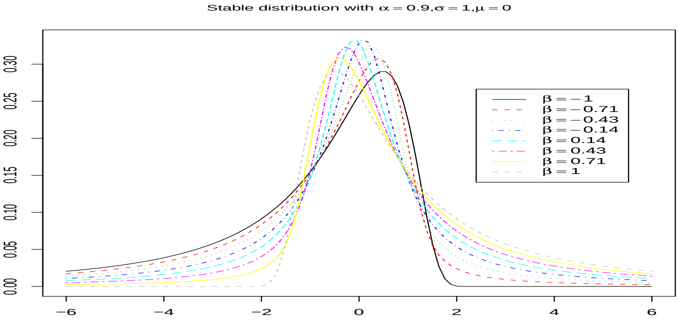
with $c_\alpha > 0$.

Moreover, when $X \sim S(\alpha, \beta, \sigma, \mu)$ with $\alpha < 2$,

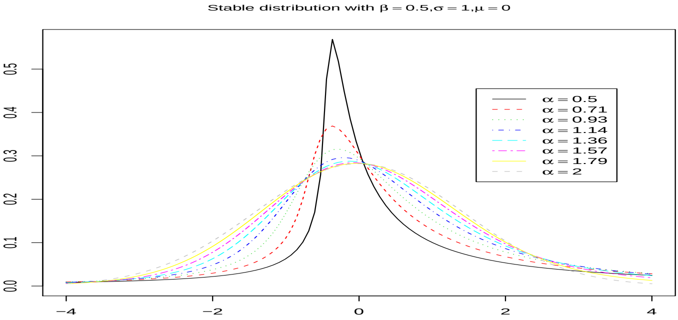
$E|X|^p < \infty$ **if and only if** $p < \alpha$

See Samorodnitsky and Taqqu (1994) *"Stable Non-Gaussian Random Processes"* for further details on stable variables.

Stable densities with different symmetry parameters



Stable densities with different tail index parameters



$$E|X|^r < \infty \text{ iff } r < \alpha \text{ (when } \alpha < 2)$$

Model : a forward looking recursion

$$Y_t = \rho Y_{t+1} + \varepsilon_t, \qquad |\rho| < 1, \qquad (\varepsilon_t) \overset{i.i.d.}{\sim} \mathcal{S}(\alpha, \beta, \sigma, 0),$$

Strict stationarity

The strictly stationary solution is given by : $Y_t = \sum_{h=0}^{\infty} \rho^h \varepsilon_{t+h}$, and the stationary distribution is stable,

$$\begin{aligned}
Y_t &\sim \mathcal{S}\left(\alpha, \beta, \frac{\sigma}{(1-|\rho|^\alpha)^{1/\alpha}}, 0\right), && \text{if} \quad \alpha \neq 1, \rho \geq 0,\\
Y_t &\sim \mathcal{S}\left(\alpha, \beta\frac{1-|\rho|^\alpha}{1+|\rho|^\alpha}, \frac{\sigma}{(1-|\rho|^\alpha)^{1/\alpha}}, 0\right), && \text{if} \quad \alpha \neq 1, \rho \leq 0,\\
Y_t &\sim \mathcal{S}\left(1, \beta\frac{1-|\rho|}{1-\rho}, \frac{\sigma}{1-|\rho|}, -\beta\sigma\frac{2}{\pi}\frac{\rho \log|\rho|}{(1-\rho)^2}\right), && \text{if} \quad \alpha = 1.
\end{aligned}$$

Proof

Forward conditional densities

We have, for the backward conditional density,

$$E(|Y_t|^p \mid Y_{t+1}) < \infty \quad \textbf{if and only if} \quad p < \alpha$$

and, for the stationary distribution,

$$E|Y_t|^p < \infty \quad \textbf{if and only if} \quad p < \alpha$$

Pareto tails of the forward transition pdf

The noncausal stable linear AR(1) process is a causal homogeneous Markov process. Let $\alpha < 2$. For any $h \geq 0$, if $|\beta| \neq 1$, we have

$$E(|Y_{t+h}|^p \mid Y_{t-1}) < \infty, \quad a.s., \qquad \text{if and only if} \quad p < 2\alpha + 1.$$

The property also holds if $|\beta| = 1$ and $\rho^{h+1} < 0$. If $|\beta| = 1$ and $\rho^{h+1} > 0$, then $E(|Y_{t+h}|^p \mid Y_{t-1}) < \infty, \quad a.s.$ for all $p > 0$. Proof

Forward conditional expectation

The forward conditional expectation $E(Y_{t+h} \mid Y_{t-1})$ always exists, whereas the unconditional and backward conditional expectations exist only when $\alpha > 1$.

Closed-form expression for symmetric stable distributions

For $\rho \neq 0$ and $\beta = 0$, we have :

$$E(Y_{t+h} \mid Y_{t-1}) \;=\; |\rho|^{(h+1)(\alpha-1)} Y_{t-1}, \quad \forall h \geq 0,$$

where, by convention, $|x|^0 = \mathrm{sign}(x)$. In particular, for Cauchy processes ($\alpha = 1$), when $\rho > 0$ we have :

$$E(Y_{t+h} \mid Y_{t-1}) \;=\; Y_{t-1}, \quad \forall h \geq 0.$$

Idea of the proof

The proof relies on

- Using the characteristic function $(u, v) \to E\left(e^{iuY_{t-1}+ivY_{t+h}}\right)$ and its derivatives to show that

$$E\left\{\left(Y_{t+h} - |\rho|^{(h+1)(\alpha-1)}Y_{t-1}\right)e^{iuY_{t-1}}\right\} = 0, \quad \text{for any } u \in \mathbb{R}.$$

- The conclusion follows from Bierens (Theorem 1, J. of Econometrics, 20, 1982).

Detailed proof*

Causal conditional density in the Cauchy case

The causal transition density, i.e. the density of Y_t given $Y_{t-1} = y$ is, by the Bayes formula,

$$g : x \mapsto f_\varepsilon\{y - \rho x\} f_Y(x)/f_Y(y),$$

where f_Y denotes the marginal density of Y_t.

Proposition

In the Cauchy case, the causal transition density is :

$$f(Y_t|Y_{t-1}) = \frac{1}{\pi} \frac{\sigma}{\sigma^2 + (Y_{t-1} - \rho Y_t)^2} \frac{\sigma^2 + (1 - |\rho|)^2 Y_{t-1}^2}{\sigma^2 + (1 - |\rho|)^2 Y_t^2}$$

1st and 2nd-order causal conditional moments in the Cauchy case

Proposition

For $\rho \neq 0$, if $\varepsilon_t \sim \mathcal{C}(0, \sigma)$ we have :

$$\begin{aligned} E(Y_t \mid Y_{t-1}) &= \operatorname{sign}(\rho) Y_{t-1}, \\ E(Y_t^2 \mid Y_{t-1}) &= \frac{1}{|\rho|} Y_{t-1}^2 + \frac{\sigma^2}{|\rho|(1-|\rho|)}, \end{aligned}$$

Proof

Remark : very unexpected result. If (X_t) is a **noncausal** AR(1)

$$X_t = \frac{1}{\rho} X_{t-1} + Z_t, \quad |\rho| < 1, \quad Z_t \overset{iid}{\sim} (0, \sigma^2)$$

then (X_t) can be expressed as a **weak causal** AR(1) :

$$X_t = \rho X_{t-1} + Z_t, \quad Z_t \overset{WWN}{\sim} (0, \sigma^2).$$

Semi-strong AR(1) representations

- **Cauchy case ($\alpha = 1$) :**

$$Y_t = \operatorname{sign}(\rho) Y_{t-1} + \eta_t \sqrt{\left(\frac{1}{|\rho|} - 1\right) Y_{t-1}^2 + \frac{\sigma^2}{|\rho|(1-|\rho|)}},$$

where :

$$E(\eta_t \mid Y_{t-1}) = 0, \quad E(\eta_t^2 \mid Y_{t-1}) = 1.$$

Unit-root when $\rho > 0$ + GARCH effect (but (η_t) is not i.i.d.)

Strong nonlinear causal representation

The Gaussian nonlinear innovations (see Rosenblatt (2000)) are defined by :

$$\varepsilon_t^* = \Phi^{-1}[F(Y_t \mid Y_{t-1})],$$

where Φ is the cdf of the standard normal, and $F(\cdot \mid Y_{t-1})$ is the conditional cdf of Y_t.
This relationship can be inverted to derive the causal strong nonlinear AR representation of process (Y_t) :

$$Y_t \;=\; G(Y_{t-1}, \varepsilon_t^*), \quad \varepsilon_t^* \sim IIN(0,1),$$

with $G(Y_{t-1}, .) = F^{-1}[\Phi(.) \mid Y_{t-1}]$.

Closed form expression of the conditional cdf

Causal conditional cdf in the Cauchy case (for $\sigma = 1$)

$$\begin{aligned}
F(Y_t|Y_{t-1}) &= \frac{\alpha(Y_{t-1},\rho)}{\pi}\log\left\{\frac{1+(1-|\rho|)^2Y_t^2}{1+(Y_{t-1}-\rho Y_t)^2}\frac{\rho^{*2}}{(1-|\rho|)^2}\right\} \\
&\quad + \frac{\beta(Y_{t-1},\rho)}{\pi}\left\{\frac{\pi}{2}-\operatorname{sign}(\rho)\tan^{-1}(Y_{t-1}-\rho Y_t)\right\} \\
&\quad + \frac{1-\beta(Y_{t-1},\rho)}{\pi}\left\{\tan^{-1}[(1-|\rho|)Y_t]+\frac{\pi}{2}\right\}, \\
\alpha(Y_{t-1},\rho) &= \frac{\rho(1-|\rho|)^2Y_{t-1}}{(1-2|\rho|)^2+(1-|\rho|)^2Y_{t-1}^2}, \\
\beta(Y_{t-1},\rho) &= \frac{|\rho|\{(1-|\rho|)^2Y_{t-1}^2-(1-2|\rho|)\}}{(1-2|\rho|)^2+(1-|\rho|)^2Y_{t-1}^2}.
\end{aligned}$$

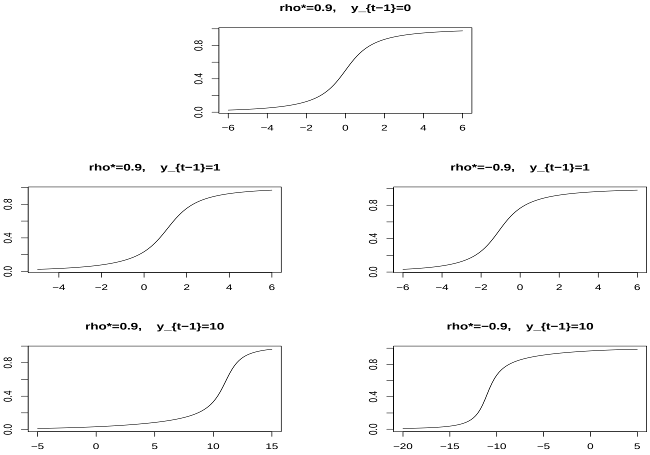


Figure 1: Examples of conditional cdf for different values of (ρ^*, Y_{t-1}).

Summary of representations in the Cauchy case

- Noncausal AR(1)

$$Y_t = \rho Y_{t+1} + \varepsilon_t, \qquad |\rho| < 1$$

- Noncausal MA(∞)

$$Y_t = \sum_{h=0}^{\infty} \rho^h \varepsilon_{t+h}$$

- Causal semi-strong

$$Y_t = \operatorname{sign}(\rho) Y_{t-1} + \eta_t \sqrt{\left(\frac{1}{|\rho|} - 1\right) Y_{t-1}^2 + \frac{\sigma^{*2}}{|\rho|(1-|\rho|)}}$$

- Causal strong nonlinear AR

$$Y_t \quad = \quad F^{-1}\{\Phi(\varepsilon_t^*) \mid Y_{t-1}\}, \quad \varepsilon_t^* \sim IIN(0,1)$$

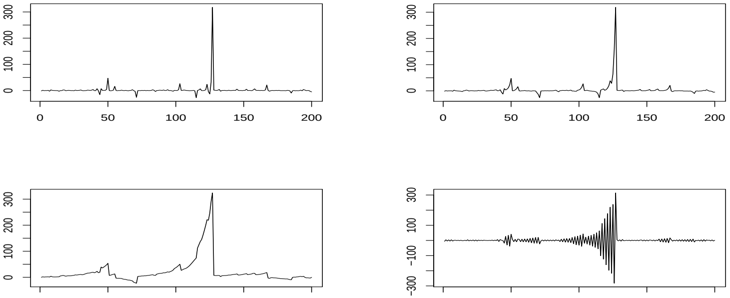


Figure 1: Simulations of the Cauchy AR(1) Model with $\sigma = 0.5$ and, from left to right and up to bottom, $\rho = 0.1$, $\rho = 0.5$, $\rho = 0.9$, $\rho = -0.9$.

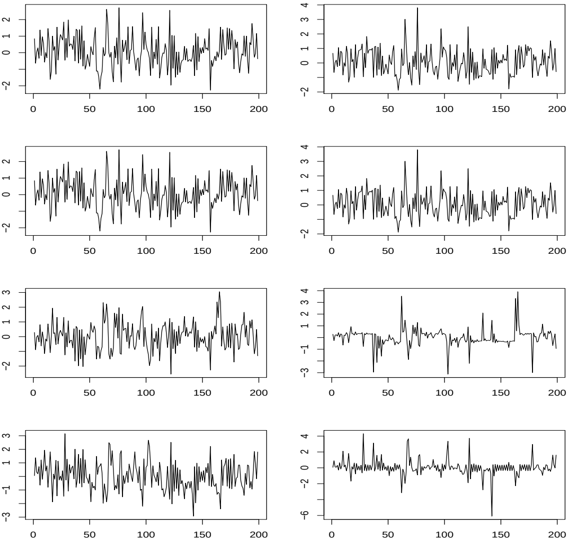


Figure 1: Nonlinear Gaussian innovations $\varepsilon_t = \Phi^{-1}[F(Y_t|Y_{t-1})]$ (left panels) and standardized causal innovations η_t (right panels) ($\rho^* = 0.1, 0.5, 0.9, -0.9$ from up to bottom).

2

Table – Descriptive statistics for the nonlinear Gaussian innovations ε_t^* and the standardized causal innovations η_t for $\rho^* = 0.9$.

	mean	stand. dev.	skewness	exc. kurtosis
ε_t^*	0.026	0.963	0.133	0.094
η_t	0.018	0.792	0.392	8.164

Two-step procedure for analyzing causal impulse responses

i) From any simulated path $(\varepsilon_t, t \geq 0)$ of the causal nonlinear Gaussian noise, and an initial value Y_0, we deduce a simulated path (Y_t) :

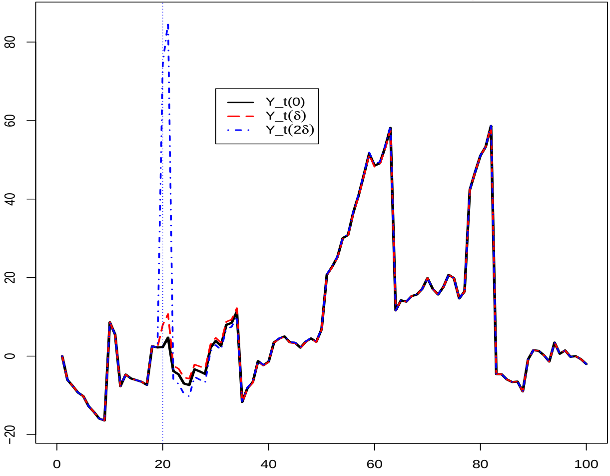
$$Y_t = G(Y_{t-1}, \varepsilon_t), \quad t = 1, \ldots .$$

ii) The effect of a transitory shock of magnitude δ at time τ is deduced by computing recursively :

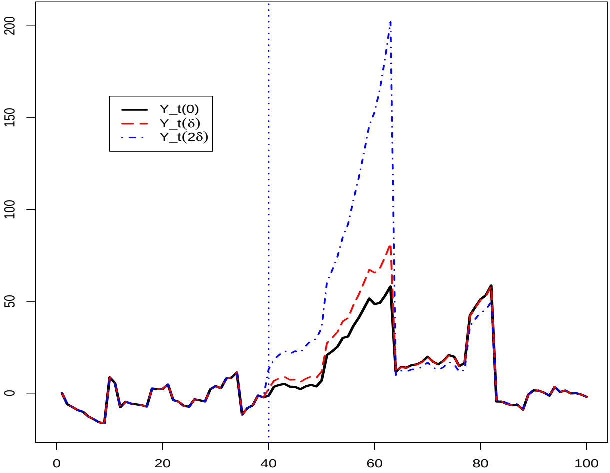
$$Y_t(\delta) = G[Y_{t-1}(\delta), \varepsilon_t(\delta)], \quad t = 1, \ldots ,$$

$$\text{with} \quad \begin{aligned} \varepsilon_t(\delta) &= \varepsilon_t, \quad t \neq \tau, \\ \varepsilon_\tau(\delta) &= \varepsilon_\tau + \delta. \end{aligned}$$

With $\rho = 0.9$, simulated sample paths of Y_t, $Y_t(\delta)$ and $Y_t(2\delta)$, with shock $\delta = 2$ at time $\tau = 20$



With $\rho = 0.9$, simulated sample paths of Y_t, $Y_t(\delta)$ and $Y_t(2\delta)$, with shock $\delta = 1$ at time $\tau = 40$



Non causal AR(1) : assumptions on the heavy-tailed errors distribution

Non-causal AR(1) process with heavy-tailed errors whose distribution is not fully specified :

$$Y_t = \rho Y_{t+1} + \varepsilon_t, \qquad |\rho| < 1, \qquad (\varepsilon_t) \text{ i.i.d.},$$

where, for simplicity, the distribution of ε_t is symmetric and the distribution of $|\varepsilon_t|$ is *regularly varying* with index $-\alpha$, that is,

$$P(|\varepsilon_t| > x) = x^{-\alpha} L(x),$$

with $\alpha \in (0, 2)$ and $L(x)$ a slowly varying function at infinity.

Estimator of the AR coefficient

An estimator of the AR coefficient ρ is the first sample autocorrelation :

$$\hat{\rho}_n = \sum_{t=2}^{n} Y_t Y_{t-1} / \sum_{t=1}^{n} Y_t^2,$$

when the observations are $Y_1, \ldots, Y_n$. More generally, we define, for $\ell \geq 0$,

$$\hat{\rho}_n(\ell) = \sum_{t=\ell+1}^{n} Y_t Y_{t-\ell} / \sum_{t=1}^{n} Y_t^2,$$

with $\hat{\rho}_n(1) = \hat{\rho}_n$. Note that the theoretical autocorrelations of (Y_t) do not exist.

Asymptotic properties

Let, for $M \geq 1$,

$$\hat{\boldsymbol{\rho}}_n = (\hat{\rho}_n(1), \dots, \hat{\rho}_n(M))', \quad \boldsymbol{\rho} = (\rho, \dots, (\rho)^M)'.$$

Asymptotic law as function of stable variables

$$\frac{a_n^2}{\tilde{a}_n}(\hat{\boldsymbol{\rho}}_n - \boldsymbol{\rho}) \xrightarrow{d} (Y_1, \dots, Y_M),$$

where $Y_\ell = \sum_{j=1}^{\infty} \{\rho^{j+\ell} - \rho^{|j-\ell|}\} S_j / S_0$, and $S_1, S_2, \dots$ are i.i.d. α-stable variables independent of the positive $\alpha/2$-stable variable S_0.

In particular, for the noncausal Cauchy process,

$$\frac{n}{\log n}(\hat{\rho}_n - \rho) \xrightarrow{d} (1+\rho) Y X,$$

where X, Y are independent with $Y \sim \mathcal{C}(0,1)$ and $X \sim \chi^2(1)$.

Remarks on the estimator of the AR coefficient

- Knowledge of α is not required for the computation of the LS estimator of ρ.
  After estimating ρ in a 1st step, the tail index α can be estimated from a standard approach in the 2nd step. For instance, the Hill estimator can be used.
- If the distribution of ϵ is completely known, a far more efficient estimator is the ML estimator.
  For α-stable causal and noncausal AR processes, see Andrews, Calder and Davis (2009) .
  With Cauchy errors, the MLE converges faster to ρ^* than the LSE (n instead of $n/\log n$), but the limiting distribution has no simple closed form.

Using residuals of the forward linear representation

To check the validity of standard AR models, a usual approach is to verify that the residuals are more or less uncorrelated.

In our model, the errors are not noises in the usual sense since their variance does not exist.

Nevertheless, we can define empirical first-order autocorrelations of the **backward residuals**, as

$$R_n^* = \sum_{t=2}^{n-1} \hat{\epsilon}_t^* \hat{\epsilon}_{t-1}^* / \sum_{t=1}^{n-1} (\hat{\epsilon}_t^*)^2,$$

where

$$\hat{\epsilon}_t^* = Y_t - \hat{\rho}_n Y_{t+1}, \quad t = 1, \dots, n-1.$$

Using residuals of a backward linear representation

Having estimated ρ, we can also consider the **forward residuals**

$$\hat{\epsilon}_t = Y_t - \hat{\rho}_n Y_{t-1}, \quad t = 2, \ldots, n,$$

corresponding to a possibly erroneous interpretation of the estimated model (backward instead of forward).

The empirical first-order autocorrelations of such residuals is

$$R_n = \sum_{t=3}^{n} \hat{\epsilon}_t \hat{\epsilon}_{t-1} / \sum_{t=2}^{n} (\hat{\epsilon}_t)^2.$$

Asymptotic behaviour of the statistics under the null

We might expect different asymptotic behaviours of R_n and R_n^*. The following result shows that it is not the case.

We have

$$\frac{a_n^2}{\tilde{a}_n} R_n^* \xrightarrow{d} \rho^2 S_1/S_0 - \{1-\rho^2\} \sum_{j=2}^{\infty} \rho^{j-1} S_j/S_0,$$

If $\alpha \geq 1$ and $|\varepsilon_t|$ is asymptotically equivalent to a Pareto, R_n and R_n^* have the same asymptotic distribution.

In the Cauchy case,

$$\frac{n}{\log n} R_n^* \xrightarrow{d} Z \quad \text{and} \quad \frac{n}{\log n} R_n \xrightarrow{d} Z \quad \text{as } n \to \infty,$$

where $Z = \rho(1+2\rho)YX$ and X, Y are independent with $Y \sim \mathcal{C}(0,1)$ and $X \sim \chi^2(1)$.

Asymptotic behaviour under a (near) random walk

Assume, for $t \geq 1$, $c \in \mathbb{R}$,

$$Y_{n,t} = a_n Y_{n,t-1} + \xi_t, \quad a_n = \exp(c/n),$$

for some random initial value Y_0 (independent of n and of (ξ_t)). The errors (ξ_t) are strictly stationary with $E(\xi_t) = 0$ and $0 < E(\xi_t^2) < \infty$.

Proposition

For the (near) random walk under mixing assumptions on (ξ_t), we have for any c :

$$\frac{n}{\log n}|R_n^*| \xrightarrow{P} \infty \quad \textit{and} \quad \frac{n}{\log n}|R_n| \xrightarrow{P} \infty, \qquad \textit{as } n \to \infty.$$

Consequences for unit roots testing

Let $\zeta_{1-\alpha}$ the $(1-\alpha)$ quantile of $|XY|$, where X, Y are independent with $Y \sim \mathcal{C}(0,1)$ and $X \sim \chi^2(1)$.

The test

$$\frac{n}{\log n}\left|\frac{R_n}{\hat{\rho}_n(1+2\hat{\rho}_n)}\right| > \zeta_{1-\alpha},$$

can distinguish a unit root due to a bubble created in a **stationary noncausal AR process** from a unit root created by a **(near) random walk model**.

Same property with R_n replaced by R_n^*.

Monte-Carlo experiments in the Cauchy case

Table – Characteristics of the empirical distribution of $\frac{n}{\log n}(\hat{\rho}_n - \rho)$ over 5,000 simulated paths. The empirical α-quantile is denoted q_α. The last column gives the frequency of $\hat{\rho}_n$ exceeding 1. The results for $n = \infty$ are obtained by simulations of the asymptotic distribution.

n	ρ	Mean	Std	$q_{0.1}$	$q_{0.25}$	Median	$q_{0.75}$	$q_{0.9}$	$P[\hat{\rho}_n > 1]$
500	0.1	0.019	3.816	-1.639	-0.444	-0.000	0.417	1.496	0.0004
	0.5	0.106	5.321	-1.817	-0.512	0.001	0.521	1.780	0.0024
	0.9	-0.045	3.214	-1.453	-0.500	-0.001	0.452	1.328	0.0142
2000	0.1	0.037	4.118	-1.636	-0.400	0.004	0.464	1.736	0.0000
	0.5	-0.220	4.718	-2.013	-0.569	-0.003	0.515	1.910	0.0000
	0.9	-0.093	3.916	-1.677	-0.543	-0.008	0.484	1.613	0.0020
5000	0.1	0.195	5.428	-1.641	-0.409	0.000	0.447	1.820	0.0000
	0.5	-0.138	8.103	-2.014	-0.549	-0.001	0.514	1.942	0.0000
	0.9	-0.053	5.057	-1.727	-0.490	0.004	0.536	1.826	0.0008
∞	0.1	-	-	-2.618	-0.426	0.000	0.432	2.644	0.0000
	0.5	-	-	-3.603	-0.584	0.000	0.581	3.562	0.0000
	0.9	-	-	-4.641	-0.741	0.000	0.741	4.506	0.0000

Monte-Carlo experiments in the Cauchy case

Table – Characteristics of the empirical distributions of $\frac{n}{\log n}R_n^*$ and $\frac{n}{\log n}R_n$ over 50,000 simulated paths for $n = 5,000$. The asymptotic distribution (the law of R) is evaluated by simulation.

	ρ^*	$q_{0.1}$	$q_{0.25}$	Median	$q_{0.75}$	$q_{0.9}$
R_n^*	0.1	-0.175	-0.045	0.000	0.047	0.195
	0.3	-0.664	-0.170	0.000	0.173	0.695
	0.5	-1.305	-0.337	0.000	0.345	1.339
	0.7	-2.058	-0.535	0.001	0.545	2.068
	0.9	-2.596	-0.724	0.001	0.749	2.731
R_n	0.1	-0.174	-0.045	0.000	0.047	0.194
	0.3	-0.663	-0.170	0.000	0.173	0.692
	0.5	-1.305	-0.337	0.000	0.341	1.327
	0.7	-2.045	-0.538	0.000	0.536	2.027
	0.9	-2.528	-0.719	0.000	0.719	2.599
R	0.1	-0.286	-0.046	0.000	0.047	0.288
	0.3	-1.142	-0.185	0.000	0.189	1.157
	0.5	-2.402	-0.390	0.000	0.387	2.375
	0.7	-4.024	-0.652	0.000	0.658	4.023
	0.9	-6.155	-0.983	0.000	0.982	5.977

Monte-Carlo experiments in the Cauchy case

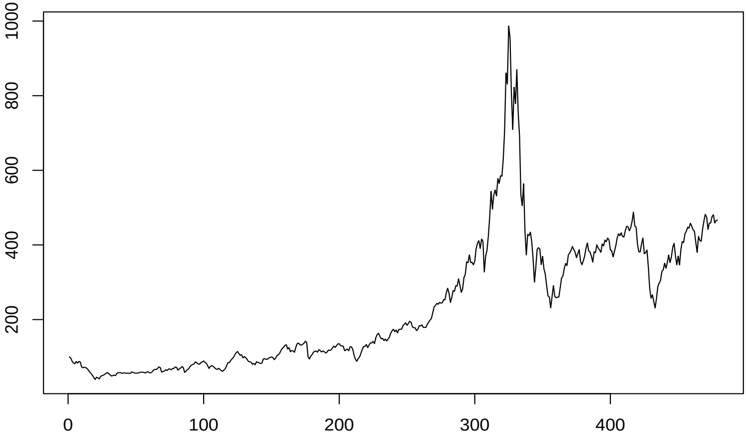
$$T_n^* = \sum_{t=2}^{n-1} \hat{\epsilon}_t^* (\hat{\epsilon}_{t-1}^*)^2 / D_n, \quad T_n = \sum_{t=3}^{n} \hat{\epsilon}_t (\hat{\epsilon}_{t-1})^2 / D_n,$$

where $D_n^2 = \sqrt{\left(\sum_{t=1}^{n-1} (\hat{\epsilon}_t^*)^2\right) \left(\sum_{t=1}^{n-1} (\hat{\epsilon}_t^*)^4\right)}$. While (ϵ_t^*) is an i.i.d. sequence, the variables $u_t = Y_t - \rho^* Y_{t-1}$ are only "empirically uncorrelated".

Table – Characteristics of the empirical distributions of $\frac{n}{\log n} T_n^*$ and $\frac{n}{\log n} T_n$.

	ρ^*	$q_{0.1}$	$q_{0.25}$	Median	$q_{0.75}$	$q_{0.9}$
T_n^*	0.1	-0.595	-0.127	0.000	0.130	0.610
	0.3	-0.606	-0.128	0.000	0.130	0.614
	0.5	-0.610	-0.130	0.000	0.129	0.616
	0.7	-0.635	-0.133	0.000	0.131	0.622
	0.9	-0.651	-0.135	0.000	0.134	0.662
T_n	0.1	-51.33	-44.54	-0.220	44.19	51.11
	0.3	-121.5	-105.3	0.139	104.8	121.0
	0.5	-144.5	-125.2	-0.162	124.5	144.0
	0.7	-82.62	-71.47	-0.272	71.07	82.16
	0.9	-8.570	-7.384	-0.054	7.348	8.518

Nasdaq composite price index : monthly, 1973 - 2012



A speculative bubble ?

Many researchers found evidence of a speculative bubble in the series of the Nasdaq composite price index (e.g. Homm and Breitung (2012), Phillips, Wu and Ju (2011)).

Let

$$Z_n^* = \frac{n}{\log n} \left| \frac{R_n^*}{\hat{\rho}_n(1+2\hat{\rho}_n)} \right|, \qquad Z_n = \frac{n}{\log n} \left| \frac{R_n}{\hat{\rho}_n(1+2\hat{\rho}_n)} \right|.$$

We have seen that the adequacy of the model is rejected at level $u \in (0,1)$ if, using for instance the statistics Z_n^*,

$$Z_n^* > \zeta_{1-u},$$

where ζ_{1-u} is the $(1-u)$ quantile of the variable $|XY|$.

Testing adequacy of the Cauchy AR(1) model

$\hat{\rho}_n$	Z_n^*	Z_n	pval(Z_n^*)	pval(Z_n)
0.9978	0.980	1.030	0.34	0.34

Table – p-values are obtained from $1,000,000$ simulations of $|Y|X$.

- The noncausal Cauchy AR(1) model cannot be rejected at any reasonable level.
- If the DGP was a unit-root or near unit-root model, the statistics Z_n and Z_n^* would converge to infinity in probability. The results do not support the UR hypothesis.

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Proof

For $\alpha \neq 1$, the characteristic function of Y_t is

$$
\begin{aligned}
& E[\exp(isY_t)] \\
= \quad & E[\exp\{is\sum_{h=0}^{\infty}\rho^h\epsilon_{t+h}\}] = \prod_{h=0}^{\infty} E[\exp\{is(\rho^h\epsilon_{t+h}\}] \\
= \quad & \exp\sum_{h=0}^{\infty}\left[-\sigma^\alpha|s|^\alpha|\rho|^{h\alpha}\left\{1 - i\beta\left(\mathsf{sign}\ s\right)\left(\mathsf{sign}(\rho)^h\right)\tan\left(\frac{\pi\alpha}{2}\right)\right\}\right] \\
= \quad & \exp\left\{-\sigma^\alpha|s|^\alpha\left(\frac{1}{1-|\rho|^\alpha} - \frac{i\beta\left(\mathsf{sign}\ s\right)}{1-\mathsf{sign}(\rho)|\rho|^\alpha}\tan\left(\frac{\pi\alpha}{2}\right)\right)\right\} \\
= \quad & \exp\left\{-\left(\frac{\sigma}{(1-|\rho|^\alpha)^{1/\alpha}}\right)^\alpha|s|^\alpha\left(1 - \frac{i\beta\left(\mathsf{sign}\ s\right)(1-|\rho|^\alpha)}{1-\mathsf{sign}(\rho)|\rho|^\alpha}\tan\left(\frac{\pi\alpha}{2}\right)\right)\right\}.
\end{aligned}
$$

This is the characteristic function of a stable distribution whose asymmetry parameter depends on the sign of ρ.

Proof (continued)

For $\alpha = 1$, we have

$$
\begin{aligned}
E[\exp(isY_t)] &= E[\exp\{is\sum_{h=0}^{\infty}\rho^h\epsilon_{t+h}\}] = \prod_{h=0}^{\infty} E[\exp\{is\rho^h\epsilon_{t+h}\}] \\
&= \exp\left\{\sum_{h=0}^{\infty} -\sigma|s||\rho|^h - is\beta\sigma\frac{2}{\pi}\sum_{h=0}^{\infty}\rho^h \log|s\rho^h|\right\} \\
&= \exp\left\{\frac{-\sigma|s|}{1-|\rho|} - is\beta\sigma\frac{2}{\pi}\left(\frac{\log|s|}{1-\rho} + \frac{\rho\log|\rho|}{(1-\rho)^2}\right)\right\}.
\end{aligned}
$$

Return

Proof

i) Let us first show that the causal Markov property holds. Denote by f_* the transition pdf in direct time and by f the transition pdf in reverse time. For any lag p, we have :

$$\begin{aligned} f_*(Y_t|Y_{t-1},\ldots,Y_{t-p}) &= \frac{f(Y_t,Y_{t-1},\ldots,Y_{t-p})}{f(Y_{t-1},\ldots,Y_{t-p})} \\ &= \frac{f(Y_t)f(Y_{t-1}|Y_t)\ldots f(Y_{t-p}|Y_{t-p+1})}{f(Y_{t-1})f(Y_{t-2}|Y_{t-1})\ldots f(Y_{t-p}|Y_{t-p+1})} \\ &= \frac{f(Y_t)f(Y_{t-1}|Y_t)}{f(Y_{t-1})}. \end{aligned}$$

Thus (Y_t) is also causal Markov, with causal transition :

$$f_*(Y_t|Y_{t-1}) = \frac{f(Y_t)f(Y_{t-1}|Y_t)}{f(Y_{t-1})}.$$

Proof (continued)

ii) Now, the forward recursive equation at horizon $h+1$ is given by

$$\begin{aligned} Y_{t-1} &= \rho^{h+1} Y_{t+h} + \varepsilon_{t-1} + \rho\varepsilon_t + \ldots + \rho^h \varepsilon_{t+h-1} \\ &= \rho^{h+1} Y_{t+h} + \varepsilon_{t-1,h}, \end{aligned}$$

where

$$\varepsilon_{t-1,h} = \varepsilon_{t-1} + \rho\varepsilon_t + \ldots + \rho^h \varepsilon_{t+h-1}.$$

The backward innovation $\varepsilon_{t-1,h}$ at lead $h+1$ follows a stable distribution with tail exponent α. Let $f_{\varepsilon,h}$ denote the pdf of $\varepsilon_{t-1,h}$. The pdf of Y_{t-1} given Y_{t+h} is thus the function

$$y \mapsto f_{\varepsilon,h}\{y - \rho^{h+1} Y_{t+h}\}.$$

By the Bayes formula, the pdf of Y_{t+h} given $Y_{t-1} = y$ is thus

$$g : x \mapsto f_{\varepsilon,h}\{y - \rho^{h+1} x\} f_Y(x)/f_Y(y),$$

where f_Y denotes the marginal pdf of Y_t.

Proof (continued)

If $|\beta| \neq 1$, the support of the stable pdf of $\varepsilon_{t-1,h}$ and Y_t is $\mathbb{R}$. It follows that when $x \to \pm\infty$,

$$g(x) \sim C(y)|x|^{-\alpha-1}|y - \rho^{h+1}x|^{-\alpha-1} \sim C^*(y)|x|^{-2(\alpha+1)},$$

where $C(y)$ and $C^*(y)$ are constants depending on y, which may change according to whether $x \to +\infty$ or $x \to -\infty$.

Now if $|\beta| = 1$, the support of the stable pdf of $\varepsilon_{t-1,h}$ and Y_t is either $\mathbb{R}^+$ or $\mathbb{R}^-$. It follows that when $\rho^{h+1} > 0$, the support of the density g is a compact ; when $\rho^{h+1} < 0$ it is bounded below or above.

Return

Proof

Recall that

$$Y_{t-1} = \rho^{h+1} Y_{t+h} + \varepsilon_{t-1,h},$$

where

$$\varepsilon_{t-1,h} = \varepsilon_{t-1} + \rho\varepsilon_t + \ldots + \rho^h \varepsilon_{t+h-1}.$$

When $\beta = 0$, we have,

$$Y_t \sim \mathcal{S}\left(\alpha, 0, \frac{\sigma}{(1-|\rho|^\alpha)^{1/\alpha}}, 0\right), \quad \varepsilon_{t-1,h} \sim \mathcal{S}\left(\alpha, 0, \sigma\left(\frac{1-|\rho|^{(h+1)\alpha}}{1-|\rho|^\alpha}\right)^{1/\alpha}\right.,$$

It follows that, for any $u \in \mathbb{R}$,

$$\begin{aligned} E\left(e^{iuY_{t-1}} \mid Y_{t+h}\right) &= e^{iu\rho^{h+1}Y_{t+h}} E\left(e^{iu\varepsilon_{t-1,h}} \mid Y_{t+h}\right) \\ &= \exp\left\{ iu\rho^{h+1}Y_{t+h} - |\sigma u|^\alpha \frac{1-|\rho|^{(h+1)\alpha}}{1-|\rho|^\alpha} \right\}. \end{aligned}$$

Proof (continued)

Thus for any $u, v \in \mathbb{R}$,

$$\begin{aligned}
& E\left(e^{iuY_{t-1}+ivY_{t+h}}\right) \\
= \; & E\left\{E\left(e^{iuY_{t-1}} \mid Y_{t+h}\right) e^{ivY_{t+h}}\right\} \\
= \; & \exp\left\{-|\sigma u|^\alpha \frac{1-|\rho|^{(h+1)\alpha}}{1-|\rho|^\alpha}\right\} E\left\{e^{i\{v+u\rho^{h+1}\}Y_{t+h}}\right\} \\
= \; & \exp\left\{-\left(|u|^\alpha(1-|\rho|^{(h+1)\alpha}) + |v+u\rho^{h+1}|^\alpha\right)\frac{\sigma^\alpha}{1-|\rho|^\alpha}\right\}.
\end{aligned}$$

Proof (continued)

Thus, for $u > 0$ and $\rho^{h+1} > 0$,

$$\begin{aligned}
&\left[\frac{\partial}{\partial u} E(e^{iuY_{t-1}+ivY_{t+h}})\right]_{v=0} \\
&= -E(e^{iuY_{t-1}}) \left(|u|^{\alpha-1}(1-|\rho|^{(h+1)\alpha}) + \rho^{h+1}|u\rho^{h+1}|^{\alpha-1} \right) \frac{\alpha\sigma^\alpha}{1-|\rho|^\alpha} \\
&= -E(e^{iuY_{t-1}})|u|^{\alpha-1} \frac{\alpha\sigma^\alpha}{1-|\rho|^\alpha}, \qquad (2)
\end{aligned}$$

and

$$\begin{aligned}
\left[\frac{\partial}{\partial v} E(e^{iuY_{t-1}+ivY_{t+h}})\right]_{v=0} &= -E(e^{iuY_{t-1}})|u\rho^{h+1}|^{\alpha-1} \frac{\alpha\sigma^\alpha}{1-|\rho|^\alpha} \qquad (3) \\
&= \rho^{(h+1)(\alpha-1)} \left[\frac{\partial}{\partial u} E(e^{iuY_{t-1}+ivY_{t+h}})\right]_{v=0}.
\end{aligned}$$

Proof (continued)

On the other hand, for $u \neq 0$,

$$\begin{aligned} \left[\frac{\partial}{\partial u} E(e^{iuY_{t-1}+ivY_{t+h}})\right]_{v=0} &= iE(Y_{t-1}e^{iuY_{t-1}}), \\ \left[\frac{\partial}{\partial v} E(e^{iuY_{t-1}+ivY_{t+h}})\right]_{v=0} &= iE(Y_{t+h}e^{iuY_{t-1}}). \end{aligned}$$

Therefore, for $u > 0$ and $\rho^{h+1} > 0$,

$$E\left\{\left(Y_{t+h} - \rho^{(h+1)(\alpha-1)}Y_{t-1}\right)e^{iuY_{t-1}}\right\} = 0. \tag{4}$$

Proof (continued)

It can be checked that for $u < 0$ and $\rho^{h+1} > 0$ both derivatives in (2) and (3) have opposite signs, thus (4) continues to hold. If now $\rho^{h+1} < 0$, we obtain

$$\left[\frac{\partial}{\partial v} E(e^{iuY_{t-1}+ivY_{t+h}})\right]_{v=0} = (-\rho)^{(h+1)(\alpha-1)} \left[\frac{\partial}{\partial u} E(e^{iuY_{t-1}+ivY_{t+h}})\right]_{v=}$$

and

$$\left[\frac{\partial}{\partial v} E(e^{iuY_{t-1}+ivY_{t+h}})\right]_{v=0} = - \left[\frac{\partial}{\partial u} E(e^{iuY_{t-1}+ivY_{t+h}})\right]_{v=0} \quad \text{if } \alpha = 1.$$

Finally, we have (with the convention $|x|^0 = \text{sign}(x)$),

$$E\left\{\left(Y_{t+h} - |\rho|^{(h+1)(\alpha-1)} Y_{t-1}\right) e^{iuY_{t-1}}\right\} = 0, \quad \text{for any } u \in \mathbb{R}.$$

The conclusion follows from Bierens (Theorem 1, 1982). Return

Proof

i) Let us compute the conditional moment of $1+(1-|\rho|)^2 Y_t^2$ in the case $\sigma = 1$. We get, using the causal transition density :

$$
\begin{aligned}
& E_{t-1}[1+(1-|\rho|)^2 Y_t^2] \\
= & \int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{1}{1+(Y_{t-1}-\rho Y_t)^2}[1+(1-|\rho|)^2 Y_{t-1}^2] dY_t \\
= & \frac{1}{\pi}[1+(1-|\rho|)^2 Y_{t-1}^2] \int_{-\infty}^{+\infty} \frac{1}{1+(Y_{t-1}-\rho Y_t)^2} dY_t \\
= & \frac{1}{|\rho|}[1+(1-|\rho|)^2 Y_{t-1}^2].
\end{aligned}
$$

$$
\Rightarrow \quad E(Y_t^2 \mid Y_{t-1}) = \frac{1}{|\rho|} Y_{t-1}^2 + \frac{1}{|\rho|(1-|\rho|)}.
$$

Proof (continued)

ii) By the same method, we retrieve the conditional mean.

$$\begin{aligned}
& E_{t-1}[1+(Y_{t-1}-\rho Y_t)^2] \\
= \quad & \frac{1}{\pi}[1+(1-|\rho|)^2 Y_{t-1}^2] \int_{-\infty}^{-\infty} \frac{1}{1+(1-[\rho|)^2 Y_t^2} dY_t \\
= \quad & \frac{1}{1-|\rho|}[1+(1-|\rho|)^2 Y_{t-1}^2] = \frac{1}{1-|\rho|} + (1-|\rho|) Y_{t-1}^2 .
\end{aligned}$$

Thus

$$\begin{aligned}
2\rho Y_{t-1} E(Y_t \mid Y_{t-1}) \quad &= \quad 1 + Y_{t-1}^2 + \rho^2 E(Y_t^2 \mid Y_{t-1}) - \frac{1}{1-|\rho|} - (1-|\rho|) Y_{t-1}^2 \\
&= \quad -\frac{|\rho|}{1-|\rho|} + |\rho| Y_{t-1}^2 + \rho^2 E(Y_t^2 \mid Y_{t-1}) = 2|\rho| Y_{t-1}^2 .
\end{aligned}$$

Therefore, we retrieve

$$E(Y_t \mid Y_{t-1}) = \operatorname{sign}(\rho) Y_{t-1} .$$

Return