Recursive Methods

Lecture 1: Dynamic optimization in discrete time

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Motivation

Macroeconomic models are designed to explain how aggregate outcomes, such as GDP, evolve over time.

Modern approach relies on micro-foundation where agents take repeated decisions in a changing environment.

To predict the behavior of agents, a natural benchmark consists in assuming that they behave optimally.

Thus most Macro models build on the premise that agents solve a dynamic optimization problem.

Motivation

The study of dynamic optimization has a long history that goes back to Euler and Lagrange's work in the mid-XVIIIth century.

Whereas early studies were devised in a continuous time framework so as to use calculus theory; the approach currently dominating in economics assumes that (i) decisions are taken at discrete points in time, and (ii) uses a recursive formulation to solve their optimization problem.

Such a solution method was pioneered by Richard Bellman in the 1950's and its definition of Dynamic Programming as a method for solving a complex problem by breaking it down into a collection of simpler subproblems.

Why Dynamic Programming?

Motivating Example

Consider the following estate planning problem.

An agent with initial assets a_0 totally finances her consumption c out of financial income so that

$$a_{t+1} = (1+r)a_t - c_t.$$

She seeks to maximize her discounted utility

$$\max_{\{c_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t U(c_t). \tag{1}$$

where $\beta \in (0,1)$ and $U : \mathbb{R}_+ \to \mathbb{R}$, with $U'(c) > 0$, $U''(c) < 0$ and $\lim_{c \to 0} U'(c) = +\infty$.

In order to rule out Ponzi schemes, we impose the terminal condition $a_T \geq 0$.

Motivating Example

Exercise 1.1 (Cake eating Problem):

1. Set $r = 0$ in the estate planning problem. Show that (i) the set of sequences $\{a_t\}_{t=0}^{T}$ is a closed, bounded and convex subset of $\mathbb{R}^{T+1}$, (ii) the objective function is continuous and strictly concave. Use these two properties to prove that problem (1) has a unique optimum.
2. Derive the optimal consumption path $c^*(t)$ under the assumption that $U(c) = ln(c)$.

Hence, the optimization problem with a finite horizon can be analyzed with standard tools.

But what about the infinite horizon problem?

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t). \tag{2}$$

Bellman's Principle of Optimality

In order to solve the infinite horizon problem, we break it down into smaller subproblems following Richard Bellman's "Principle of Optimality".

Principle of Optimality

"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

Value Function

Suppose that the optimization problem has been solved for all possible values of the state variable a.

Then we can define a **value function** $V : \mathbb{R}_+ \to \mathbb{R}$ associating to each value of a the maximized value of the objective function.

Using the principle of optimality we get

$$V(a_t) = \max_{0 \leq a_{t+1} \leq (1+r)a_t} \{U((1+r)a_t - a_{t+1}) + \beta V(a_{t+1})\}.$$

Roadmap

We will now formalize this intuition and show that Sequential Problems (SP) of the form

$$(SP) : V^*(x_0) = \max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}),$$

where $x_t \in X$ and $x_{t+1} \in \Gamma(x_t)$, can be solved using a recursive Functional Equation (FE)

$$(FE) : V(x) = \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta V(x') \right\}.$$

Principle of Optimality

A feasible plan is a sequence $\{x_t\}_{t=0}^{\infty}$ such that $x_{t+1} \in \Gamma(x_t)$ for all $t = 0, 1, ...$ Let $\Pi(x_0)$ be the set of all plans that are feasible from x_0, and let $\underline{x} = (x_0, x_1, ...)$ denote a typical element of $\Pi(x_0)$.

Assumption 1.1. $\Gamma(x)$ is non empty, for all $x \in X$.

Assumption 1.2. $\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1})$ exists, for all $x_0 \in X$ and $\underline{x} \in \Pi(x_0)$.

Principle of Optimality

Theorem 1.1.

Assume that Assumptions 1.1 and 1.2 hold and that $V^*(x)$ resulting from (SP) is well defined for all $x \in X$, then V^* satisfies the Bellman equation

$$V^*(x) = \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta V^*(x') \right\}$$

PROOF: Take any $x_0 \in X$ and let $\{x_t^*\}_{t=0}^{\infty} \in \Pi(x_0)$ be an optimal feasible plan for x_0. By definition

$$V^*(x_0) = F(x_0, x_1^*) + \beta \sum_{t=1}^{\infty} \beta^{t-1} F(x_t^*, x_{t+1}^*) \geq F(x_0, x_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} F(x_t, x_{t+1}),$$

for all $\underline{x} \in \Pi(x_0)$. Thus for all $x_1 \in \Gamma(x_0)$ we have

$$V^*(x_0) = F(x_0, x_1^*) + \beta \sum_{t=1}^{\infty} \beta^{t-1} F(x_t^*, x_{t+1}^*) \geq F(x_0, x_1) + \beta V^*(x_1).$$

Since $x_1^* \in \Gamma(x_0)$, we have $\sum_{t=1}^{\infty} \beta^{t-1} F(x_t^*, x_{t+1}^*) \geq V^*(x_1^*)$. But since $\{x_t^*\}_{t=1}^{\infty} \in \Pi(x_1^*)$ this must holds as an equality, and so

$$F(x_0, x_1^*) + \beta V^*(x_1^*) \geq F(x_0, x_1) + \beta V^*(x_1), \text{ for all } x_1 \in \Gamma(x_0). \blacksquare$$

Principle of Optimality

Exercise 1.2: Generalize Theorem 1.1 without assuming that a maximum exists. Instead let the solution of (SP) be defined as a supremum and prove that it satisfies (FE).

Theorem 1.2.

Assume that $V^*(x)$ is well defined for all $x \in X$. Suppose that the plan $\{x_t^*\}_{t=0}^{\infty} \in \Pi(x_0)$ is optimal for x_0, then V^* satisfies the Bellman equation

$$V^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*), \text{ for all } t = 0, 1, 2, \ldots$$

PROOF: Follow the same steps as in the proof of Theorem 1.1 and apply induction.■

Policy Function

Theorem 1.2 proves that any optimal paths $\{x_t^*\}_{t=0}^{\infty}$ is generated by the optimal policy correspondence

$$G^*(x) = \arg\max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta V^*(x') \right\}$$

so that $x_{t+1}^* \in G^*(x_t^*)$.

But the reverse is not always true. Here is an example from SLP where a plan generated by the policy function is not optimal.

Exercise 1.3: Consider the Estate Planning problem (1) when $U(c) = c$ and $\beta = 1/(1+r)$. Show that the path $x_{t+1} = \beta^{-1} x_t$ for all $t = 0, 1, \ldots$ is generated by G^* and explain why it is suboptimal.

Transversality Condition

To rule out suboptimal paths we need to exclude "Ponzi schemes".

Theorem 1.3.

A feasible plan $\{x_t^*\}_{t=0}^{\infty}$ is optimal if and only if

$$V^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*), \text{ for all } t = 0, 1, ..., \tag{3}$$

and $\lim_{t\to\infty} \beta^t V^*(x_t^*) = 0$.

PROOF: (Necessity) We have already shown that (3) must hold, substituting it repeatedly we obtain

$$V^*(x_0) = \sum_{t=0}^{T} \beta^t F(x_t^*, x_{t+1}^*) + \beta^{T+1} V^*(x_{T+1}^*). \tag{4}$$

According to Assumption 1, $\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t^*, x_{t+1}^*)$ exists and is by definition equal to $V^*(x_0^*)$. Hence $\lim_{t\to\infty} \beta^t V^*(x_t^*) = 0$.

(Sufficiency) If $\lim_{t\to\infty} \beta^t V^*(x_t^*) = 0$, (4) implies that $\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t^*, x_{t+1}^*) = V^*(x_0)$ and so

$\{x_t^*\}_{t=0}^{\infty}$ is optimal. ■

Transversality Condition

We still need to prove that solving (FE) yields V^*. For that we need to strengthen the transversality condition, as shown by the example below.

Exercise 1.4: Consider the Estate Planning problem (1) when $U(c) = c$, $\beta = 1/(1+r)$ and modify the savings technology so that

$$\Gamma(x) = \begin{cases} -x/\beta, \text{ if } \; x > 0 \\ x/\beta, \text{ if } \; x \leq 0 \end{cases}$$

Show that $V(x) = (1+r)x$ solves the Bellman equation although $V^*(x) \neq (1+r)x$.

Policy Function

To ensure that a solution to (FE) is indeed the function value V^*, we need to impose an additional transversality condition on the value function itself.

Theorem 1.4.
Suppose that V is a finite value function that solves (FE), and that for each x_0 there exists a feasible plan $\{x_t^*\}_{t=0}^{\infty}$ with $V(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V(x_{t+1}^*)$, for t=0,1,..., and $\lim_{t\to\infty} \beta^t V(x_t^*) = 0$. Suppose also that

$$\limsup_{t\to\infty} \beta^t V(x_t) \geq 0,$$

for all feasible plans with $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) > -\infty$. Then $V = V^*$ where V^* solves (SP).

Policy Function

PROOF Theorem 1.4: Optimality of V is equivalent to

$$V(x_0) \geq \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}),$$

for all $\underline{x} \in \Pi(x_0)$. Pick a $x_0 \in X$ and a plan $\{x_t^*\}_{t=0}^{\infty}$ that satisfies the theorem's condition. If $\sum_{t=0}^{\infty} \beta^t F(x_t^*, x_{t+1}^*) = -\infty$, the optimality condition above is immediately satisfied. Otherwise, iterating on the Bellman equation we get

$$V(x_0) = \sum_{t=0}^{T-1} \beta^t F(x_t^*, x_{t+1}^*) + \beta^T V(x_T^*) \geq \sum_{t=0}^{T-1} \beta^t F(x_t, x_{t+1}) + \beta^T V(x_T).$$

Taking the supremum limit on both sides, we get

$$V(x_0) = \sum_{t=0}^{\infty} \beta^t F(x_t^*, x_{t+1}^*) \geq \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) + \limsup_{T} \beta^T V(x_T),$$

which gives us the desired inequality because the limit on the right side is greater or equal to zero. Since x_0 was arbitrary, global optimality is established.■

Conclusion

To summarize, we have shown in this lecture that

1. Theorem 1.1.: V^* solves the sequential problem (SP) $\Rightarrow$ V^* solves the recursive problem (FE).

2. Theorem 1.2.: $\{x_t^*\}_{t=0}^{\infty} \in \Pi(x_0)$ is optimal $\Rightarrow$ $V^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*)$.

3. Theorem 1.3.: $\{x_t^*\}_{t=0}^{\infty} \in \Pi(x_0)$ satisfies $V^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*)$ and transversality condition $\Rightarrow$ $\{x_t^*\}_{t=0}^{\infty}$ is optimal.

4. Theorem 1.4.: V solves recursive problem (FE) and transversality condition $\Rightarrow$ V solves sequential problem, i.e., $V = V^*$.

"An interesting question is, where did the name, dynamic programming, come from? The 1950s were not good years for mathematical research. We had a very interesting gentleman in Washington named Wilson. He was Secretary of Defense, and he actually had a pathological fear and hatred of the word research. Im not using the term lightly; Im using it precisely. His face would suffuse, he would turn red, and he would get violent if people used the term research in his presence. You can imagine how he felt, then, about the term mathematical. (...) What title, what name, could I choose? In the first place I was interested in planning, in decision making, in thinking. But planning, is not a good word for various reasons. I decided therefore to use the word programming. I wanted to get across the idea that this was dynamic, this was multistage, this was time-varying. I thought, let's kill two birds with one stone. Let's take a word that has an absolutely precise meaning, namely dynamic, in the classical physical sense. It also has a very interesting property as an adjective, and that it's impossible to use the word dynamic in a pejorative sense. Try thinking of some combination that will possibly give it a pejorative meaning. It's impossible. Thus, I thought dynamic programming was a good name. It was something not even a Congressman could object to. So I used it as an umbrella for my activities."— Richard Bellman, *Eye of the Hurricane: An Autobiography*. Back to main.