Dynamic models with latent variables

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Introduction : Definitions and examples

Outline

Outline (continued)

Chapter IV: Bubble modeling

1. Rational bubbles
2. Nonlinear models (time-varying parameters, thresholds)
3. Noncausal linear models for local explosions

Chapter V: Bayesian and simulated methods

1. Simulation by acceptance-rejection
2. The Metropolis-Hastings and Gibbs algorithms
3. Examples: STAR model, stochastic volatility model

Textbooks

- Harvey, A.C. (1989) *Forecasting, structural time series models and the Kalman filter*, Cambridge University Press.
- Brockwell, P.J. and R.A. Davis (1991) *Time Series: Theory and Methods*, Springer-Verlag (2nd edition).
- Hamilton, J. D. (1994) *Time Series Analysis*, Princeton University Press.
- Gouriéroux, C. and A. Monfort (1997) *Time Series and Dynamic Models*, Cambridge University Press, Cambridge.
- Fruhwirth-Schnatter, S. (2006) *Finite Mixture and Markov Switching Models*, Springer.
- Douc, R., Moulines, E. and D.S. Stoffer (2014)*Nonlinear Time Series*, CRC Press.
- Durlauf, S. and L. Blume (Eds) (2016) Macroeconometrics and time series analysis. Springer
- Francq, C. and J-M. Zakoian (2019) *GARCH Models*. Structure, Statistical Inference and Financial Applications. 2nd Edition, Wiley.

Dynamic models, hidden variables

- Dynamic → Time series course
- Hidden → Variables that are not statistically observable

Synonyms for hidden: latent, unobservable

HV in econometrics

Several types of variables appear in the standard multivariate linear regression model

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{U}$$

- $\boldsymbol{Y} \in \mathbb{R}^N$: Endogenous/dependent variable (regressand)
- $\boldsymbol{X}$: $N \times K$ Matrix, whose columns are the exogenous/explanatory variables (regressors)
- $\boldsymbol{\beta} \in \mathbb{R}^K$: vector of parameters
- $\boldsymbol{U} \in \mathbb{R}^N$: error/disturbance term. An unobservable variable which is generally interpreted as a measurement error

Time series models

Many models are of the form:

$$\boldsymbol{Y}_t = \phi(\boldsymbol{Y}_{t-1}, \ldots, ; \boldsymbol{U}_t)$$

where $\boldsymbol{U}_t$ is an unobervable error term.

Exogenous variables can also be included:

$$\boldsymbol{Y}_t = \phi(\boldsymbol{Y}_{t-1}, \boldsymbol{Y}_{t-2}, \ldots, ; \boldsymbol{X}_t; \boldsymbol{U}_t)$$

Latent (unobserved or partially unobserved) variables models:

$$\boldsymbol{Y}_t = \phi(\boldsymbol{Y}_{t-1}, \boldsymbol{Y}_{t-2}, \ldots, ; \boldsymbol{\alpha}_t; \boldsymbol{U}_t),$$
$$\boldsymbol{\alpha}_t = \phi^*(\boldsymbol{\alpha}_{t-1}, \boldsymbol{\alpha}_{t-2}, \ldots, ; \boldsymbol{V}_t),$$

where $\boldsymbol{U}_t, \boldsymbol{V}_t$ are error terms.

Why introducing hidden variables in TS models?

- Dynamic properties of real series may be difficult to capture with classical models
- Latent variables may have an economic interpretation
- Allows to reduce the dimension of a statistical problem

Entails two types of difficulties:

- **Probabilistic properties** of such models can be difficult to derive (existence of solutions?...).
- Standard **statistical tools** can be inadequate.

Strict stationarity

$(\boldsymbol{Y}_t)_{t\in\mathbb{Z}}$: a (discrete time) stochastic process, valued in $\mathbb{R}^d$.

Definition

The process $(\boldsymbol{Y}_t)$ *is called* **strictly stationary** *if*

$$(\boldsymbol{Y}_{t_1},\ldots,\boldsymbol{Y}_{t_k})' \stackrel{d}{=} (\boldsymbol{Y}_{t_1+h},\ldots,\boldsymbol{Y}_{t_k+h})',$$

for all integers $k\in\mathbb{N}$ *and* $h,t_1,\ldots,t_k\in\mathbb{Z}$.

Second-order stationarity

Definition

The process $(\boldsymbol{Y}_t)$ *is called* **second-order stationary** *if*

(i) $\boldsymbol{Y}_t \in L^2 \; (E\|\boldsymbol{Y}_t\|^2 < \infty) \quad \forall t \in \mathbb{Z}$,

(ii) $E\boldsymbol{Y}_t = \boldsymbol{\mu} \quad \forall t \in \mathbb{Z}$,

(iii) $Cov(\boldsymbol{Y}_t, \boldsymbol{Y}_{t+h}) = \boldsymbol{\Gamma}(h) \quad \forall t, h \in \mathbb{Z}$.

$\Gamma(\cdot)$ is the **autocovariance function of** $(\boldsymbol{Y}_t)$.

$\rho(\cdot) := \frac{\gamma(\cdot)}{\gamma(0)}$ **is the autocorrelation function of** (X_t).

Different concepts of noise

A (weak) white noise process $(\boldsymbol{\epsilon}_t)$ is a sequence of centered and uncorrelated variables:

$$\boldsymbol{\epsilon}_t \in L^2, \quad E(\boldsymbol{\epsilon}_t) = \mathbf{0}, \quad \text{Cov}(\boldsymbol{\epsilon}_t, \boldsymbol{\epsilon}_{t+h}) = \mathbf{0}, \quad \forall h \neq 0.$$

A strong white noise process $(\boldsymbol{\epsilon}_t)$ is a sequence of centered, independent variables belonging to L^2.

An iid (independent and identically distributed) noise is a strong white noise where the $\boldsymbol{\epsilon}_t$ have the same distribution.

White noise are useful to construct more complex stationary processes.

Two concepts of prediction

- Conditional expectation: If $(\boldsymbol{Y}_t)$ is second-order stationary,

$$E(\boldsymbol{Y}_t \mid \boldsymbol{Y}_{t-1}, \boldsymbol{Y}_{t-2}, \ldots)$$

is the best approximation of $\boldsymbol{Y}_t$ (in the L^2 sense) as a function of its past.

- Linear conditional expectation:

$$EL(\boldsymbol{Y}_t \mid \boldsymbol{Y}_{t-1}, \boldsymbol{Y}_{t-2}, \ldots)$$

is the best approximation of $\boldsymbol{Y}_t$ as a linear function of its past.

Notation: $\boldsymbol{Y}_{\underline{t-1}}$ the past of $\boldsymbol{Y}_t$ and $\mathscr{H}_{\boldsymbol{Y}}(t-1)$ the linear past of $\boldsymbol{Y}_t$.

Two concepts of innovations

- Strong innovation:

$$\boldsymbol{\epsilon}_t = \boldsymbol{Y}_t - E(\boldsymbol{Y}_t \mid \boldsymbol{Y}_{t-1}, \boldsymbol{Y}_{t-2}, \ldots)$$

is "orthogonal" to any function of the past of $\boldsymbol{Y}_t$

$$E(\boldsymbol{\epsilon}_t' \boldsymbol{Z}_{t-1}) = 0, \quad \forall \boldsymbol{Z}_{t-1} \in \boldsymbol{Y}_{\underline{t-1}}.$$

- Linear innovation:

$$\epsilon_t^* = \boldsymbol{Y}_t - EL(\boldsymbol{Y}_t \mid \boldsymbol{Y}_{t-1}, \boldsymbol{Y}_{t-2}, \ldots)$$

$(\boldsymbol{\epsilon}_t^*)$ is a white noise and $\boldsymbol{\epsilon}_t^*$ is orthogonal to any linear function of the past of $\boldsymbol{Y}_t$

$$E(\boldsymbol{Z}_{t-1}' \boldsymbol{\epsilon}_t^*) = 0, \quad \forall \boldsymbol{Z}_{t-1} \in \mathscr{H}_{\boldsymbol{Y}}(t-1).$$

Vector Autoregressive Moving Average (ARMA) models

The most general class of linear models

A VARMA process is any stationary solution (when existing) $(\boldsymbol{Y}_t)$ of the stochastic recurrence equation

$$\boldsymbol{Y}_t + \sum_{i=1}^{p} \boldsymbol{A}_i \boldsymbol{Y}_{t-i} = \boldsymbol{c} + \boldsymbol{\epsilon}_t + \sum_{j=1}^{q} \boldsymbol{B}_j \boldsymbol{\epsilon}_{t-j},$$

where $(\boldsymbol{\epsilon}_t)$ is a white noise, $\boldsymbol{A}_i, \boldsymbol{B}_j$ are real $d \times d$ matrices, $\boldsymbol{c}$ is a d-vector.

The model can be written in more a compact way using the lag operator B:

$$\boldsymbol{\Phi}(B)\boldsymbol{Y}_t = \boldsymbol{c} + \boldsymbol{\Psi}(B)\boldsymbol{\epsilon}_t$$

where $\boldsymbol{\Phi}(B)$ and $\boldsymbol{\Psi}(B)$ are lag polynomials.

Existence of a nonanticipative solution

A solution $(\boldsymbol{Y}_t)$ of is called causal, or nonanticipative, if $\boldsymbol{Y}_t$ can be written as a measurable function of the $\boldsymbol{\epsilon}_s$, $\quad s \leq t$.

Proposition

If

$$\det \boldsymbol{\Phi}(z) = 0 \quad \Rightarrow \quad |z| > 1,$$

then the VARMA model admits a unique nonanticipative stationary solution $(\boldsymbol{Y}_t)$ of the form,

$$\boldsymbol{Y}_t = \boldsymbol{d} + \sum_{j=1}^{\infty} \boldsymbol{C}_j \boldsymbol{\epsilon}_{t-j}.$$

Strengths and weaknesses of VARMA

VARMA models remain very much used (mainly VAR models in econometrics, because VARMA seem too complex).

The class is very flexible, especially when weak noise are used for the errors (cf the Wold rep.).

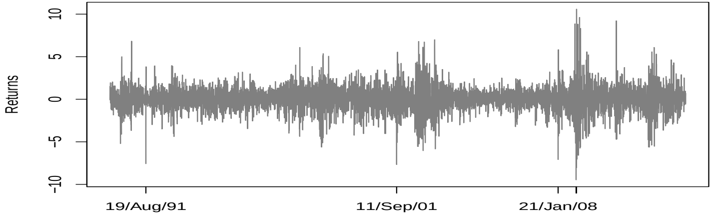
VARMA provide adequate fits of many vector series, and can be extended to incorporate seasonalities (SARMA) and even some types of non stationarity (ARIMA).

However, they are unable to capture some phenomena called **nonlinear**: volatility effects, change of regimes, bubbles...

Various alternative models have been proposed, some of which including hidden variables (apart from the noise).

1. Classical dynamic models
2. Dynamic models with latent variables
   - Volatility models
   - Switching-regime models
   - State-space models

The sample paths of financial returns often look like that



The autocorrelations are those of a white noise (i.e. close to zero)

$$\epsilon_t = \sigma \eta_t$$

where $\sigma > 0$ is the standard deviation of ϵ_t and η_t is another noise (with unit variance).
But this model is inappropriate to represent the strong dependencies that appear on the graph.

Random variance

The idea is to make the standard-deviation random:

$$\epsilon_t = \sigma_t \eta_t$$

where

- (η_t) is an iid (0,1) process
- (σ_t) is a process called volatility, $\sigma_t > 0$
- the variables σ_t and η_t are independent

Two classes of models: $\epsilon_t = \sigma_t \eta_t$

- GARCH (Generalized AutoRegressive Conditional Heteroskedasticity): $\sigma_t \in \epsilon_{\underline{t-1}}$ (past of ϵ_t).
  For instance
  $$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2$$
- Stochastic volatility (SV) models: $\sigma_t \notin \epsilon_{\underline{t-1}}$.
  For instance $\log \sigma_t^2 \sim \mathrm{AR}(1)$:
  $$\log \sigma_t^2 = \omega + \beta \log \sigma_{t-1}^2 + \tau v_t,$$
  where (v_t) is an iid process, with v_t independent of η_t.

In SV models, the volatility is **never observed**.

1. Classical dynamic models

2. Dynamic models with latent variables
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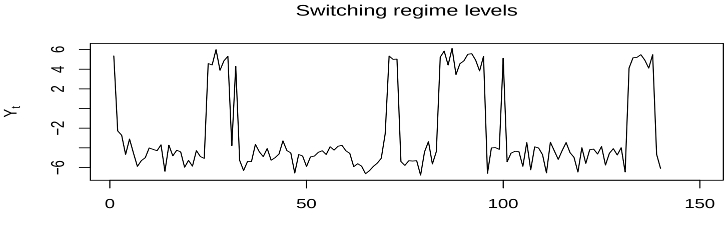
If a time series looks like that

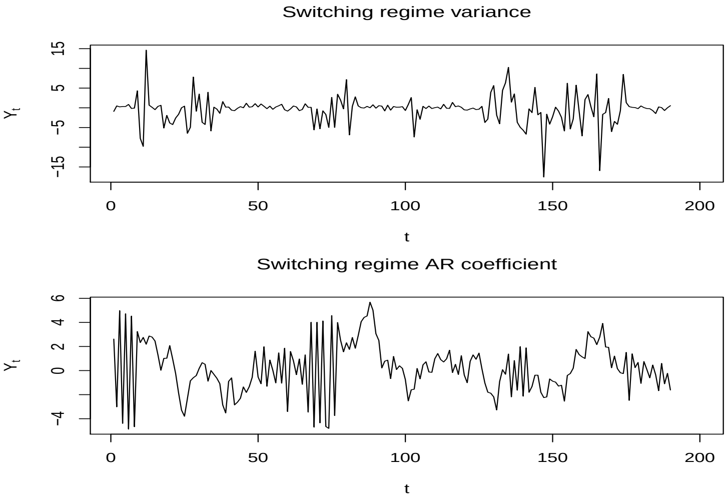


it seems sensible to introduce **several regimes** in the model.

This is different from the situation where a **break** occurs (in this case, the stationarity assumption cannot be made: the model changes after some date t_0).

On the contrary, if the regimes are **recurrent**, the series may have been generated by a stationary model.

Other examples



How to model regime switches?

If d regimes have to be accounted, a latent variable $\Delta_t \in \{1,\ldots,d\}$ can be introduced:

$$Y_t = \phi(Y_{t-1},\ldots,;\epsilon_t;\Delta_t).$$

For instance, a switching-regime AR(1):

$$Y_t = \phi(\Delta_t)Y_{t-1} + \epsilon_t.$$

The model has to be completed by specifying the dynamics of the process (Δ_t):

- Could be iid (but this assumption is often too restrictive)
- A Markov chain
- More complex processes? not much used

Models used for the simulations

$\epsilon_t \sim \mathcal{N}(0,1)$, 2 regimes, $p(1,1) = p(2,2) = 0.95$

- Switching regime level

$$X_t = \begin{cases} 5 + \epsilon_t & \text{if} \quad \Delta_t = 1 \\ -5 + \epsilon_t & \text{if} \quad \Delta_t = 2 \end{cases}$$

- Switching regime variance

$$X_t = \begin{cases} 0.5\epsilon_t & \text{if} \quad \Delta_t = 1 \\ 5\epsilon_t & \text{if} \quad \Delta_t = 2 \end{cases}$$

- Switching regime AR coefficient

$$X_t = \begin{cases} 0.9X_{t-1} + \epsilon_t & \text{if} \quad \Delta_t = 1 \\ -0.9X_{t-1} + \epsilon_t & \text{if} \quad \Delta_t = 2 \end{cases}$$

1. Classical dynamic models

2. Dynamic models with latent variables
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State-space models

Many models can be written under the form

$$\begin{cases} \boldsymbol{y}_t = \boldsymbol{M}_t \boldsymbol{\alpha}_t + \boldsymbol{d}_t + \boldsymbol{u}_t & \text{Measurement equation} \\ \boldsymbol{\alpha}_t = \boldsymbol{T}_t \boldsymbol{\alpha}_{t-1} + \boldsymbol{c}_t + \boldsymbol{R}_t \boldsymbol{v}_t, & \text{Transition equation} \end{cases}$$

where $\boldsymbol{y}_t \in \mathbb{R}^N$, $\boldsymbol{\alpha}_t \in \mathbb{R}^m$ (**state vector**), $(\boldsymbol{u}_t)$ and $(\boldsymbol{v}_t)$ are two sequences of independent variables, valued in $\mathbb{R}^N$ and $\mathbb{R}^K$ such that

$$E(\boldsymbol{u}_t) = \boldsymbol{0}_N, \quad E(\boldsymbol{v}_t) = \boldsymbol{0}_K, \quad \mathsf{Var}(\boldsymbol{u}_t) = \boldsymbol{H}_t, \quad \mathsf{Var}(\boldsymbol{v}_t) = \boldsymbol{Q}_t,$$

$\boldsymbol{M}_t, \boldsymbol{T}_t$ and $\boldsymbol{R}_t$ are non-random $N \times m$, $m \times m$ and $m \times K$ matrices, $\boldsymbol{d}_t \in \mathbb{R}^N$, $\boldsymbol{c}_t \in \mathbb{R}^m$ are non-random vectors.

Example: MA model

$$y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}.$$

State-space representation:

$$\begin{cases} y_t & = & [1, \theta_1, \ldots, \theta_q] \boldsymbol{\alpha}_t \\ \boldsymbol{\alpha}_t & = & \boldsymbol{T} \boldsymbol{\alpha}_{t-1} + (\epsilon_t, 0, \ldots, 0)' \end{cases}$$

$$\boldsymbol{\alpha}_t = \begin{pmatrix} \epsilon_t \\ \epsilon_{t-1} \\ \vdots \\ \vdots \\ \epsilon_{t-q} \end{pmatrix}, \qquad \boldsymbol{T} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

Example: MA model

State-space representation of smaller dimension:

$$\begin{cases} y_t &= [\theta_1,\dots,\theta_q]\boldsymbol{\alpha}_t + \epsilon_t \\ \boldsymbol{\alpha}_t &= \boldsymbol{T}\boldsymbol{\alpha}_{t-1} + (\epsilon_{t-1},0,\dots,0)', \end{cases}$$

$$\boldsymbol{\alpha}_t = \begin{pmatrix} \epsilon_{t-1} \\ \epsilon_{t-2} \\ \vdots \\ \vdots \\ \epsilon_{t-q} \end{pmatrix}, \qquad \boldsymbol{T} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Example: VAR model

$$\boldsymbol{y}_t - \boldsymbol{\mu} = \boldsymbol{\phi}_1(\boldsymbol{y}_{t-1} - \boldsymbol{\mu}) + \cdots + \boldsymbol{\phi}_p(\boldsymbol{y}_{t-p} - \boldsymbol{\mu}) + \boldsymbol{\epsilon}_t$$

Vector representation:

$$\underbrace{\begin{pmatrix} \boldsymbol{y}_t - \boldsymbol{\mu} \\ \boldsymbol{y}_{t-1} - \boldsymbol{\mu} \\ \vdots \\ \vdots \\ \boldsymbol{y}_{t-p+1} - \boldsymbol{\mu} \end{pmatrix}}_{\boldsymbol{\alpha}_t} \begin{matrix} = \\ \\ = \end{matrix} \underbrace{\begin{pmatrix} \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \dots & \boldsymbol{\phi}_{p-1} & \boldsymbol{\phi}_p \\ \boldsymbol{I} & \boldsymbol{0} & \dots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \dots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{I} & \boldsymbol{0} \end{pmatrix}}_{\boldsymbol{\Phi}} \underbrace{\begin{pmatrix} \boldsymbol{y}_{t-1} - \boldsymbol{\mu} \\ \boldsymbol{y}_{t-2} - \boldsymbol{\mu} \\ \vdots \\ \vdots \\ \boldsymbol{y}_{t-p} - \boldsymbol{\mu} \end{pmatrix}}_{\boldsymbol{\alpha}_{t-1}} + \underbrace{\begin{pmatrix} \boldsymbol{\epsilon}_t \\ \boldsymbol{0} \\ \vdots \\ \vdots \\ \boldsymbol{0} \end{pmatrix}}_{\boldsymbol{v}_t}.$$

Measurement equation:

$$\boldsymbol{y}_t \quad = \quad [\boldsymbol{I}, \boldsymbol{0}, \dots, \boldsymbol{0}]\boldsymbol{\alpha}_t + \boldsymbol{\mu}.$$

Example: ARMA model

Consider the state-space model, for $y_t \in \mathbb{R}$, $p > 1$,

$$\begin{cases} y_t &= [1, \theta_1, \dots, \theta_{p-1}] \boldsymbol{\alpha}_t + \mu \\ \boldsymbol{\alpha}_t &= \boldsymbol{\Phi} \boldsymbol{\alpha}_{t-1} + \boldsymbol{v}_t \end{cases}$$

where

$$\boldsymbol{\alpha}_t = \begin{pmatrix} \alpha_{1t} \\ \alpha_{2t} \\ \vdots \\ \alpha_{pt} \end{pmatrix}, \qquad \boldsymbol{\Phi} = \begin{pmatrix} \phi_1 & \phi_2 & \dots & \phi_{p-1} & \phi_p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{v}_t = \begin{pmatrix} \epsilon_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Example: ARMA model

It follows that

$$\alpha_{it} = B^{i-1}\alpha_{1t}, \qquad \phi(B)\alpha_{1t} = \epsilon_t,$$

where $\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p$. Then

$$y_t = (1 + \theta_1 B + \ldots + \theta_{p-1} B^{p-1})\alpha_{1t} + \mu := \theta(B)\alpha_{1t} + \mu,$$

and thus

$$\phi(B)(y_t - \mu) = \theta(B)\epsilon_t.$$

Finally

$$y_t \sim \mathrm{ARMA}(p, p-1), \quad \alpha_t \in \mathbb{R}^p.$$

Example: ARMA model

Let $\theta(z) = 1 + \sum_{i=1}^{p} \theta_i z^i$ and $\phi(z) = 1 - \sum_{i=1}^{p} \phi_i z^i$. Suppose that

$$\phi(B)(y_t - \mu) = \theta(B)\epsilon_t,$$

where the roots of $\phi(z)$ are outside the unit circle. Let

$$\alpha_{1t} = \phi^{-1}(B)\epsilon_t.$$

Then

$$\begin{aligned} y_t - \mu &= \phi^{-1}(B)\theta(B)\epsilon_t \\ &= \theta(B)\alpha_{1t} = [1, \theta_1, \ldots, \theta_q]\boldsymbol{\alpha}_t, \qquad \boldsymbol{\alpha}_t = \begin{pmatrix} \alpha_{1t} \\ \alpha_{1,t-1} \\ \vdots \\ \vdots \\ \alpha_{1,t-q} \end{pmatrix}. \end{aligned}$$

Example: ARMA model (continued)

It can be assumed that $p \leq q+1$. Thus, α_{1t} can be expressed in terms of the components of $\boldsymbol{\alpha}_{t-1}$ and we have

$$\boldsymbol{\alpha}_t = \boldsymbol{\Phi}\boldsymbol{\alpha}_{t-1} + \boldsymbol{v}_t,$$

where

$$\boldsymbol{\Phi} = \begin{pmatrix} \phi_1 & \phi_2 & \dots & \phi_{p-1} & \phi_p & 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & & & & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & & & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & & & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & & & & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & & & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & & & \dots & 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{v}_t = \begin{pmatrix} \epsilon_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Remark: It can be shown that, for an ARMA(p,q), the minimum dimension of a state vector $\boldsymbol{\alpha}_t$ is Max(p,q).

Stochastic-trend models

Structural model: involves unobservable variables which can be interpreted.

For instance, the series can be decomposed as the sum of a trend and a noise:

$$y_t = \mu_t + \epsilon_t, \quad \text{where} \quad (\mu_t) \perp (\epsilon_t).$$

The trend can be modelled as

$$\begin{cases} \mu_t &= \mu_{t-1} + \beta_{t-1} + \eta_t, \\ \beta_t &= \beta_{t-1} + \xi_t, \end{cases}$$

where $(\eta_t) \perp (\xi_t)$ are white noise with variances σ_η^2 and σ_ξ^2.
The 2nd equation introduces a stochastic slope in the random walk followed by μ_t.

Stochastic-trend models

State-space representation of the model:

$$\begin{cases} y_t = (1,0)\begin{pmatrix} \mu_t \\ \beta_t \end{pmatrix} + \epsilon_t, \\ \begin{pmatrix} \mu_t \\ \beta_t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mu_{t-1} \\ \beta_{t-1} \end{pmatrix} + \begin{pmatrix} \eta_t \\ \xi_t \end{pmatrix}. \end{cases}$$

Let $\Delta = 1 - B$ the difference operator. The latent variables μ_t and β_t can be eliminated from the representation:

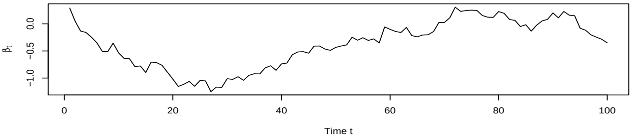
$$\Delta^2 y_t = \xi_t + \Delta\eta_t + \Delta^2\epsilon_t.$$

Symbolic notation:

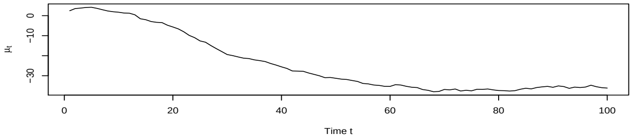
$$y_t = \frac{\xi_t}{\Delta^2} + \frac{\eta_t}{\Delta} + \epsilon_t.$$

Sample paths of a stochastic-trend process

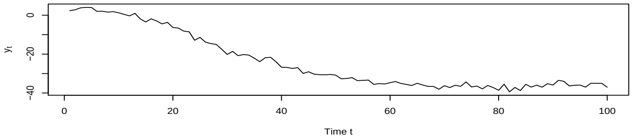
(a) Simulation of the slope $\beta_t = \beta_{t-1} + \xi_t$, Var (ξ_t) =0.1



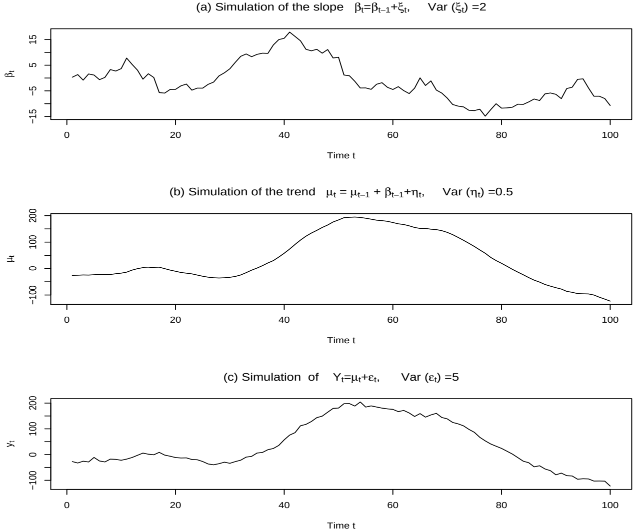
(b) Simulation of the trend $\mu_t = \mu_{t-1} + \beta_{t-1} + \eta_t$, Var (η_t) =0.5



(c) Simulation of $Y_t = \mu_t + \varepsilon_t$, Var (ε_t) =1



Sample paths of a stochastic-trend process



Stochastic trend, cycles and seasonality

Let

$$\left\{\begin{array}{lcl} y_t & = & y_{1t} + y_{2t} + \epsilon_t \\ y_{1t} & = & \delta + y_{1,t-1} + \eta_t \\ y_{2t} & = & \phi_1 y_{2,t-1} + \phi_2 y_{2,t-2} + \xi_t \end{array}\right.$$

where the roots of the AR(2) polynomial are outside the unit circle, (e_{1t}) and (e_{2t}) are white noises.

State-space representation:

$$\left\{\begin{array}{lcl} y_t & = & (1,1,0)\begin{pmatrix} y_{1t} \\ y_{2t} \\ y_{2,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_t \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} y_{1t} \\ y_{2t} \\ y_{2,t-1} \end{pmatrix} & = & \begin{pmatrix} 1 & 0 & 0 \\ 0 & \phi_1 & \phi_2 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \\ y_{2,t-2} \end{pmatrix} + \begin{pmatrix} \delta \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \eta_t \\ \xi_t \\ 0 \end{pmatrix} \end{array}\right.$$

Stochastic trend, cycles and seasonality

A seasonal component y_{3t} can be added.
Suppose for instance that

$$y_{3t} = \psi y_{3,t-m} + \zeta_t$$

where $m = 4$ for a quarterly series, $m = 12$ for a monthly series.

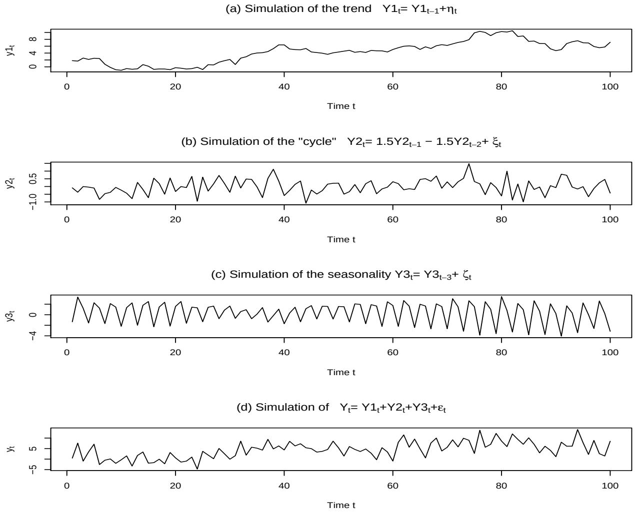
The state-space vector can be completed as

$$\alpha_t = (y_{1t}, y_{2t}, y_{2,t-1}, y_{3,t}, \ldots, y_{3,t-m+1})'$$

and the transition matrix can be completed accordingly.

Sample paths with stochastic trend, cycle and seasonality

$\mathrm{Var}(\epsilon_t) = 2, \mathrm{Var}(\eta_t) = \mathrm{Var}(\xi_t) = \mathrm{Var}(\zeta_t) = 0.5$



Random coefficients models

With temporal data, the classical linear model writes

$$\boldsymbol{Y}_t = \boldsymbol{X}_t \boldsymbol{\beta} + \boldsymbol{U}_t$$

where $\boldsymbol{X}_t \in \mathbb{R}^K$ is a vector of exogenous variables.
Structural changes can motivate the introduction of random coefficients indexed by time:

$$\boldsymbol{Y}_t = \boldsymbol{X}_t' \boldsymbol{\beta}_t + \boldsymbol{U}_t$$

For instance an AR(1) model can be set on the coefficients:

$$\boldsymbol{\beta}_t = \boldsymbol{\Phi} \boldsymbol{\beta}_{t-1} + \boldsymbol{V}_t$$

The latter equation is the transition equation in a state-space model.

Canonical stochastic volatility model

$$\begin{cases} \epsilon_t = \sqrt{h_t}\eta_t, & (\eta_t) \overset{iid}{\sim} (0,1), \\ \\ \log h_t = \omega + \beta \log h_{t-1} + \sigma v_t, & (v_t) \overset{iid}{\sim} (0,1). \end{cases}$$

- Similar to diffusion models used in the financial literature.
- Positivity of h_t does not entail constraints on the coefficients.
- Interpretation of the coefficients:
  - ω level parameter,
  - β persistence parameter, in general > 0.
  - $\sigma > 0$ without generality loss. Volatility of the volatility.

Assuming $P(\eta_t = 0) = 0$, we get the state-space model

$$\begin{cases} \log \epsilon_t^2 = \log h_t + \log \eta_t^2 \\ \\ \log h_t = \omega + \beta \log h_{t-1} + \sigma v_t \end{cases}$$

Sample paths of a stochastic volatility process