Recursive Methods

Lecture 4: Dynamic Factor Demand

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Motivation

Until now, we have focused on deterministic models, so that the trajectory of the state variables could be perfectly predicted.

Much of macroeconomics is concerned with the impact of random shocks.

It is straightforward to extend the analysis of previous lectures to cases where the random state variable z takes value over $Z = \{z_1, z_2, ..., z_n\}$ with associated probabilities $(\pi_1, \pi_2, ..., \pi_n)$. Then the Bellman equation reads

$$
\begin{aligned}
(FE) : V(x, z) &= \max_{x' \in \Gamma(x)} \left\{ F(x, x', z) + \beta \sum_{i=1}^{n} V(x', z_i') \pi_i \right\} \\
&= \max_{x' \in \Gamma(x)} \left\{ F(x, x', z) + \beta E \left[V(x', z') \right] \right\}.
\end{aligned}
$$

Motivation

As an illustration, we will study the theory of optimal capital investment.

Basic Keynesian Investment Equation (see IS curve):

$$I = a_0 + a_1 Y_t - a_2 r_t.$$

However, neoclassical theory of investment predicts that interest rate determines the desired capital stock.

Investment being the derivative of the target, one should not expect a smooth relationship between I and r.

Optimization problem

Consider a risk-neutral firm which seeks to maximize its discounted profits

$$
\begin{aligned}
V(k_0) = & \max_{\{k_{t+1}, n_t\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \pi(k_{t+1}, n_t | k_t, w_t, a_t) \right], \\
s.t. \quad & \pi(k_{t+1}, n_t | k_t, w_t, a_t) = a_t f(k_t, n_t) - w_t n_t - I_t, \\
& I_t = k_{t+1} - (1-\delta) k_t,
\end{aligned}
$$

where δ is the depreciation rate of capital and $k_0 \geq 0$.

Problem in recursive form

Recursive formulation reads

$$
\begin{aligned}
& V(k_t) = \max_{\{k_{t+1}, n_t\}} \pi(k_{t+1}, n_t | k_t, w_t, a_t) + \left(\frac{1}{1+r}\right) E_t \left[V(k_{t+1})\right], \\
s.t. \quad & \pi(k_{t+1}, n_t | k_t, w_t, a_t) = a_t f(k_t, n_t) - w_t n_t - I_t, \\
& I_t = k_{t+1} - (1-\delta) k_t.
\end{aligned}
$$

Combining the FOCs and envelope condition, we get

$$
\begin{aligned}
& a_t f_n(k_t, n_t) = w_t, \\
& E_t \left[a_{t+1} f_k(k_{t+1}, n_{t+1})\right] = r + \delta.
\end{aligned}
$$

Optimality conditions

For simplicity, assume that employment is fixed over time, i.e., $n_t = \bar{n}$ for all t (e.g., full employment equilibrium).

Then wages are endogenous, with an equilibrium value determined by the FOC for labor.

Differentiating the FOC for capital we find that

$$\frac{\partial k_{t+1}}{\partial r_t} = \frac{1}{E_t\left[a_{t+1} f_{kk}(k_{t+1}, \bar{n})\right]} < 0.$$

- Interest rate determines the desired capital stock and not the level of investment, as postulated by Keynesian theory (See Haavelmo's critique, *A study in the Theory of Investment*, 1960).
- In other word, investment is determined by changes in r and not by the level of r.

Adjustment costs

The basic neoclassical model is not without its flaws either:

Since capital is a smooth function of r, discrete changes in r should lead to discrete changes in k. Taking the continuous time limit, we get an infinite rate of investment. Neoclassical model generate too much volatility in investment.

In order to smooth investment, we introduce an adjustment cost function, such that investing I entails spending $I + \phi(I, k)$, with $\phi : \mathbb{R} \rightarrow \mathbb{R}+$ such that

$$\phi(0, K) = 0, \ \phi_I(0, K) = 0, \ \phi_{II}(I, K) > 0, \ \phi_K(I, K) \leq 0.$$

Adjustment costs are convex, infinitesimally small at 0 and positive when $I < 0$ (e.g, U-shaped curve).

Investment with adjustment costs

It will prove convenient to rewrite the problem with adjustment costs as a Lagrangien

$$\mathcal{L}_t = \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} E_t \left[a_s f(k_s, n_s) - w_s n_s - I_s - \phi\left(I_s, k_s\right) + q_s(I_s + (1-\delta)k_s - k_{s+1})\right]$$

q is the Lagrange multiplier, and so measures the shadow value of relaxing the capital accumulation constraint.

The FOCs read

$$\frac{\partial \mathcal{L}_t}{\partial I_t} = 0 \Leftrightarrow q_t = 1 + \phi_I\left(I_t, k_t\right),$$

$$\frac{\partial \mathcal{L}_t}{\partial k_{t+1}} = 0 \Leftrightarrow q_t = \frac{1}{1+r} E_t \left[a_{t+1} f_k(k_{t+1}, n_{t+1}) - \phi_k\left(I_{t+1}, k_{t+1}\right) + (1-\delta) q_{t+1}\right].$$

Exercise 4.1: Use recursive formulation to derive the optimality condition.

Lagrange multiplier

Using the transversality condition

$$TV: \lim_{T\to\infty}\left(\frac{1}{1+r}\right)^{T} E_t\left[q_{t+T}k_{t+T+1}\right]=0,$$

we can rewrite the FOC above as

$$q_t=\tfrac{1}{1-\delta}\textstyle\sum_{j=1}^{\infty}\left(\tfrac{1-\delta}{1+r}\right)^{j} E_t\left[a_{t+j}f_k(k_{t+j},n_{t+j})-\phi_k\left(I_{t+j},k_{t+j}\right)\right].$$

This equation shows that q is the present discounted value associated to a marginal increase in k, which sums up:

1. Additional marginal productivity: $a_{t+j}f_k(k_{t+j},n_{t+j})$.
2. Reduction in future adjustment costs: $\phi_k\left(I_{t+j},k_{t+j}\right)$.

Q and investment

The FOC

$$\phi_I(I_t, k_t) = q_t - 1$$

implies that investment is positive (negative) when q_t is above (below) one.

Hence the marginal value of capital q is a sufficient statistics for the level of investment. Can it be related to an empirically observable measure?

A commonly advanced statistics is Tobin's average q, $q_t^a = V_t / k_t$, which is simply the ratio of the firm's value (equity+debt) to the replacement value of its capital stock (remember that the price of capital has been normalized to 1).

Marginal and average q

Although Tobin's average q is conceptually different from the marginal q arising from the FOC, $q_t = \partial V_t / \partial k_t$, there is a deep connection between the two measures.

As shown by Hayashi (1982, *Econometrica*) when the production and adjustment cost functions are both homogenous of degree one, marginal and average q are one and the same.

We refer to the paper for a formal proof and, instead, briefly describe the intuition.

Exercise 4.2: Assume that: (i) labor is constant and equal to $\bar{n}$, (ii) adjustment costs are quadratic so that $\phi(I, k) = (I^2/k)\chi/2$ with $\chi > 0$ and (iii) technology of production $f(\cdot)$ is homogenous of degree one. Show that $q = q^a$.

Intuition behind identity between marginal and average q

Let technology be Cobb-Douglas, i.e., $Y = ak^{\alpha}n^{1-\alpha}$ with $\alpha \in (0,1)$.

Then FOC w.r.t. labor implies that $n_t = k_t \left[a_t(1-\alpha)/w_t\right]^{1/\alpha}$, so that n is linearly proportional to k.

Similarly, the FOC $q_t = 1 + \phi_I(I_t, k_t) = 1 + \varphi'(I_t/k_t)$ implies that, for a fixed q, if you double k, the firm will double I.

But then profits $\pi_t = a_t f(k_t, n_t) - w_t n_t - I_t - \varphi(I_t/k_t)\, k_t$ will also be linear in k due to the homogeneity of the production function f.

As a result the value of the firm will be of the form $V(k) = \nu k$, and so marginal and average q are indeed one and the same.

Evidence on Tobin's q and investment

The model predicts that Tobin's q should be a sufficient statistics for investment. Although micro and macro evidence yields a positive correlation, Tobin's q predictive power is far from perfect. For example, cash flows matter too.

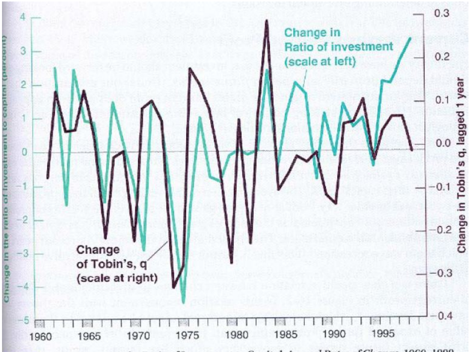


Figure 1 *Tobin's q versus the Ratio of Investment to Capital: Annual Rates of Change, 1960–1999*

Source: Figure 16-1, Blanchard, *Macroeconomics*, chapter 16.

Joint dynamics of K and q

To characterize the model's dynamics, let's assume that the depreciation rate $\delta = 0$, labor is constant and equal to $\bar{n}$, TFP $a_t = a$ is also constant and that adjustment costs are quadratic so that

$$\phi(I, k) = \frac{\chi}{2}\left(\frac{I^2}{k}\right).$$

Then the law of motions of the K and q read

$$\begin{aligned} \Delta K_{t+1} &= (q_t - 1)\frac{k_t}{\chi}, && (1) \\ \Delta q_{t+1} &= r q_t - a f_k\left(k_t\left(1 + \frac{q_t - 1}{\chi}\right), \bar{n}\right) - \frac{(q_{t+1} - 1)^2}{2\chi}. && (2) \end{aligned}$$

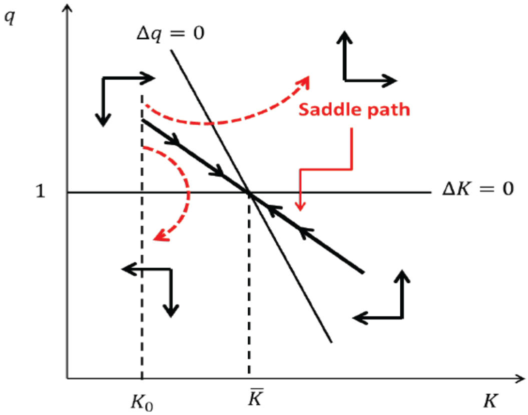
Steady state and linearization

Setting $\Delta K = \Delta q = 0$, we find that at the steady-state $\bar{q} = 1$ and $f_k\left(\bar{k}, \bar{n}\right) = r$.

We linearize (1) and (2) around the steady-state to study the system's dynamics

$$
\begin{aligned}
\Delta K_{t+1} &\approx \left(\frac{q_t - 1}{\chi}\right)\bar{k}, \\
\Delta q_{t+1} &\approx \left(r - \frac{a\bar{k} f_{kk}\left(\bar{k}, \bar{n}\right)}{\chi}\right)(q_t - 1) - a f_{kk}\left(\bar{k}, \bar{n}\right)(k_t - \bar{k}).
\end{aligned}
$$

Phase portrait



Exercise 4.3: Illustrate the effect of an unexpected increase in TFP.

Plant level evidence

The model with convex adjustment costs imply that firms continuously adjust their capital stocks and that they try to avoid big changes. Neither of these predictions is supported by the data (Source: Dums and Donne, *Review of Economic Dynamics*, 1998).

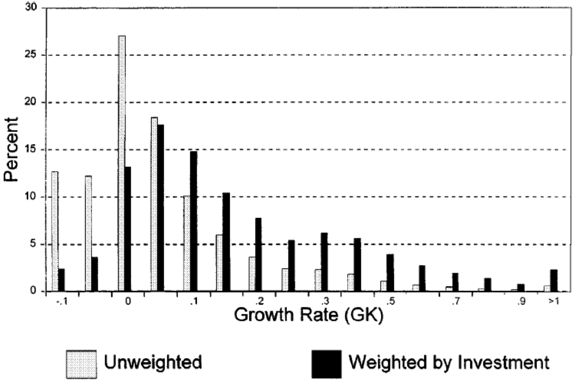


FIG. 1. Capital growth rate (*GK*) distributions: Unweighted and weighted by investment.

Non-convex adjustment costs

To reconcile our model with evidence on lumpy investment, we have to introduce non convex adjustment costs. The simplest specification is linear

$$\phi(I) = \begin{cases} -FI, & \text{if } I < 0, \\ HI, & \text{if } I \geq 0. \end{cases}$$

where F and H are positive constants.

The kink in the adjustment cost function creates an incentive for firms to leave their capital stock unchanged, as documented in the data.

Inaction region

Whenever $1 - F < q_t < 1 + H$, the FOC $\phi_I(I_t, k_t) = q_t - 1$ cannot hold as an equality. Expected returns are too low to justify investing and yet too high to motivate lowering the stock of capital. This generates an inaction range where the firm let its capital depreciate freely.

We simplify the analysis by assuming that capital is the only factor of production and that there is no depreciation, i.e., $\delta = 0$. Then, since $\phi_k(I_{t+j}, k_{t+j}) = 0$, the forward-looking expression for q reads

$$
\begin{aligned}
q_t &= \sum_{j=1}^{\infty} \left(\frac{1}{1+r}\right)^j E_t\left[a_{t+j} f'(k_{t+j})\right] \\
&= \left(\frac{1}{1+r}\right) E_t\left[a_{t+1} f'(k_{t+1}) + q_{t+1}\right],
\end{aligned}
$$

where the second equality follows from the law of iterated expectations.

(s,S) policy

Optimal policy of the firm ensure that $1 - F \leq q_t \leq 1 + H$ otherwise it could increase its value by lowering or increasing its capital stock.

Solving for the optimal policy remains complicated because q_t is forward-looking and thus depends on the optimal level of capital as a function of the realization of the stochastic variable a_t. The problem is circular and has to be solved as a fixed-point.

Restricting the state space for a_t to two realizations $\{a_b, a_g\}$, with $a_b < a_g$, makes it possible to analyze the problem analytically.

We assume that a_t is a Markov Process with symmetric transition matrix, so that a_t remains constant with probability p and changes state with the complementarity probability $1 - p$.

(s,S) policy

We obtain a system of two equations

$$\begin{aligned}(1+r)q_b &= p\left[a_b f'(k_b)+q_b\right]+(1-p)\left[a_g f'(k_g)+q_g\right],\\(1+r)q_g &= p\left[a_g f'(k_g)+q_g\right]+(1-p)\left[a_b f'(k_b)+q_b\right].\end{aligned}$$

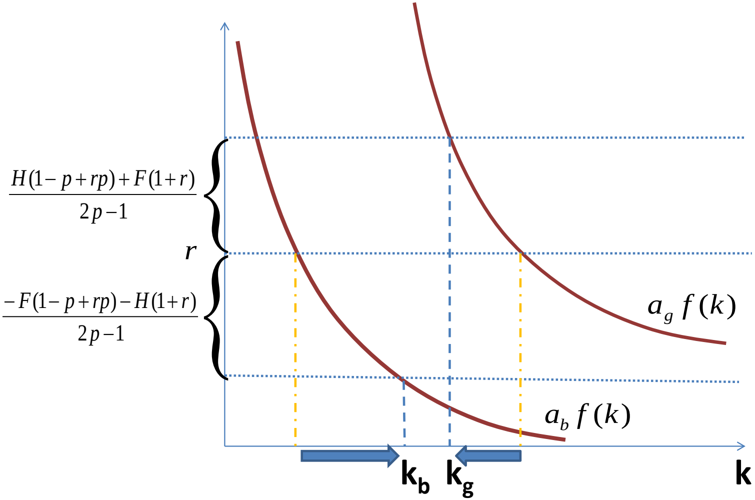
Assuming that changes in a are wide enough that the firm invests in the good state and decumulates in the bad state, we have $q_b = 1-F$ and $q_g = 1+H$. Reinserting these two equalities in the system above and simplifying, we find that

$$a_b f'(k_b) = r - \frac{1}{2p-1}\left[F(1-p+rp)+H(1+r)\right],$$

$$a_g f'(k_g) = r + \frac{1}{2p-1}\left[H(1-p+rp)+F(1+r)\right].$$

Notice that we need some persistence ($p \in (1/2, 1)$) for the states to be well ranked (Why?). Then the Marginal productivity of capital is indeed higher in the good than in the bad state, with the frictionless value r in between.

(s,S) policy



(s,S) policy

Exercise 4.4:

1. Solve the (s,S) problem under the assumption that $f(k) = k - k^2$.
2. Assigning a probability of one-half to each of the two first-order conditions, calculate the average value of the marginal productivity of capital. What is the effect of F and H?
3. Set $r = 0$. What is the effect of F and H on the average stock of capital? Do you think that a similar result holds when $f(k) = k^{(1-\alpha)}/(1-\alpha)$ with $\alpha \in (0, 1)$?
4. Change the model's interpretation and consider that the factor of production, instead of capital, is labor. What are the model's implications about the effect of employment protection?

Additional Reference: Chapters 2 and 3 in *Models for Dynamic Macroeconomics*, Fabio-Cesare Bagliano and Giuseppe Bertola,