SVAR Statistical Identification

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Common Statistical Identifications

Recursive Identification

- A common identification scheme is the recursive identification
- It amounts to imposing a Cholesky decomposition $SS' = \Sigma$ (hence $H = I$)
- It creates a recursive contemporaneous ordering among variables since S^{-1} is triangular
- Variables in the vector Y_t do not depend contemporaneously on the variables ordered after
- Results depend on the particular ordering of the variables

Recursive Identification

Example – Recursive Identification

Consider a bivariate VAR. We have a total of

$$n^2 = 4$$

parameters to fix.

$$\frac{n(n+1)}{2} = 3$$

are pinned down by the orthonormality restrictions so that there are

$$\frac{n(n-1)}{2} = 1$$

free parameters

Recursive Identification

Example – Recursive Identification

Suppose that the theory tells us that:

- Shock 2 has no effect on impact (contemporaneously) on Y_1
- Hence $F_{0,12} = 0$.
- This is the additional restriction that allows us to identify the shocks

In particular we will have the following restrictions (S is the Cholesky factor):

$$HH' = I$$
$$F_{0,12} = S_{11}H_{12} + S_{12}H_{22} = 0$$

Since $S_{12} = 0$ the solution is $H_{11} = H_{22} = 1$ and $H_{12} = H_{21} = 0$

Contemporaneous Restrictions

Sources of identifying restrictions:

- Economic models
- Information delays
- Physical constraints
- Institutional knowledge
- Assumptions about market structure (e.g. no feedback from a small open economy to the rest of the world)
- Extraneous parameter estimates
- ...

Contemporaneous Restrictions

Potential problems with the recursive identification:

- Recursive identification requires **strong identifying assumptions about the timing** of responses of the variables in the VAR
- The ordering is not unique: for a VAR with n variables, there are $n!$ orderings
- Often there is no reason for the model to be recursive: **contemporaneous effects** on all of the variables!

Long Run Restrictions

- An identification scheme based on zero long run restrictions is a scheme which imposes restrictions on the matrix $F(1) = F_0 + F_1 + F_2 + \ldots$, the matrix of the long run coefficients

Example – Long Run Restrictions

Let us consider a bivariate VAR. Suppose that the theory tells us that shock 2 does not affect Y_1 in the long run, i.e. $F_{12}(1) = 0$. This is the additional restriction that allows us to identify the shocks. In particular we will have the following restrictions:

$$
\begin{aligned}
HH' &= I \\
F_{12}(1) = D_{11}(1)H_{12} + D_{12}(1)H_{22} &= 0
\end{aligned}
$$

where $D(1) = C(1)S$ represents the long run effects of the Cholesky shocks

Long Run Restrictions

Remark: When LR restrictions can be thought of as producing a total impact matrix F(1) estimation becomes particularly easy. Using the relation:

$$F(1) = (I_n - A_1 - \cdots - A_p)^{-1} SH$$

and observing that

$$F(1)F(1)' = (I_n - A_1 - \cdots - A_p)^{-1} \Sigma (I_n - A_1' - \cdots - A_p')^{-1}$$

the matrix SH can be estimated by premultiplying a Choleski decomposition of

$$(I_n - A_1 - \cdots - A_p)^{-1} \Sigma (I_n - A_1' - \cdots - A_p')^{-1}$$

by $(I_n - A_1 - \cdots - A_p)$ This procedure works only if the VAR is stable and the process is stationary

Long Run Restrictions

Problems with Long-Run Restrictions:

- They require an accurate estimate of the impulse responses at the infinite horizon
- Numerical estimates of the responses in VAR models identified by long-run restrictions are identified only up to their sign
- The system has to be estimated with stationary variables only
- Results are sensitive to the assumptions about the stationarity of the variables of interest, i.e. whether the variables of interest are entered in levels or differences

Partial Identification

- In many cases we might be interested in identifying just a single shock and not all the n shocks
- Since the shock are orthogonal we can also partially identify the model, i.e. fix just one (or a subset of) column of H
- Fix $n-1$ elements of H: all but one elements of a column of the identifying matrix. The additional restriction is provided by the norm of the vector equal one

Signs Restrictions

- The previous two examples yield just identification in the sense that the shock is uniquely identified, there exists a unique matrix H yielding the structural shocks (model identification)
- Sign identification is based on qualitative restriction involving the sign of some shocks on some variables
- In this case we will have sets of consistent impulse response functions (model set identification)

Signs Restrictions

Example – Signs Restrictions

Consider a bivariate VAR. Suppose that the theory tells us that shock 2, which is the interesting one, has a positive effect on Y_1 for k periods after the shock, i.e.

$$F_j^{12} > 0 \qquad j = 0, 1, ..., k$$

We have the following restrictions:

$$\begin{aligned} HH' &= I \\ S_{11}H_{12} + S_{12}H_{22} &> 0 \\ D_{j,12}H_{12} + D_{j,22}H_{22} &> 0 \quad j = 1, ..., k \end{aligned}$$

where $D_j = C_j S$ represents the effects at horizon j

Signs Restrictions

- In a classical statistics approach this delivers not exact identification since there can be many H consistent with such a restriction
- For each parameter of the impulse response functions we will have an admissible set of values
- Increasing the number of restrictions can be helpful in reducing the number of H consistent with such restrictions

Rotation Matrices

- A useful way to parametrise the matrix H in order to include orthonormality restrictions is using rotation matrices.
- Let us consider the bivariate case, a rotation matrix in this case is the unity matrix

$$H = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

- Note that such a matrix incorporates the orthonormality conditions
- The parameter θ will be found by imposing the additional economic restriction

Rotation Matrices – Givens matrices

- For $n = 3$ the rotation matrix can be found as the product of the following three matrices

$$H_1 = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad H_2 = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}$$

$$H_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & \sin\theta_3 \\ 0 & -\sin\theta_3 & \cos\theta_3 \end{pmatrix}$$

- In general the rotation matrix will be found as the product of $\frac{n(n-1)}{2}$ rotation matrices, also called **Givens matrices**

Sign Restrictions

Example – Signs Restrictions

Suppose that $n = 2$ and the restriction we want to impose is that the effect of the first shock on the second variable has a positive sign, i.e.

$$S_{21}H_{11} + S_{22}H_{21} > 0$$

Using the parametrisation seen before the restriction becomes

$$S_{21}\cos(\theta) - S_{22}\sin(\theta) > 0$$

$$\implies tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} < \frac{S_{21}}{S_{22}}$$

If $S_{21} = 0.5$ and $S_{22} = 1$ then <u>all the impulse response functions obtained with $\theta < arctan(0.5)$ satisfy the restriction and should be kept</u>

Partial Identification

Example – Signs Restrictions

Suppose $n = 3$. We want to identify a single shock assuming that it has

(i) no effects on the first variable on impact

(ii) a positive effect on the second variable

(iii) negative on the third variable

Notice that the first column of the product of the rotation matrices

$$\begin{pmatrix} \cos\theta_1 \cos\theta_2 \\ -\sin\theta_1 \cos\theta_2 \\ -\sin\theta_2 \end{pmatrix}$$

Partial Identification

Example – Signs Restrictions

We have that the impact effects of the first shock are given by

$$\begin{pmatrix} S_{11} & 0 & 0 \\ S_{21} & S_{22} & 0 \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \begin{pmatrix} \cos\theta_1 \cos\theta_2 \\ -\sin\theta_1 \cos\theta_2 \\ -\sin\theta_2 \end{pmatrix}$$

To implement the first restriction we can set $\theta_1 = \pi/2$, i.e. $\cos\theta_1 = 0$. This implies that

$$\begin{pmatrix} S_{11} & 0 & 0 \\ S_{21} & S_{22} & 0 \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \begin{pmatrix} 0 \\ -\cos\theta_2 \\ -\sin\theta_2 \end{pmatrix}$$

Partial Identification

Example – Signs Restrictions

The second restriction implies that

$$-S_{22}\cos\theta_2 > 0$$

and the third

$$-S_{32}\cos\theta_2 - S_{33}\sin\theta_2 < 0$$

All the values of θ_2 satisfying the two restrictions yield impulse response functions consistent with the identification scheme

Efficient Sign Restrictions

Rubio-Ramirez, Waggoner and Zha (2010)

Let S denote the lower triangular Cholesky decomposition that satisfies $SS' = \Sigma$

1. Draw an $n \times n$ matrix X of $NID(0, I_n)$ random variables. Derive the QR decomposition of X such that $X = QR$ and Q is orthogonal matrix $QQ' = I_n$, while R is an upper triangular matrix

2. Let $H = Q$. Compute impulse responses using the orthogonalisation $F_0 = SH$. If all implied impulse response functions satisfy the identifying restrictions, retain H. Otherwise discard H

Efficient Sign Restrictions

Rubio-Ramirez, Waggoner and Zha (2010)

3. Repeat the first two steps a large number of times, recording each H that satisfies the restrictions (and the corresponding impulse response functions)

The resulting set F_0 in conjunction with the reduced-form estimates characterises the set of admissible structural VAR models

Remark: The fraction of the initial candidate models that satisfy the identifying restriction may be viewed as an indicator of how informative the identifying restrictions are about the structural parameters

A Critique of Efficient Sign Restrictions

Baumeister, Hamilton (2015)

The RWZ algorithm can be viewed as generating draws from a prior distribution for $B_0^{-1} = SH$ conditional on Σ

- The first column of Q is simply the first column of X normalised to have unit length

$$\begin{bmatrix} q_{11} \\ \vdots \\ q_{n1} \end{bmatrix} = \begin{bmatrix} \frac{x_{11}}{\sqrt{x_{11}^2 + \ldots x_{n1}^2}} \\ \vdots \\ \frac{x_{n1}}{\sqrt{x_{11}^2 + \ldots x_{n1}^2}} \end{bmatrix}$$

- Each element of the vector has a marginal density given by

$$p(q_{i1}) = \begin{cases} \frac{\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)} \left(1 - q_{i1}^2\right)^{(n-3)/2} & \text{if } q_{i1} \in [-1, 1] \ , \\ 0 & \text{otherwise} \end{cases}$$

A Critique of Efficient Sign Restrictions

Baumeister, Hamilton (2015)

- This implies a prior distribution for the effect of a 1-standard-deviation increase in structural shock number 1 on variable number 1 that is characterised by the random variable $f_{11} = \sqrt{\Sigma_{11}} q_{11}$ for Σ_{11} the element (1,1) in Σ
- More in general one finds

$$p(f_{ij}|\Sigma) = \begin{cases} \frac{\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)} \frac{1}{\sqrt{\Sigma_{ii}}} \left(1 - \frac{h_{ij}^2}{\Sigma_{ii}}\right)^{(n-3)/2} & \text{if } h_{ij} \in [-\sqrt{\Sigma_{ii}}, \sqrt{\Sigma_{ii}}] \\ 0 & \text{otherwise} \end{cases},$$

A Critique of Efficient Sign Restrictions

Baumeister, Hamilton (2015)

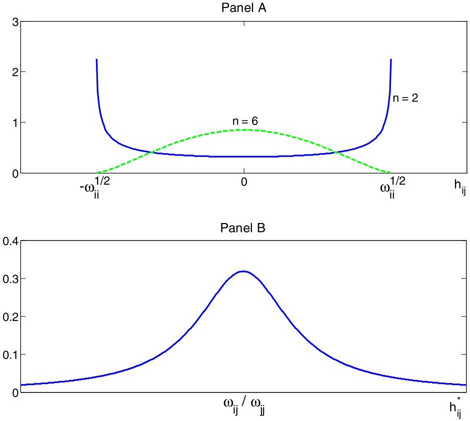


FIGURE 1.—Prior densities for the initial effect of shocks implicit in the traditional approach to sign-restricted VAR. Panel A: Response of variable i to 1-standard-deviation increase of any structural shock when the number of variables in the VAR is 2 (solid) or 6 (dashed). Panel B: Response of variable i to a structural shock that increases variable j by one unit.

Identification by Heteroskedasticity

Let's consider a bivariate VAR's residuals

$$\begin{pmatrix} e_t^a \\ e_t^b \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} u_t^1 \\ u_t^2 \end{pmatrix}$$

Under unconditional homoskedasticity

$$\Sigma = \frac{1}{(1-\alpha\beta)^2} \begin{pmatrix} \beta^2\sigma_2^2 + \sigma_1^2 & \beta\sigma_2^2 + \alpha\sigma_1^2 \\ \cdot & \sigma_2^2 + \alpha^2\sigma_1^2 \end{pmatrix}$$

We have three moments in four unknowns $(\alpha, \beta, \sigma_2^2, \sigma_1^2)$, hence we need some assumption (e.g. $\alpha = 0$)

Identification by Heteroskedasticity

Let's consider that case of heteroskedasticity.

Suppose that there are two regimes $r = \{1, 2\}$ with different variances. Also suppose that only the (relative) variance of the two structural shocks changes in the two regimes, while parameters α, β remains unchanged

$$\Sigma_r = \frac{1}{(1-\alpha\beta)^2} \begin{pmatrix} \beta^2\sigma^2_{2,r} + \sigma^2_{1,r} & \beta\sigma^2_{2,r} + \alpha\sigma^2_{1,r} \\ \cdot & \sigma^2_{2,r} + \alpha^2\sigma^2_{1,r} \end{pmatrix}$$

Six moments in six unknowns $(\alpha, \beta, \sigma^2_{1,1}, \sigma^2_{2,1}, \sigma^2_{1,2}, \sigma^2_{2,2})$, hence the system can be identified

Identification by Heteroskedasticity

- Changes in the conditional or unconditional volatility of the VAR errors (and hence of the observed variables) can be used to assist in the identification of structural shocks

- Rigobon (2003) applies this idea to identify demand and supply shocks in South American bond markets

4. Monetary Policy Shocks (I)

Monetary Policy Shocks

What is a **Monetary Policy Shock**?

- Monetary policy shocks is the unexpected part of the equation for the monetary policy instrument (i_t)

$$i_t = f(\mathcal{I}_t) + u_t^{mp}$$

- $f(\mathcal{I}_t)$: systematic response of the monetary policy to economic conditions
- $\mathcal{I}_t$: central bank's information set at time t
- u_t^{mp}: monetary policy shock

Question: Why is the Central Bank injecting volatility into the Economy?

Recursive Identification of MP Shocks

Monetary Policy Shocks

Christiano, Eichenbaum and Evans (1999, 2005)

- The ‘classic’ way to identify monetary policy shock is through zero contemporaneous restrictions (**recursive identification**)
- The $\mathbf{Y}_t$ vector of endogenous variables in a standard monetary VAR includes output, inflation and the federal funds rate, together with other macro variables

$$\mathbf{Y}'_t = [\mathbf{Y}'_{1t} \;\; i_t \;\; \mathbf{Y}'_{2t}]$$

Monetary Policy Shocks

Christiano, Eichenbaum and Evans (1999)

- The vector $\mathbf{Y}_{1t}$ is composed of the variables whose time t values are contained in $\mathcal{I}_t$ and that are assumed not to respond contemporaneously to a monetary policy shock
- i_t is a measure of the policy rate
- The vector $\mathbf{Y}_{2t}$ consists of the time t values of all the other variables in $\mathbf{Y}_t$.

Monetary Policy Shocks

Christiano, Eichenbaum and Evans (1999, 2005)

Example

- $\mathbf{Y}_{1t}$: real gross domestic product, real consumption, GDP deflator, real investment, real wage, and labor productivity.
- i_t: Federal Funds Rate
- $\mathbf{Y}_{2t}$: real profits and the growth rate of M2
- With one exception (the growth rate of money), all the variables in $\mathbf{Y}_t$ are included in levels

Monetary Policy Shocks

Christiano, Eichenbaum and Evans (1999)

- The ordering of the variables in embodies two key identifying assumptions
  - Variables in $\mathbf{Y}_{1t}$ do not respond contemporaneously to a monetary policy shock
  - The monetary authority's time t information set consists of current and lagged values of the variables in $\mathbf{Y}_{1t}$ and only past values of the variables in $\mathbf{Y}_{2t}$

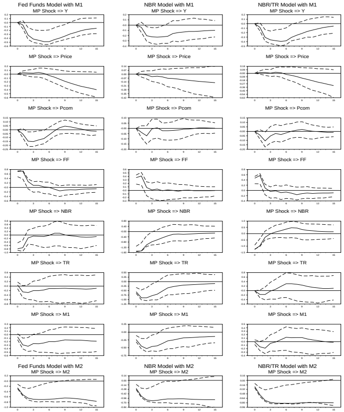
  Remark: Results are invariant to changes in the ordering of $\mathbf{Y}_{1t}$ or $\mathbf{Y}_{2t}$

- These two restrictions are not sufficient to identify all the shocks but are sufficient to identify the monetary policy shock

- A simple way to implement the restrictions is to take simply the Cholesky decomposition of the VAR residuals' variance covariance matrix

Monetary Policy Shocks

Christiano, Eichenbaum and Evans (1999)



Monetary Policy Shocks

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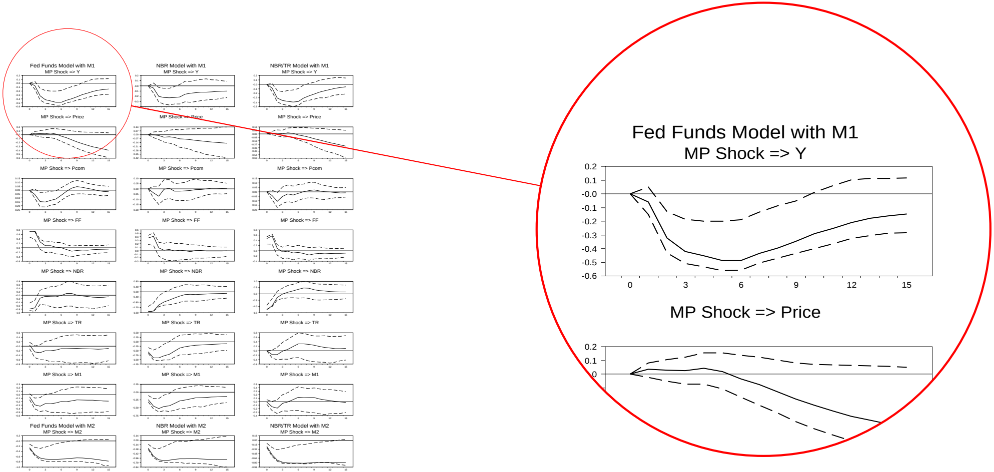

Monetary Policy Shocks

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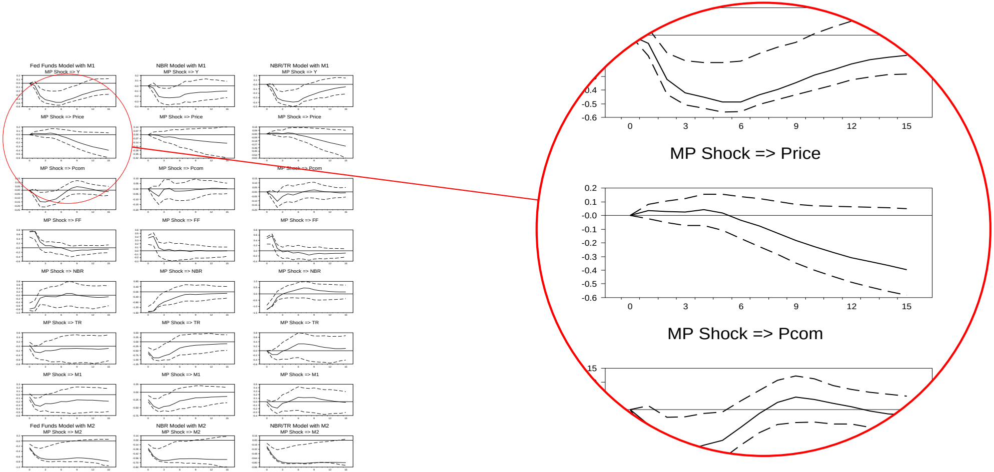

Monetary Policy Shocks

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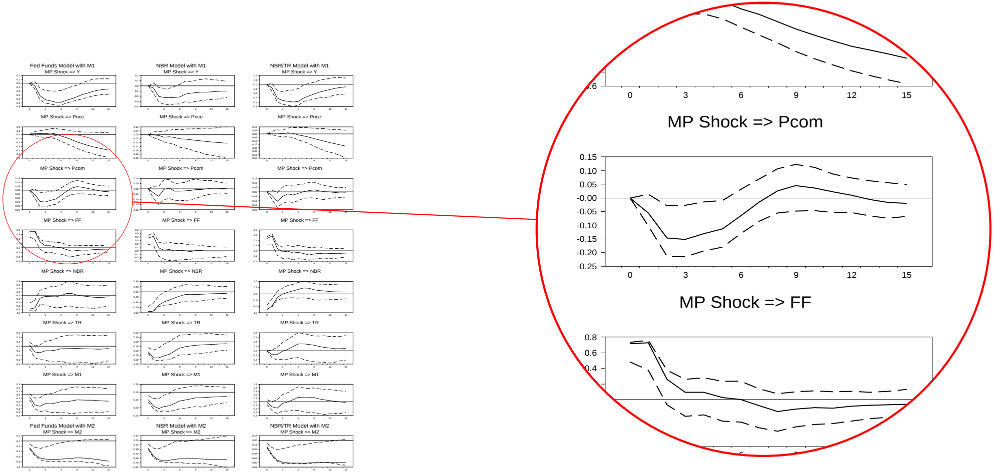

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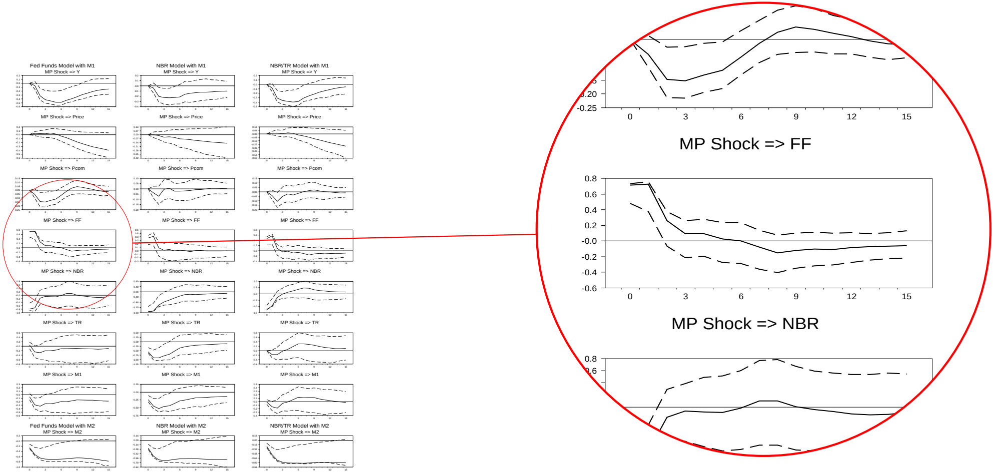

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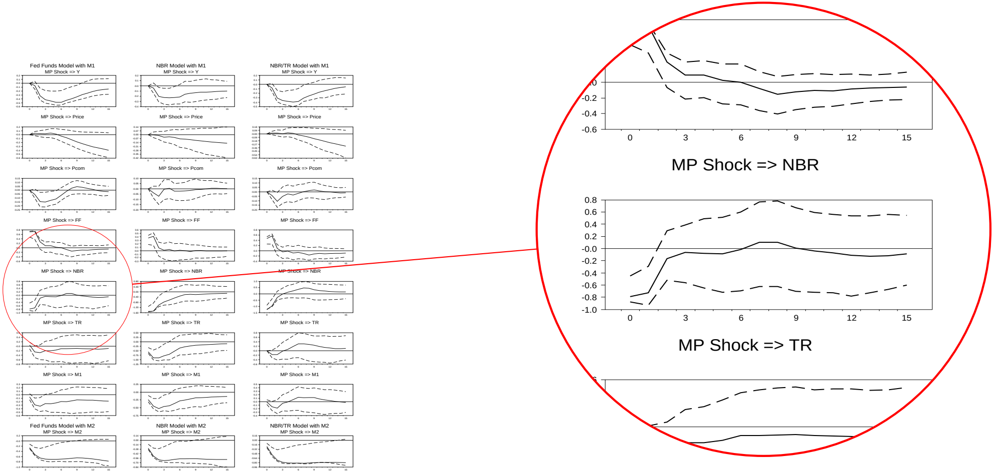

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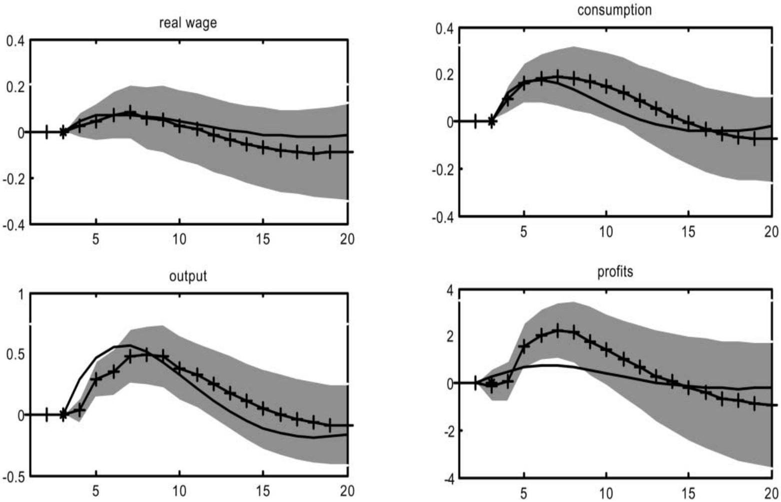
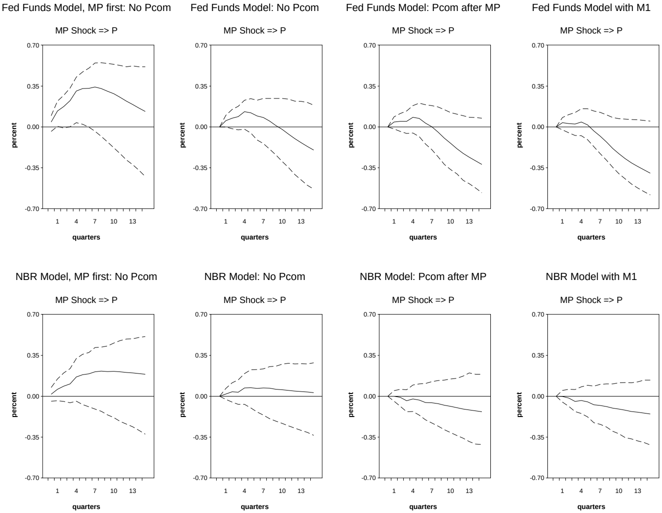


Monetary Policy Shocks

Price puzzle – Christiano, Eichenbaum and Evans (1999)



Monetary Policy Shocks

Price Puzzle – Sims (1992)

- Sims (1992) conjectured that prices puzzles were due to VAR specifications that did not include information about future inflation available to the Fed
- The puzzle is due to confounding policy shocks with with non-policy news shocks that signal future inflation
- Including commodity prices in the VARs, a forward looking variable, helps solving the puzzle

Sign Restrictions for MP Shocks

Monetary Policy Shocks

Uhlig (2005)

- Uhlig (2005) proposes an ‘agnostic identification’ using sign restrictions instead of zero restrictions
- Monetary policy shocks identified by using restrictions common to several economic models

A contractionary monetary policy shock (assumptions):

1. Prices do not increase for k periods after the shock
2. Money or monetary aggregates (i.e. reserves) do not rise for k periods after the shock
3. Short term interest stays above pre-shock level for k periods after the shock

Monetary Policy Shocks

Uhlig (2005)

Example

We order the variables in vector Y_t as follows:

- GDP
- Inflation
- The interest rate
- Money growth

using the notation of the previous slides, the restrictions imply

$$F_k^{i1} < 0 \qquad i = 2, 4$$

and

$$F_k^{31} > 0$$

Monetary Policy Shocks

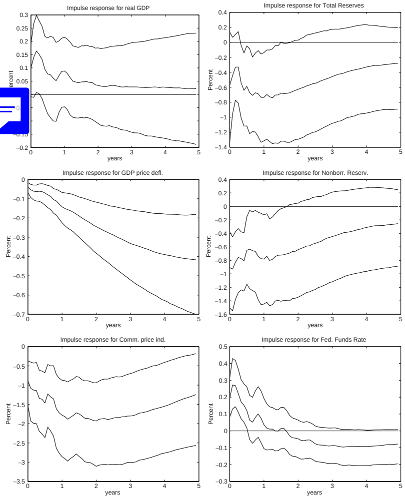


Figure: Contractionary MP Shock $k = 5$, Uhlig (2005)