Recursive Methods

Pierre Rouillard

1. Dynamic optimization in discrete time

> x_t state variable at time t
> x_{t+1} determined through at t $f(x_t, a_t)$ with a_t control/action variable (consumption in classic model)

DEFINITION 1. ***Sequential problem***

$$\textbf{\textit{(SP)}}: \quad V^*(x_0) = \max_{\{x_t\}_{t\geqslant 0}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \qquad x_t \in X, x_{t+1} \in \Gamma(x_t)$$

DEFINITION 2. ***Functional equation***

$$\textbf{\textit{(FE)}}: \quad V(x) = \max_{x' \in \Gamma(x)} \{F(x, x') + \beta.V(x')\}$$

$\Gamma(x_t)$

ASSUMPTION 1. *For all $x \in X$: $\Gamma(x)$ is non empty*

ASSUMPTION 2. *For all $x_0 \in X$ and $\underline{x} \in \Pi(x_0)$: $\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1})$ exists*

> THEOREM 1. *Bellman Equation (1)*
>
> - *Assumption 1 & 2 hold*
> - *$V^*(x)$ solution of **(SP)** is well defined for all $x \in X$*
>
> $$\Longrightarrow V^*(x) = \max_{x' \in \Gamma(x)} \{F(x, x') + \beta.V^*(x')\}$$

Bellman equation Theorem.1 : V^* solves (SP) $\Rightarrow$ V^* solves (FE)

> THEOREM 2. *Bellman Equation (2)*
>
> - *Assumption 1 & 2 hold*
> - *$V^*(x)$ solution of **(SP)** is well defined for all $x \in X$*
> - *$\{x_t^*\}_{t\geqslant 0}$ optimal path for x_0, $\{x_t^*\} \in \Pi(x_0)$*
>
> $$\Longrightarrow V^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta.V^*(x_{t+1}^*) \qquad \forall t \geqslant 0$$

Bellman equation Theorem.2 : any optimal path $\{x_t^*\}_{t\geqslant 0}$ is generated by the optimal policy correspondence **but** reverse not always true!

$$G^*(x) = \underset{x'\in\Gamma(x)}{\operatorname{argmax}}\{F(x,x') + \beta . V^*(x')\} \qquad \Rightarrow x_{t+1}^* \in G^*(x_t) \quad \forall t \geqslant 0$$