Recursive Methods

Lecture 2: Analyzing the Bellman Equation

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Fixed point formulation

We have shown that (under some transversality conditions) the original (SP) objective has a recursive formulation.

Why is this progress?

- Breaks down original problem into series of 2-period problems whose optimality conditions are intuitive and economically meaningful.
- Fixed point analysis allows us to prove existence, uniqueness and to establish properties of optimal policy.
- Recursive problem is "easy" to solve with numerical methods.

Fixed point formulation

Think of the LHS of

$$(FE) : V(x) = \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta V(x') \right\}.$$

as a functional $T : \mathcal{C}(X) \to \mathcal{C}(X)$ where $\mathcal{C}(X)$ is the space of bounded continuous function $f : X \to \mathbb{R}$.

Then (FE) is equivalent to

$$Tv(x) = \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta v(x') \right\}.$$

Our problem therefore boils down to finding a fixed-point of T.

Example

Exercise 2.1: Consider the estate planning problem

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t U(c_t)$$

s.t. $a_{t+1} = (1+r)a_t - c_t$. Assume that $U(c) = log(c)$. Write the associated-fixed point problem. Then show that the value function is of the form $V(a) = K + D\ log(a)$ with $D = 1/(1-\beta)$.

We are now going to prove this result in a convoluted manner which outlines the general approach to problems that cannot be solved analytically.

Example

1. Contraction: Define the function $T(D) = 1 + \beta D$. Then use the fact that, $|T(D'') - T(D')| = \beta|D'' - D'|$, to show that if T has a fixed-point, it is unique.

2. Convergence: Define the sequence $D_n = 1 + \beta D_{n-1}$. Show that D_n is a Cauchy sequence, so that $\lim_{n\to\infty} D_n$ exists.

3. Fixed-point: Show that if $D = \lim_{n\to\infty} D_n$, D is a fixed-point of T.

Steps 1 and 2 establish uniqueness and existence. Step 3 provides a way to compute the fixed-point.

We now generalize this approach.

Metric Spaces

A norm $||\cdot||$ is a real-valued function on $\mathcal{C}$ which captures the notion of distance between functions. It satisfies the following properties

1. Positive definite $||y|| > 0$ if $y \neq 0$,
2. Homogeneous $||\lambda y|| = |\lambda| \cdot ||y||$ for all $\lambda \in \mathbb{R}, y \in V$,
3. Triangle Inequality $||y+z|| \leq ||y|| + ||z||$.

The norm allows us to define a metric $d(y,z) \equiv ||y-z||$.

On the space $\mathcal{C}(X)$, the most common norm is

$$||y|| \equiv \max_{\{x \in X\}} |y(x)|,$$

where $|\cdot|$ is the standard Euclidean norm on $\mathbb{R}^n$.

Complete Metric Spaces

Definition 2.1: A sequence $\{x_n\}_{n=0}^{\infty}$ in a vector space S converges to $x \in S$, if for each $\varepsilon > 0$, there exists N_ε such that $||x_n - x|| < \varepsilon$ for all $n \geq N_\varepsilon$.

Definition 2.2: A sequence $\{x_n\}_{n=0}^{\infty}$ in S is a Cauchy sequence if for each $\varepsilon > 0$, there exists N_ε such that $||x_n - x_m|| < \varepsilon$ for all $n, m \geq N_\varepsilon$.

Definition 2.3: A metric space $(S, ||\cdot||)$ is complete if every Cauchy sequence in S converges to an element in S.

Exercise 2.2: Prove that the set $\mathcal{C}(X)$ of bounded continuous functions $f : X \to \mathbb{R}$ equipped with the sup norm $||y|| \equiv \max_{\{x \in X\}} |y(x)|$ is a complete normed vector space.

Contraction Mapping

Definition 2.4: Let $(S, ||\cdot||)$ be a metric space and $T : S \to S$ be a function mapping S into itself. T is a contraction mapping if for some $\beta \in (0, 1)$, $||Tx - Ty|| \leq \beta||x - y||$, for all $x, y \in S$.

Theorem 2.1. (Contraction Mapping Theorem)

If $(S, ||\cdot||)$ is a complete metric space and $T : S \to S$ is a contraction mapping, then T has exactly one fixed point v in S. Furthermore, for any $v_0 \in S$, $||T^n v_0 - v|| \leq \beta^n ||v_0 - v||$ where $T^{n+1}(v) = T(T^n(v))$ and $n = 0, 1, 2, ...$

Proof of Contraction Mapping Theorem

PROOF of Theorem 2.1. <u>**(Contraction Mapping Theorem)**</u>**:**

Step 1. Take any $v_0 \in S$ and let $v_{n+1} \equiv Tv_n$. Then

$$||v_{n+1} - v_n|| = ||Tv_n - Tv_{n-1}|| \leq \beta||v_n - v_{n-1}|| \leq \beta^n||v_1 - v_0||$$

and so, for $m > n$,

$$\begin{aligned} ||v_m - v_n|| &\leq ||v_m - v_{m-1}|| + ||v_{m-1} - v_{m-2}|| + \ldots + ||v_{n+1} - v_n|| \\ &\leq (\beta^{m-1} + \beta^{m-2} + \ldots + \beta^n)||v_1 - v_0|| \\ &\leq \beta^n(\beta^{m-n-1} + \beta^{m-n-2} + \ldots + 1)||v_1 - v_0|| \leq \frac{\beta^n}{1-\beta}||v_1 - v_0||. \end{aligned}$$

Thus $\{v_n\}$ is a Cauchy sequence and $v_n \to v$.

Step 2. To show that $v = Tv$ notice that

$$||Tv - v|| \leq ||Tv - v_n|| + ||v_n - v|| \leq \beta||v - v_{n-1}|| + ||v_n - v|| \to 0.$$

Step 3. Finally, we proceed by contradiction to prove that v is unique. Assume that there are two fixed points v^1 and v^2. Then

$$0 \leq a = ||v^1 - v^2|| = ||Tv^1 - Tv^2|| \leq \beta||v^1 - v^2|| = \beta a,$$

which is only possible if $a = 0$, i.e., if $v^1 = v^2$.

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Contraction Mapping

Theorem 2.2. (Blackwell's Sufficient Condition)

Let $X \subseteq \mathbb{R}^n$, $T : \mathcal{C}(X) \to \mathbb{R}$ is a contraction mapping if it satisfies:

1. Monotonicity: $f(x) \leq g(x)$ for all $x \in X$ and $f, g \in \mathcal{C}(X)$, implies $Tf(x) \leq Tg(x)$, for all $x \in X$.
2. Discounting: There exists some $\beta \in (0, 1)$ such that $T(f + a)(x) \leq Tf(x) + \beta a$ for all $f \in \mathcal{C}(X)$, $a \geq 0$, $x \in X$.

PROOF: See theorem 3.3 in SLP. ■

Contraction Mapping

We now apply the contraction mapping theorem to our functional equation

$$Tv(x) = \max_{x' \in \Gamma(x)} \left\{ F(x, x') + \beta v(x') \right\}.$$

We need to show that:

1. T maps the set of continuous and bounded functions into itself.
2. T is a contraction.

We first prove 2 assuming 1, and then establish 1.

Contraction Mapping

Theorem 2.3.

Let $Tv(x) = \max_{x' \in \Gamma(x)} \{F(x, x') + \beta v(x')\}$. T satisfies Balckwell's sufficient conditions. => T is a contraction mapping

PROOF:

1. Monotonicity: For $f \geq v$

$$
\begin{aligned}
Tv(x) &= \max_{x' \in \Gamma(x)} \left\{F(x, x') + \beta v(x')\right\} = F(x, g(x)) + \beta v(g(x)) \\
&\leq F(x, g(x)) + \beta f(g(x)) \leq \max_{x' \in \Gamma(x)} \left\{F(x, x') + \beta f(x')\right\} = Tf(x).
\end{aligned}
$$

2. Discounting: For $a > 0$

$$
\begin{aligned}
T(v + a)(x) &= \max_{x' \in \Gamma(x)} \left\{F(x, x') + \beta(v(x') + a)\right\} \\
&= \max_{x' \in \Gamma(x)} \left\{F(x, x') + \beta v(x')\right\} + \beta a = Tv(x) + \beta a.
\end{aligned}
$$

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Theorem of the Maximum

We still have to identify the restrictions on the correspondence Γ and on the return function F under which T maps the set of continuous and bounded functions into itself.

Our optimization problem is of the form

$$h(x) = \max_{x' \in \Gamma(x)} f(x, x'). \tag{1}$$

The max is attained when $f(x, \cdot)$ is continuous in x' and $\Gamma(x)$ is nonempty and compact. Then the function $h(x)$ is well defined as is the policy correspondence

$$G(x) = \left\{ x' \in \Gamma(x) : f(x, x') = h(x) \right\}. \tag{2}$$

Theorem of the Maximum

Theorem 2.4. (Theorem of the Maximum)

Let $X \subseteq \mathbb{R}^n$ and $f : X \times X \to \mathbb{R}$ be a continuous function, and let $\Gamma : X \to X$ be a compact valued and continuous correspondence. Then the function h defined in (1) is continuous, and the correspondence G defined in (2) is nonempty, compact valued and upper hemi-continuous.

PROOF: See theorem 3.6 in SLP. ■

Corollary 2.1.: If Γ is convex-valued and f is strictly concave in y, then the policy correspondence G is single-valued and continuous.

Summary

We are now in a position to study our original problem. If we assume that:

Assumption 2.1. X is a convex subset of $\mathbb{R}^n$, and Γ is a non-empty, continuous and compact-valued correspondence.

Assumption 2.2. $F : X \times X \to \mathbb{R}$ is continuous and bounded.

Then we can combine the theorems above to study (FE) in a similar way as the basic estate planning problem:

1. Theorem of the maximum shows that T maps $\mathcal{C}(X)$ into itself;
2. Then Blackwell's Sufficient Conditions shows that T is a contraction;
3. Contraction Mapping Theorem shows that, for any initial guess v_0, T generates a Cauchy sequence of functions v_n;
4. Since $\mathcal{C}(X)$ is a complete metric space, v_n converges to the unique value function v.

Additional Assumptions

To further characterize the value and policy functions, we have to impose more stringent assumptions on the fundamentals.

Assumption 2.3. F is strictly concave and, for each x', $F(\cdot, x')$ is strictly increasing in each of its first arguments.

Assumption 2.4. Γ is convex and monotone in the sense that $x \leq y$ implies $\Gamma(x) \subseteq \Gamma(y)$.

Concavity of Value Function

Theorem 2.5.

When assumptions 2.1-2.4 hold, the value function v is strictly increasing.

PROOF: We prove a stronger version, namely Tf is increasing if f is non-decreasing. Pick $x_1, x_2 \in X$ with $x_2 > x_1$. The optimal policy $g(x_1) \in \Gamma(x_2)$ by monotonicity of Γ, so

$$Tf(x_2) = \max_{x' \in \Gamma(x_2)} \left\{ F(x_2, x') + \beta f(x') \right\} \geq F(x_2, g(x_1)) + \beta f(g(x_1)) > F(x_1, g(x_1)) + \beta f(g(x_1)) = Tf(x_1),$$

where the last inequality holds because F is increasing. Since v is the limit of $T^n f_0$, and the space of non-decreasing function is the closure of the space of increasing functions, v must be non-decreasing. Furthermore, since $v = Tv$, the equation above implies that V is actually increasing.

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Exercise 2.3: Use an inductive argument similar to the one in the proof of Theorem 2.5. to establish that, when assumptions 2.1-2.4 hold, v is concave.

Differentiability of Value Function

It is often insightful to look at the FOC of the problem, in our case

$$F_{x'}(x, x') + \beta V'(x') = 0.$$

To do so, however, we first need to establish that the value function is indeed differentiable.

Example of non differentiable value function: Two period problem

$$v(x) = \max_{y \in [0,1]} y^2 - xy.$$

Then $v(x) = 1 - min(x, 1)$ and the value function is not differentiable at 1.

This example suggests that non-differentiability is likely to originate from non-concavity.

Differentiability of Value Function

The approach used to prove concavity does not work because the space of differentiable functions is not closed.

We use instead the notion of subgradient:

1. If a function $f : X \to \mathbb{R}$ is concave, with X a convex subset of $\mathbb{R}^n$, it admits a subgradient $p \in \mathbb{R}^n$ so that

$$f(x) - f(x_0) \leq p \cdot (x - x_0), \text{for all } x \in X.$$

2. If f is differentiable, then p is unique and is the gradient of f at x_0.

3. The converse of 2 holds, that is if f is concave with a unique subgradient, it is differentiable (See Rockafellar, 1970, Th.25.1 for a proof).

Differentiability of Value Function

Theorem 2.5. (Benveniste and Sheinkman)

Suppose that F is differentiable in x and that assumptions 2.1-2.4 hold. If $x_0 \in int\ X$ and $g(x_0) \in int\ \Gamma(x_0)$, then v is differentiable at x_0 and $\bigtriangledown v(x_0) = \bigtriangledown F_x(x_0, g(x_0))$.

PROOF: Consider the following lower approximation of v in the neighborhood of x_0

$$w(x) = F(x, g(x_0)) + \beta v(g(x_0)).$$

Since F is differentiable so is w. Given continuity of Γ and the fact that $g(x_0) \in int\Gamma(x_0)$, there exists a neighborhood D of x_0 such that $g(x_0) \in \Gamma(x)$ for all $x \in D$. By definition of v, we have

$$w(x) \leq v(x) \text{ for all } x \in D.$$

Since v is concave, it has a subgradient p and so

$$w(x) - w(x_0) \leq v(x) - v(x_0) \leq p \cdot (x - x_0) \text{ for all } x \in D.$$

But remember that w is differentiable. Hence $p = \bigtriangledown w(x_0)$ and the subgradient is unique, which by point 3 in the previous slide, proves that v is differentiable.

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Conclusion

To summarize, we have identified in this lecture the conditions under which

1. The Functional Equation is the unique solution of a fixed point problem;
2. The value function can be approximated by an iterative procedure;
3. The value function is concave and differentiable.