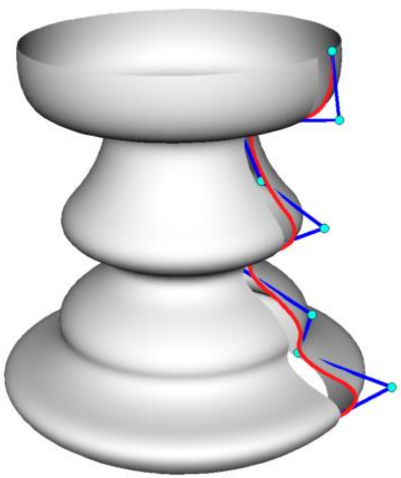
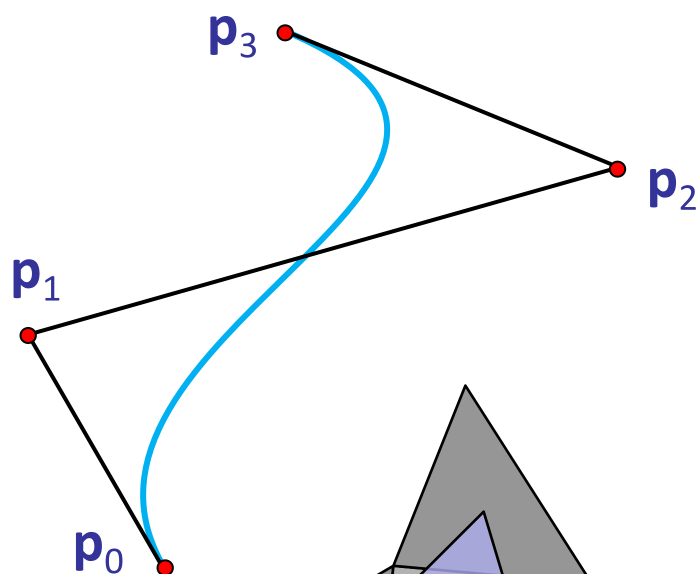
Introduction to Visualization and Computer Graphics
DH2320

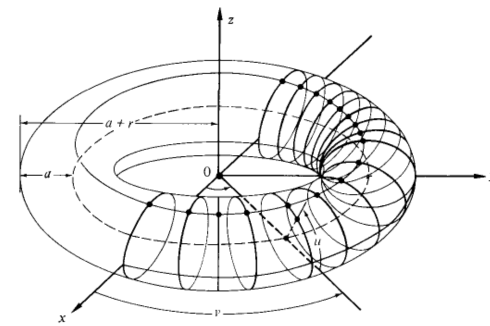
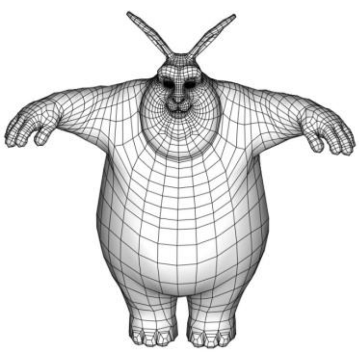
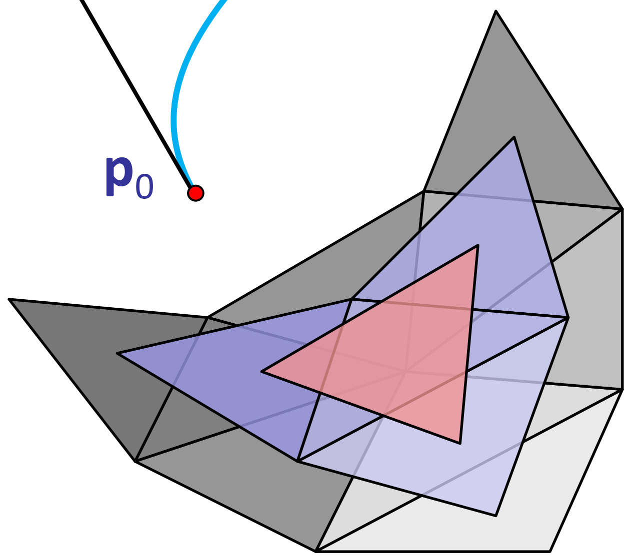
Prof. Dr. Tino Weinkauf

Geometric Modeling

Introduction

- There are many ways for creating graphical data.
- Classic way: **Geometric Modeling**

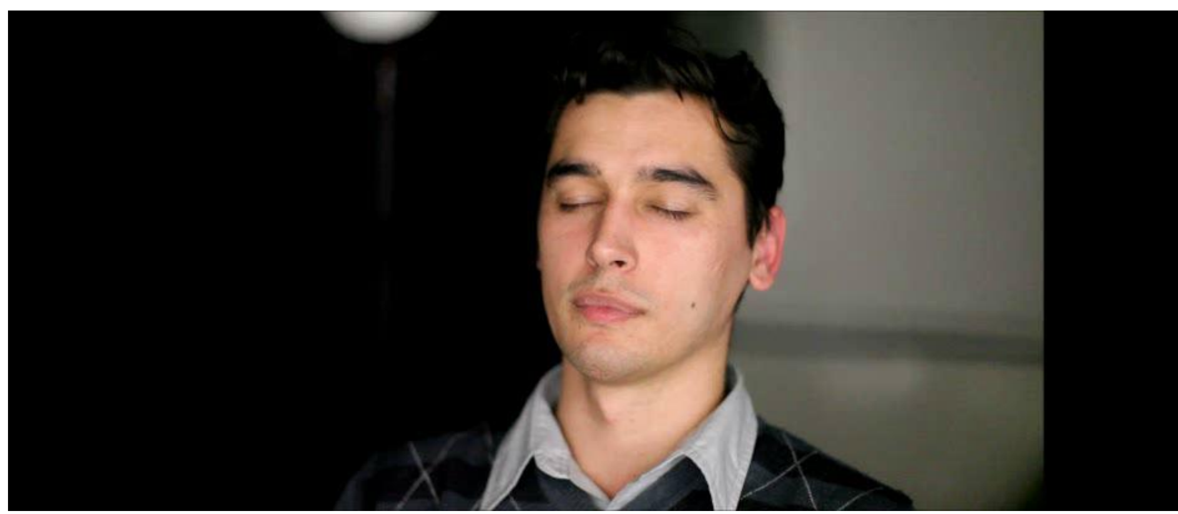



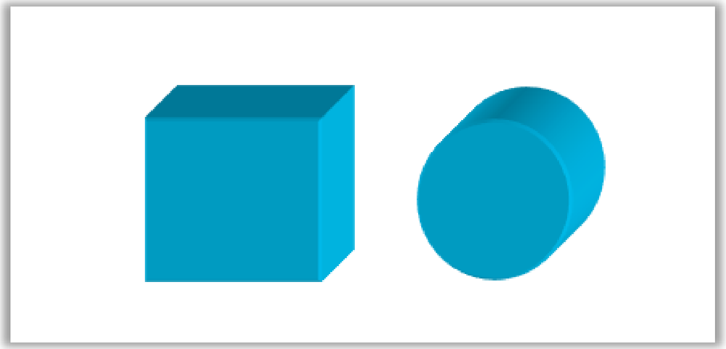

- There are many ways for creating graphical data.

- Other approaches:
  - **3D scanners**
  - Photography for measuring optical properties
  - Simulations, e.g., for flow data

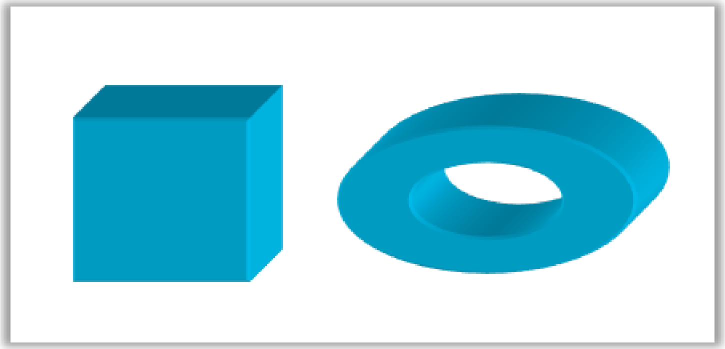
3D Scanning



- Geometric objects convey a part of the real or theoretical world; often, something tangible
- They are described by their **geometric** and **topological** properties:
  - Geometry describes the form and the position/orientation in a coordinate system.
  - Topology defines the fundamental structure that is invariant against continuous transformations.



Different geometry
Same topology



Different geometry
Different topology

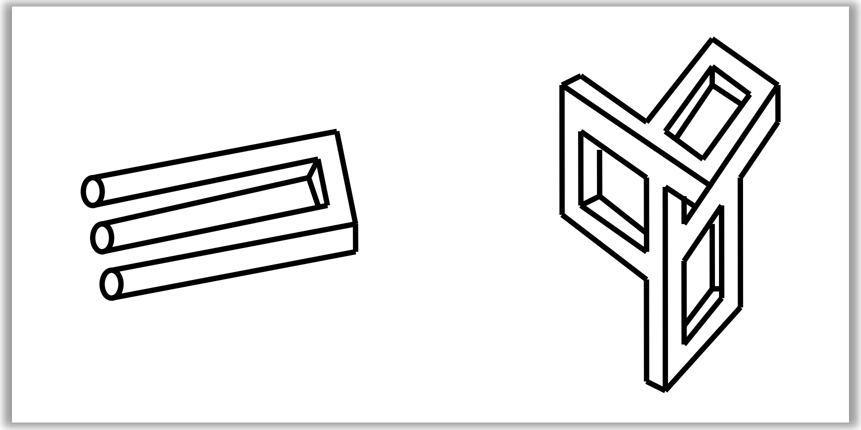
- Geometric Modeling is the computer-aided design and manipulation of geometric objects. (CAD)

- It is the basis for:
  - computation of geometric properties
  - rendering of geometric objects
  - physics computations (if some physical attributes are given)

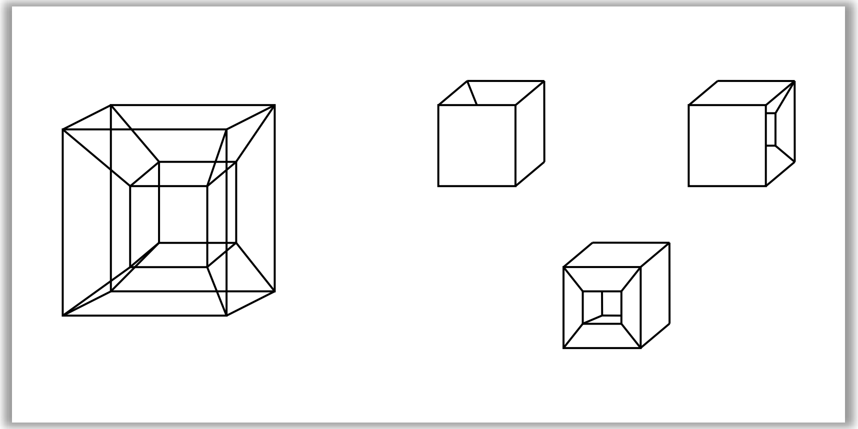
- 3D models are geometric representations of 3D objects with a certain level of abstraction.
- We distinguish between three types of models:
- Wire Frame Models
  - describe an object using boundary lines
- Surface Models
  - describe an object using boundary surfaces
- Solid Models
  - describe an object as a solid

Wire Frame Models

- Describe an object using boundary curves
- No relationship between these curves
  - Surfaces between them are not defined
- Properties:
  - simple, traditional
  - non-sense objects possible
  - visibility of curves cannot be decided
  - solid object intersection cannot be computed
  - surfaces between the curves cannot be computed automatically
  - not useable for CAD/CAM



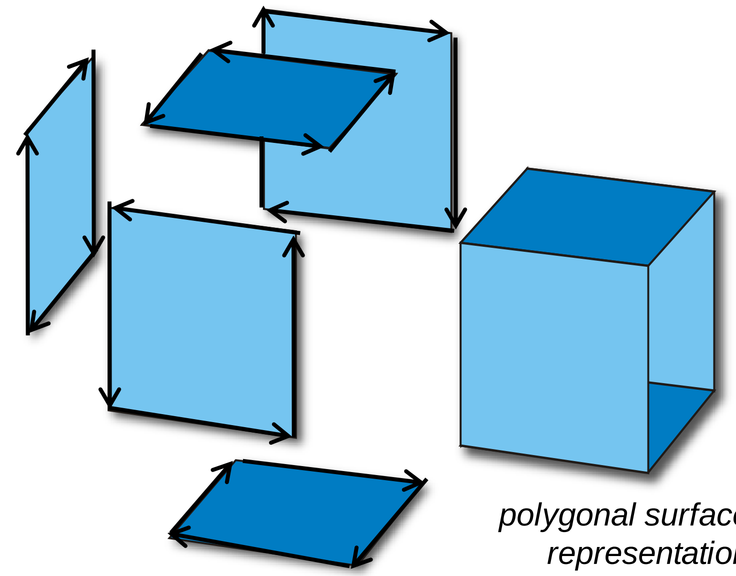
non-sense objects (Ernst, 1987)



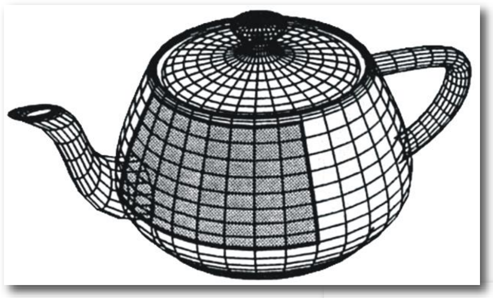
ambiguity of wire frame models

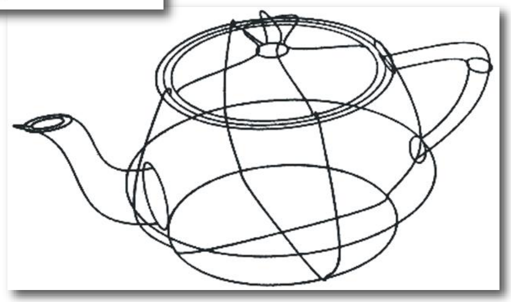
Surface Models

- Defines surfaces between boundary curves
- Describes the hull, but not the interior of an object
- Often implemented using polygons, hull of a sphere or ellipsoid, free-form surfaces, …
- No relationship between the surfaces
  - The interior between them is not defined
- Visibility computations: yes
  Solid intersection comp.: no
- Most often used type of model



polygonal surface representation

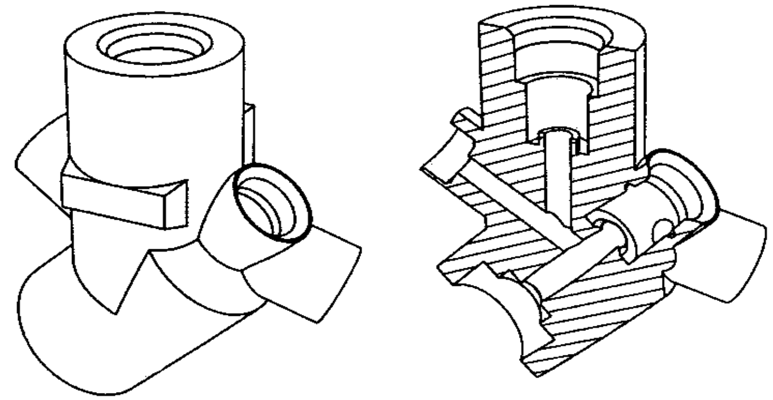




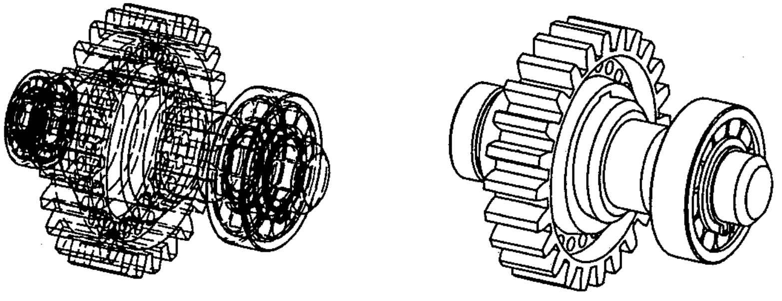
parametric surface representation using 32 Bezier patches

Solid Models

- Describe the 3D object completely by covering the solid
- For every point in 3D, we can decide whether it is inside or outside of the solid.
- Visibility and intersection computations are fully supported
- Basis for creating solid objects using computer-aided manufacturing



solid model and a cut through it (Werkbild Strässle, from Ockert, 1993)



visibility computation for lines using a solid model

Chrome-cobalt disc with crowns for dental implants, manufactured using WorkNC CAM Sescoi CAD/CAM

Prof. Dr. Tino Weinkauf

Geometric Modeling

Bezier Curves and Splines

de Casteljau Algorithm
Bernstein Form
Bezier Splines

Bezier Curves
de Casteljau algorithm

- Paul de Casteljau (1959) @ Citroën
- Pierre Bezier (1963) @ Renault

Meine Zeit bei Citroën / My time at Citroën
see the PDF deCasteljau_de.pdf and deCasteljau_en.pdf in the download area of the webpage

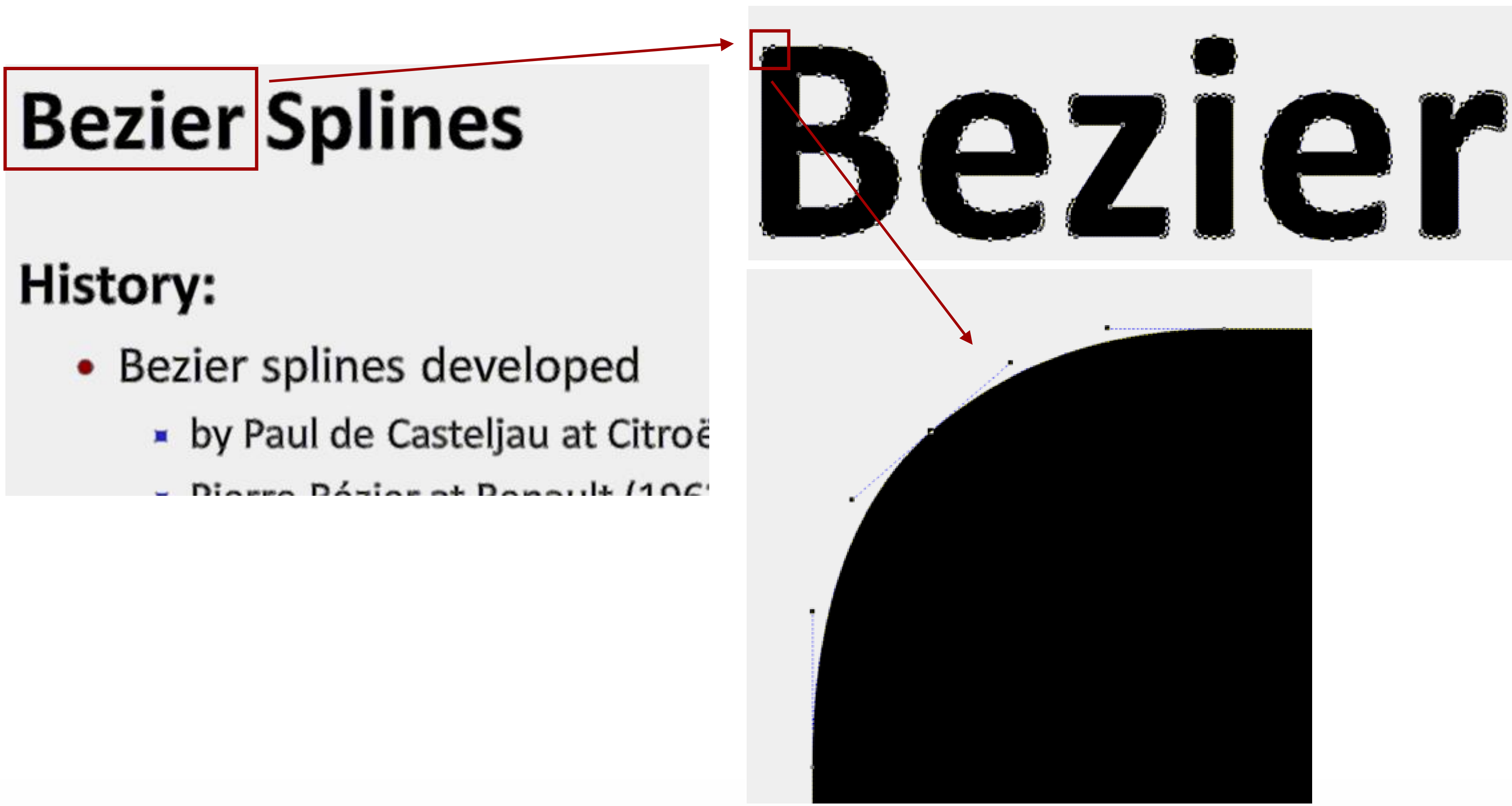
Bezier curves

History:

- Bezier curves/splines developed by
  - Paul de Casteljau at Citroën (1959)
  - Pierre Bézier at Renault (1963)

  for free-form parts in automotive design
- Today: Standard tool for 2D curve editing
- Cubic 2D Bezier curves are everywhere:
  - Postscript, PDF, Truetype (quadratic curves), Windows GDI...
  - Inkscape, Corel Draw, Adobe Illustrator, Powerpoint, ...
- Widely used in 3D curve & surface modeling as well

All You See is Bezier Curves...

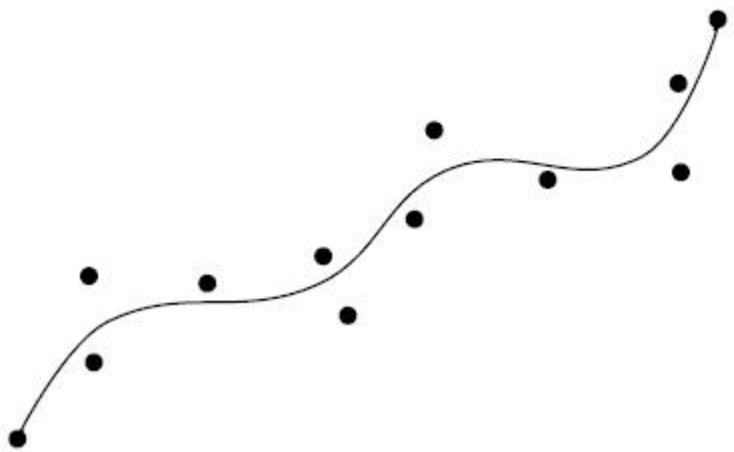


De Casteljau algorithm

Approximation setting:

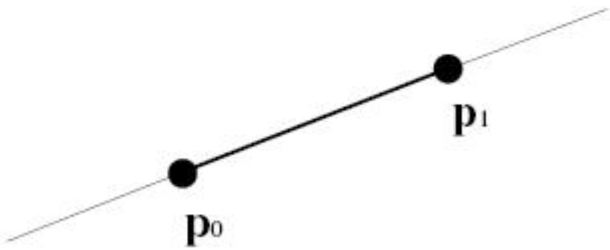
Given: p_0, …, p_n

Wanted: smooth, approximating curve



De Casteljau algorithm

Linear interpolation



$$\mathbf{x}(t) = (1-t)\cdot\mathbf{p}_0 + t\cdot\mathbf{p}_1$$

De Casteljau algorithm

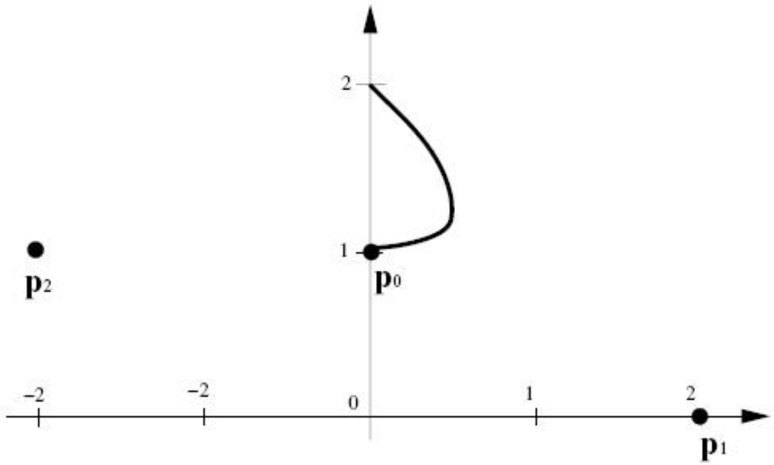
Parabolas

$$\mathbf{x}(t) = \mathbf{p}_0 + t \cdot \mathbf{p}_1 + t^2 \cdot \mathbf{p}_2$$

➔ planar curve, even if defined in R³

Example:

$$\mathbf{p}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{p}_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{p}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$



De Casteljau algorithm

Another parabola construction

given: 3 points b_0, b_1, b_2

$$\mathbf{b}_0^1 = (1-t)\cdot\mathbf{b}_0 + t\cdot\mathbf{b}_1$$
$$\mathbf{b}_1^1 = (1-t)\cdot\mathbf{b}_1 + t\cdot\mathbf{b}_2$$

$$\mathbf{b}_0^2 = (1-t)\cdot\mathbf{b}_0^1 + t\cdot\mathbf{b}_1^1$$

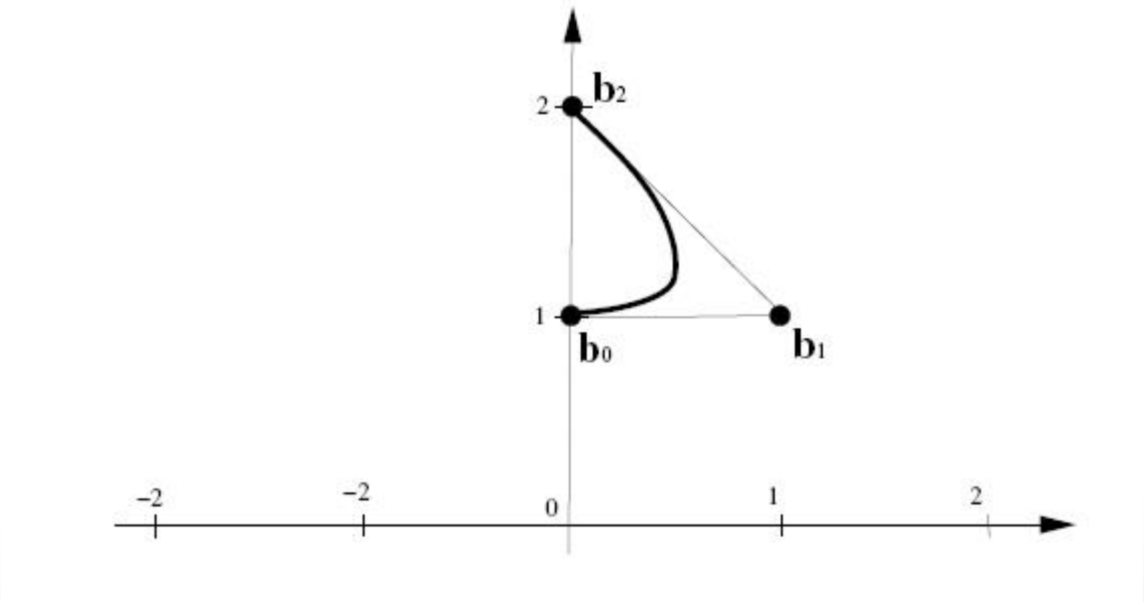
parabola x(t)

$$\mathbf{x}(t) = (1-t)^2\cdot\mathbf{b}_0 + 2\cdot t\cdot(1-t)\cdot\mathbf{b}_1 + t^2\cdot\mathbf{b}_2$$

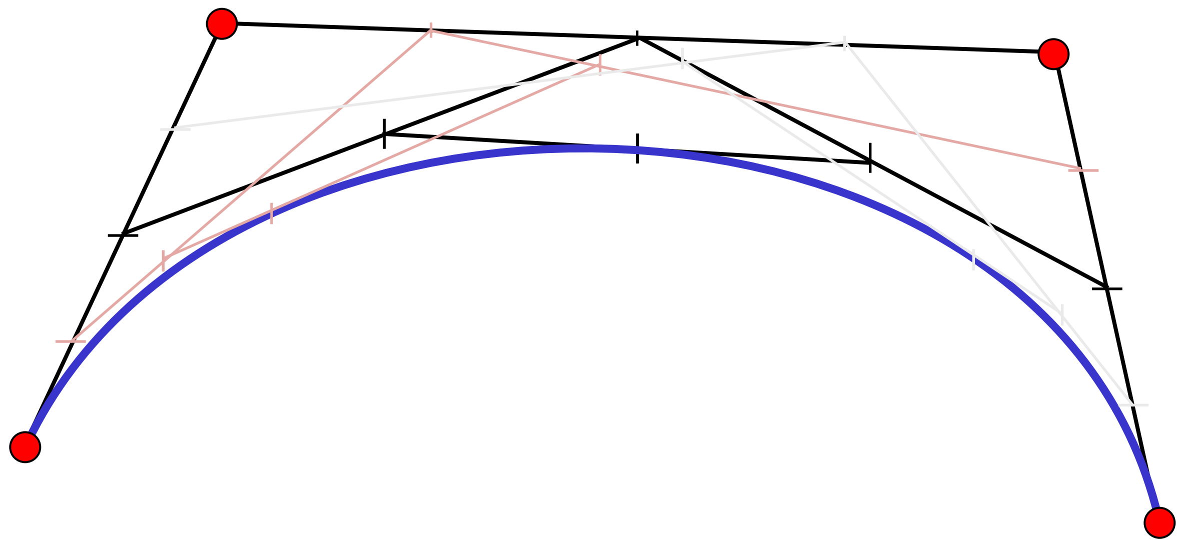
De Casteljau algorithm

Example

$$\mathbf{b}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \;,\; \mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \;,\; \mathbf{b}_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$



De Casteljau algorithm



De Casteljau Algorithm: Computes $x(t)$ for given t

- Bisect control polygon in ratio $t : (1 - t)$
- Connect the new dots with lines (adjacent segments)
- Interpolate again with the same ratio
- Iterate, until only one point is left

De Casteljau algorithm

Description of the de Casteljau algorithm

- given: points $\mathbf{b}_0, \mathbf{b}_1, ..., \mathbf{b}_n \in \mathbb{R}^3$
- wanted: curve $\mathbf{x}(t), \quad t \in [0, 1]$

- geometric construction of the point x(t) for given t:

$$\mathbf{b}_i^0(t) = \mathbf{b}_i \qquad \text{für } i = 0, ..., n$$

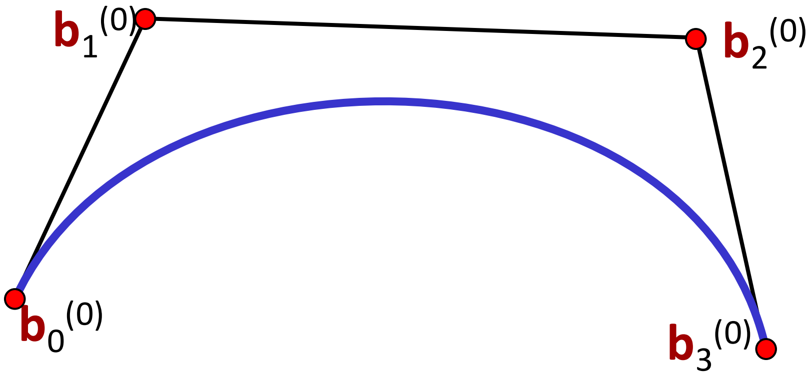
$$\mathbf{b}_i^r(t) = (1-t) \cdot \mathbf{b}_i^{r-1}(t) \; + \; t \cdot \mathbf{b}_{i+1}^{r-1}(t)$$
$$\text{für } r = 1, ..., n \quad ; \quad i = 0, ..., n-r.$$

- Then, $\mathbf{b}_0^n(t)$ is the searched curve point x(t) at the parameter value t

De Casteljau algorithm

repeated convex combination of control points

$$\mathbf{b}_i^{(r)} = (1-t) \cdot \mathbf{b}_i^{(r-1)} + t \cdot \mathbf{b}_{i+1}^{(r-1)}$$



$\mathbf{b}_0^{(0)}$

$\mathbf{b}_1^{(0)}$

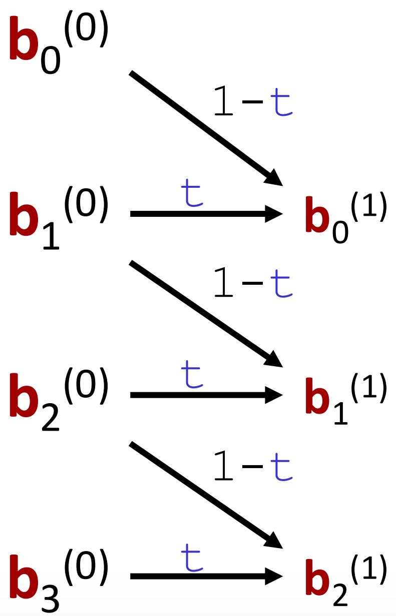
$\mathbf{b}_2^{(0)}$

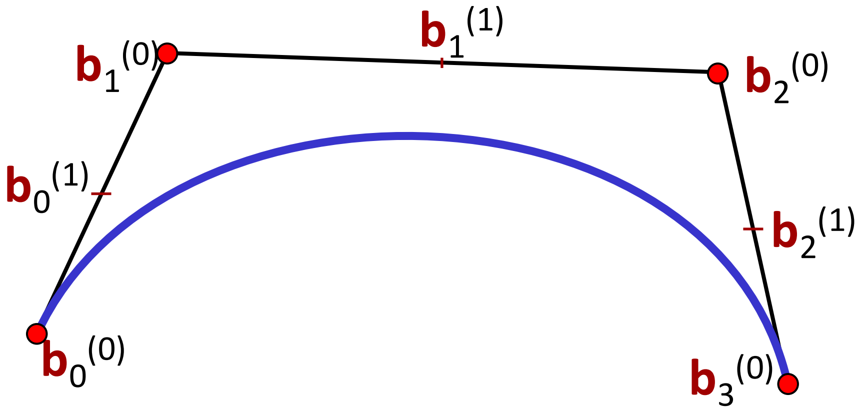
$\mathbf{b}_3^{(0)}$

De Casteljau algorithm

repeated convex combination of control points

$$\mathbf{b}_i^{(r)} = (1-t)\cdot\mathbf{b}_i^{(r-1)} + t\cdot\mathbf{b}_{i+1}^{(r-1)}$$



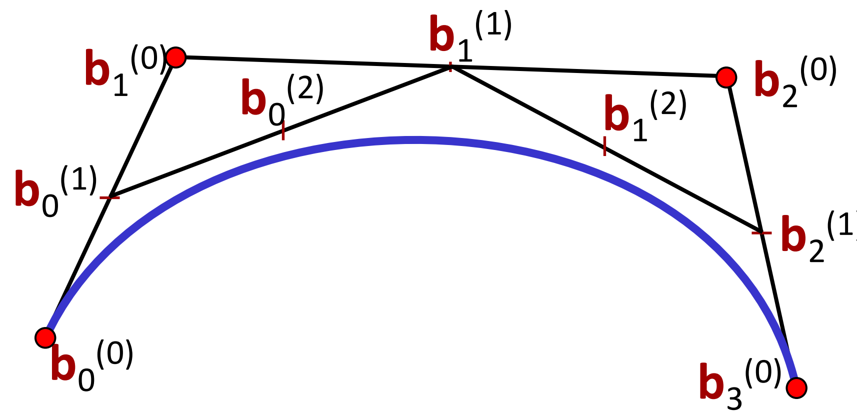


De Casteljau algorithm

repeated convex combination of control points

$$\mathbf{b}_i^{(r)} = (1-t)\cdot\mathbf{b}_i^{(r-1)} + t\cdot\mathbf{b}_{i+1}^{(r-1)}$$

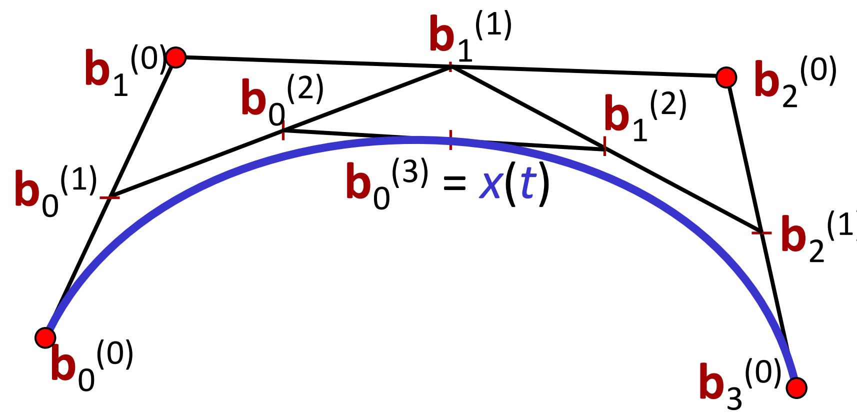
$$\begin{tikzcd}
\mathbf{b}_0^{(0)} \arrow[dr, "1-t"] & & \\
\mathbf{b}_1^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \mathbf{b}_0^{(1)} \arrow[dr, "1-t"] & \\
\mathbf{b}_2^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \mathbf{b}_1^{(1)} \arrow[r, "t"] \arrow[dr, "1-t"] & \mathbf{b}_0^{(2)} \\
\mathbf{b}_3^{(0)} \arrow[r, "t"] & \mathbf{b}_2^{(1)} \arrow[r, "t"] & \mathbf{b}_1^{(2)}
\end{tikzcd}$$



De Casteljau algorithm

repeated convex combination of control points

$$\mathbf{b}_i^{(r)} = (1-t)\cdot\mathbf{b}_i^{(r-1)} + t\cdot\mathbf{b}_{i+1}^{(r-1)}$$



$$\begin{tikzcd}
\mathbf{b}_0^{(0)} \arrow[dr, "1-t"] & & & \\
\mathbf{b}_1^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \mathbf{b}_0^{(1)} \arrow[dr, "1-t"] & & \\
\mathbf{b}_2^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \mathbf{b}_1^{(1)} \arrow[r, "t"] \arrow[dr, "1-t"] & \mathbf{b}_0^{(2)} \arrow[dr, "1-t"] & \\
\mathbf{b}_3^{(0)} \arrow[r, "t"] & \mathbf{b}_2^{(1)} \arrow[r, "t"] & \mathbf{b}_1^{(2)} \arrow[r, "t"] & \mathbf{b}_0^{(3)} = x(t)
\end{tikzcd}$$

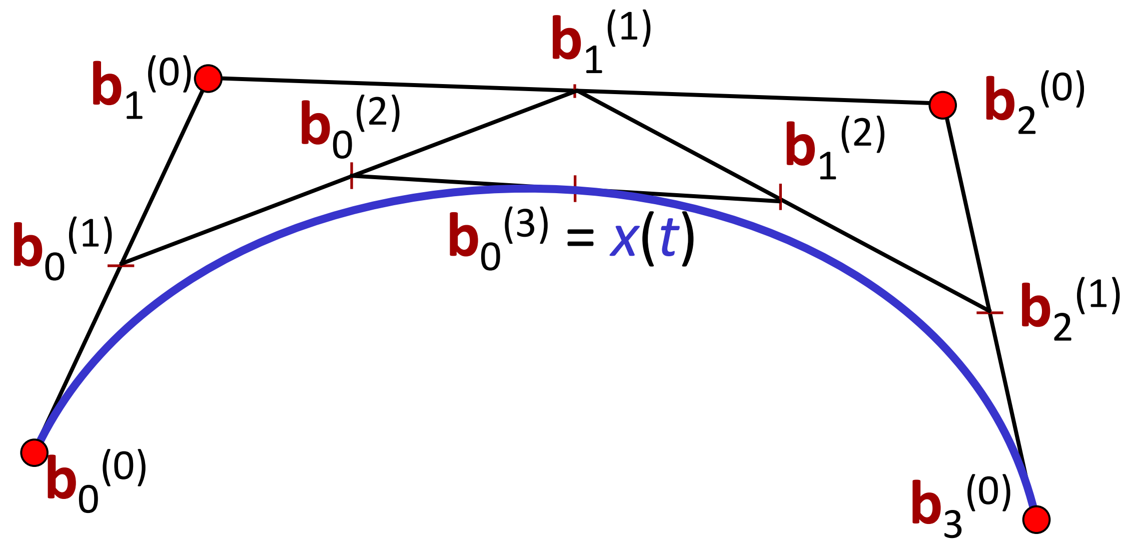
de Casteljau scheme

De Casteljau algorithm

The intermediate coefficients $b_i^r(t)$ can be written in a triangular matrix: the de Casteljau scheme:

$$
\begin{array}{lllllll}
\mathrm{b}_0 = \mathrm{b}_0^0 & & & & & & \\
\mathrm{b}_1 = \mathrm{b}_1^0 & \mathrm{b}_0^1 & & & & & \\
\mathrm{b}_2 = \mathrm{b}_2^0 & \mathrm{b}_1^1 & \mathrm{b}_0^2 & & & & \\
\mathrm{b}_3 = \mathrm{b}_3^0 & \mathrm{b}_2^1 & \mathrm{b}_1^2 & \mathrm{b}_0^3 & & & \\
\vdots & & & & & & \\
\mathrm{b}_{n-1} = \mathrm{b}_{n-1}^0 & \mathrm{b}_{n-2}^1 & \dots & \mathrm{b}_0^{n-1} & & & \\
\mathrm{b}_n = \mathrm{b}_n^0 & \mathrm{b}_{n-1}^1 & \dots & \mathrm{b}_1^{n-1} & \mathrm{b}_0^n = \mathbf{x}(t) & &
\end{array}
$$

De Casteljau algorithm



Algorithm:

for $r = 1..n$ **do**

  for $i = 0..n-r$ **do**

    $\mathbf{b}_i^{(r)} = (1-t) \cdot \mathbf{b}_i^{(r-1)} + t \cdot \mathbf{b}_{i+1}^{(r-1)}$

  end for

end for

return $\mathbf{b}_0^{(n)}$

The whole algorithm consists only of repeated linear interpolations.

De Casteljau algorithm

The polygon consisting of the points $b_0, \dots, b_n$ is called Bezier polygon. The points b_i are called Bezier points.

The curve defined by the Bezier points $b_0, \dots, b_n$ and the de Casteljau algorithm is called Bezier curve.

The de Casteljau algorithm is numerically stable, since only convex combinations are applied.

Complexity of the de Casteljau algorithm

- $O(n^2)$ time
- $O(n)$ memory
- with n being the number of Bezier points

De Casteljau algorithm

Properties of Bezier curves:

- given: Bezier points $\mathbf{b}_0, \ldots, \mathbf{b}_n$

  Bezier curve $\mathbf{x}(t)$
- Bezier curve is polynomial curve of degree n.
- End point interpolation: $\mathbf{x}(0) = \mathbf{b}_0$, $\mathbf{x}(1) = \mathbf{b}_n$. The remaining Bezier points are only generally approximated.
- Convex hull property:

  Bezier curve is completely inside the convex hull of its Bezier polygon.

De Casteljau algorithm

- **Variation diminishing**
  no line intersects the Bezier curve more often than its Bezier polygon.

- **Influence of Bezier points: global, but pseudo-local**
  - *global:* moving a Bezier point changes the whole curve progression
  - *pseudo-local:* $\mathbf{b}_i$ has its maximal influence on $\mathbf{x}(t)$ at $t = i/n$.

- **Affine invariance:**
  Bezier curve and Bezier polygon are invariant under affine transformations

- **Invariance under affine parameter transformations**

De Casteljau algorithm

- **Symmetry:**
  The following two Bezier curves coincide, they are only traversed in opposite directions:

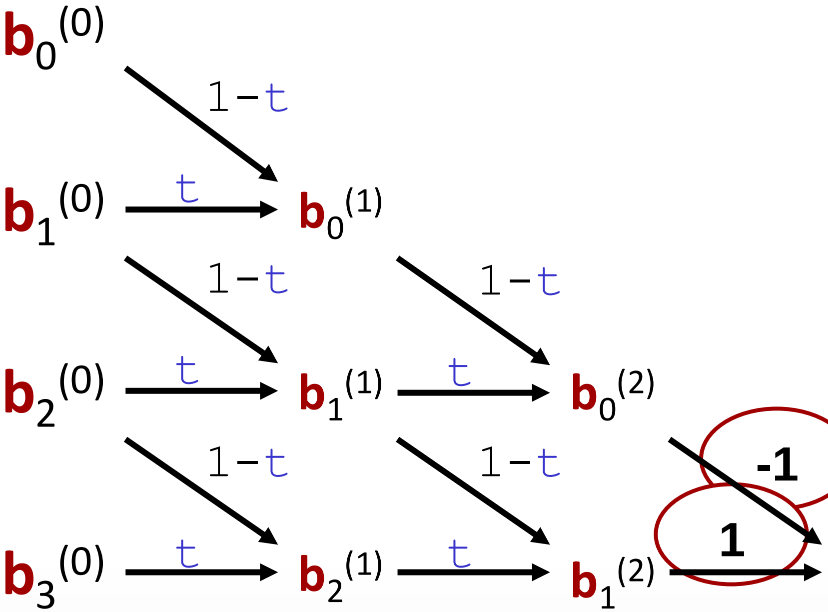
$$\mathbf{x}(t) = [\mathbf{b}_0, \ldots, \mathbf{b}_n] \qquad \mathbf{x}'(t) = [\mathbf{b}_n, \ldots, \mathbf{b}_0]$$

- **Linear precision:**
  Bezier curve is line segment, if $\mathbf{b}_0, \ldots, \mathbf{b}_n$ are collinear

- **Invariant under barycentric combinations**

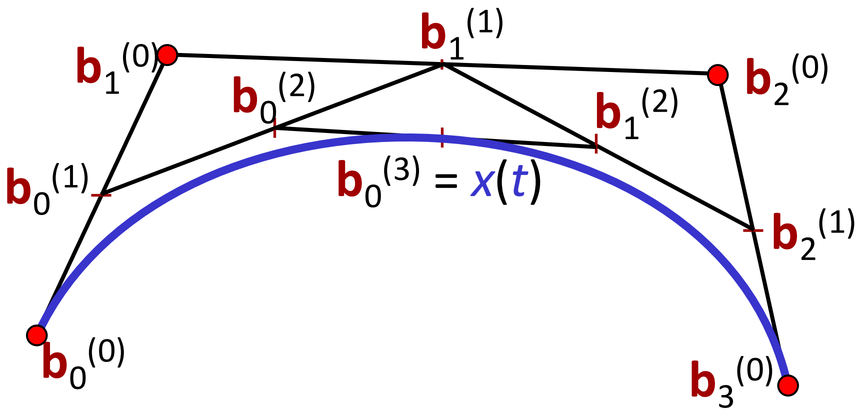
De Casteljau algorithm

- **First derivative of a Bezier curve**
  - **Endpoints:** $\dot{\mathbf{x}}(0) = n \cdot (\mathbf{b}_1 - \mathbf{b}_0)$ $\dot{\mathbf{x}}(1) = n \cdot (\mathbf{b}_n - \mathbf{b}_{n-1})$ $t = 0, t = 1$:



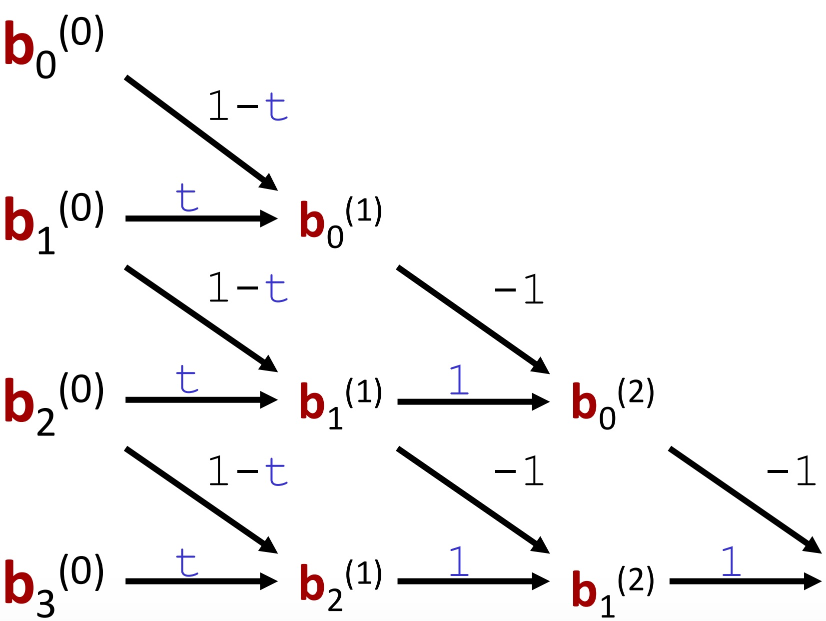
$$\dot{\mathbf{x}}(t) = n\left(\mathbf{b}_1^{(n-1)} - \mathbf{b}_0^{(n-1)}\right)$$

de Casteljau scheme

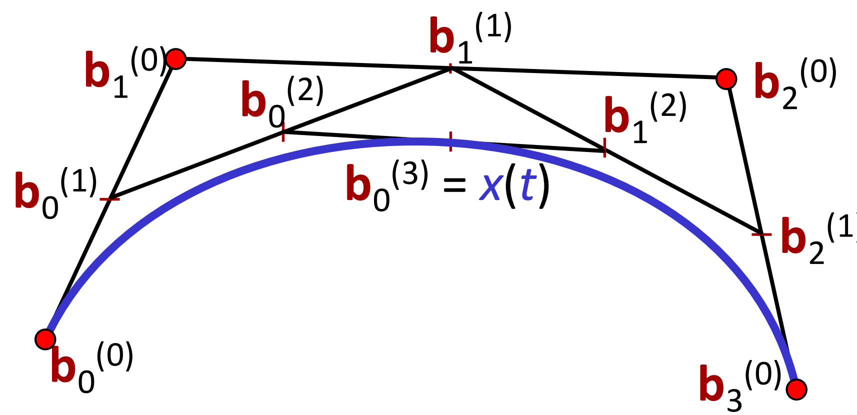


De Casteljau algorithm

- **Second derivative of a Bezier curve**



de Casteljau scheme



$$\ddot{\mathbf{x}}(t) = n(n-1)\left(\mathbf{b}_2^{(n-2)} - 2\mathbf{b}_1^{(n-2)} + \mathbf{b}_0^{(n-2)}\right)$$

Bezier Curves

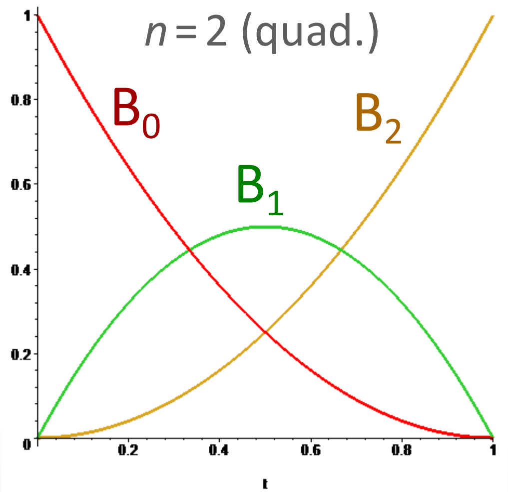
Bernstein form

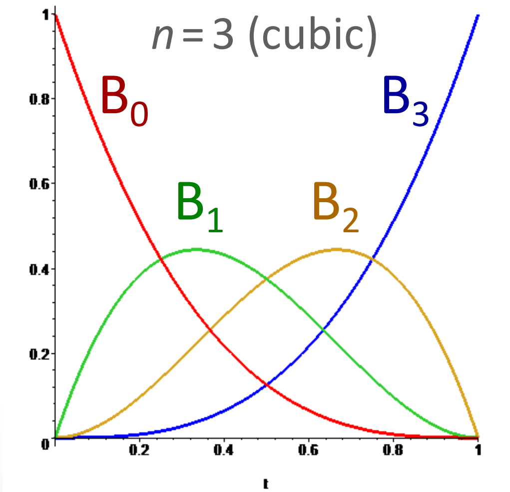
Bernstein Basis

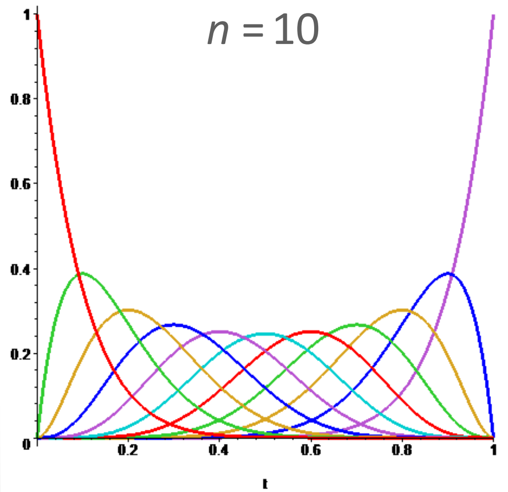
Bezier curves are algebraically defined using the Bernstein basis:

- Bernstein basis of degree *n*: $$B = \left\{ B_0^{(n)}, B_1^{(n)}, \ldots, B_n^{(n)} \right\}$$

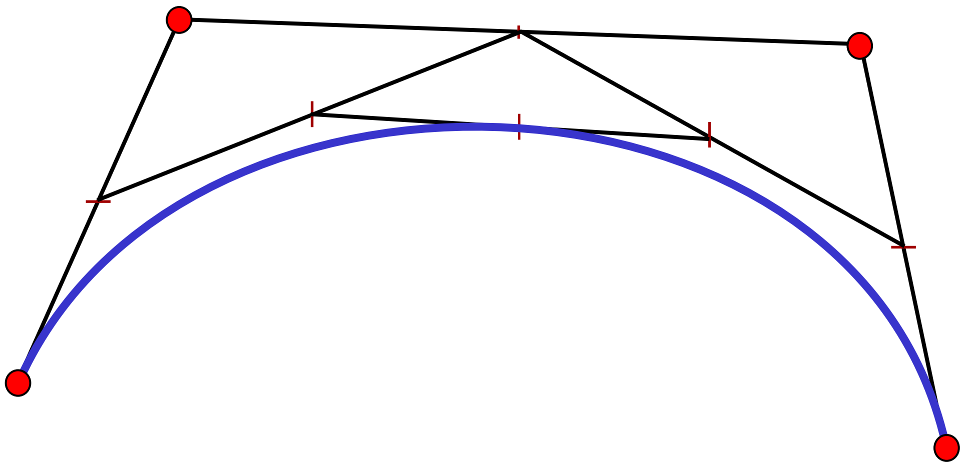
$$B_i^{(n)}(t) := \binom{n}{i} t^i (1-t)^{n-i}$$



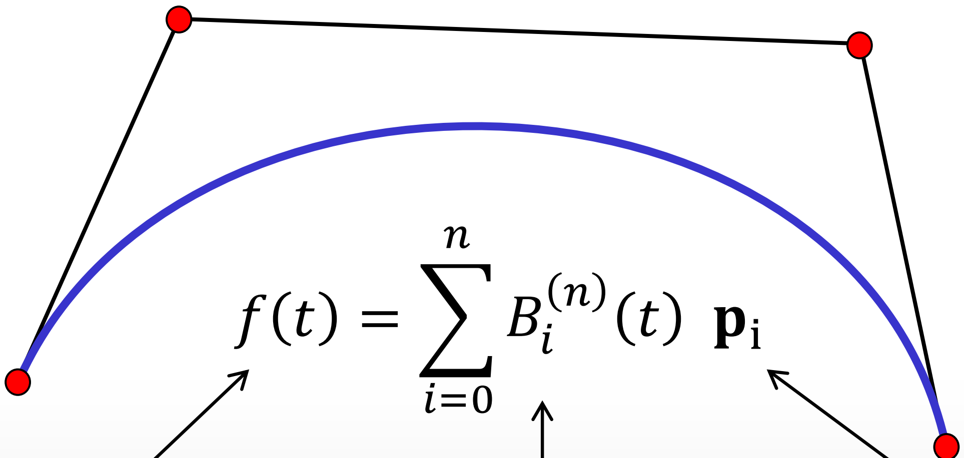




Bernstein Basis



de Casteljau algorithm



Bernstein form

$$f(t) = \sum_{i=0}^{n} B_i^{(n)}(t)\ \mathbf{p}_\mathrm{i}$$

curve　　basis function　　control point

Examples

The first three Bernstein bases:

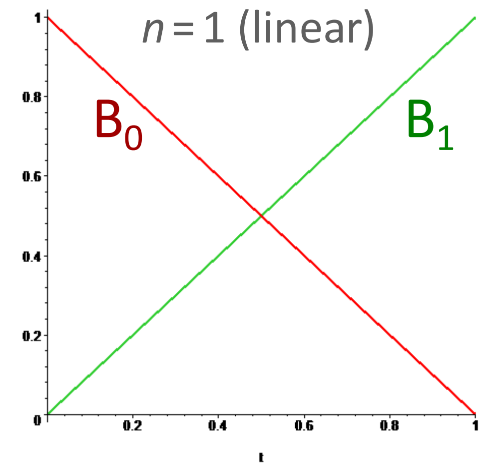
$$B_0^{(0)} := 1$$

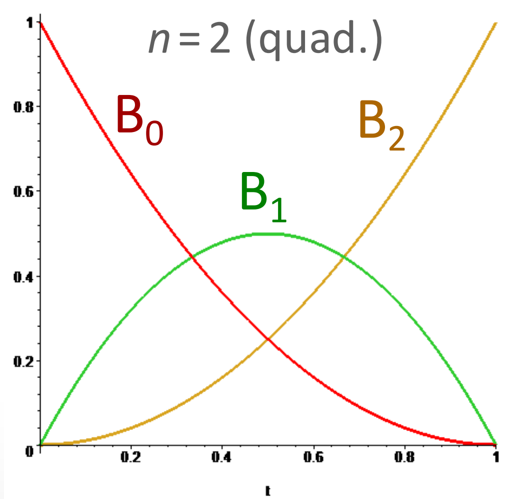
$$B_0^{(1)} := (1-t) \qquad B_1^{(1)} := t$$

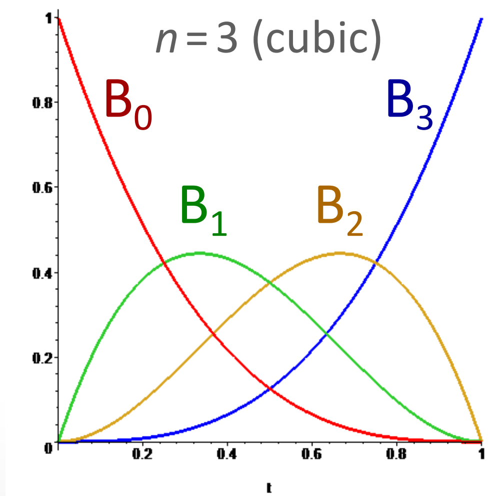
$$B_0^{(2)} := (1-t)^2 \qquad B_1^{(2)} := 2t(1-t) \qquad B_2^{(2)} := t^2$$

$$B_0^{(3)} := (1-t)^3 \qquad B_1^{(3)} := 3t(1-t)^2$$

$$B_2^{(3)} := 3t^2(1-t) \qquad B_3^{(3)} := t^3$$







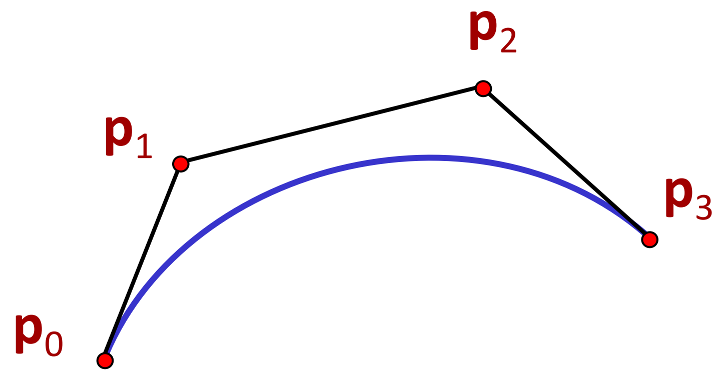
$$B_i^{(n)}(t) := \binom{n}{i} t^i (1-t)^{n-i}$$

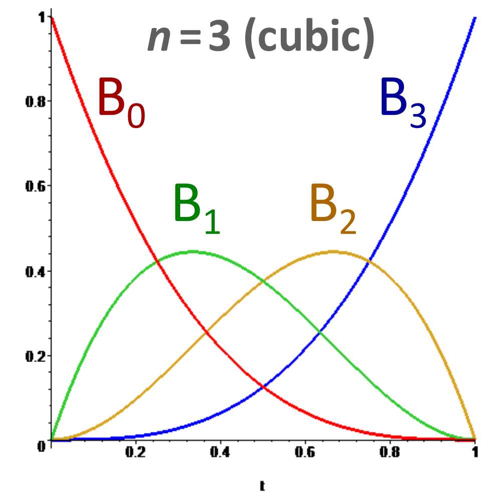
Bezier Curves in Bernstein form

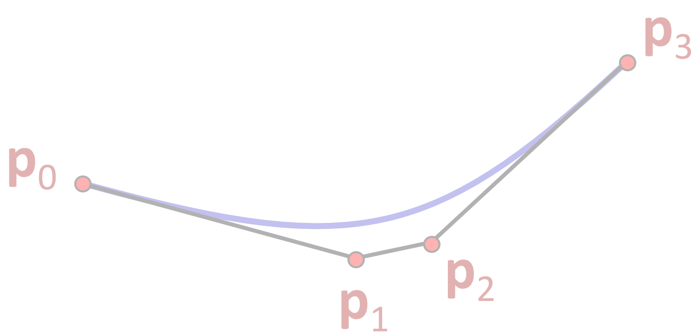
Bezier Curves:

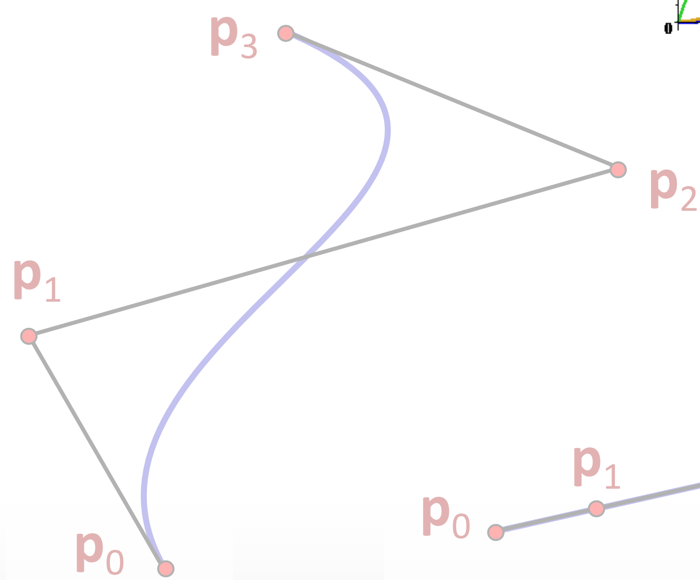
- $\mathbf{f}(t) = \sum_{i=0}^{n} \mathbf{p}_i B_i^{(n)}$

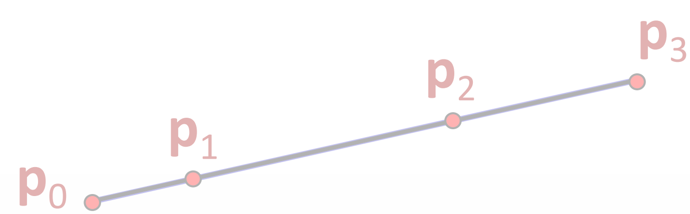
  $t \in [0..1]$







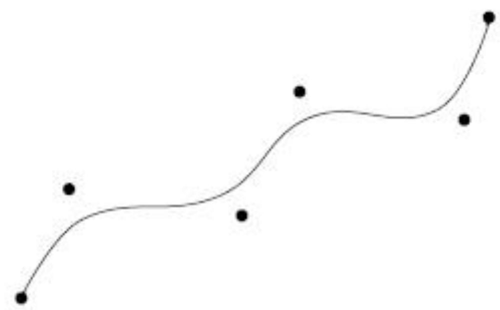




Summary for Bezier Curves

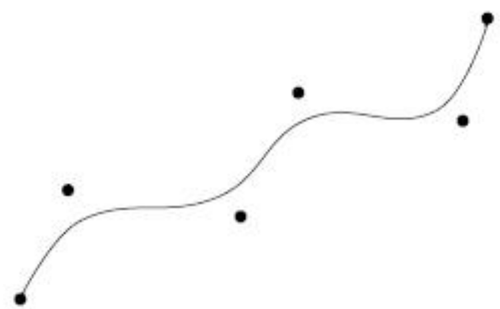
Bezier curves and curve design:

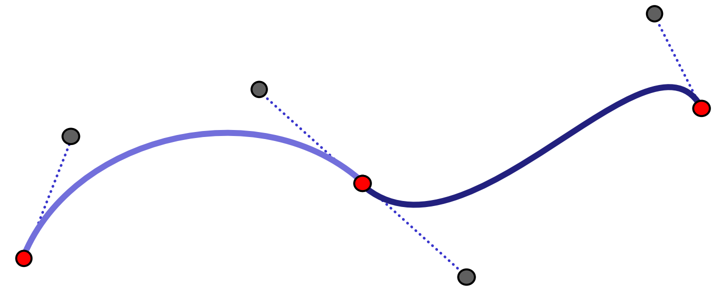
- The rough form is specified by the position of the control points
- Result: smooth curve approximating the control points
- Computation / Representation:
  - de Casteljau algorithm
  - Bernstein form



- Problems:
  - high polynomial degree
  - moving a control point can change the whole curve
  - interpolation of points
  - → **Bezier splines**

Towards Bezier Splines





Approximation Interpolation

Towards Bezier Splines

Interpolation problem:

- given:

  $\mathbf{k}_0, \ldots, \mathbf{k}_n \in \mathbb{R}^3$ control points

  $t_0, \ldots, t_n \in \mathbb{R}$ knot sequence

  $t_i < t_{i+1}$ für $i = 0, \ldots, n-1$

- wanted:

  interpolating curve $\mathbf{x}(t)$, i.e., $\mathbf{x}(t_i) = \mathbf{k}_i$ for $i = 0, \ldots, n$

- Approach:
  "Joining" of n Bezier curves with certain intersection conditions

Towards Bezier Splines

The following issues arise when stitching together Bezier curves:

- Continuity
- Degree
- (Parameterization)

Bezier Splines

Parametric and Geometric Continuity

Continuity

Joining of curves - continuity

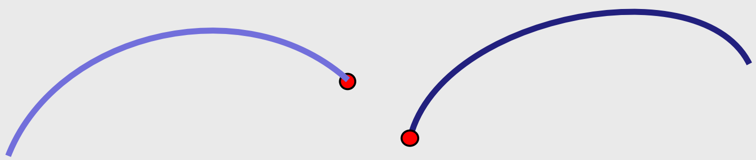
- given: 2 curves

  $\mathbf{x}_1(t)$ over $[t_0, t_1]$

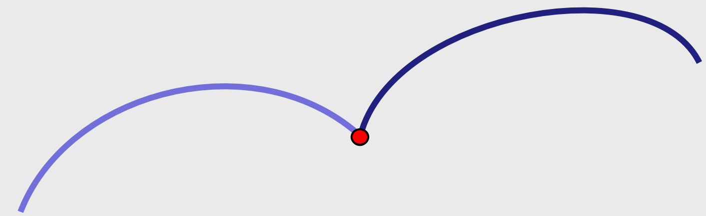
  $\mathbf{x}_2(t)$ over $[t_1, t_2]$

- $\mathbf{x}_1$ and $\mathbf{x}_2$ are C^r continuous in t_1, if they coincide in $0^{th} - r^{th}$ derivative vector in t_1.

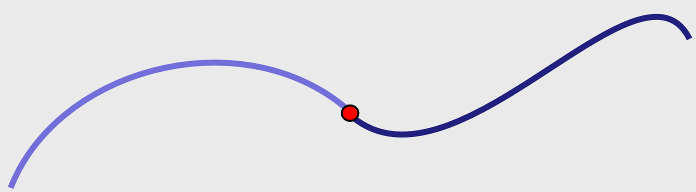
Continuity



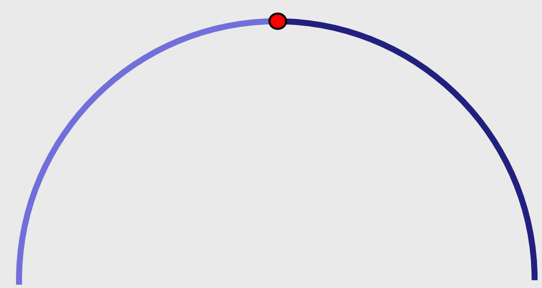
C^{-1} continuity



C^{0} continuity



C^{1} continuity



C^{2} continuity

Continuity

Parametric Continuity C^r**:**

- C^0, C^1, C^2... continuity.
- Does a particle moving on this curve have a smooth trajectory (position, velocity, acceleration,...)?
- Useful for animation (object movement, camera paths)
- **Depends** on parameterization

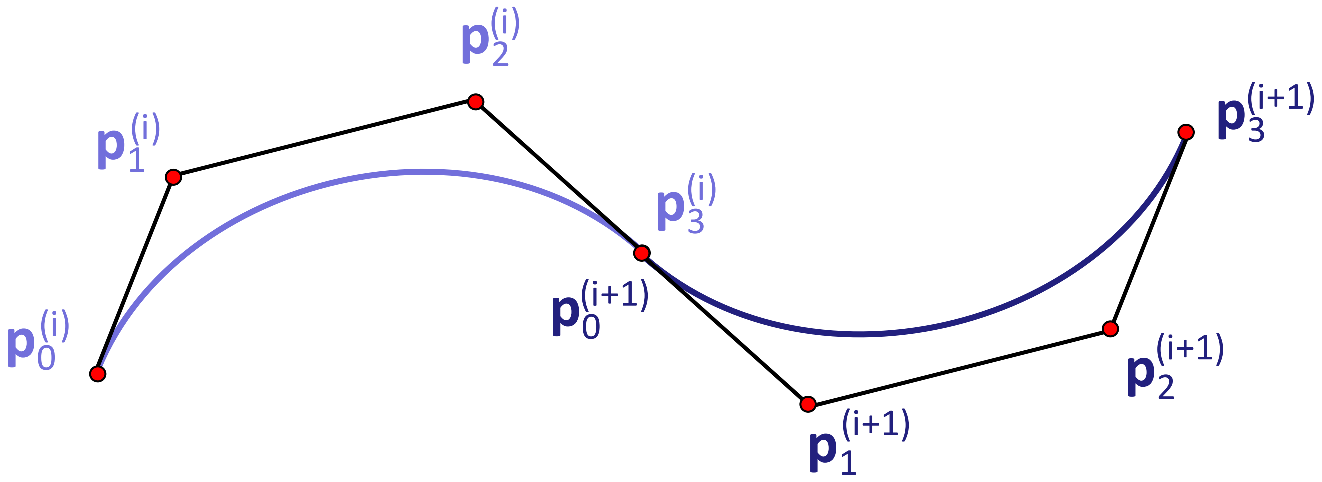
Geometric Continuity G^r**:**

- **Independent** of parameterization
- Is the curve itself smooth?
- More relevant for modeling (curve design)

Bezier Splines

Local control: Bezier splines

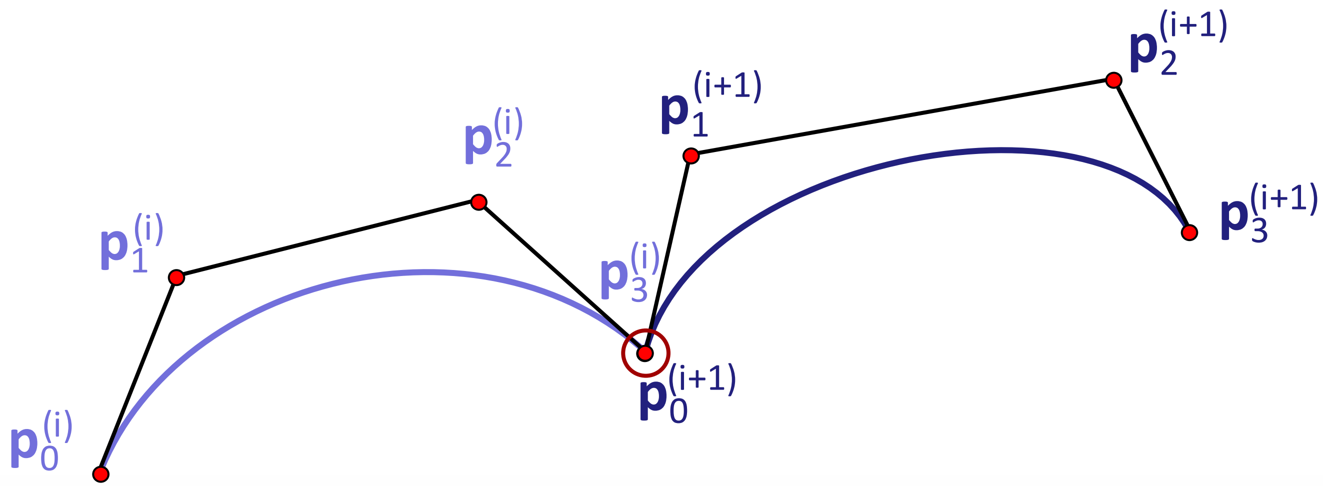
- Concatenate several curve segments
- Question: Which constraints to place upon the control points in order to get C^{-1}, C^0, C^1, C^2 continuity?



Bezier Spline Continuity

Rules for Bezier spline continuity:

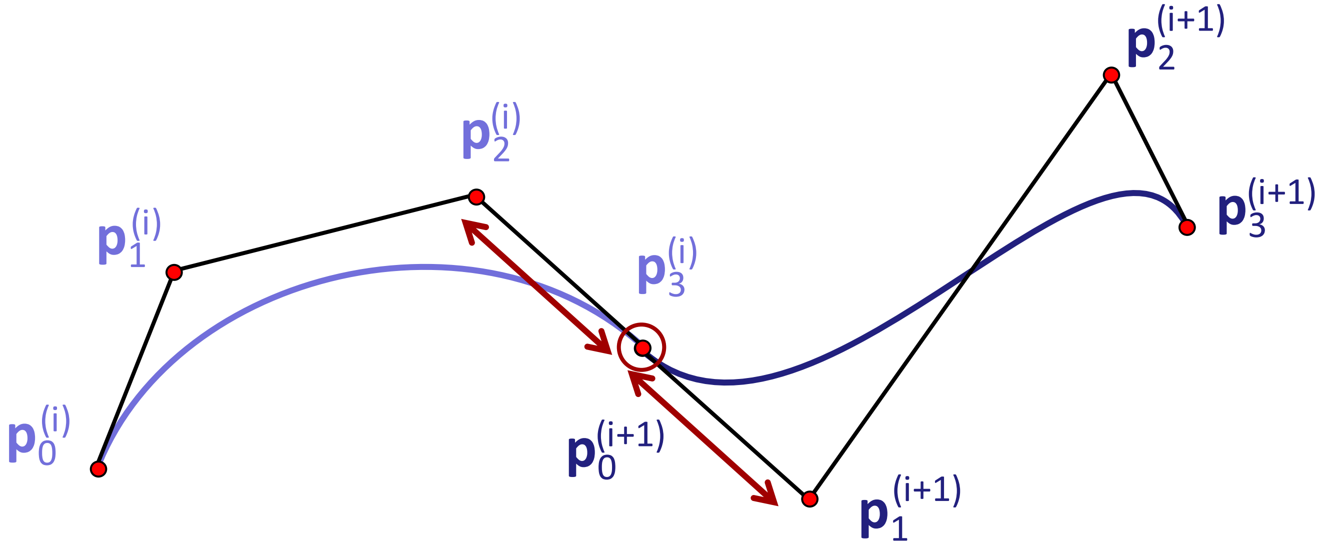
- C^0 continuity:
  - Each spline segment interpolates the first and last control point
  - Therefore: Points of neighboring segments have to coincide for C^0 continuity.



Bezier Spline Continuity

Rules for Bezier spline continuity:

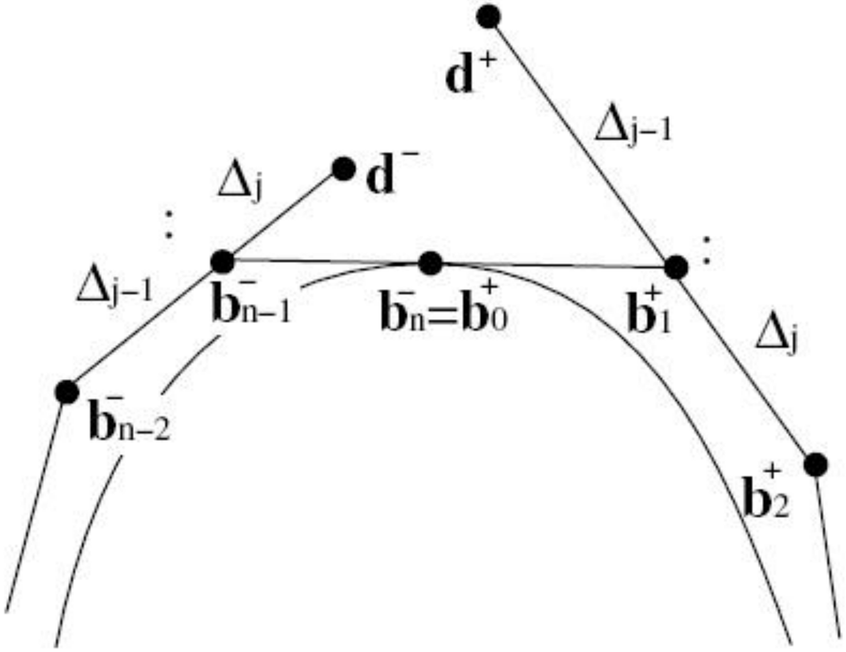
- Additional requirement for C^1 continuity:
  - Tangent vectors are proportional to differences $\mathbf{p}_1 - \mathbf{p}_0$, $\mathbf{p}_n - \mathbf{p}_{n\text{-}1}$
  - Therefore: These vectors must be identical for C^1 continuity



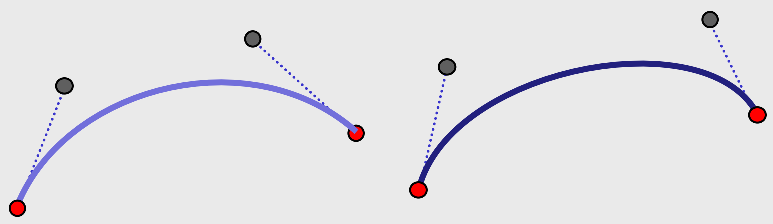
Bezier Spline Continuity

Rules for Bezier spline continuity:

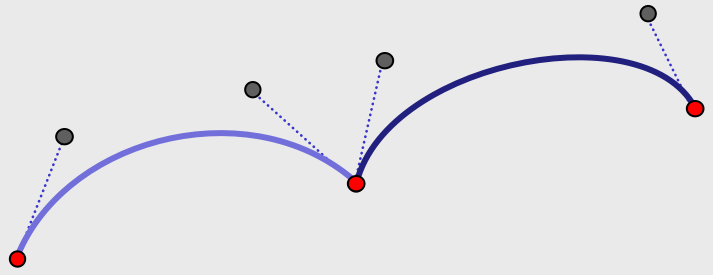
- Additional requirement for C^2 continuity:
  - $\mathbf{d}^- = \mathbf{d}^+$



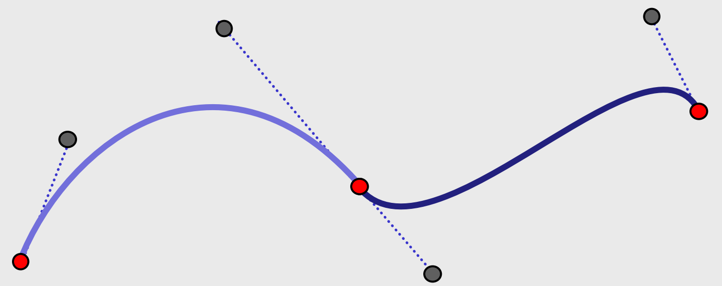
Continuity



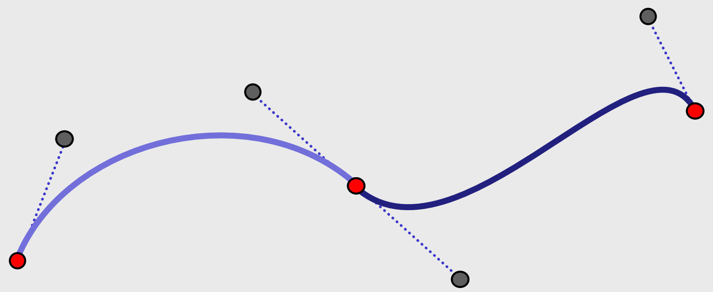
C^{-1} continuity



C^0 continuity



G^1 continuity



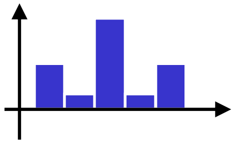
C^1 continuity

Bezier Splines
Choosing the degree

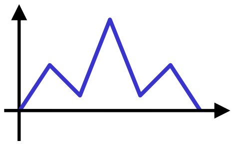
Choosing the Degree...

Candidates:

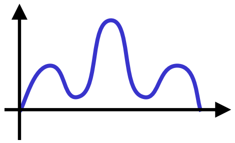
- d = 0 (piecewise constant): not smooth



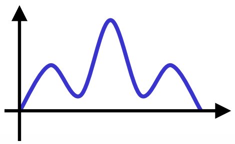
- d = 1 (piecewise linear): not smooth enough



- d = 2 (piecewise quadratic): constant 2nd derivative, still too inflexible



- d = 3 (piecewise cubic): degree of choice for computer graphics applications



Cubic Splines

Cubic piecewise polynomials:

- We can attain C^2 continuity without fixing the second derivative throughout the curve
- C^2 continuity is perceptually important
  - We can see second order shading discontinuities (esp.: reflective objects)
  - Motion: continuous *position*, *velocity* & *acceleration*
    Discontinuous acceleration noticeable (object/camera motion)
- One more argument for cubics:
  - Among all C^2 curves that interpolate a set of points (and obey to the same end conditions), a piecewise cubic curve has the least integral acceleration ("smoothest curve you can get").
    - see `AdditionalMaterial/CubicsMinimizeAcceleration.pdf`

Summary

- Bezier Curves
  - de Casteljau algorithm
  - Bernstein form
- Bezier Splines

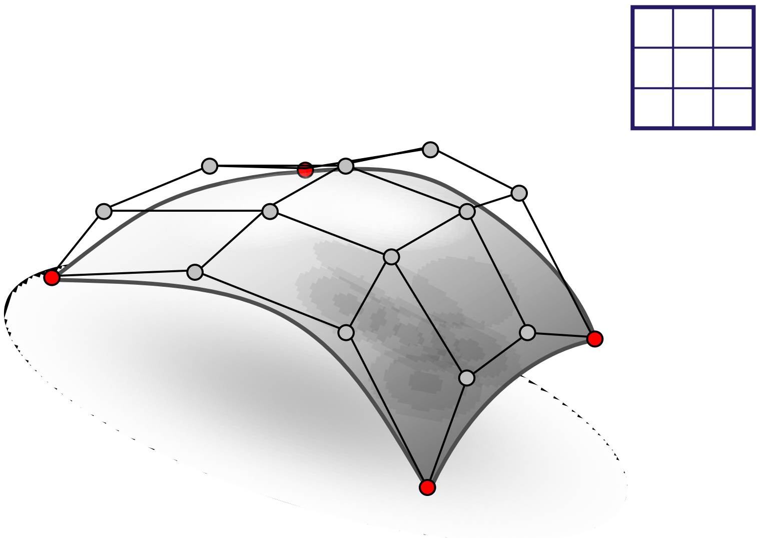
Spline Surfaces

next time

Spline Surfaces

Two different approaches

- Tensor product surfaces
  - Simple construction
  - Everything carries over from curve case
  - Quad patches
  - Degree anisotropy



- Total degree surfaces
  - Not as straightforward
  - Isotropic degree
  - Triangle patches
  - “Natural” generalization of curves