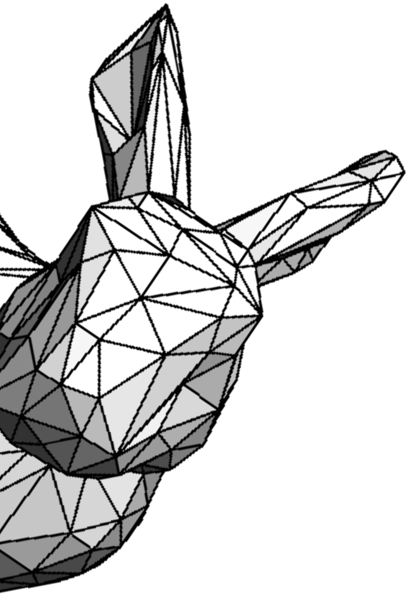
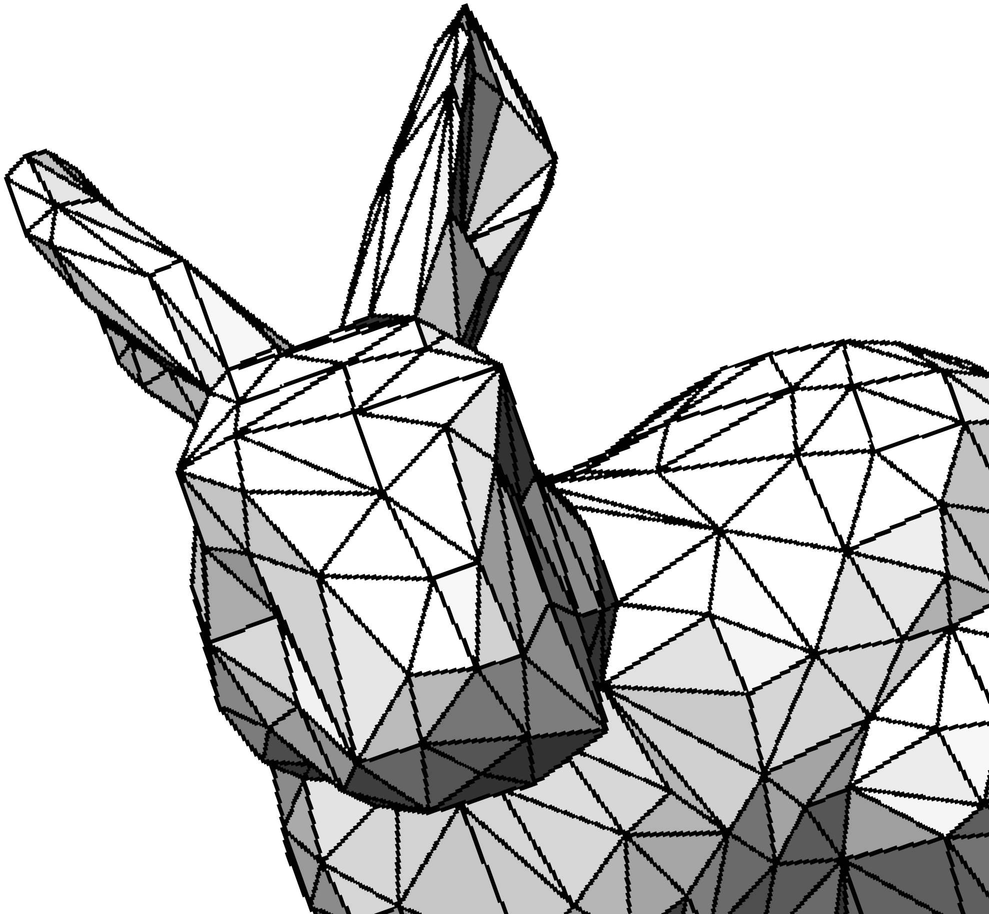
Introduction to Visualization and Computer Graphics

Projection

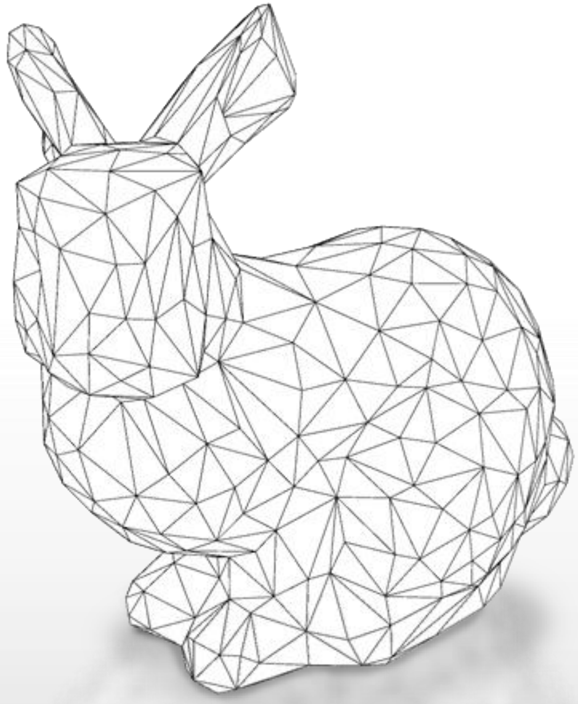
Now for

3D Rendering





3D Rendering

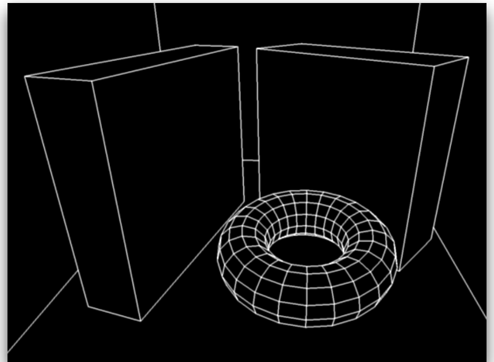


Assumption

- 3D Model is given
- Triangle mesh
  (for simplicity)

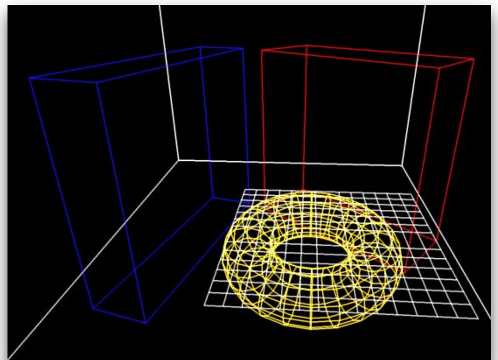
How do we get it to the screen?

3D Rendering

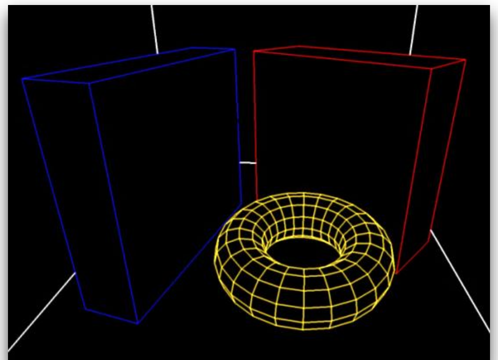


Geometric Model

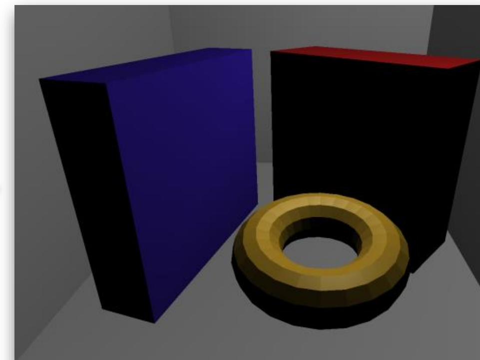
Color



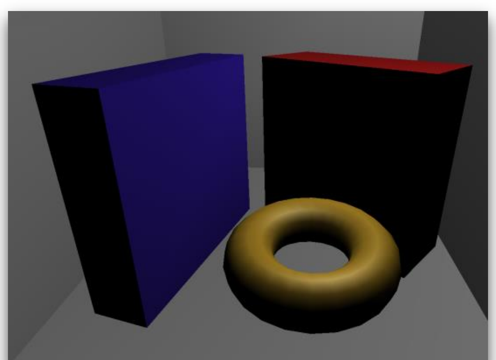
Perspective



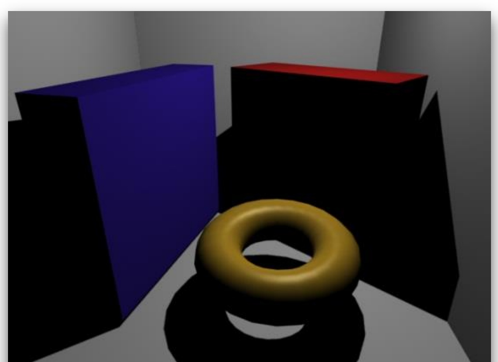
Visibility



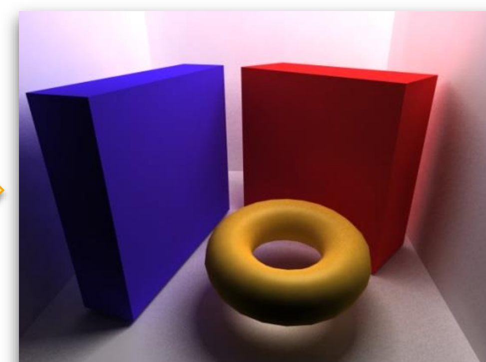
Local Illumination



Smooth Shading

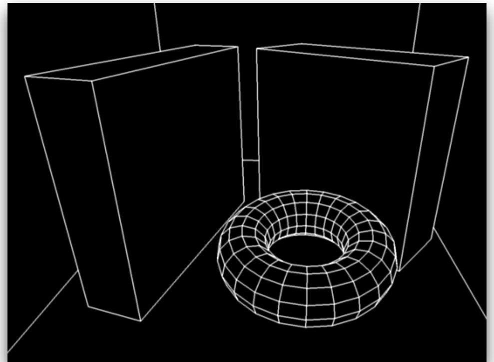


Simple Shadows



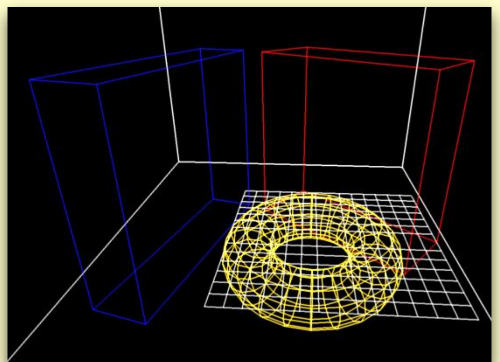
Global Illumination

3D Rendering



Geometric Model

Color



Perspective

Visibility

Local Illumination

Smooth Shading

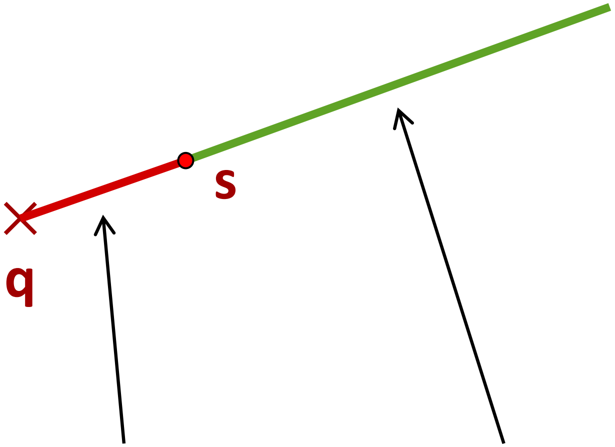
Simple Shadows

Global Illumination

More about homogenous coordinates

Projective Geometry

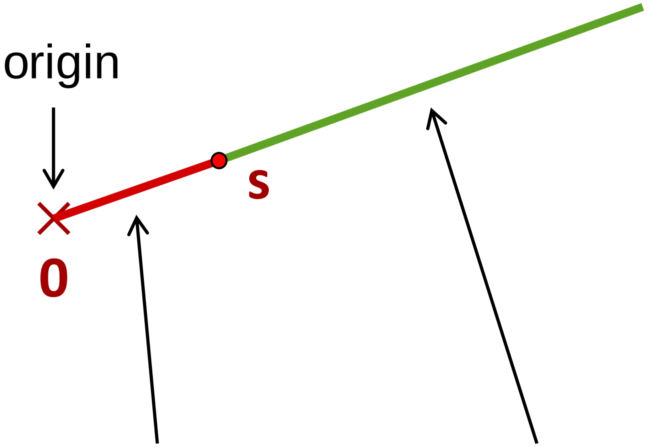
Constructing Projective Spaces



$(1-t)\mathbf{q}+t\mathbf{s}$
for $t \in [0,1]$

$(1-t)\mathbf{q}+t\mathbf{s}$
for $t > 1$

Constructing Projective Spaces



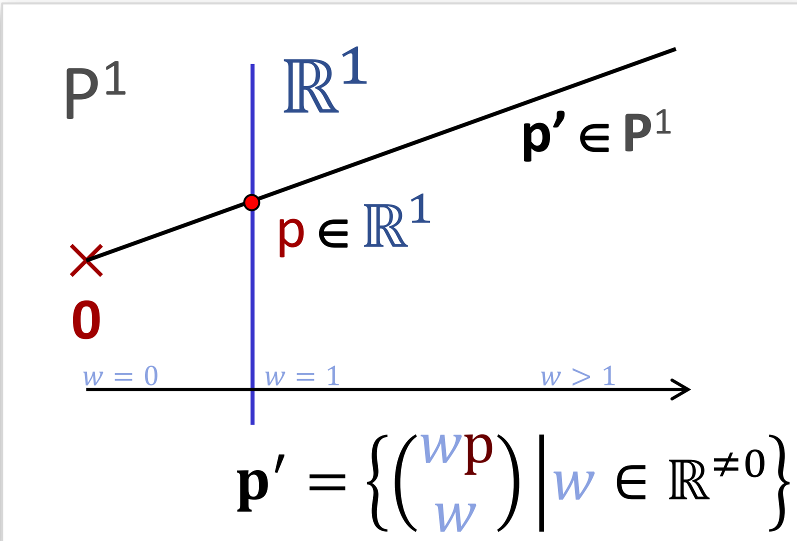
$(1-t)\mathbf{0}+t\mathbf{s}$
for $t \in [0,1]$

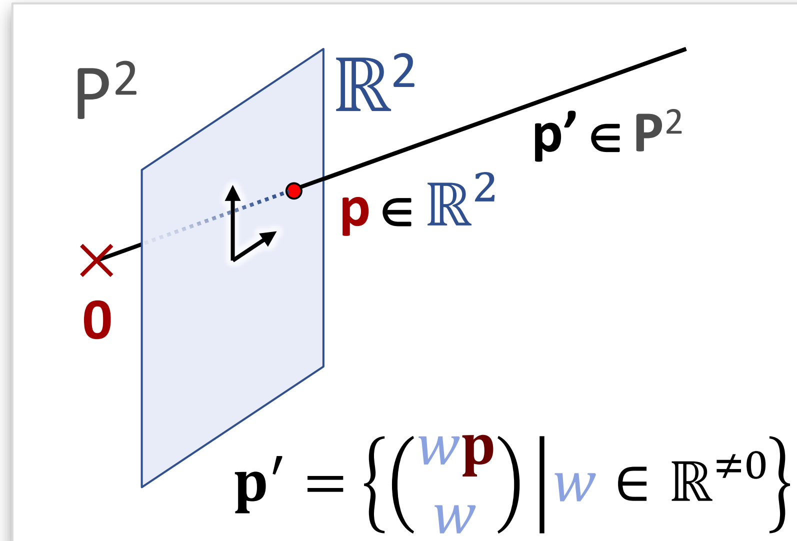
$(1-t)\mathbf{0}+t\mathbf{s}$
for $t > 1$

Since the first point is the origin,
we just have for all points along the ray:

$$\mathbf{s}' = t\mathbf{s} = \begin{pmatrix} ts_x \\ ts_y \end{pmatrix}$$

Constructing Projective Spaces

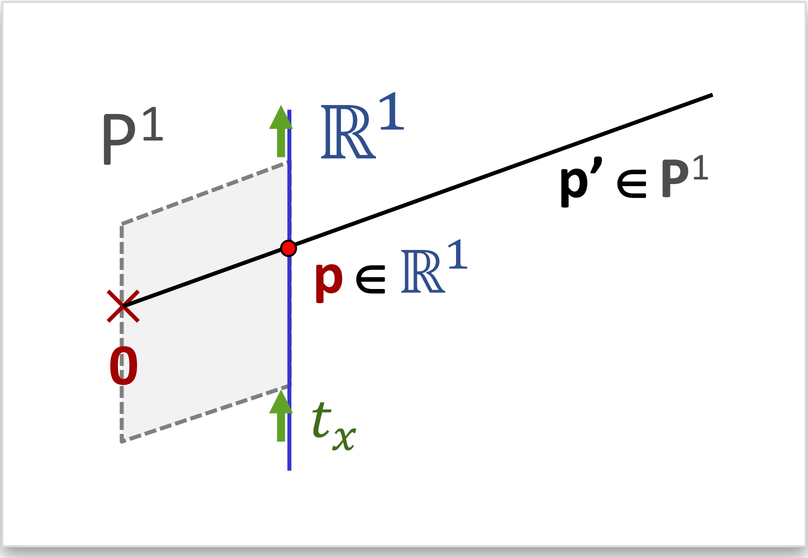




Projective Space P^d:

- Euclidean ("affine") space $\mathbb{R}^d$ embedded in $\mathbb{R}^{d+1}$
- At $w = 1$
- Identify all points on lines through the origin
  - *Represented* by the same Euclidean point

Constructing Projective Spaces



Translations:

- Sheering of the projective space

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

= Translation of the embedded affine space

Normalization

Conversion between

- Cartesian coordinates (Euclidian space)
- Homogeneous coordinates (projective space)

$$\mathbf{x} \rightarrow \begin{pmatrix} w\mathbf{x} \\ w \end{pmatrix}$$

Cartesian coordinates
(Euclidian space)

homogenous coordinates
(projective space)

$$\frac{1}{w}\mathbf{x} \leftarrow \begin{pmatrix} \mathbf{x} \\ w \end{pmatrix}$$

normalization*)

*) overloaded name
do not confuse with $\mathbf{x}/||\mathbf{x}||$

Vectors & Points

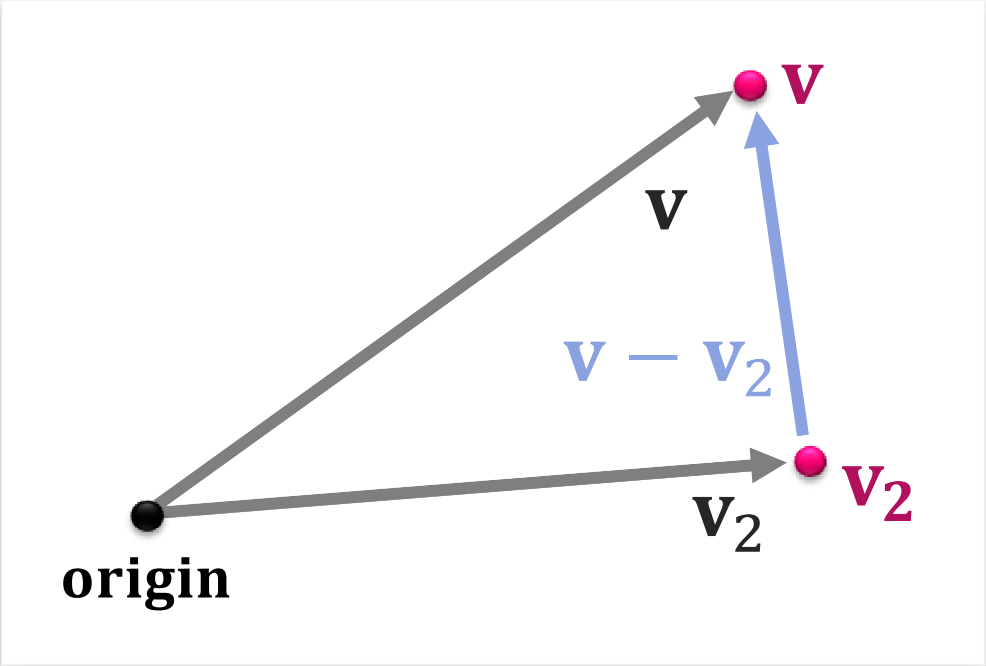
Interpretation

- Points: $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, w \neq 0$
- Vectors: $\begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}$ – "pure directions"

Vectors & Points

Rules

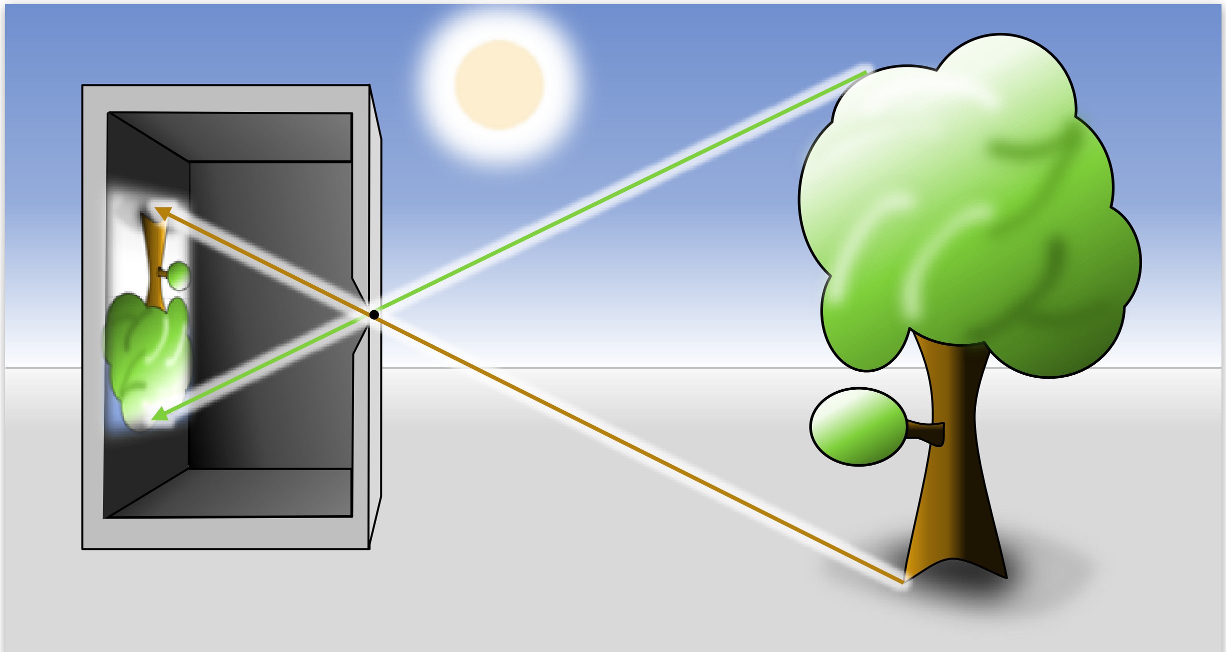
- Substracting points yields vectors
  - Normalize first!
- Vectors can be added to
  - Other vectors
  - Points (normalize first!)



Physics

Perspective Projection

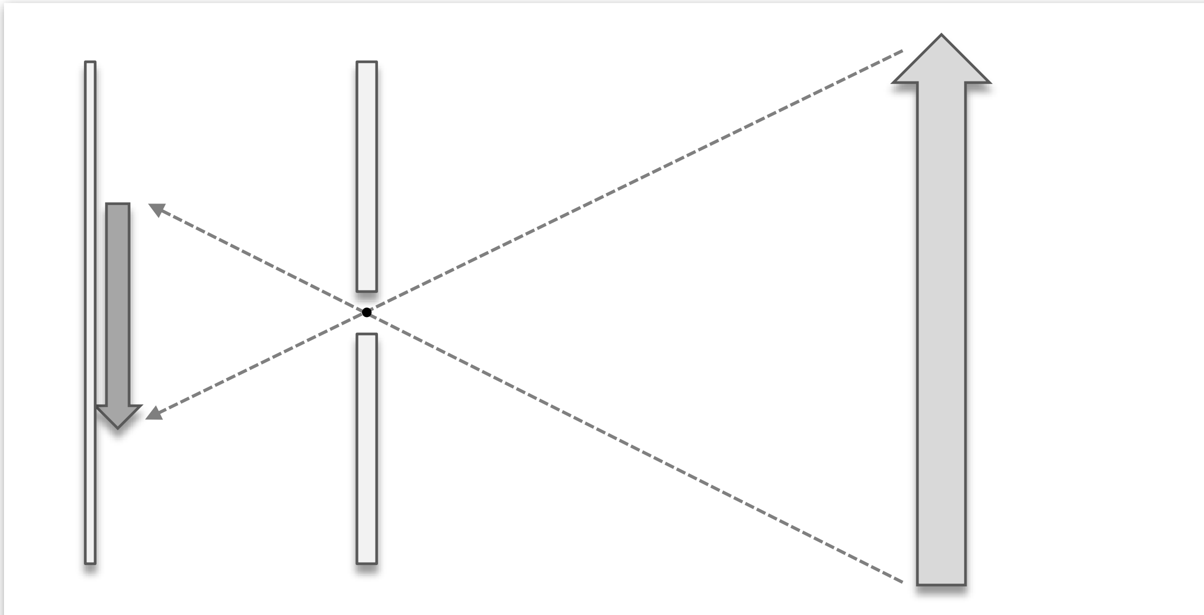
Pinhole Camera



Pinhole camera

- Create image by selecting rays of specific angles
- Low efficiency (small holes for sharp images)

Pinhole Camera



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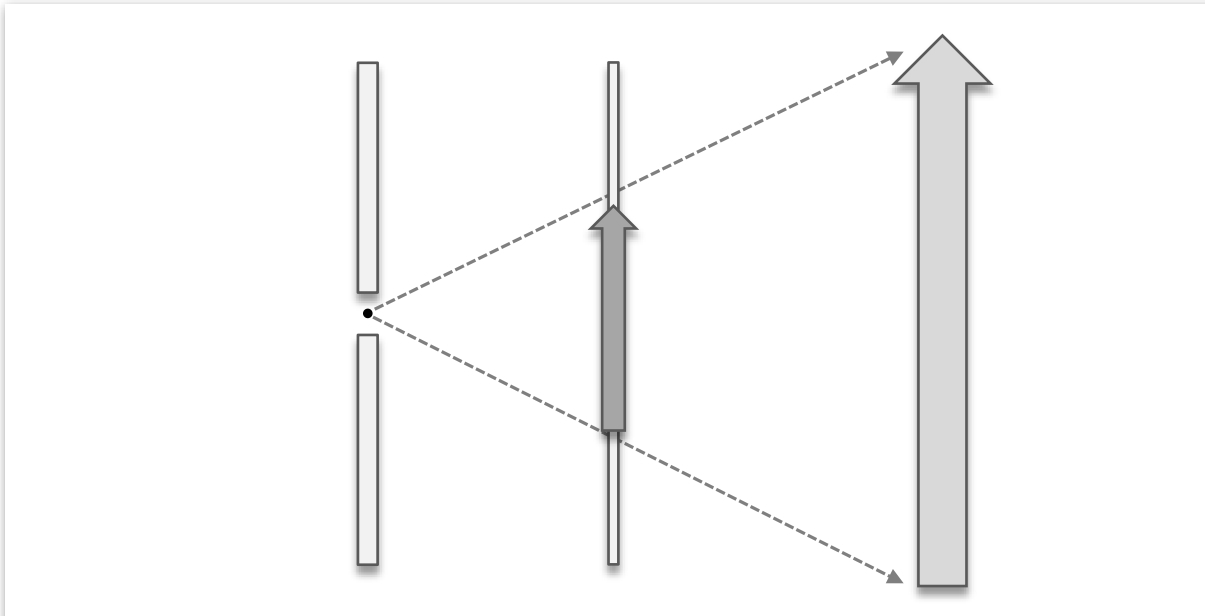
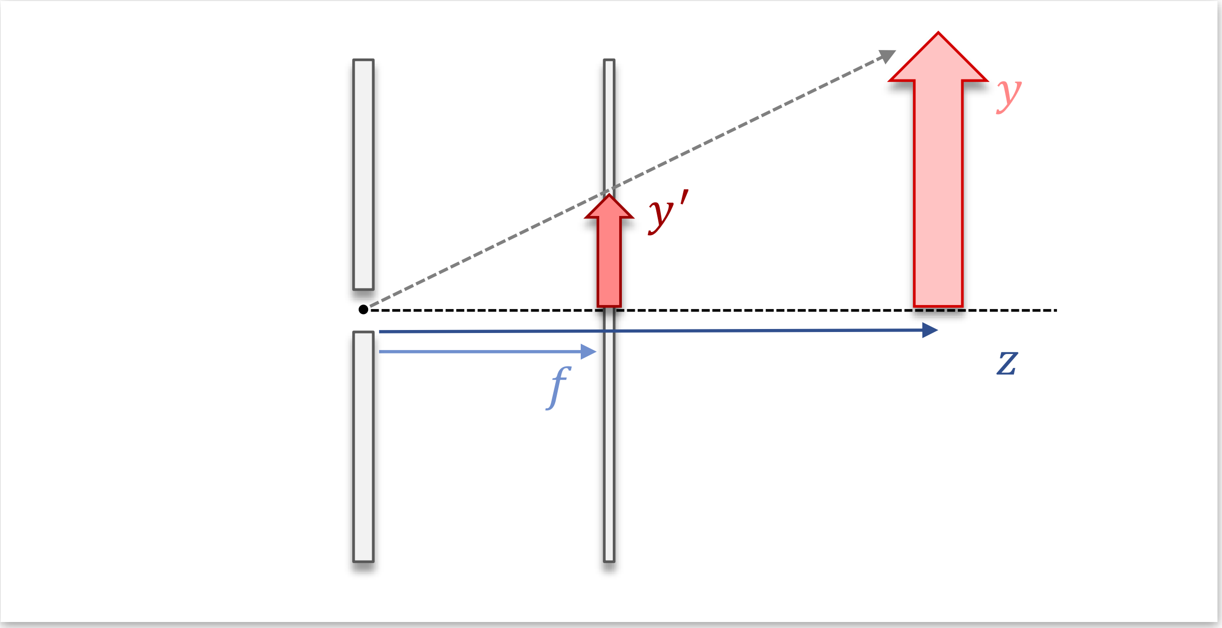


Central Projection

Pinhole Camera

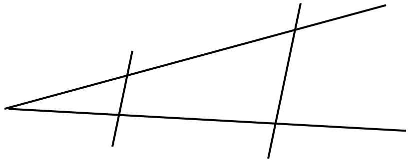


Central projection

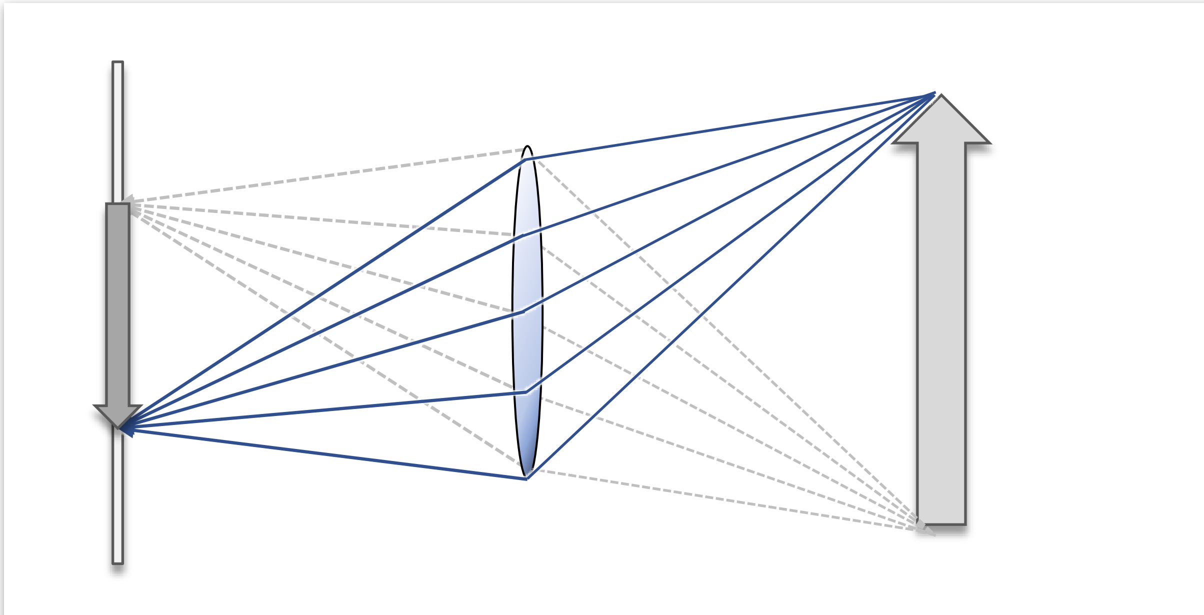
$$x' = f\frac{x}{z}$$

$$y' = f\frac{y}{z}$$

Proof:
Intercept theorem!



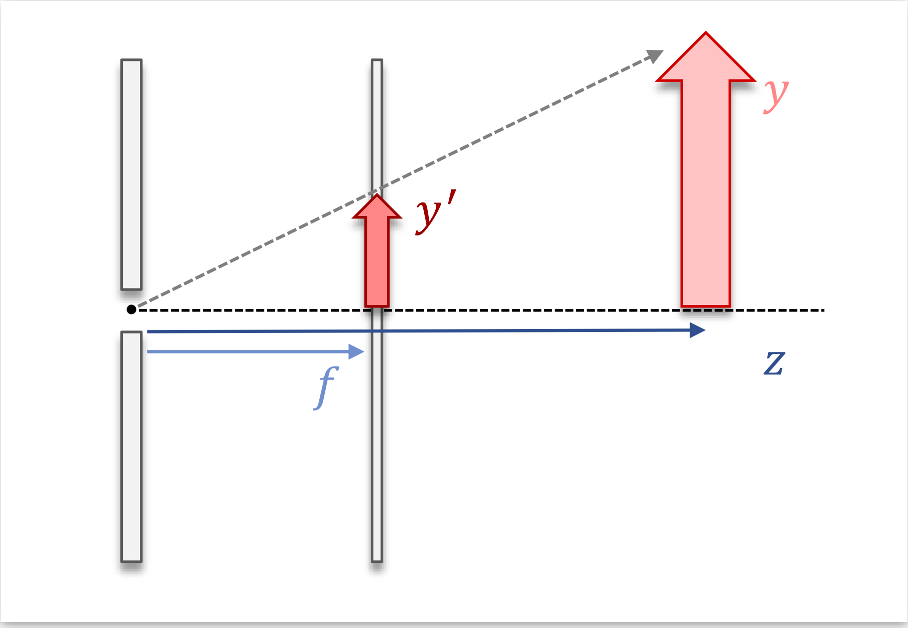
(Actual Camera)



Camera with Lens

- Higher efficiency (bundles many rays)
- Finite Depth of field
- We will consider pinhole cameras only.

Pinhole Camera



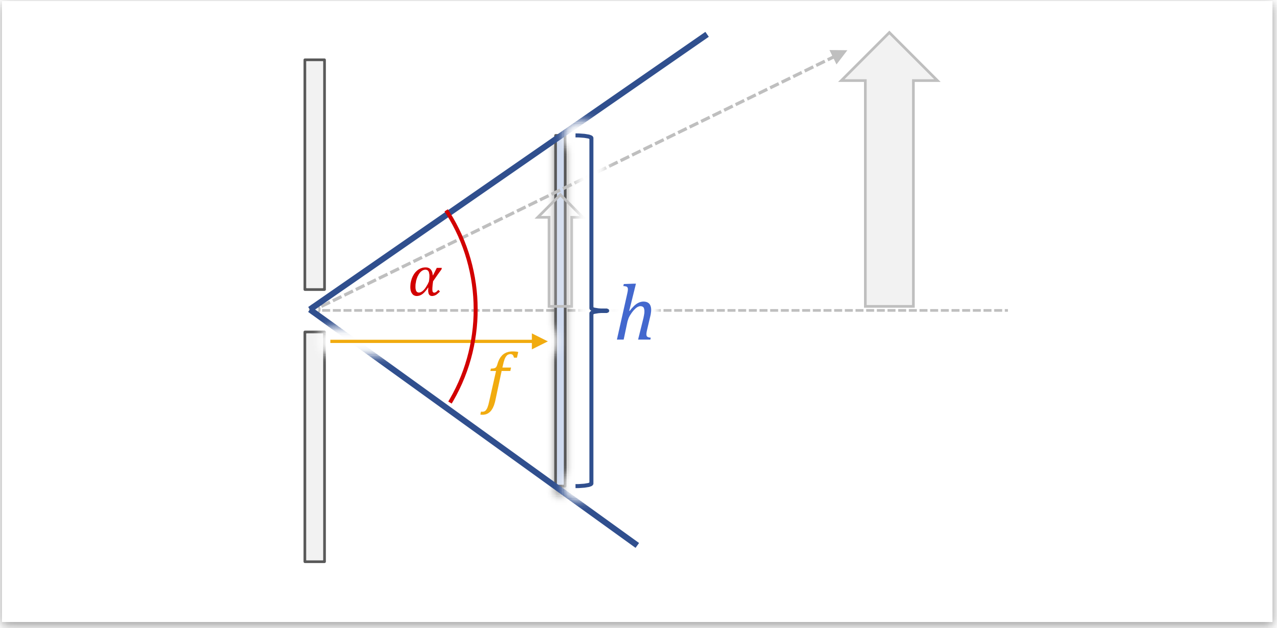
$$x' = f\frac{x}{z}$$

$$y' = f\frac{y}{z}$$

Undetermined degree of freedom

- Focal length vs. image size
- Source of a lot of confusion!

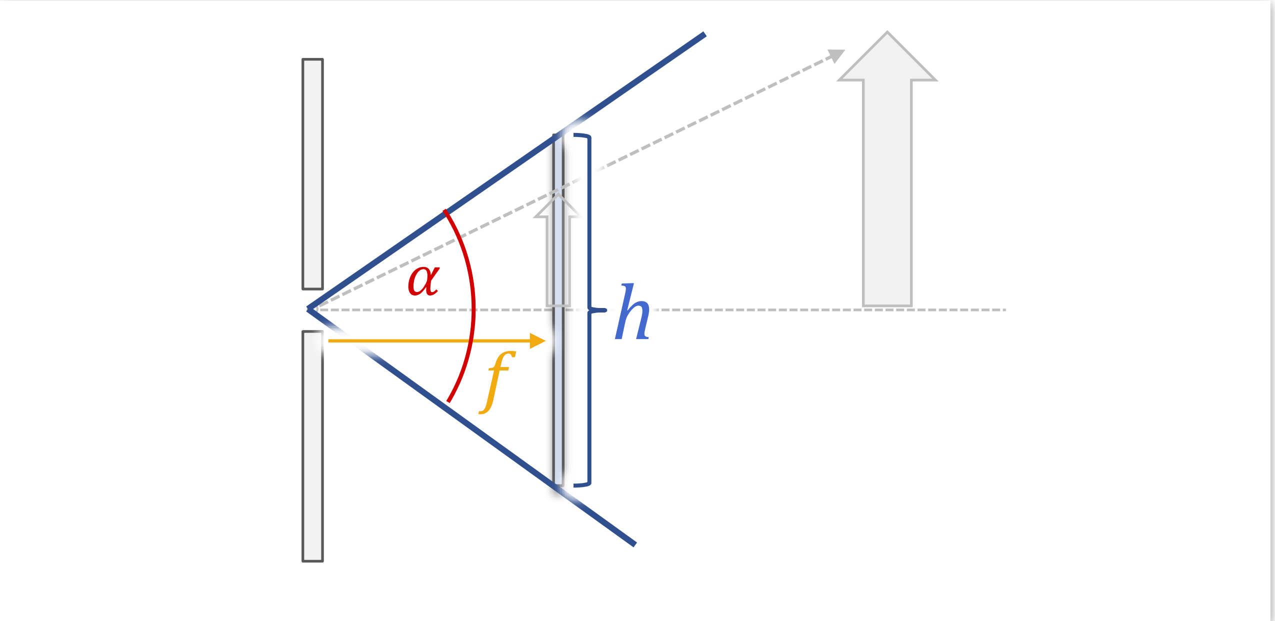
Pinhole Camera



Parameters

- h - size of the screen (pixels, cm, ± 1.0,...)
- f – focal length (classical photography)
- Meaningful parameter: α – viewing angle

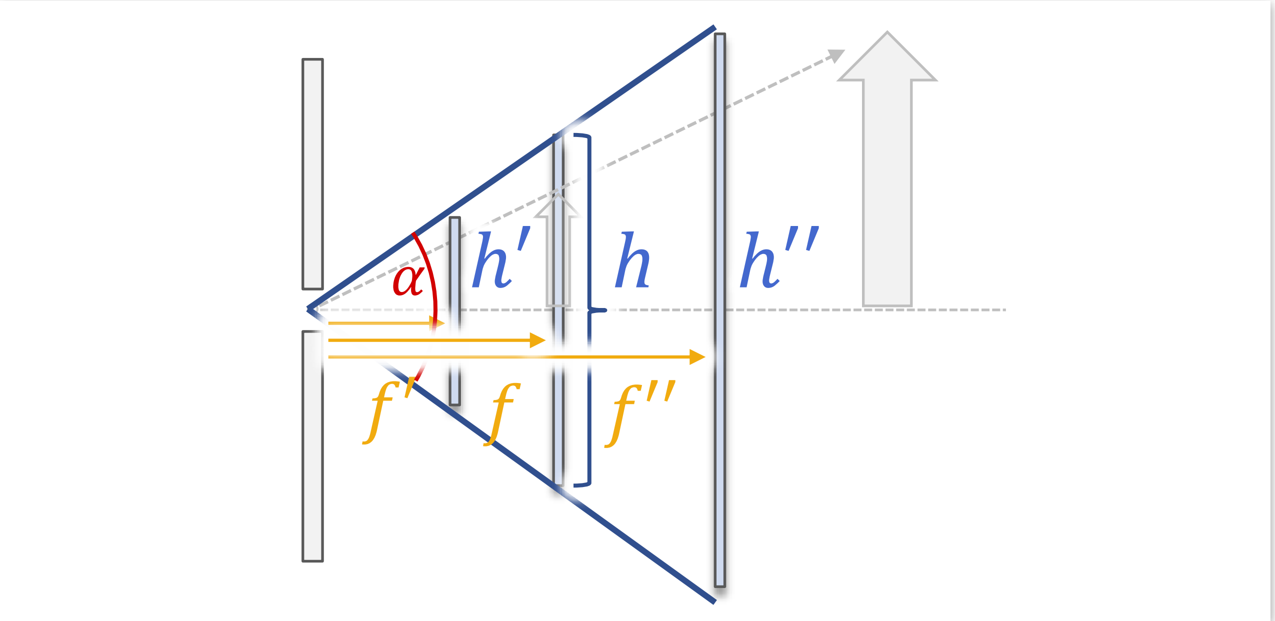
Pinhole Camera



Relation:

$$\tan\frac{\alpha}{2} = \frac{h}{2f}$$

Pinhole Camera

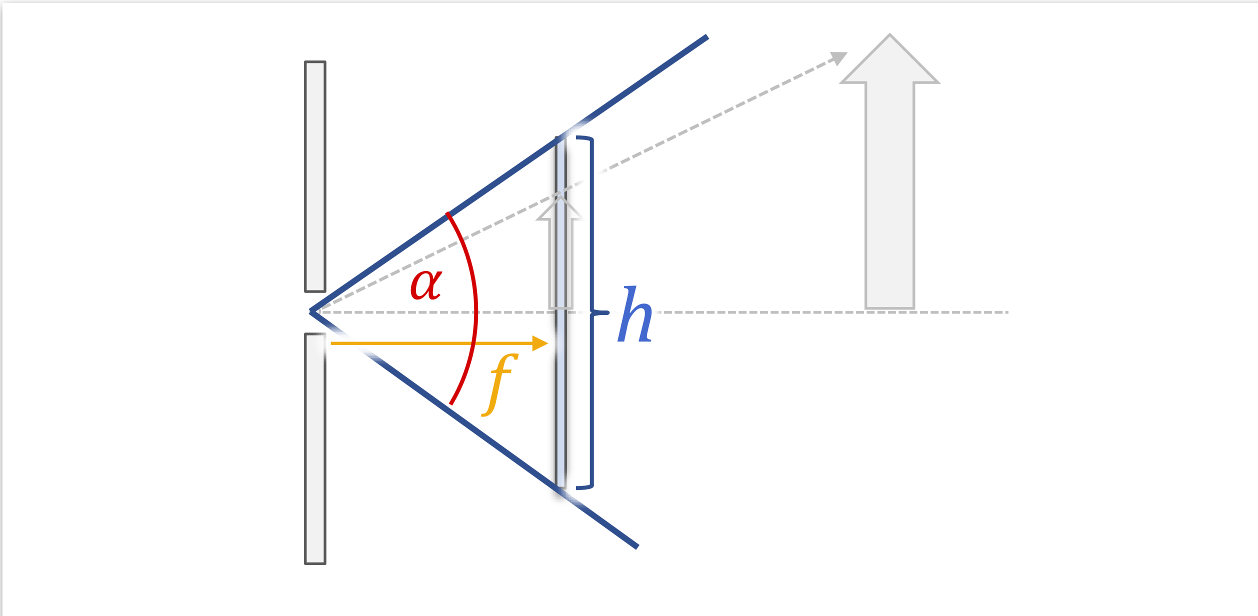


Invariance

$$\tan\frac{\alpha}{2} = \frac{h}{2f} = \frac{h'}{2f'} = \frac{h''}{2f''}$$

- Scaling h and f by a common factor: *no change*

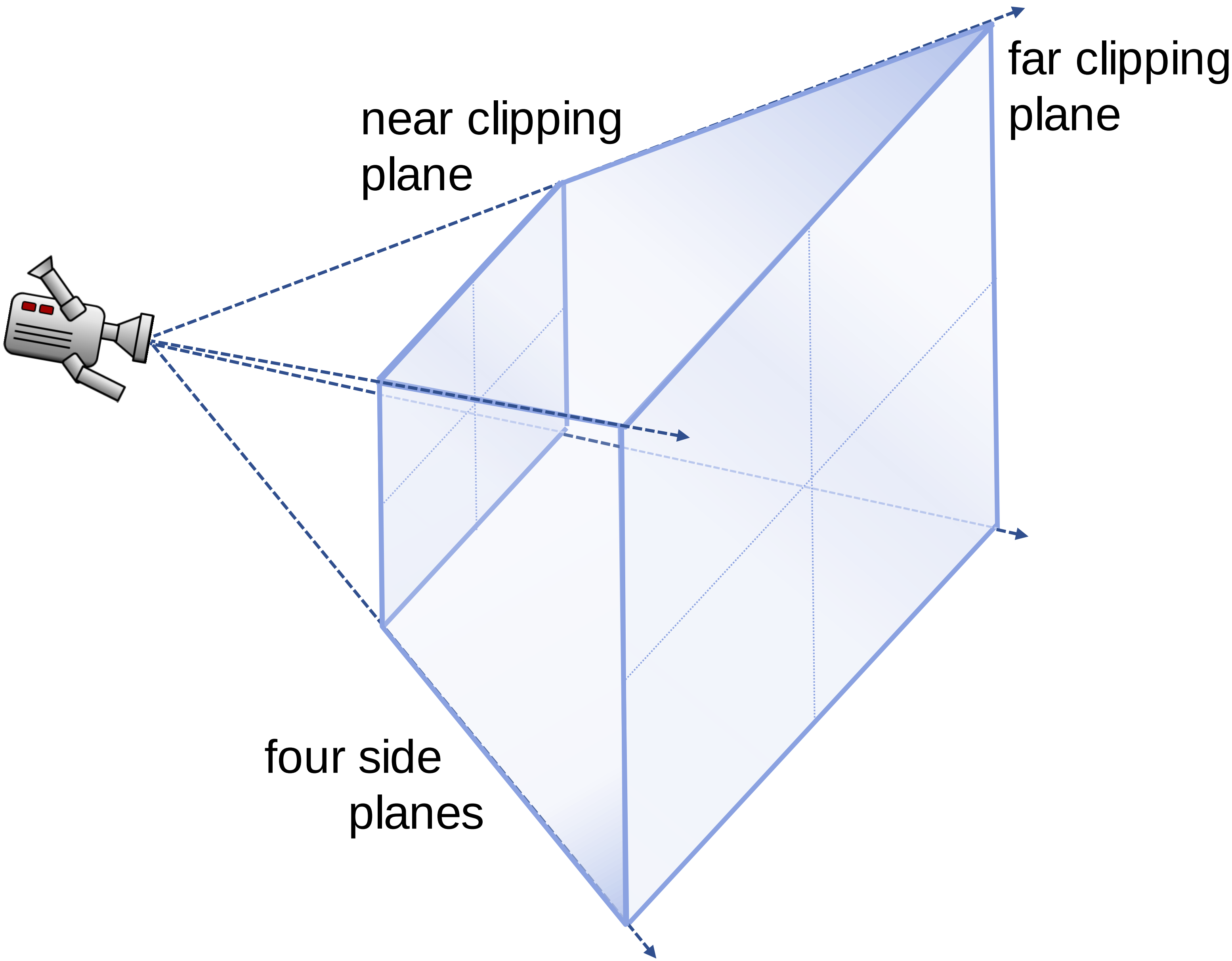
Pinhole Camera



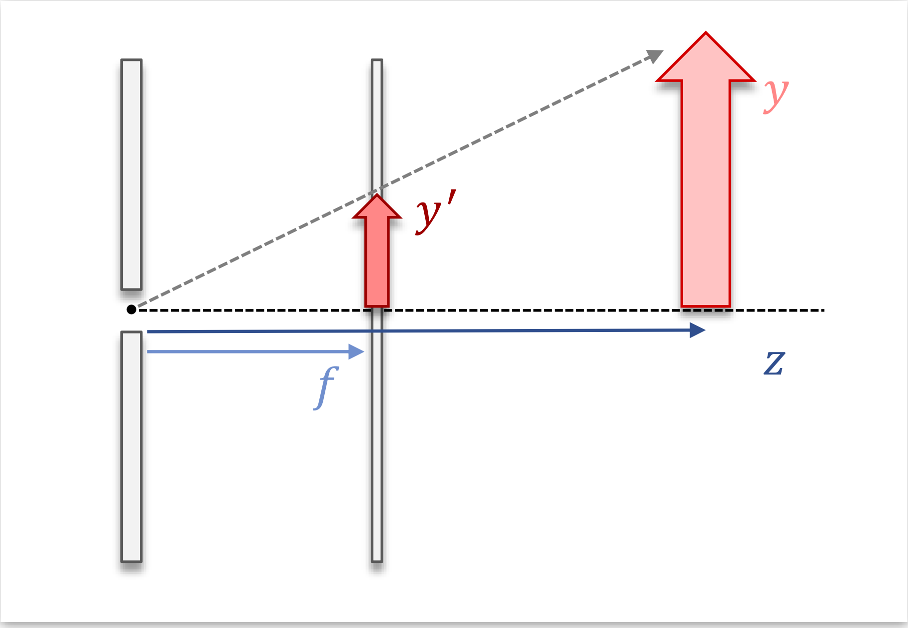
Typical choices (vertical angles)

- "Normal" perspective: $\alpha \approx 30°$ ("50mm" lens: 27°)
- Tele photography: $\alpha \approx 5° - 20°$ (275–70mm)
- Wide angle lens: $\alpha \approx 45° - 90°$ (28–12mm)

View Volume



Pinhole Camera



$$x' = f\,\frac{x}{z}$$

$$y' = f\,\frac{y}{z}$$

Our camera:

- Focus point: origin
- View direction: z-axis

Homogeneous Coordinates

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \underbrace{\begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{\text{Projection Matrix } \mathbf{P}} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$x' = fx$$
$$y' = fy$$
$$z' = z - 1$$
$$w' = z$$

before normalization

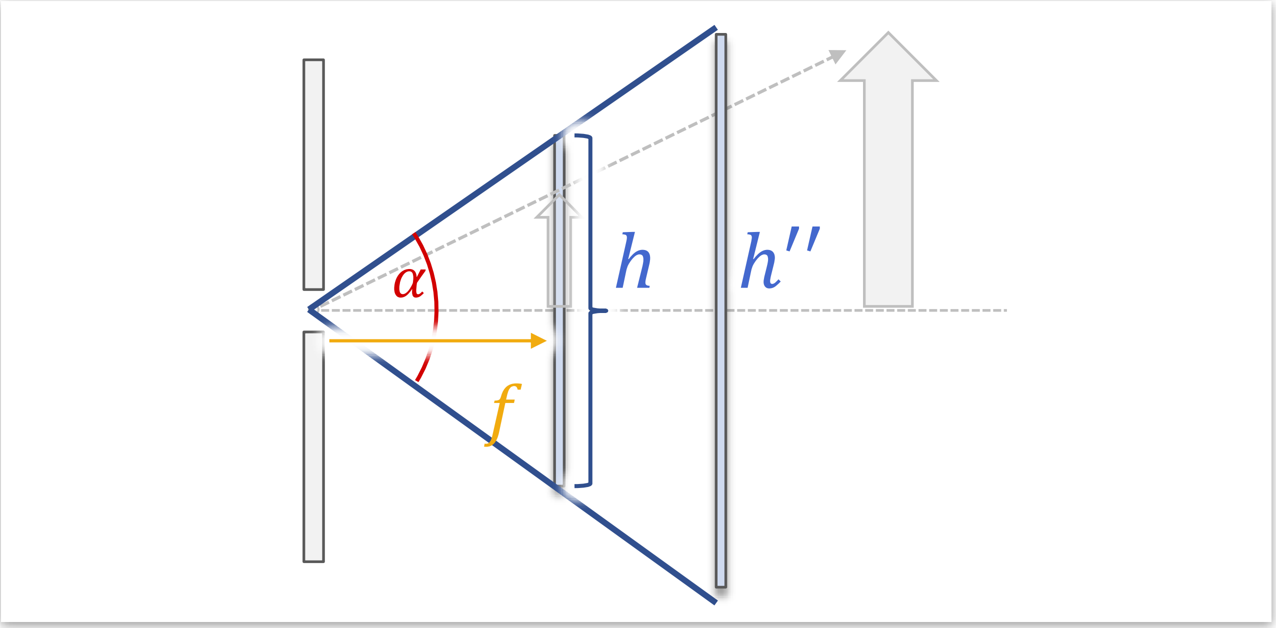
$$x' = f\frac{x}{z}$$
$$y' = f\frac{y}{z}$$
$$z' = \frac{z-1}{z}$$
$$w' = 1$$

after normalization

Write in homogeneous coordinates

- Third row is arbitrary (for now), not used.

View transform

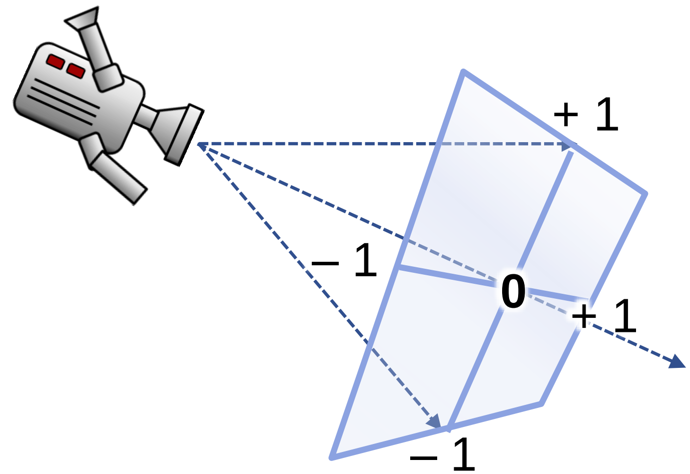


Reminder:

$$\tan\frac{\alpha}{2} = \frac{h}{2f}$$

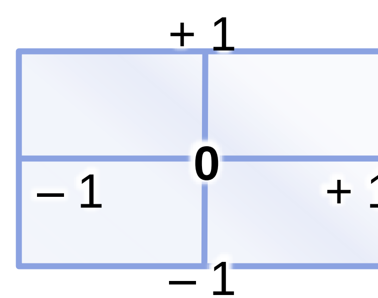
To Screen Coordinates

$$\begin{pmatrix} \frac{1}{\tan\left(\frac{\alpha}{2}\right)} & 0 & 0 & 0 \\ 0 & \frac{1}{\tan\left(\frac{\alpha}{2}\right)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Scale to unit screen coordinates

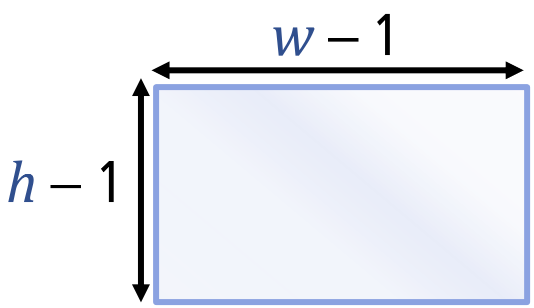
- We set f to 1 in previous matrix
- Third row is arbitrary (for now), not used.



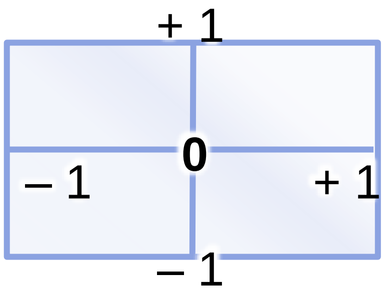
normalized screen coordinates

Aspect Ratio

$$\begin{pmatrix} \frac{1}{\frac{w}{h} \cdot \tan\left(\frac{\alpha}{2}\right)} & 0 & 0 & 0 \\ 0 & \frac{1}{\tan\left(\frac{\alpha}{2}\right)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



non-square screen



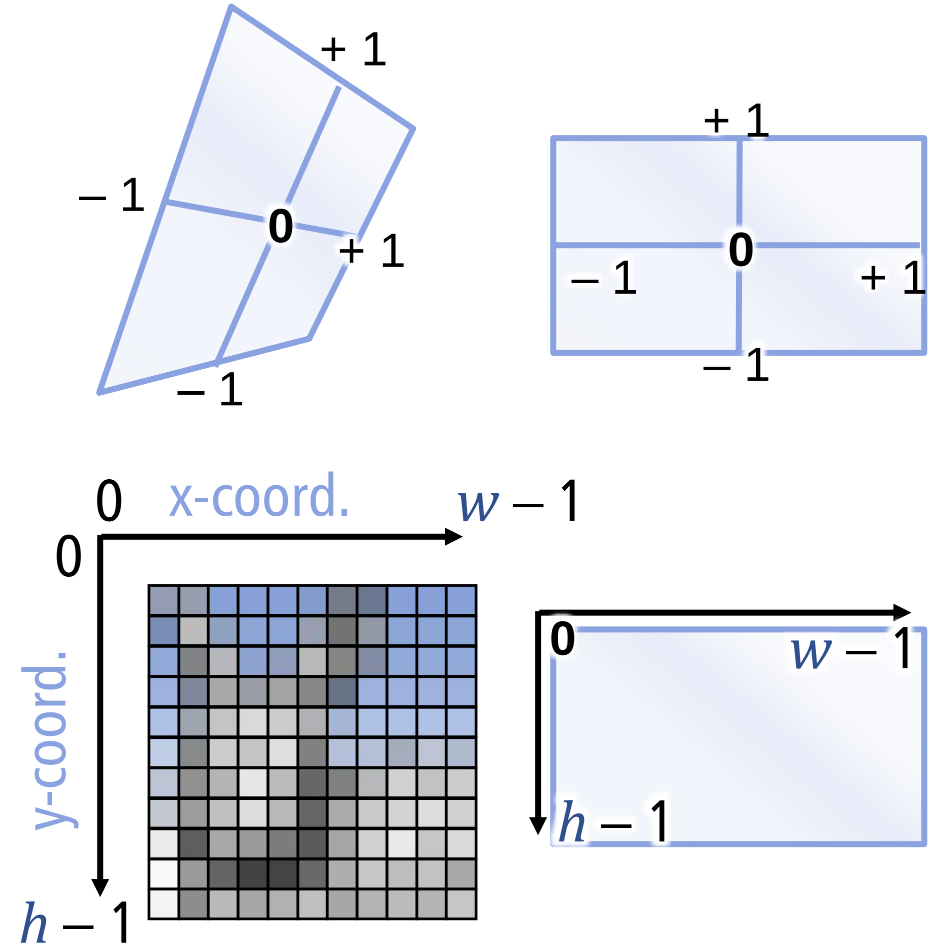
normalized screen coordinates

Non-square screens?

- Screen: $w \times h$ pixels
- Aspect ratio $\frac{w}{h}$
- Different horizontal angle!

To Screen Coordinates

$$\begin{pmatrix} w/2 & 0 & 0 & w/2 \\ 0 & -h/2 & 0 & h/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Scale to pixels

- Third row is arbitrary (for now), not used.

To Screen Coordinates

$$\begin{pmatrix} \frac{h/2}{\tan\left(\frac{\alpha}{2}\right)} & 0 & 0 & \frac{w/2}{\tan\left(\frac{\alpha}{2}\right)} \\ 0 & -\frac{h/2}{\tan\left(\frac{\alpha}{2}\right)} & 0 & \frac{h/2}{\tan\left(\frac{\alpha}{2}\right)} \\ 0 & 0 & a & b \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Overall

- Multiply both

$$a = \frac{z_{far} + z_{near}}{z_{near} - z_{far}}$$

$$b = \frac{2 \cdot z_{near} \cdot z_{far}}{z_{near} - z_{far}}$$

Additionally:
Also scale + shift such that

$$z' = \frac{z - 1}{z}$$

are in value [0..1] for inputs

$$z \in [z_{near}, z_{far}]$$

Summary

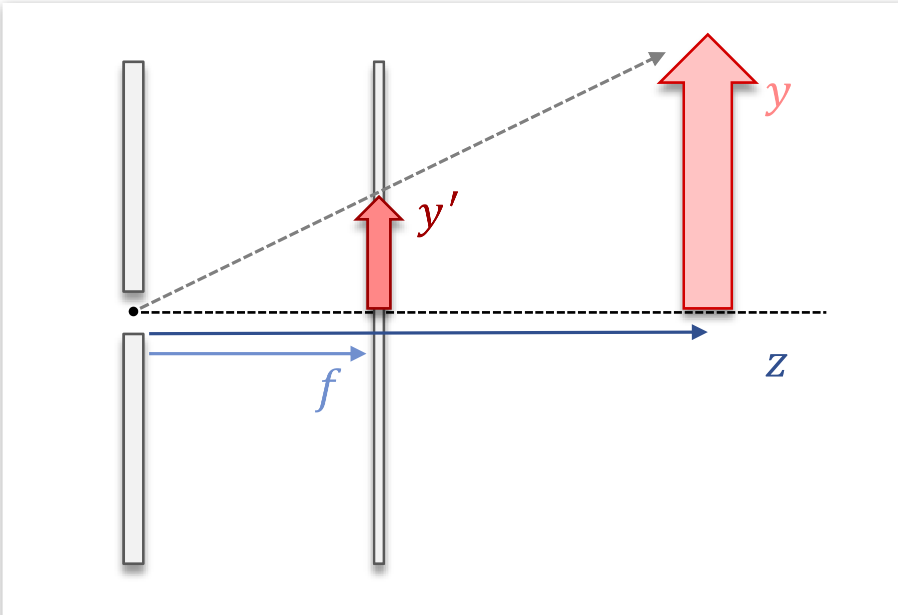
Projection matrix

$$\mathbf{P} = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Projection & conversion to screen coords

$$\mathbf{P}_s = \underbrace{\begin{pmatrix} w/2 & 0 & 0 & w/2 \\ 0 & -h/2 & 0 & h/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{scaling to pixels, upper left origin}} \cdot \underbrace{\begin{pmatrix} \frac{1}{\frac{w}{h}\tan\left(\frac{\alpha}{2}\right)} & 0 & 0 & 0 \\ 0 & \frac{1}{\tan\left(\frac{\alpha}{2}\right)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{normalized screen coord's}} \cdot \underbrace{\overset{(f = 1)}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}}_{\text{projection matrix}}$$

General Camera



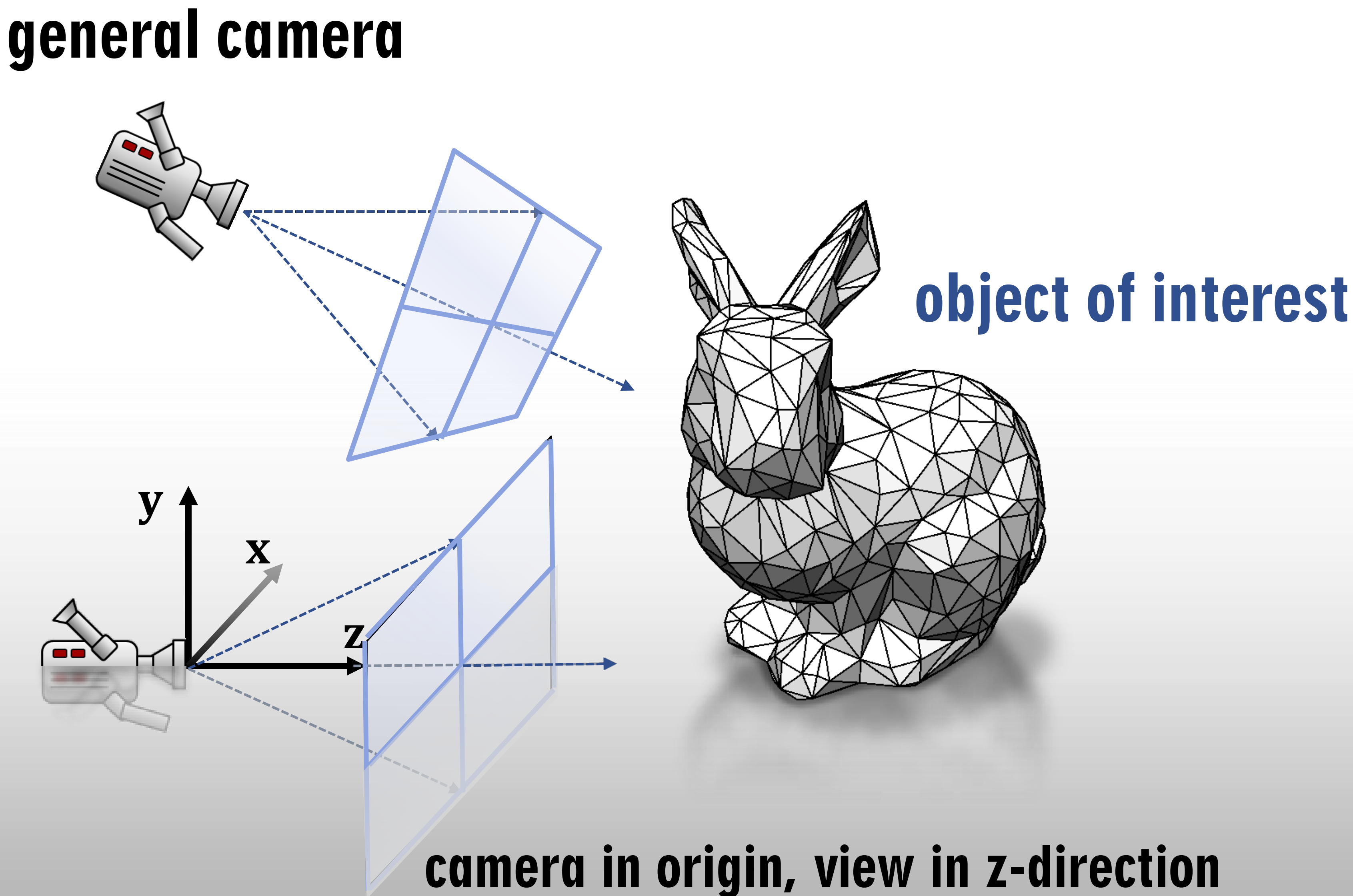
$$x' = f\,\frac{x}{z}$$

$$y' = f\,\frac{y}{z}$$

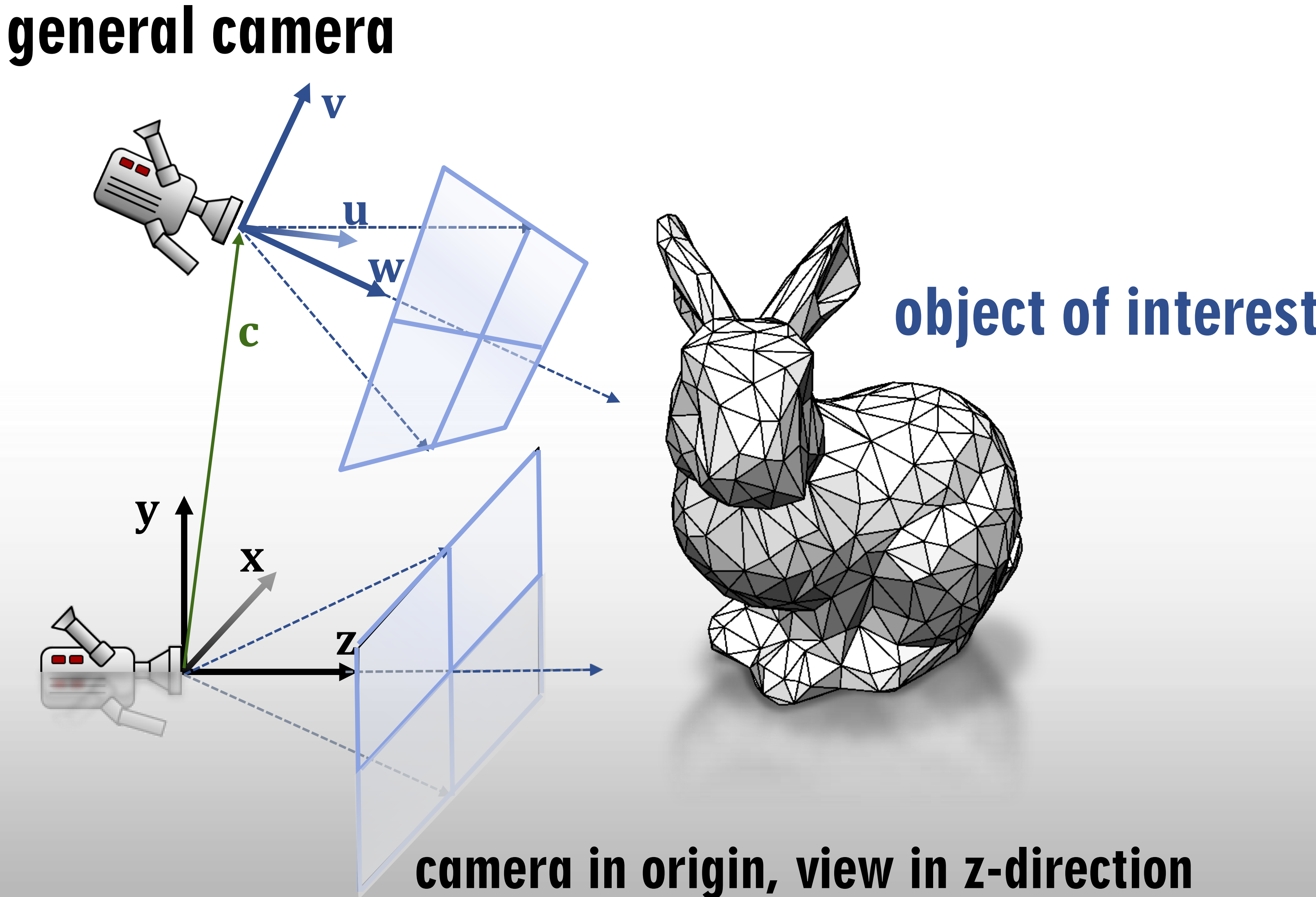
Our camera so far:

- Focus point: origin
- View direction: z-axis
- General position/orientation?

General Camera

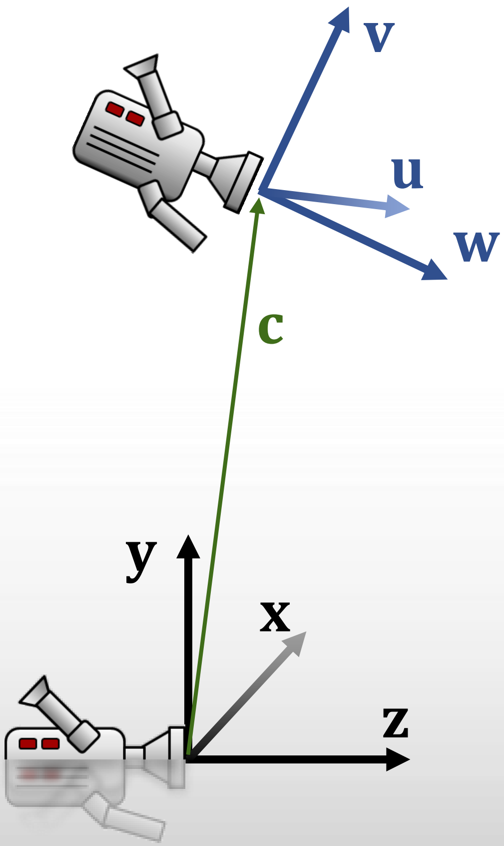


General Camera



General Camera

general camera

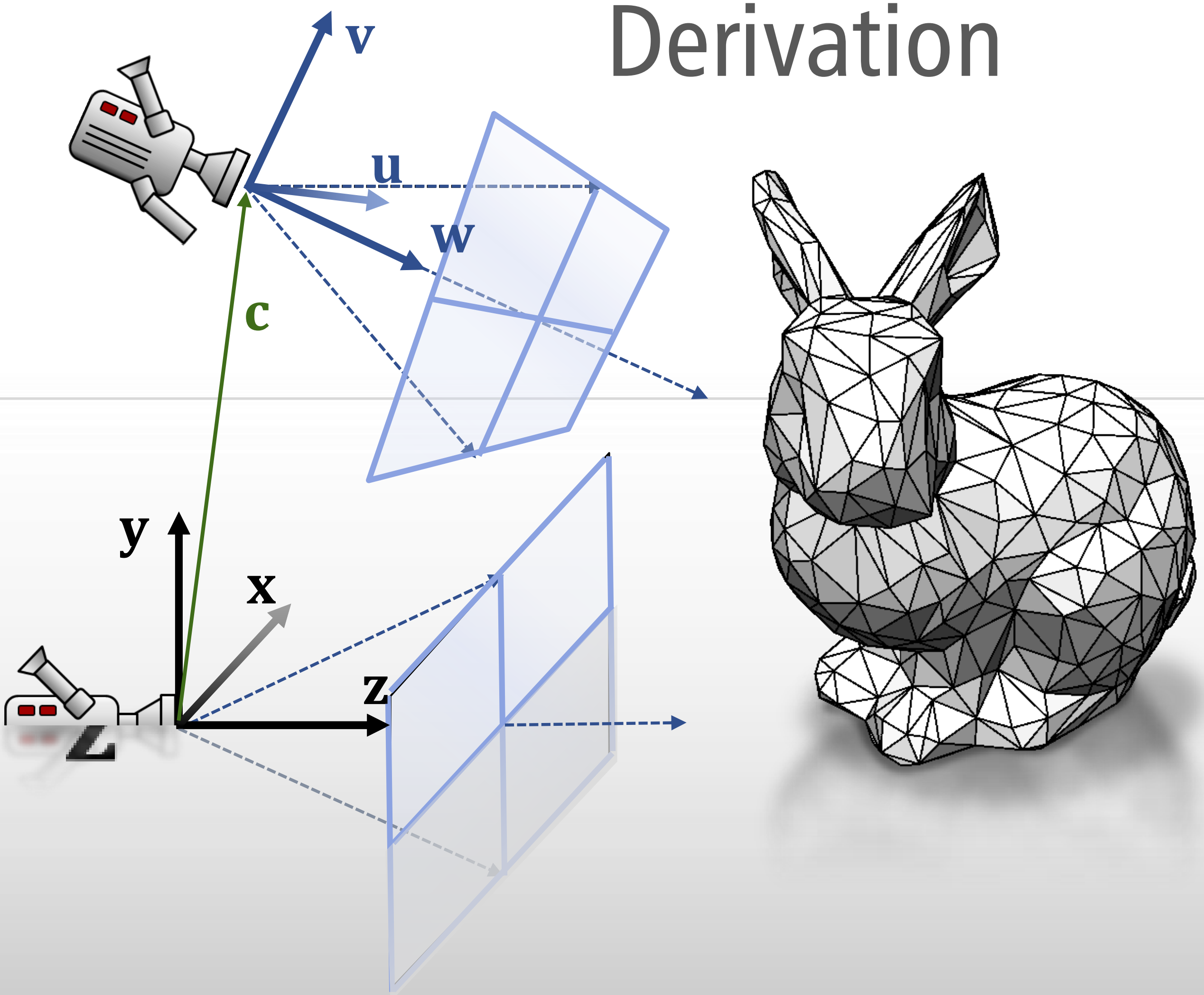


Camera coordinate system $(\mathbf{u}, \mathbf{v}, \mathbf{w})$
Origin: $\mathbf{c}$

Standard coordinates $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$

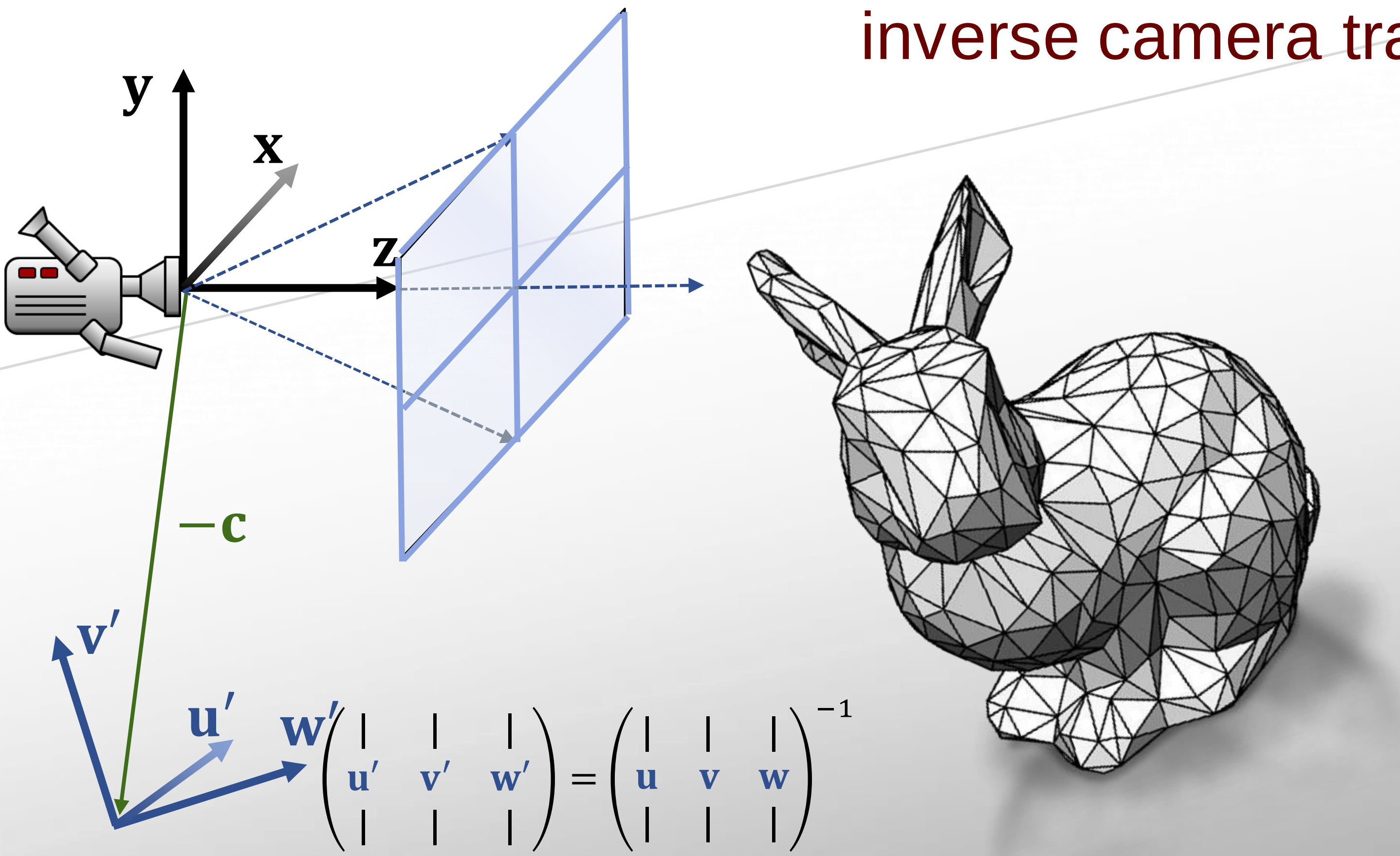
camera in origin,
view: z-direction

Derivation



Derivation

Same effect:
Transform the world with inverse camera transform

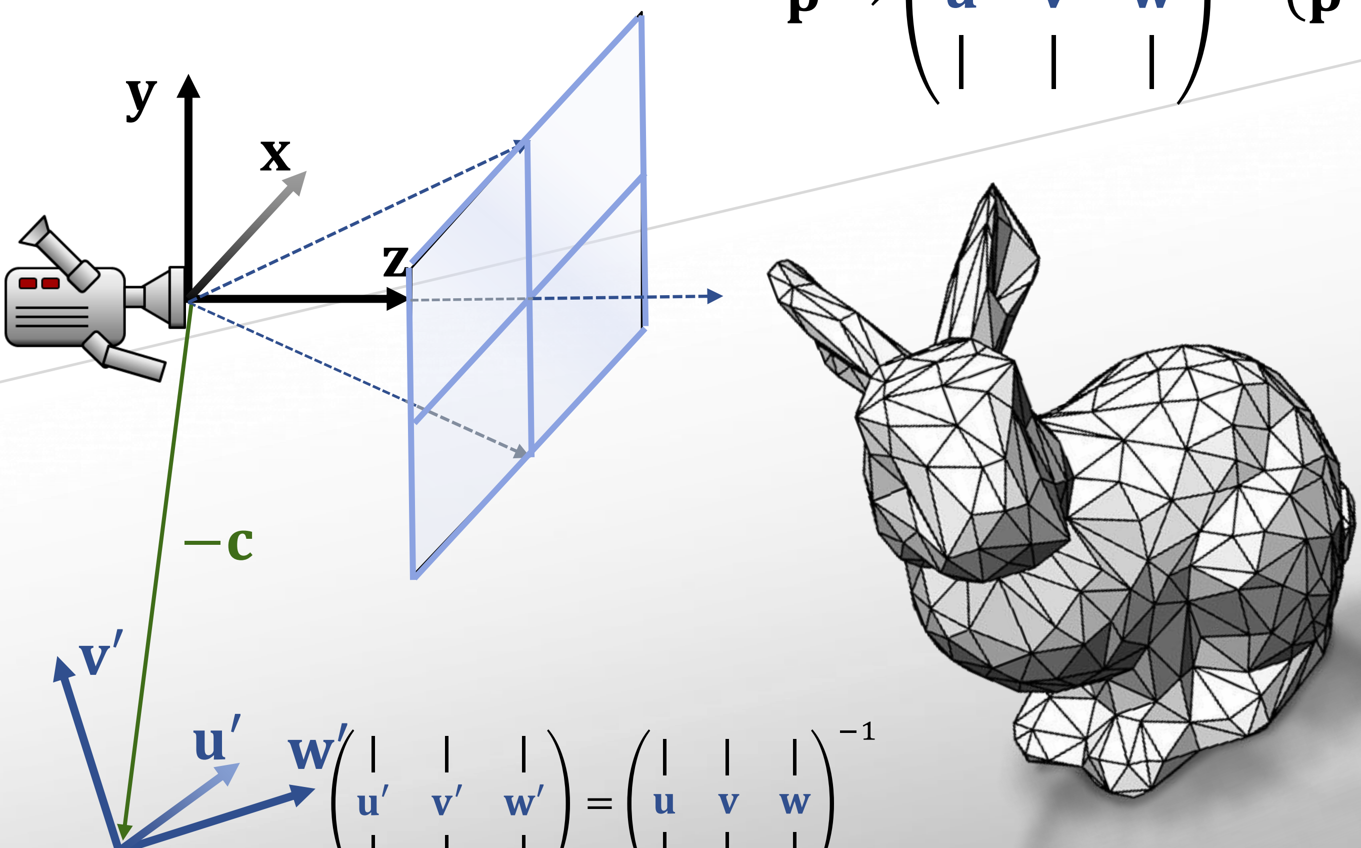


$$\begin{pmatrix} | & | & | \\ \mathbf{u}' & \mathbf{v}' & \mathbf{w}' \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ \mathbf{u} & \mathbf{v} & \mathbf{w} \\ | & | & | \end{pmatrix}^{-1}$$

Derivation

Transform:

$$\mathbf{p} \rightarrow \begin{pmatrix} | & | & | \\ \mathbf{u} & \mathbf{v} & \mathbf{w} \\ | & | & | \end{pmatrix}^{-1} (\mathbf{p} - \mathbf{c})$$

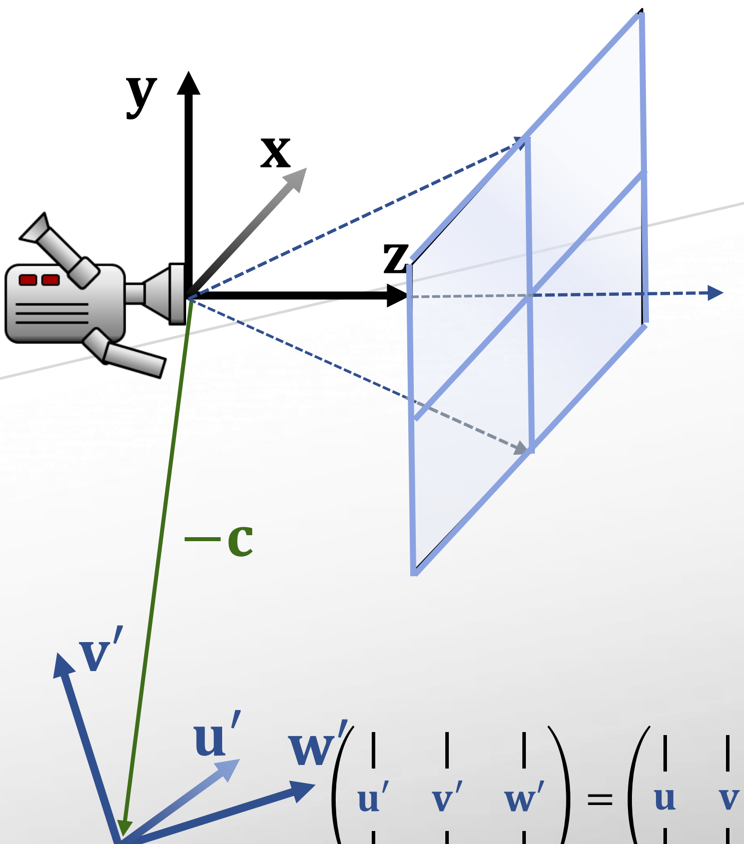


$$\begin{pmatrix} | & | & | \\ \mathbf{u}' & \mathbf{v}' & \mathbf{w}' \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ \mathbf{u} & \mathbf{v} & \mathbf{w} \\ | & | & | \end{pmatrix}^{-1}$$

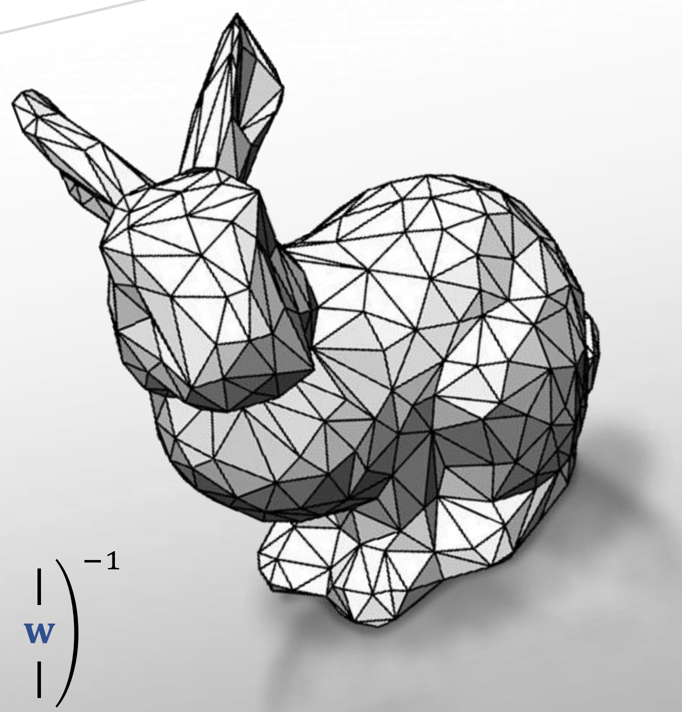
Derivation

Transform: $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ orthogonal!

$$\mathbf{p} \to \begin{pmatrix} - & \mathbf{u} & - \\ - & \mathbf{v} & - \\ - & \mathbf{w} & - \end{pmatrix} (\mathbf{p} - \mathbf{c})$$

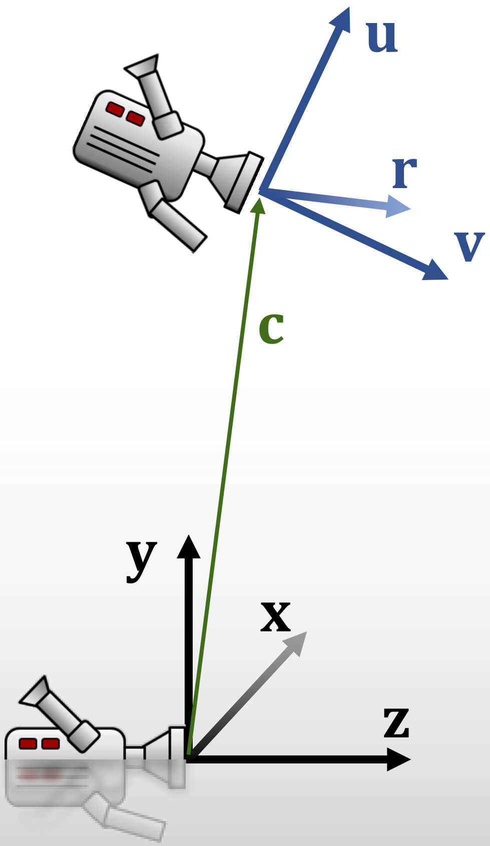


$$\begin{pmatrix} | & | & | \\ \mathbf{u}' & \mathbf{v}' & \mathbf{w}' \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ \mathbf{u} & \mathbf{v} & \mathbf{w} \\ | & | & | \end{pmatrix}^{-1}$$



General Camera

general camera



Camera coordinate system $(\mathbf{u}, \mathbf{r}, \mathbf{v})$

Origin: $\mathbf{c}$

Transform:

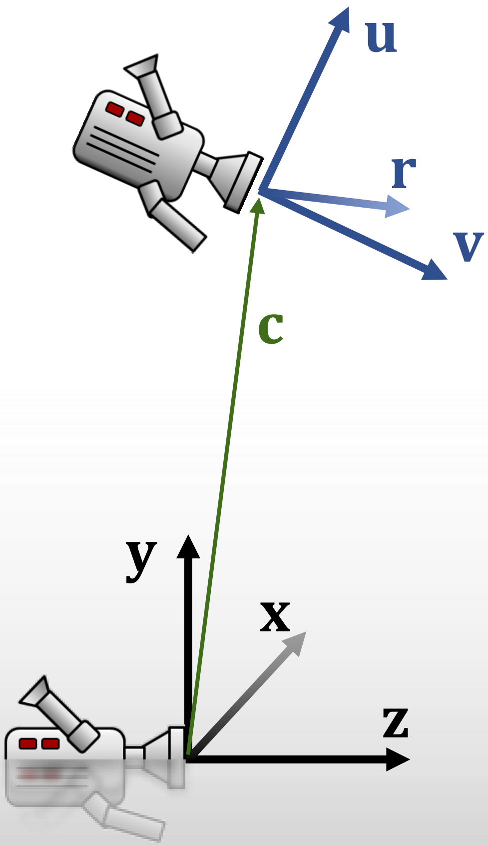
$$\mathbf{p} \to \begin{pmatrix} - & \mathbf{u} & - \\ - & \mathbf{v} & - \\ - & \mathbf{w} & - \end{pmatrix} (\mathbf{p} - \mathbf{c})$$

Standard coordinates $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$

camera in origin, view: z-direction

General Camera

general camera



Camera coordinate system $(\mathbf{u}, \mathbf{r}, \mathbf{v})$
Origin: $\mathbf{c}$

Homogeneous:

$$\mathbf{p} \to \begin{pmatrix} - & \mathbf{u} & - & | \\ - & \mathbf{v} & - & -\mathbf{c}' \\ - & \mathbf{w} & - & | \\ 0 & 0 & 0 & 1 \end{pmatrix} (\mathbf{p})$$

$$\mathbf{c}' = \begin{pmatrix} - & \mathbf{u} & - \\ - & \mathbf{v} & - \\ - & \mathbf{w} & - \end{pmatrix} \mathbf{c}$$

Summary

Projection (screen coord's)

$$\mathbf{P}_s = \begin{pmatrix} h/2 & 0 & 0 & w/2 \\ 0 & -h/2 & 0 & h/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\tan\left(\frac{\alpha}{2}\right)} & 0 & 0 & 0 \\ 0 & \frac{1}{\tan\left(\frac{\alpha}{2}\right)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$(f = 1)$

Add View Matrix

Benefit:

Still only one overall
4×4 matrix
to multiply with!

$$\mathbf{P}_s \cdot \begin{pmatrix} - & \mathbf{u} & - & | \\ - & \mathbf{v} & - & -\mathbf{c}' \\ - & \mathbf{w} & - & | \\ 0 & 0 & 0 & 1 \end{pmatrix}$$