Pricing American options under B-S assumptions using projected SOR

We discretise the time interval $[0, T]$ into m sub-intervals of length Δt such that

$$0 = t_0 < t_1 < ... < t_m = T$$

and we discretise the space interval $[0, R]$ into N intervals of length h such that

$$0 = x_0 < x_1 < ... < x_N = R.$$

Given the parabolic partial differential inequality problem

$$v_t - \frac{1}{2}\sigma^2 x^2 v_{xx} - rxv_x + rv \geq 0 \qquad \text{and} \qquad v \geq g$$

where $\sigma, r \in \mathbb{R}, \sigma > 0$, with initial condition $v(0, x) = g(x)$ and boundary conditions $v(t, 0) = f_0(t)$ and $v(t, R) = f_R(t)$, we approximate $v(x_i, t_{n+1})$ with v_i^{n+1} where

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} - \frac{1}{2}\sigma^2 x_i^2 \Big(\frac{v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1}}{h^2}\Big) - rx_i \Big(\frac{v_{i+1}^{n+1} - v_i^{n+1}}{h}\Big) + rv_i^{n+1} \geq 0$$

and

$$v_i^{n+1} \geq g(x_i).$$

The first discretised inequality can be expressed in matrix form as $\mathbf{D}\mathbf{v}^{n+1} - \mathbf{v}^n \geq 0$:

$$\begin{bmatrix} d_1 & u_1 & 0 & \cdots & 0 \\ l_1 & d_2 & u_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & l_{N-3} & d_{N-2} & u_{N-2} \\ 0 & \cdots & 0 & l_{N-2} & d_{N-1} \end{bmatrix} \begin{bmatrix} v_1^{n+1} \\ v_2^{n+1} \\ \vdots \\ v_{N-2}^{n+1} \\ v_{N-1}^{n+1} \end{bmatrix} - \begin{bmatrix} v_1^n + (\lambda_1 x_1^2) f_0(t_{n+1}) \\ v_2^n \\ \vdots \\ v_{N-2}^n \\ v_{N-1}^n + (\lambda_1 x_{N-1}^2 + \lambda_2 x_{N-1}) f_R(t_{n+1}) \end{bmatrix} \geq \mathbf{0}$$

This is a tri-diagonal system for each time step. Introduce the following notation:

$$\lambda_1 := \frac{\sigma^2 \Delta t}{2h^2} \qquad \lambda_2 := \frac{r\Delta t}{h}$$

Then, the leading diagonal of the matrix is

$$d_i = 1 + r\Delta t + 2\lambda_1 x_i^2 + \lambda_2 x_i$$

the upper diagonal is

$$u_i = -\lambda_1 x_i^2 - \lambda_2 x_i$$

and the lower diagonal is

$$l_i = -\lambda_1 x_{i+1}^2$$

Simple rearranging gives us the following problem:

$$-\mathbf{D}\mathbf{v}^{n+1} + \mathbf{v}^n \leq \mathbf{0} \tag{1}$$

$$-\mathbf{v}^{n+1} + \mathbf{g} \leq \mathbf{0} \tag{2}$$

$$(-\mathbf{D}\mathbf{v}^{n+1} + \mathbf{v}^n)^\top (-\mathbf{v}^{n+1} + \mathbf{g}) = 0 \tag{3}$$

This problem can be solved by the projected SOR method: at each time step we solve for $-\mathbf{v}^{n+1}$.