The β-Function in the Binary-Sympletic Representation

Piers Lillystone

I. BACKGROUND

This document is intended as a companion to the mathematica code written to simulate the 2-qubit contextual ψ-epistemic model. Here we will derive the closed form expression for the β-function in the binary symplectic representation used in the code. For full generality we shall derive the expression for n-qubits.

II. THE BINARY-SYMPLECTIC REPRESENTATION

Any n-qubit stabilizer operator can be represented as a $2n+1$ binary vector, where the $2n$ bits represents which members of a generating set for the Pauli group generates the given stabilizer operator, and the final bit determines the ± 1 phase of the stabilizer operator. So suppose we have some generating set, $\mathcal{G} = \{G_i\}_{i=1,..,2n}$, for the Pauli group;

$$\langle G_i \rangle_{i=1,..,2n} = \mathcal{P}_n.$$

Then any stabilizer operator can be expressed as;

$$S = (-1)^s i^{f(\{g_i\})} \prod_{i=1}^{2n} G_i^{g_i}, \ \forall S \in \mathcal{S}_n \,|\, s, g_i \in \mathbb{Z}_2, \tag{1}$$

where we note that the product is necessarily ordered as the G_i do not mutually commute. From this we can represent any stabilizer operator as a binary vector;

$$\mathcal{B}(S) = (g_1, g_2, ..., g_{2n}|s).$$

In standard quantum information literature it is common practise to take the generating set to be the single-qubit X and Z operators, so;

$$\mathcal{G}_{BS} = \{X_i\}_{i=1,..,n} \cup \{Z_i\}_{i=1,...,n}\,,$$

where $X_i = \mathbb{I}_{j\neq i} \otimes X$ and $Z_i = \mathbb{I}_{j\neq i} \otimes Z$ are used to suppress tensor notation. With this generating set the binary vector representing a stabilizer operator can be broken into an X vector and a Z vector;

$$\mathcal{B}(S) = (s_{x_1}, s_{x_2}, ..., s_{x_n}|s_{z_1}, s_{z_2}, ..., s_{z_n}|s) = (\vec{s}_x|\vec{s}_z|s)$$

From this we can reconstruct the stabilizer operator via;

$$S = (-1)^s i^{\vec{s}_x\cdot\vec{s}_z} X(\vec{s}_x) Z(\vec{s}_z) = (-1)^s \bigotimes_i^n i^{s_{x_i} s_{z_i}} X^{s_{x_i}} Z^{s_{z_i}} \tag{2}$$

which is the analogue of eq. (1).

III. THE β-FUNCTION IN THE BINARY SYMPLECTIC REPRESENTATION

In the ψ-epistemic model of the stabilizer formalism we are usually interested in the projective Pauli operators. These are the set of Pauli operators that are composed of tensor products of $\mathbb{I}, X, Y, Z$. From this we can represent any stabilizer operator, S_a, as a projective Pauli operator, P_a, with some ± 1 phase;

$$S_a = (-1)^{s_a} P_a, \tag{3}$$
$$P_a = i^{a_x . a_z} X(a_x) Z(a_z) = \bigotimes_i i^{a_{x_i} a_{z_i}} X^{a_{x_i}} Z^{a_{z_i}}, \tag{4}$$

where we have suppressed vector notation for simplicity.

The β-function is then defined as follows;

$$P_a P_b = (-1)^{\beta(a,b)} P_{a+b} | [a,b] = 0, \forall P_a, P_b, P_{a+b} \in \widetilde{\mathcal{P}}_n. \tag{5}$$

However, to derive an expression for the function it will be useful to drop the commuting assumption and use;

$$P_a P_b = i^{\gamma(a,b)} P_{a+b},$$

From this we can directly derive an expression for this γ function;

$$\begin{aligned} P_a P_b &= i^{a_x . a_z + b_x . b_z} \bigotimes_i^n X^{a_{x_i}} Z^{a_{z_i}} X^{b_{x_i}} Z^{b_{z_i}}, \\ &= i^{a_x . a_z + b_x . b_z} (-1)^{a_z . b_x} \bigotimes X^{a_{x_i} + b_{x_i}} Z^{a_{z_i} + b_{z_i}}, \\ &= i^{a_x . a_z + b_x . b_z + 2 a_z . b_x - (a_x + b_x).(a_z + b_z)} \bigotimes i^{(a_{x_i} + b_{x_i}).(a_{z_i} + b_{z_i})} X^{a_{x_i} + b_{x_i}} Z^{a_{z_i} + b_{z_i}}, \\ &= i^{a_z . b_x - a_x . b_z} \bigotimes i^{(a_{x_i} + b_{x_i}).(a_{z_i} + b_{z_i})} X^{a_{x_i} + b_{x_i}} Z^{a_{z_i} + b_{z_i}} \end{aligned}$$

Now if we re-investigate eq. (4) we note that by assumption $a_{x_i}, a_{z_i} \in \mathbb{Z}_2$ and therefore $a_{x_i} a_{z_i} \in \mathbb{Z}_2$. However, in the above we have $(a_{x_i} + b_{x_i}).(a_{z_i} + b_{z_i}) \in \{0, 1, 2\}$, to convert it to the correct form we use;

$$a + b \bmod 2 = a + b - 2ab, \; a, b \in \mathbb{Z}_2.$$

Focusing on the i exponent in the tensor product we have;

$$\begin{aligned} (a_{x_i} + b_{x_i}).(a_{z_i} + b_{z_i}) &= ([a_{x_i} + b_{x_i}] \bmod 2 + 2 a_{x_i} b_{x_i}).([a_{z_i} + b_{z_i}] \bmod 2 + 2 a_{z_i} b_{z_i}), \\ &= ([a_{x_i} + b_{x_i}] \bmod 2).([a_{z_i} + b_{z_i}] \bmod 2) \\ &\quad + 2 a_{x_i} b_{x_i} [a_{z_i} + b_{z_i}] \bmod 2 + 2 a_{z_i} b_{z_i} [a_{x_i} + b_{x_i}] \bmod 2, \\ &= ([a_{x_i} + b_{x_i}] \bmod 2).([a_{z_i} + b_{z_i}] \bmod 2) \\ &\quad + 2 a_{x_i} b_{x_i} (a_{z_i} + b_{z_i} - 2 a_{z_i} b_{z_i}) + 2 a_{z_i} b_{z_i} (a_{x_i} + b_{x_i} - 2 a_{x_i} b_{x_i}), \\ &= ([a_{x_i} + b_{x_i}] \bmod 2).([a_{z_i} + b_{z_i}] \bmod 2) \\ &\quad + 2 a_{x_i} b_{x_i} (a_{z_i} + b_{z_i}) + 2 a_{z_i} b_{z_i} (a_{x_i} + b_{x_i}). \end{aligned} \tag{6, 7}$$

Where we have repeatedly used the fact that anything with a factor of 4 disappears.

Finally re-substituting the above into the previous derivation, noting that $a_z . b_x - a_x . b_z = [a, b]$ which is the binary symplectic inner product, and defining a triple product of vectors as $a.b.c = \sum_i a_i b_i c_i$, we get;

$$P_a P_b = i^{[a,b] + 2 a_x . b_x . (a_z + b_z) + 2 a_z . b_z . (a_x + b_x)} P_c, \tag{8}$$

and therefore;

$$\gamma(a,b) = [a,b] + 2a_x.b_x.(a_z + b_z) + 2a_z.b_z.(a_x + b_x), \tag{9}$$

$$\beta(a,b) = \frac{1}{2}[a,b] + a_x.b_x.(a_z + b_z) + a_z.b_z.(a_x + b_x) \mid [a,b] \in 0, 2. \tag{10}$$

Where we note the binary sympletic product indicates two Pauli operators commute if it is even.

Finally we note that if we define $P_a P_b P_c = (-1)^{\beta(a,b,c)} \mathbb{I} \mid [a,b] = 0$, where $c_x = a_x + b_x \bmod 2$ and $c_z = a_z + b_z \bmod 2$, the β function can be expressed as, via eq. (6), as;

$$\beta(a,b,c) = \frac{1}{2}[a,b] + a_x.b_x.c_z + a_z.b_z.c_x,$$

which could have a useful operational interpretation.