

Mass Transfer in Binary Stars

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Abstract

In this paper, I will investigate the properties of mass transfer in binary stars, including the Roche Lobe model, evolutionary stages, evolution, rate, progenitors, and resultant stars. I will use the Roche Lobe model to catalyze the stars into Detached (Wind Accretion), Roche Lobe Overflow (RLO), and Contact Binaries (CB). I will then look at stars that fit these stages and investigate them further using data from example systems corroborated with data from POSYDON.

I found that mass transfer effected the stars by ...

Introduction

Most of the stars we see in the night sky are not actually single stars, instead consisting of multiple stars orbiting a common center of mass. A pair of such stars is called a binary. These binaries are formed in the same nature of single stars, that being through a molecular. However, due to instability in the formation process, two stars are formed instead of one [1]. These stars orbit a common center of mass (COM). The stars in these binary pairs will have a different evolutionary track than a single star, as it is likely for the two stars to interact at some point during their life [2] because of their proximity in formation. As these stars evolve and interact with each other, it will drastically affect their evolution.(see fig. 1)

1.1 Mass Transfer in Common Binaries

In systems with small orbital separations and a large mass ratio a process called mass transfer (MT) can occur [3]. This is where one star called a donor will

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“donate” mass to the other star, called the accretor. This is likely to occur at some point during the binaries’ lifespan, leading to different evolutionary outcomes as compared to single star evolution [3] (see fig. 1). However, depending on the duration of mass transfer and the star type, this process can be hard to detect observationally. Because of this, a large of studies is conducted on systems with more extreme star, such as neutron stars (NS) or black holes (BH), where the process of mass transfer leads to much more pronounced effects. This is because as the mass is transferred to the BH or NS the process leads to a large spike in X-ray emissions (sect. 1.6.1), which can more easily measured as compared to the mass transfer process between a red giant (RG) a main sequence (MS) star, which have little or no x-ray emission.

1.2 Roche Lobe Model

The Roche Lobe (RL) model was proposed by Édouard Roche and defines the gravitational potential of a binary through a simple model. Simply put, it defines the region around a star where it can hold onto its mass (i.e. great enough gravitational potential). If one of the stars in the binaries' mass overflows said lobe, it will transfer mass to its binary pair. This model can be used to classify binary star populations into various populations, including **Detached Binaries** (where neither star has filled their potential), **Contact Binaries** (where both stars have filled their potentials), and **Roche Lobe Overflow (RLO)** systems (where one star has filled its potential, leading to mass transfer to an accretor).

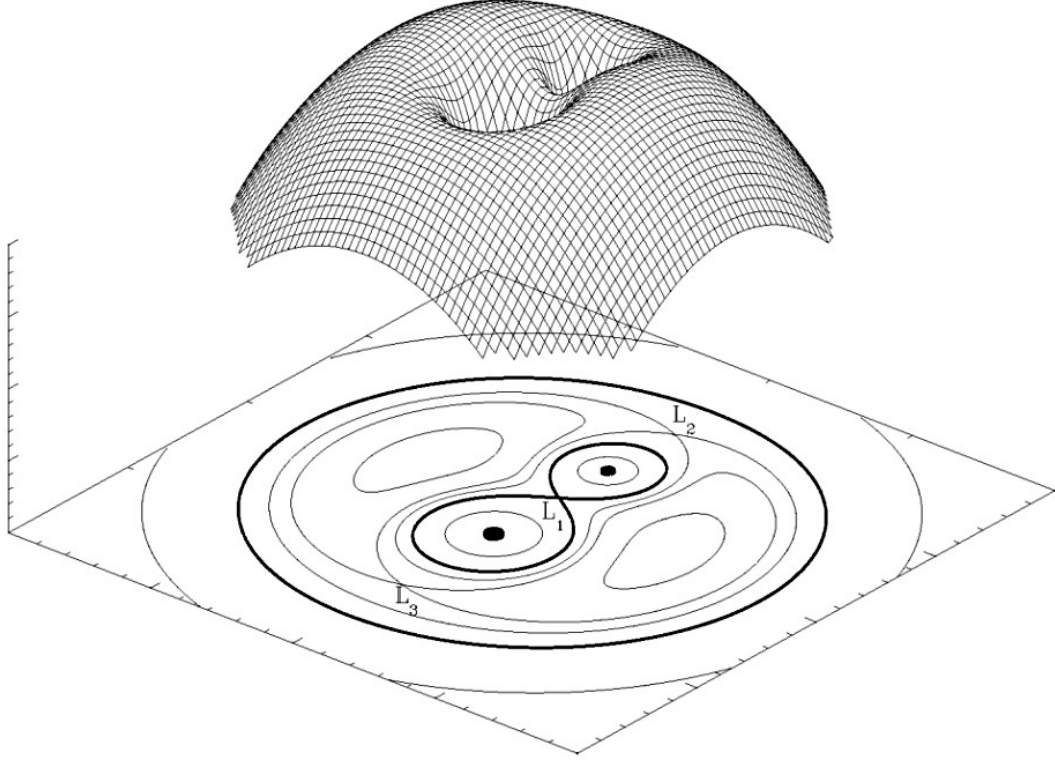


Figure 2: *A 3D representation of the gradient of the Roche lobe*
Reprint from [5]

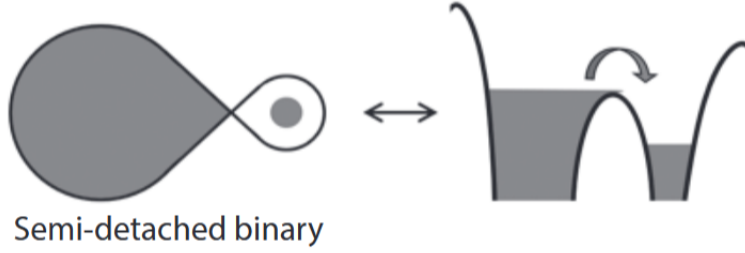


Figure 3: *Reprint from [3]*

1.3 Roche Lobe Overflow

In systems where one star fully fills its RL (sect. 1.2, fig. 3) mass begins to be transferred to its binary partner. This process is called Roche Lobe Overflow (RLO) and is defined by a donor and accretor star (sect. 1.1). This is incredibly likely to happen at some point in a binaries' lifespan drastically affecting the course of evolution [3][4]. Depending on the systems star types, masses, and orbital eccentricity, this process of mass transfer will either be stable or unstable. When this process is unstable, it leads to either another stage called common envelope (sect. 1.5.1) or a rapid merger [3]. However, if the process of mass transfer is stable, the two stars will remain detached, slowly exchanging mass [4] [3].

We are able to observe this very commonly in Low Mass X-ray Binaries, where a donor star transferred mass to an accretor which is a compact object (a black hole or neutron star). This process produces X-rays (sect. 1.6.1) which we can measure to understand the processes within the binary in greater detail.

This process, similar to section 1.5.1, can be either stable or unstable. A large factor which determines this stability is the evolutionary stage of the donor star [3], as upon the onset RLO, some donors stars' radius will increase, leading to even more rapid RLO. This is primarily due to the nature of the stars' envelope, with radiative envelopes shrinking upon mass loss and convective envelope swelling [3].

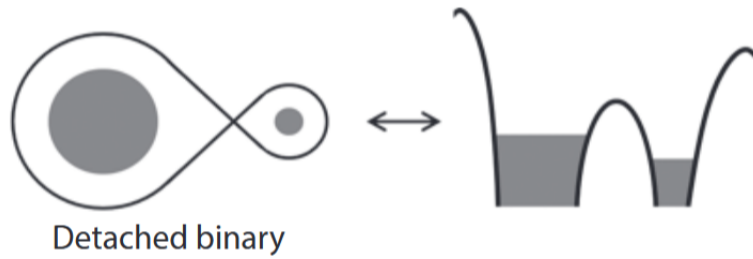
In intermediate mass x-ray binaries, a system will survive this process of mass transfer if the unevolved or early stage donor star mass is between $2 < M_2/M_\odot < 5$ [3]. In low mass x-ray binaries (LMXBs) it is $M_2 \gtrsim 1.8M_\odot$. Systems with $M_2 > 2$ Will be unstable for giant donors in later stage RLO [3].

I used the system V404 Cygni (11) with data from [6] and [7] as an example of this behavior, as its magnitude, proximity to Earth, and quantity of studies make it an ideal choice to understand how systems like it behave.

1.3.1 Mass transfer through L_1

In this process of mass transfer, the mass will be transferred through the Lagrangian point L_1 , as it is the point of lowest potential between them, as seen in figure 2 [3]. The mass at L_1 feels an equal gravitational pull from both stars. The mass is then transferred through a pressure differential, where the donor star pushes the mass through.

1.4 Detached binaries



Reprint from [3]

Figure 4: *Graphic of the RL model with regard to how much potential is being filled in a detached binary. Note here that has both a ‘top-down’ perspective and from the side.*

Systems where neither star fills its potential fully (see fig. 4) are called detached binaries. Despite the fact that these systems have not filled their Roche Lobes, mass transfer is still possible through a processes called wind accretion (see 1.4.1). We see this predominantly in systems called *High Mass X-ray Binaries*, where a supergiant star transfers mass to a compact object via wind accretion. This process leads to an increase in X-ray Emission (sect. 1.6.1), which we can easily measure [3]. It is important to note the majority these systems are not experiencing full-blown RLO (sect1.3), however, they tend to be incredibly close to doing so [3]. It is important to note that these systems transfer can transfer mass through wind accretion, atmospheric Roche Lobe Overflow, and full-blown RLO.

I used the system Vela X-1 [8] as an example of this, as it is generally regarded as the “archetypical wind accretor” [8]

1.4.1 Wind Accretion Mass Transfer

Wind accretion is very different from normal mass transfer in a binary. All stars produce ‘wind,’ i.e. mass which is pushed away from the star. Most stars cause a

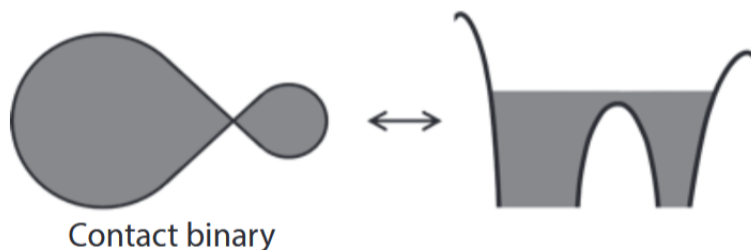


Figure 5: *Reprint from [3]*

process called stellar wind, occurring when mass from the outer envelope is ejected at speed from the star. All stars have experience varying strengths of stellar wind, with some winds having very high velocities, and others much lower. [9] In binary stars this process allows mass to be transferred from a donor to accretor.

1.5 Contact Binary

In binaries with prominent RLO it is possible for the donor star to fill up not just its own Roche Lobe, but the accretors' as well. This means that both the donors potential and the gravitational accretors potential are filled.

If both of the binaries' potentials are full, then donor star will then begin to fill up the systems total gravitational potential (i.e. the area between the two ridges in fig. 5). This can lead to mass being loss from the system completely.

This process generally is not stable, as most stars this in stage generally are experiencing a brief stage of their evolution called common envelope (CE). (See section 1.5.1) However, in cases where it is stable, the star will continue to evolve as one body (sect. 1.5.2).

1.5.1 Common Envelope

The CE stage of binary evolution has two distinct outcomes, both of which heavily depend on the systems initial conditions. The two stars can either merge, in most cases becoming a single star, or that one star can be ejected, reverting the system back to a detached state (sect. 1.4) In this process the potential of the systems is completely filled, at which point mass stars being pushed out of the entire system through L_2 . This mass then begins to form a disk around the stars, orbiting at a reduced rate compared to the stars [3]. This mass that has been ejected from the system entirely will create ionized gas around the stars, which can be observed from Earth. [3]

1.5.2 Stable Evolution

In systems which are stable the stars will share mass and that their shells will evolve in tandem, this process is called “homogenous chemical evolution”. (See fig 1 [4]). As these stars transfer mass, their mass ratio (q) will oscillate around a value of $q = 1$, eventually reaching q_{min} at which point the stars will rapidly merge (see fig. 6). [10]

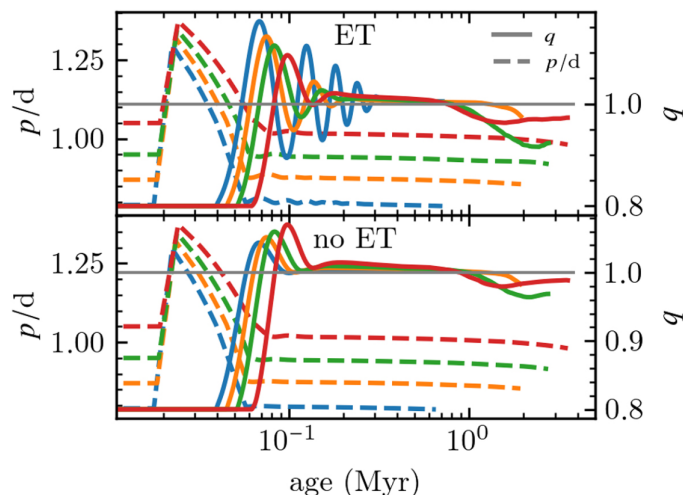


Figure 6: Reprinted from [2]. These graphs show the effects of modeling energy transfer(ET) on the systems’ evolution.

Oscillation with mass transfer

I used W Ursae Majoris (W UMa) as an example of these systems, as it is used an example system in order to categorize these contact binaries as a whole.

1.6 X-ray Binaries

1.6.1 X-rays caused by accretion onto a compact object

In 1962 the first X-ray binary was discovered by Riccardo Giacconi and colleagues. This system, Scorpius X-1, is so bright in x-ray that it actively raises ionization levels in Earths atmosphere when above the horizon. [3] [11] In the years following, it was discovered that that Scorpius X-1 consisted of a normal star and neutron star. Since then, many thousands more have been discovered[12].

These x-rays are produced from the friction of in-spiralling matter, which causes it to become incredibly hot, with the inner disk reaching temperatures of ≥ 10 million K, causing it to emit a large amount of x-rays. In systems with a NS

accretor, the surface of the NS itself will also emit a large amount of x-rays [3] (fig. 7).

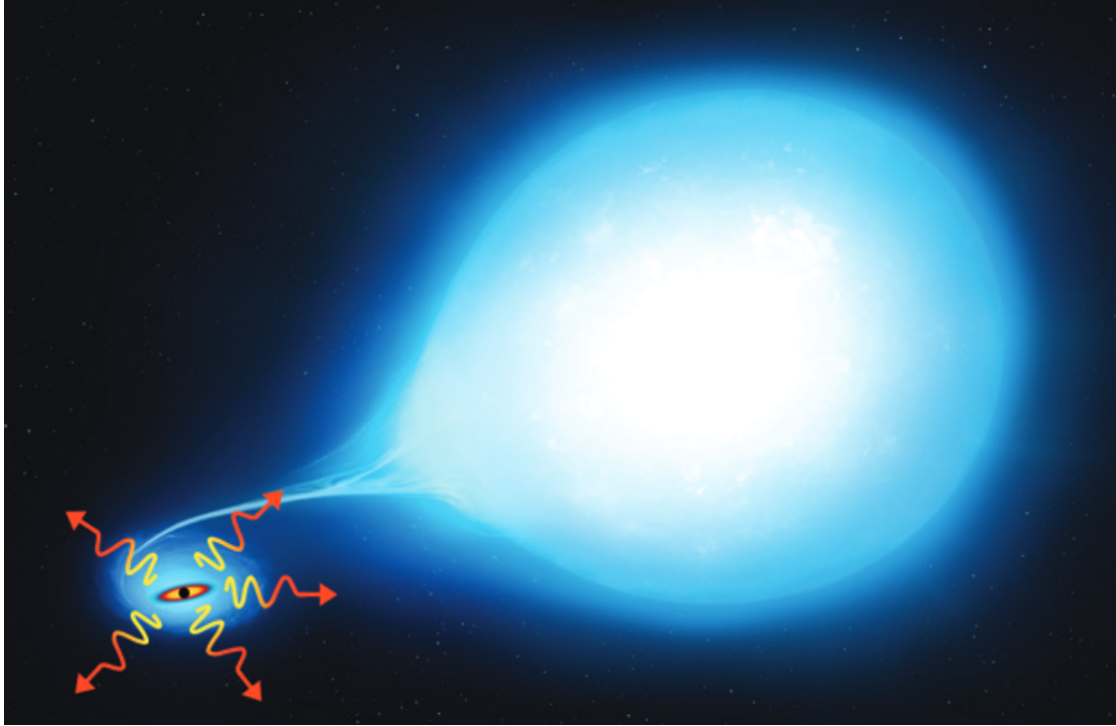


Figure 7: Reprinted from [3], original work by Mark Garlick, ©Mark Garlick. Here we see matter from a normal star falling onto a compact object, with the compact object being the source of x-rays

Data Acquisition

2.1 Vela X-1 (Detached)

2.1.1 Known properties

	Vela X-1 A	Vela X-1 B
Star Type	Neutron Star	Supergiant [8]
Masses M_{\odot}	≥ 1.8 [8]	20–30 [8]
Radius	11-12.5 $_{KM}$ [8]	30 R_{\odot} [8]
Temperature		$33.7 \pm 5.2kK$ [8]
Luminosity		5.8-6.2 $L_{M_{\odot}}$ [8]
Separation	2 kpc [8]	
Mass Loss Rate	$10^{-6}M_{\odot}yr^{-1}$ [8]	
Eccentricity	$e \approx 0.0898$) [8]	

Table 1: *Properties of Vela X-1*

Vela X-1 consists of a Neutron Star and Supergiant and is an Eclipsing and pulsing high mass x-ray binary (HMXB) (sect. ??). This means that the Neutron star passed behind the Supergiant every 8.94 days [13], leading to a variable luminosity between $10^{36} \text{ erg s}^{-1}$ and $10^{37} \text{ erg s}^{-1}$. Additionally, the neutron star itself is spinning every 293 seconds. [8].

Vela X-1 is described as an archetypical wind accretor, as it is a system which is undergoing wind accretion in a stable, predictable, and easy to measure way. The x-ray emission is persistent as well having prominent broadband spectra. Having both x-ray and broadband allows both to be measured and compared. Astronomers use Vela X-1 as examples when looking at other systems with comparable x-ray emissions [8].

The wind accretion (see 1.4.1) comes in the form of wind from a supergiant star (Vela X-1B) falling onto the neutron star. This wind does not have a very high velocity, but because the supergiant has almost filled it RL [8], the wind mass can easily escape, falling onto the NS. This accretion process (Sect. 1.6.1) is what creates the prominent X-ray emission.

2.2 W Ursae Majoris (Contact Binary)

2.2.1 Known properties

	W UMa A	W UMa B
Masses M_{\odot}	$1.139 \pm 0.019[14]$	$0.551 \pm 0.006[14]$
Radius R_{\odot}	$1.092 \pm 0.016[14]$	$0.792 \pm 0.015[14]$
Temperature K	$6450 \pm 100 [14]$	$6170 \pm 21 [14]$
Luminosity L_{\odot}	$1.557 \pm 0.166[14]$	$0.978 \pm 0.071[14]$
Distance	$52pc [15]$	
Max Magnitude	$7.75 [16]$	
Min Magnitude	$8.48 [16]$	
Period	$.3336 \text{ days } [14]$	
Inclination Plane	$88.4 \pm 0.8^{\circ} [14]$	

Table 2: *Properties of W Ursae Majoris*

W UMa is a contact binary, meaning that the two stars are physically ‘connected’ by their mass. This system is known as an archetype because it has a high magnitude at 7.75 at peak and 8.48 at minimum (table 2), meaning that its fairly easy to observe the variability. We can measure said variability in the form of light curves, which reveal a distinct nature different which is different from non-contact binaries (fig. 8). This magnitude variability is due to the fact that the binary is eclipsing due to its low inclination plane (table 2), meaning that the stars will pass behind each other in their orbit relative to the Earth.

Because of the prominent nature of this binary, similar contact binaries are called referred to as ‘UWMa type’ if they also possess said eclipsing nature.

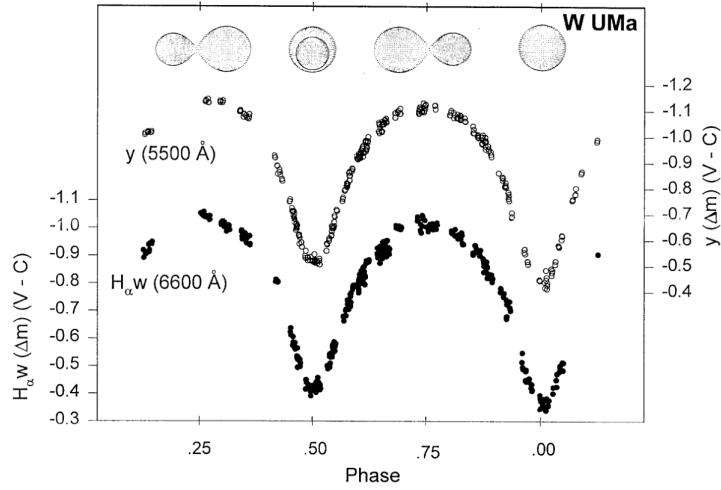


Figure 8: Reprint from [17]

2.3 Roche Lobe overflow in V404 Cygni

2.3.1 Known properties

	V404 Cygni B (Donor)	V404 Cygni A (Black Hole)
Star Type	Early K-type Giant	Black Hole
Masses	$.7M_{\odot}$ [6]	$9M_{\odot}$ [7]
Radius	$6.0R_{\odot}$ [7]	
Temperature	$4800K$ [7]	
Luminosity	$10.2L_{\odot}$ [7]	
Distance	2390_{pc} [6]	

Table 3: Properties of V404 Cygni

V404 is a LMXB, meaning that the donor star has a relatively low mass. In this system the material being accreted by V404 Cygni A forms an accretion disk, which greatly increases the luminosity of the system (**sect. 1.6.1**). This mass is transferred directly through L_1 (**sect. 1.3.1**) [18]. V404 is also a variable star system, with its magnitude varying overtime [6]. The reason why is currently under great discussion, there are two current likely causes. That being that either the donor star begins to release mass at a greater rate [19], or due to instability in the accretion disk itself [20]. This is a property which is commonly observed in LMXBs.

Binary ID	System State	Orbital Period (days)	\log_{10} Mass Transfer Rate	Donor State	Donor Mass M_{\odot}	Accretor State	Accretor Mass M_{\odot}
54	Detached	0.047520	-99.00000	NS	1.196033	stripped He Core He burning	≈ 1.002
183	Detached	0.0429883	≈ -80.8	NS	1.196033	stripped He Core He burning	$\approx .9957$

Table 4: *Example of POSYDON data, heavily modified for readability*

2.4 POSYDON Simulations

I used data generated by POSYDON [21] in corroboration with three observed systems in order to fully understand the depth of the process of mass transfer in Binary Systems. This is because while catalogues of contact binaries, HMXBS and LMXBs exist, there is not enough of them to get a true grasp of the full picture. Hence, I used POSYDON. This dataset was simulated on the NU Super computing Cluster, QUEST. POSYDON is developed and maintained by a team of astrophysicists and computer scientists working at the Université de Genève and Northwestern University. POSYDON uses an additional script called MESA, which is dedicated to single star and binary evolution. POSYDON utilizes MESA on a much larger scale in order to simulate full populations. The data was stored in the form of a .h5 file, containing a total of ≈ 6.1 million rows and 83 columns. (See greatly reduced example of the data frame in (table 4) and an HR Diagram of the full dataset in Fig. 9)

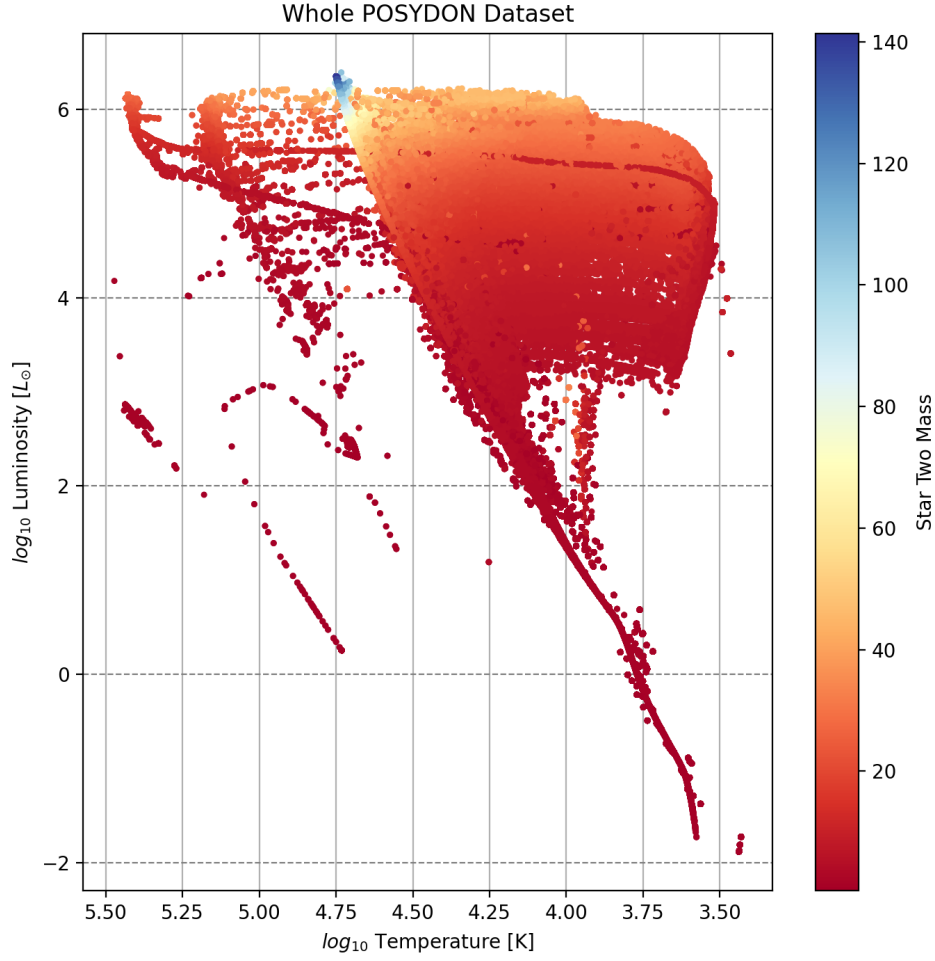


Figure 9: *HR Diagram of the donor star for the full (~ 6.4 million points) POSYDON dataset. Note that the color of the plotted points correspond to the \log_{10} solar mass of the donor star. Generated with Matplotlib*

This dataset looks very similar to a standard HR diagram, however, there are some key population differences. In the center we can see the Main sequence

2.4.1 Data Processing

With my research I sought to contextualize these very specific systems, allowing one to better understand how these systems play into the larger picture of binaries.

In order to do this I utilized a large dataset generated from POSYDON. In order to properly analyze this data I utilized Python with a large quantity of packages. These packages included Pandas [22], Matplotlib [23], and NumPy [24]. These tools allowed me to rapidly and efficiently analyze the large amount of data, something this paper would not have been possible without.

Results

Through my research I found a number of interesting discoveries, both regarding the populations of binaries as a whole, but also some applicable to systems themselves.

3.1 Roche Lobe Overflow

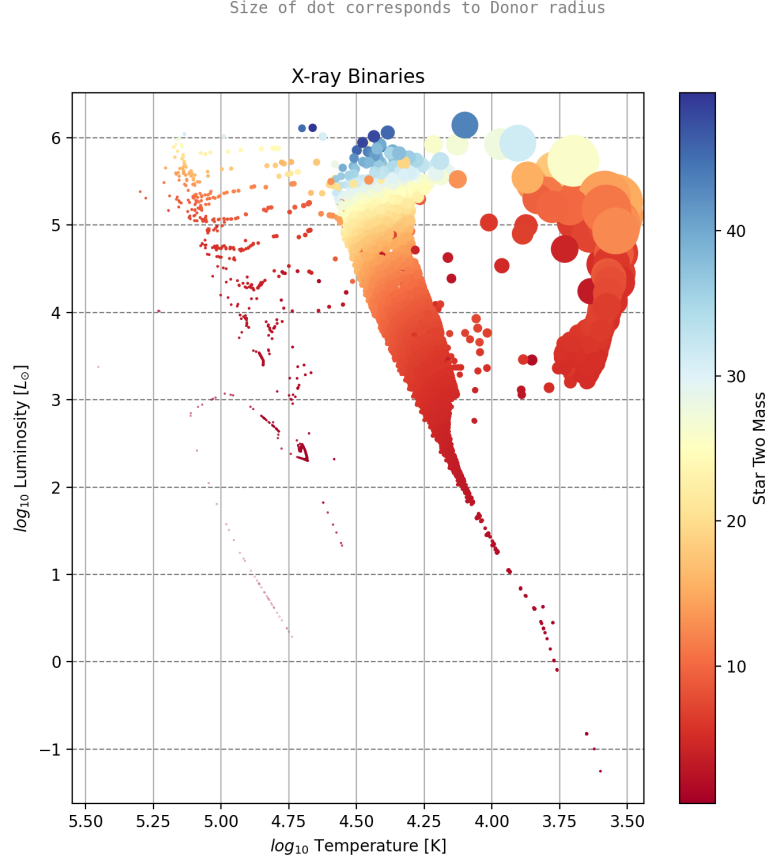


Figure 10: *HR diagram of the donor star in X-ray Binaries. Size of the plotted dot corresponds to the size of the star.*

In figure 10 we can see that X-ray binaries' donor stars generally follow a semi-standard HR diagram.

3.1.1 V404 Cygni Results

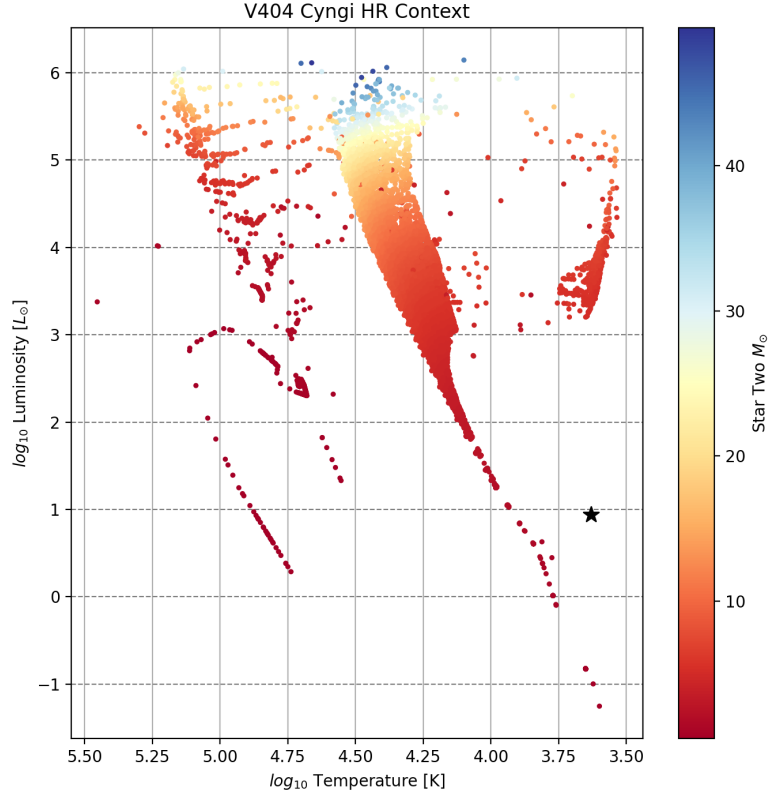


Figure 11: *HR Diagram with reference for V404 Cygni. Utilizing data from [18]*

I discovered that does not fall onto the two main simulated evolutionary tracks for LMXBs. I believe this is because V404 Cygni B is a stripped giant. [12]

3.2 Contact Binaries

Through my analysis I discovered that contact binaries had some of the most interesting results.

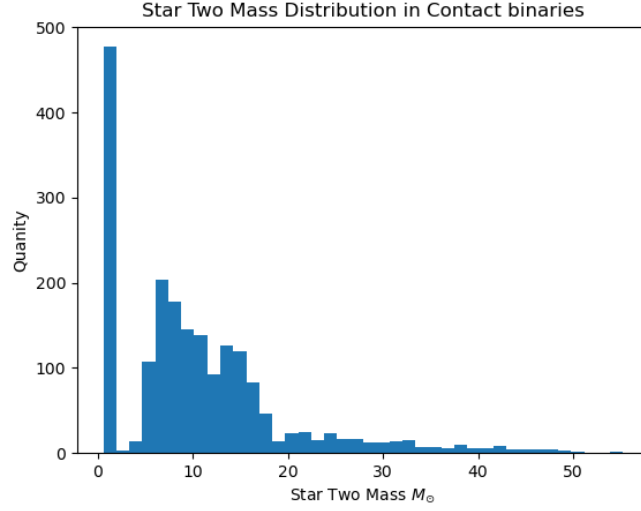


Figure 12: *Star two mass distribution for contact Binaries*

In figure 12 we can see that contact binaries mass distribution feature a prominent spike at around one solar mass, with a distribution centered around 10, and then a scattered amount afterward.

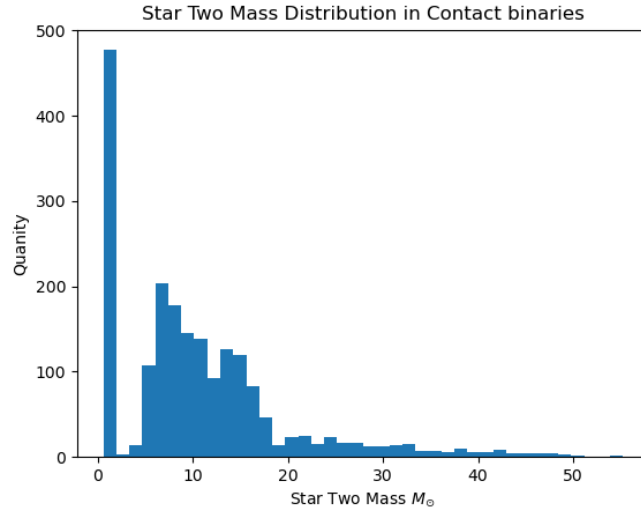


Figure 13: *Star one mass distribution of contact Binaries*

Note how similar this is to the star two distribution. From the POSYDON data, I found the mean mass of star one in contact binaries to be ≈ 10.703 and star two to be ≈ 10.6548 . This is congruent with what previous papers have found

([2]). This is due to the nature of mass transfer in contact binaries leading to a stabilization in mass. As a contact system evolves, the mass transfer causes q (the mass ratio between stars) to stabilize to a value of $q = 1$ [2]. We can clearly see this oscillation and then stabilization in **figure 6**.

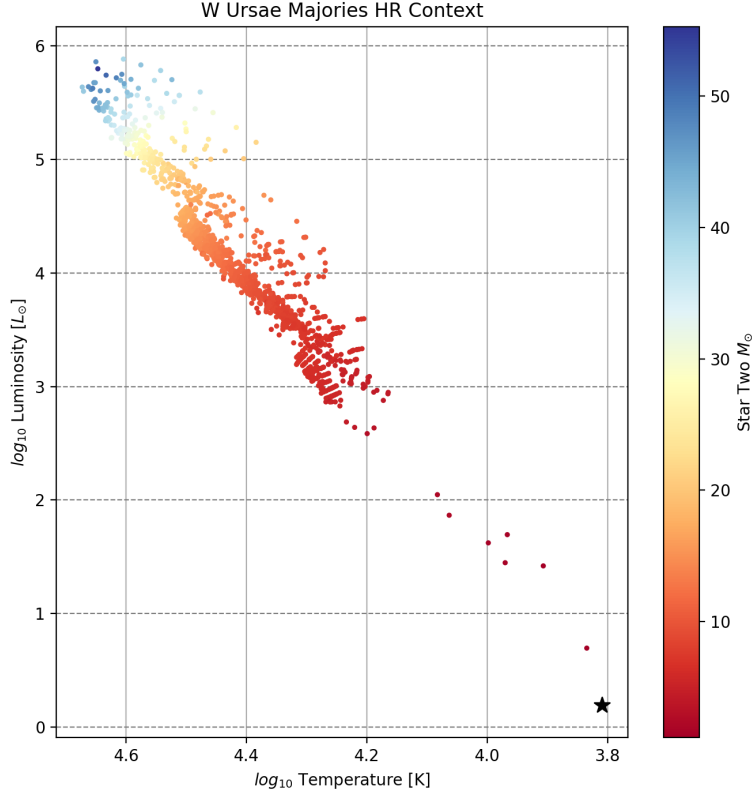


Figure 14: *HR diagram of contact Binaries from POSYDON data, generated with Matplotlib.*

This is one of the more interesting results, as we can see that contact binaries form a very specific population on the HR diagram, falling on specific linear with a linear relationship (fig 14) I believe this is because of the stabilizing natures regarding mass ratios in contact binaries. It is important to note that these values are much above what we observe in contact binaries. Again, there are many reasons this could be, including the inherent skewing of the grid, the nature of observing contact binaries, and more.

3.2.1 W UMa Cygni Results

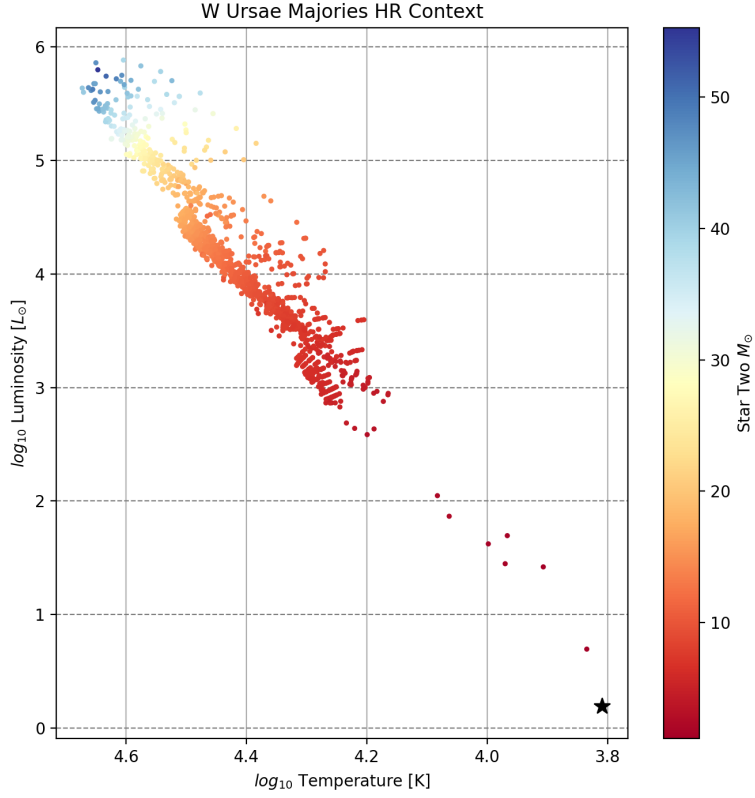


Figure 15: *HR Diagram with reference for W UMa using W UMa A. Plotted using data from*

In **figure 15** we see that W UMa also follows this linear relationship, however, it has a much lower temperature and luminosity than most of the population. This is a known discrepancy with current simulated models, where simulated values are both typically more luminous and massive than observed. [2]. There are many in line with other observed contact binaries.

3.3 Detached Binaries

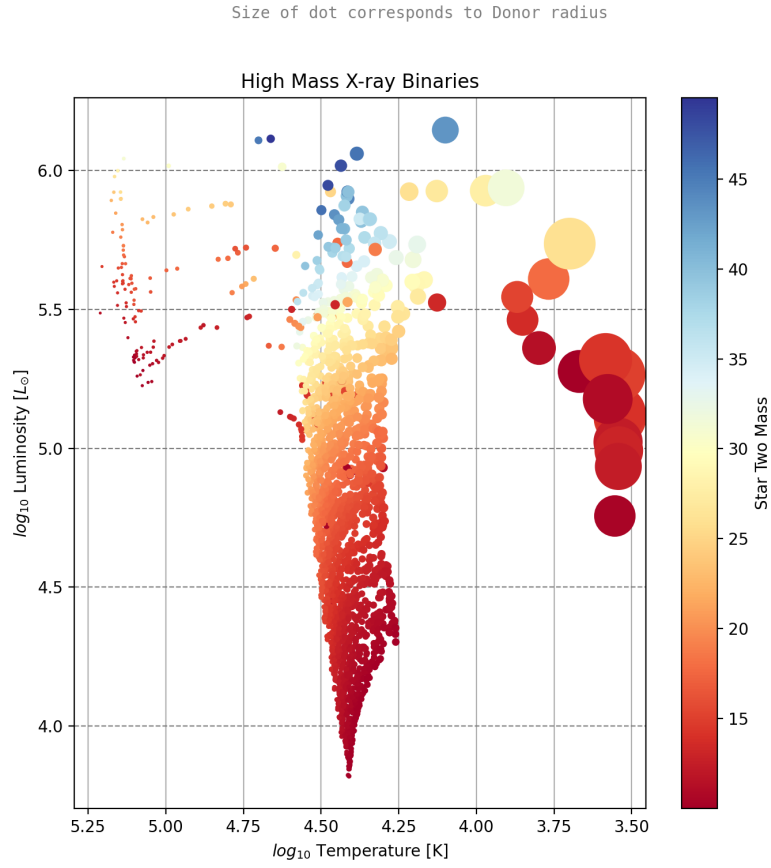


Figure 16: *HR Diagram of HMXBS*

3.3.1 Vela X-1 Results

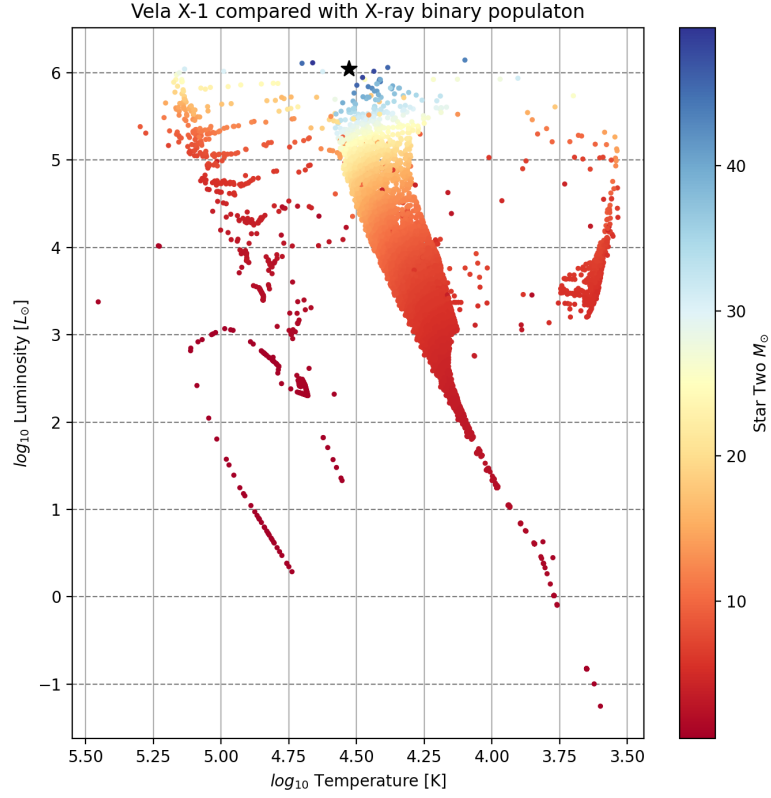


Figure 17: *HR Diagram with reference for Vela X-1. The star is the mean value of observation range, box is overlap of the max and min temperature and luminosity values*

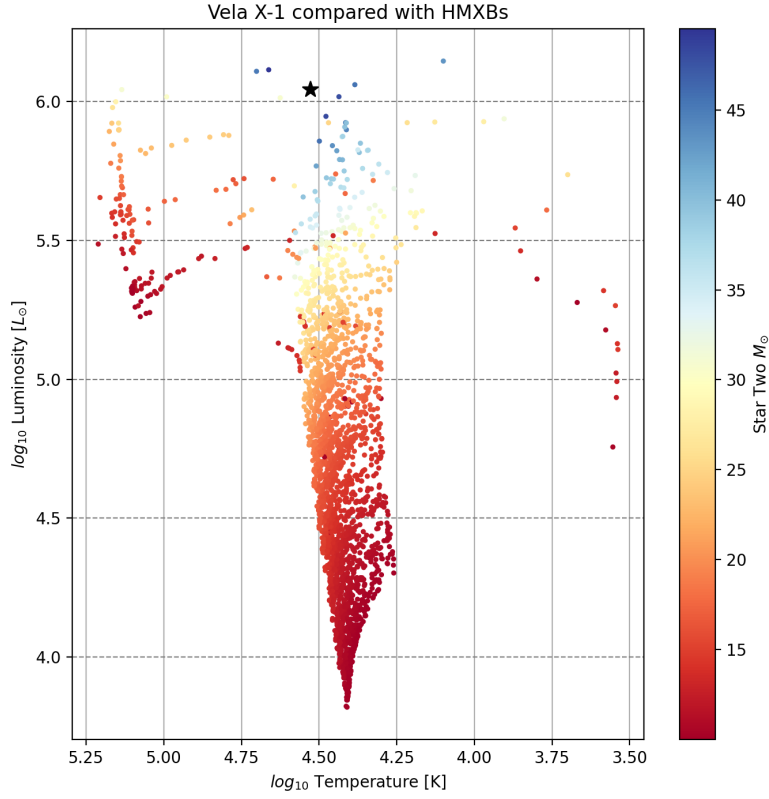


Figure 18: *HR Diagram of HMXBs with reference point for Vela X-1. The star is the mean value of observation range, box is overlap of the max and min temperature and luminosity values*

In figures 17 and 18 we see that Vela X-1 is one of the more extreme example of HMXBs, as it has a higher temperature and luminosity then a lot of its similar stars, the majority of which are in main sequence.

Conclusion

In conclusion mass transfer heavily affects the evolution of binary stars, leading them to evolve in a uniquely different way then to single star evolution.

Discussion

Originally I planned on simulating grids for all the types of systems, however, due to time constraints I chose to only focus on one type.

5.0.1 Disclaimers

Due to time constraints, I was not able to cite all the sources within the textbook [3] which I used. This is because it was quite tedious to find the paper itself the textbook references.

5.1 Git Galore

Originally, I started working on this project using Overleaf, however, due the amount of graphs, the time to compile started to rapidly climb up, and eventually I just decided to switch to compiling it locally. On-top of this, I figured I would set up a GitHub in order to additionally push the code and graphs as well. This turned out to absolutely be the right decision, giving me way more freedom.

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