Calc II Notes

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Ratio Test

Suppose that $\sum_{n=1}^{\infty} a_n$ is ***any*** series,
and that
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r$$
Then:
The series converges absolutely if $r < 1$
The series diverges if $r > 1$

Root Test

Suppose that $\sum_{n=1}^{\infty} a_n$ is *any* series,
and that
$$\lim_{n\to\infty} (|a_n|)^{\frac{1}{n}} = L$$
Then:
The series converges absolutely if $L < 1$
The series diverges if $L > 1$

Absolute Convergence

Limit Comparison Test

$\lim_{a=\infty} \frac{A_n}{B_n}$, Where B_n is a series with known behavior

If $\lim_{a=\infty} \frac{A_n}{B_n} \neq 0$, then B_n and A_n must either both converge or diverge
If $\lim_{a=\infty} \frac{A_n}{B_n} = 0$ and B_n is a convergent series, then A_N must also converge
If $\lim_{a=\infty} \frac{A_n}{B_n}$ diverges and B_n is a divergent series, then A_N must diverge

Comparison Tests

Direct Comparison Test

If B_n is a known series which converges and
$A_n \leq B_n$
then A_n must also converge

If B_n is a known series which diverges and
$A_n \geq B_n$
then A_n must also diverge

Gen. NTTFDO

If condition (II) fails...

If $\lim_{n\to\infty} |a_n| \neq 0$
then $\sum a_n$ diverges

Alternating Series Test

Suppose $\sum_{n=1}^{\infty} (-1)^n a_n$ is an alternating series with

I) a_n must is eventually non-increasing
II) a_n must goes to zero

Hence, a_n must converge

Converge/Diverge

N^{th} test for divergnce (NTTFDO)

$\lim_{n\to\infty} a_n \neq 0$
then the series must diverge

Integral Test

$\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

and

$\sum_{n=1}^{\infty} a_n$ diverges if and only if $\int_1^{\infty} f(x)\,dx$ diverges.

Conditions for the Integral Test

I) Must be continous for all values of x
II) Must be positive
III) Must be decreasing

$$\sum_{n=1}^{\infty} a_n = \lim_{a\to\infty} \sum_{n=1}^{a} a_n$$

P-Series Test

$$\sum_{n=1}^{\infty} \frac{1}{x^p} \begin{cases} Converge & x \leq 1 \\ Diverge & 1 > 1 \end{cases}$$

By the integral test

Geometric Series

$\sum_{k=0}^{\infty} (r)^k$ *converges to* $\frac{1}{1-r}$, $|r| < 1$

or

$\sum_{k=N}^{\infty} r^k$ *converges to* $\frac{r^N}{1-r}$, $|r| < 1$

Diverges if $|r| \geq 1$

A series which increases by the same amount for each value of N. Ie.
$\frac{A_{n+1}}{A_n} = L$, where L is a number

Evaluate Exactly

Telescoping Series

$$\sum_{n=1}^{\infty} \frac{1}{n(n-1)}$$

A series where intermediate terms cancel, leaving a finite sum
- Partial Fraction Decomposition
- Find Equation for S_n
- $\lim_{n\to\infty} S_n$

1 Absolute Convergence $\boxed{\textbf{10.5}}$

1.1 Good Practice Problems

$\boxed{\textit{57}}$

1.2 Ratio Test

The Ratio Test[1] states that

$$\text{Suppose that } \sum_{n=1}^{\infty} a_n \text{ is } \textit{\textbf{any}} \text{ series,}$$

and that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r$$

Then:

The series converges absolutely if $r < 1$
The series diverges if $r > 1$

1.2.1 "Keys"

$$\frac{a^{n+1}}{a^n} = a, \qquad \frac{a^n}{a^{n+1}} = \frac{1}{a}$$

$$\frac{(n+1)^a}{n^a} = \left(1 + \frac{1}{n}\right)^a$$

$$\frac{(n+1)!}{n!} = (n+1)$$

Note that these can be combined

$\boxed{\text{Example}}$ $\frac{2^{n+1}(n+1)!}{2^n(n!)} = 2(n+1)$

[1]Very important regarding factorials

1.2.2 Examples

Example 1

$$\boxed{\text{Simplify } \frac{(n+2)!}{(n-1)!}}$$

Ratio of factorials

$$= \frac{(n+2)!}{(n-1)!}$$

$$= \frac{(n+1)(n+1)(n)(n-1)\cdots(1)}{(n-1)(n-2)(n-3)\cdots(1)}$$

$$= \frac{(n+1)(n+1)(n)(n-1)}{(n-1)}$$

$$= (n+1)(n+1)(n)$$

$\boxed{\textbf{example 2}}$

Problem 57 from book

$$\boxed{\sum_{n=1}^{\infty} \frac{2^n (n!)^2}{2n!}}$$

We apply the Ratio test

$$\lim_{n\to\infty} \frac{\frac{(2^{n+1})(n+1)!^2}{2(n+1)!}}{\frac{2^n n!^2}{2n!}}$$

$$\lim_{n\to\infty} \frac{2^{n+1}(n+1)^2}{2(n+1)} \times \frac{2n!}{2n(n!)^2}$$

$$\lim_{n\to\infty} \frac{2^{n+1}(n+1)^2}{\mathbf{2n+2}} \times \frac{2n!}{2n(n!)^2}, \text{ note the +2}$$

$$\lim_{n\to\infty} \frac{2(n+1)^2}{\mathbf{(2n+2)(2n+1)}}$$

$$= \tfrac{1}{2} < 1$$

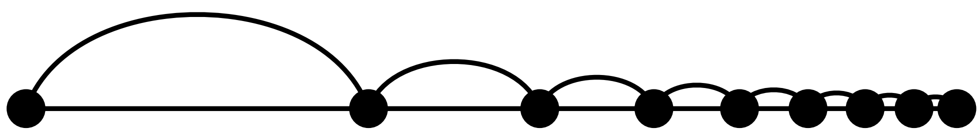
Hence, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (n!)^2}{2n!}$ converges

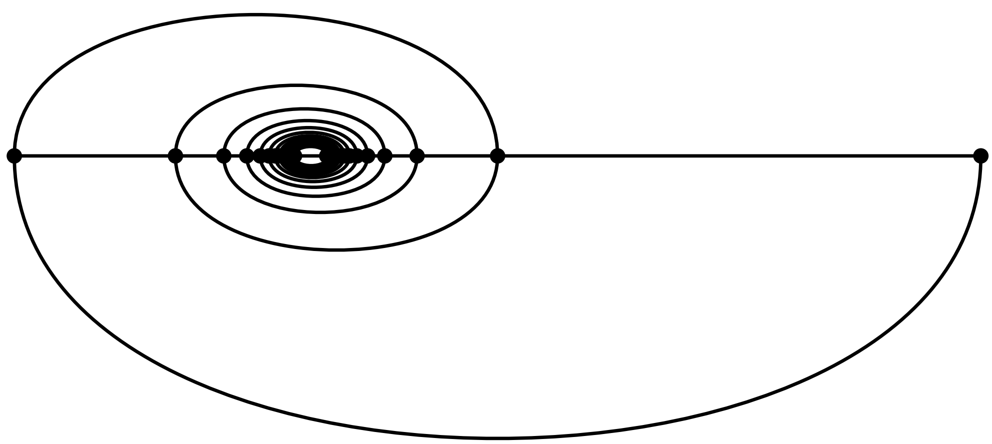
2 Alternating Series

Suppose $\sum_{n=1}^{\infty}(-1)^n a_n$ is an alternating series with

I) a_n eventually non-increasing
II) a_n going to zero

Then, a_n must converge





2.1 Examples

$\boxed{\text{Example 1}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, Converge or diverge?

We apply the Alternating Series Test (AST)

$$a_n = \tfrac{1}{n}$$

$$1)\ \tfrac{1}{n+1} < \tfrac{1}{n} \rightarrow \textit{Decreasing}$$

$$2)\ \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

Hence, by AST, the series converges **conditionally**

3 Misc

3.1 Factorials

ex.1

$$\boxed{4! = 4 \text{ x } 3! = 4\text{x}3\text{x}2! = 4\text{x}3\text{x}2\text{x}1! = 4\text{x}3\text{x}2\text{x}1\text{x}0!}$$

$$\rightarrow$$

$$\boxed{0! = 1}$$

$$\rightarrow$$

$$\boxed{4! = 24}$$

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ex. 2

$$\boxed{\text{Simplify } \frac{(n+2)!}{(n-1)!}}$$

Ratio of factorials

$$= \frac{(n+2)!}{(n-1)!}$$

$$= \frac{(n+1)(n+1)(n)(n-1)\cdots(1)}{(n-1)(n-2)(n-3)\cdots(1)}$$

$$= \frac{(n+1)(n+1)(n)(n-1)}{(n-1)}$$

$$= (n+1)(n+1)(n)$$