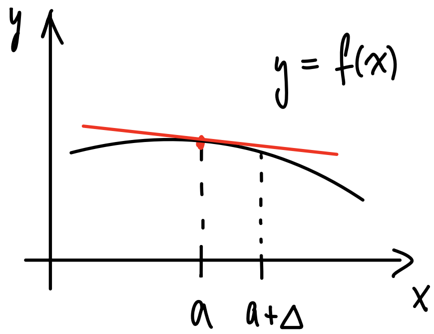
14.4 Tangent Planes and Linear Approximations

Fns of 1 variable:



tangent line to $f(x)$ at $(a, f(a))$

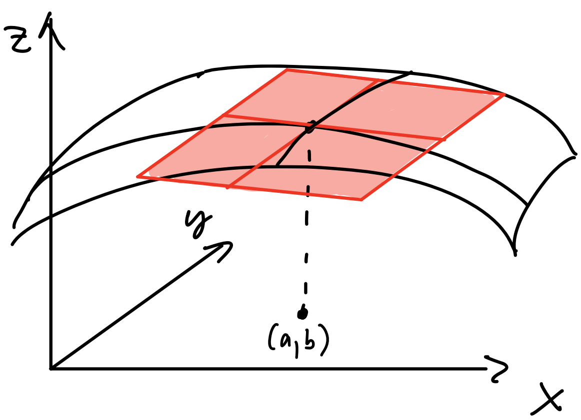
$$\boxed{y = f(a) + f'(a)(x-a)}$$

– linear approx. of $f(x)$ for x around a

$y - y_0 = m(x - x_0)$

$y - f(a) = f'(a)(x - f(a))$

Fns of 2 variables



tangent plane to $f(x,y)$ at $(a, b, f(a,b))$

$$\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

– linear approx. of $f(x,y)$ around (a,b)

$z - z_0 = m_1(x - x_0) + m_2(y - y_0)$

$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(x-b)$

EX: $z = f(x,y) = \frac{x^2}{16} + \frac{y^2}{9}$. Find tangent plane at $(2, 3, \frac{5}{4})$

$f_x = \frac{2x}{16} = \frac{x}{8}$ $\qquad f_x(2,3) = \frac{2}{8} = \frac{1}{4}$

$f_y = \frac{2y}{9}$ $\qquad f_y(2,3) = \frac{2 \cdot 3}{9} = \frac{2}{3}$

tangent plane: $z - \frac{5}{4} = \frac{1}{4}(x-2) + \frac{2}{3}(y-3)$

$$z = \frac{5}{4} + \frac{1}{4}(x-2) + \frac{2}{3}(y-3)$$

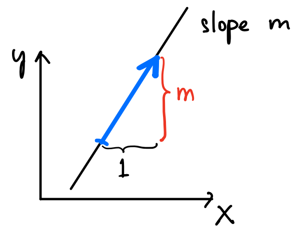
Ex: Approximate $f(x,y) = \frac{x^2}{16} + \frac{y^2}{9}$ at $(x,y) = (1.9, 3.2)$ Plug in $x = 1.9$ into
$y = 3.2$

$$f(1.9, 3.2) \approx \frac{5}{4} + \frac{1}{4}(1.9-2) + \frac{2}{3}(3.2-3)$$

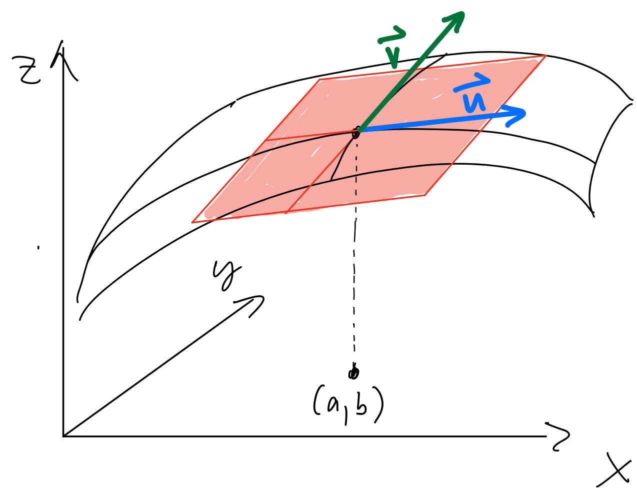
$$\approx \frac{5}{4} + \frac{1}{4}\cdot\left(-\frac{1}{10}\right) + \frac{2}{3}\cdot\frac{2}{10} = \frac{30\cdot 5 - 3 + 16}{120} = \frac{163}{120}$$

Proof of formula

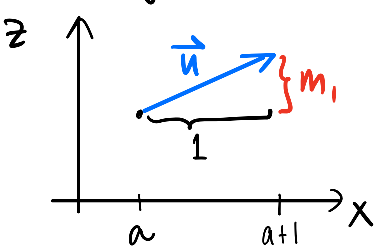
Recall: In $\mathbb{R}^2$, a line with slope m has a direction vector $\langle 1, m\rangle$

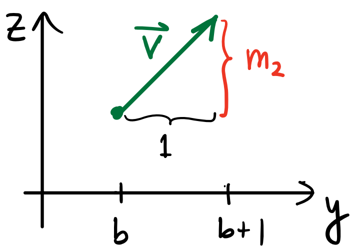


In $\mathbb{R}^3$: Let $m_1 = f_x(a,b)$ and $m_2 = f_y(a,b)$.



Find direction vectors $\vec{u}$ and $\vec{v}$ of the two tangent lines with slopes m_1 & m_2.





$\vec{u} = \langle 1, 0, m_1\rangle$ $\qquad$ $\vec{v} = \langle 0, 1, m_2\rangle$

$$\vec{n} = \vec{u}\times\vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & m_1 \\ 0 & 1 & m_2 \end{vmatrix} = \langle -m_1, -m_2, 1\rangle$$

Plane:

$$-m_1(x-x_0) - m_2(y-y_0) + 1\cdot(z-z_0) = 0$$

$$z - z_0 = m_1(x-x_0) + m_2(y-y_0)$$

$$z - f(a,b) = f_x(a,b)\cdot(x-a) + f_y(a,b)\cdot(y-b)$$

Linear Approximation of f at (a,b):

$$\boxed{L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)} \leftarrow \text{tangent plane}$$

$f(x,y) \approx L(x,y)$ for points near (x,y)

$\uparrow$ also called linearization

EX: Find linear approximation of $f(x,y) = e^x \cos(xy)$ at $(0,0)$

$f_x = e^x \cos(xy) + e^x \cdot (-\sin(xy)) \cdot y \Rightarrow f_x(0,0) = 1$

$f_y = e^x \cdot (-\sin(xy)) \cdot x \Rightarrow f_y(0,0) = 0$

$f(0,0) = 1$

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$L(0,0) = 1 + 1 \cdot (x-0) + 0 \cdot (y-0) = 1 + x$

$e^x \cos(xy) \approx 1 + x$

$e^{0.1} \cdot \cos(0.1 \cdot 0.2) \approx 1 + 0.1 = 1.1$

Alternative notation:

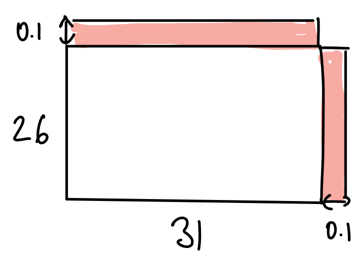
$$\left.\begin{aligned} \Delta x &= x - a \\ \Delta y &= y - b \\ \Delta z &= f(x,y) - f(a,b) \end{aligned}\right\} \Rightarrow \boxed{\Delta z \approx f_x(a,b) \cdot \Delta x + f_y(a,b)\,\Delta y}$$

EX: The length and width of a rectangle are measured as 31 cm and 26 cm, with an error in measurement of at most 0.1 cm in each. Estimate the maximum error in the calculated area of the rectange.

$A(x,y) = x \cdot y$

$A_x(x,y) = y \Rightarrow A_x(31,26) = 26$

$A_y(x,y) = x \Rightarrow A_y(31,26) = 31$



$\Delta A \approx A_x(31,26)\,\Delta x + A_y(31,26)\,\Delta y$

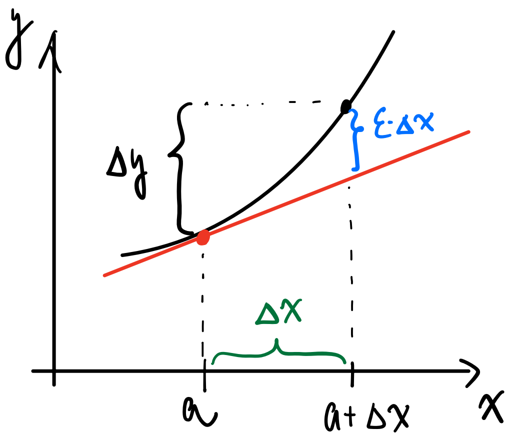
$\approx 26 \cdot 0.1 + 31 \cdot 0.1 = 5.7 \text{ cm}^2$

Differentiability

Fns of 1 variable: $\boxed{y = f(x)}$

Differentiable at a if limit exists:

$$f'(a) = \lim_{\Delta x \to 0} \frac{\overbrace{f(a+\Delta x) - f(a)}^{\Delta y}}{\Delta x}$$



$$f'(a) \approx \frac{\Delta y}{\Delta x}$$

$$\Delta y \approx f'(a)\cdot \Delta x$$

$$\boxed{\Delta y = f'(a)\,\Delta x + \underbrace{\varepsilon \Delta x}_{\text{ERROR}}}\,, \quad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

Think: Tangent line gives a good approx. to f around a.

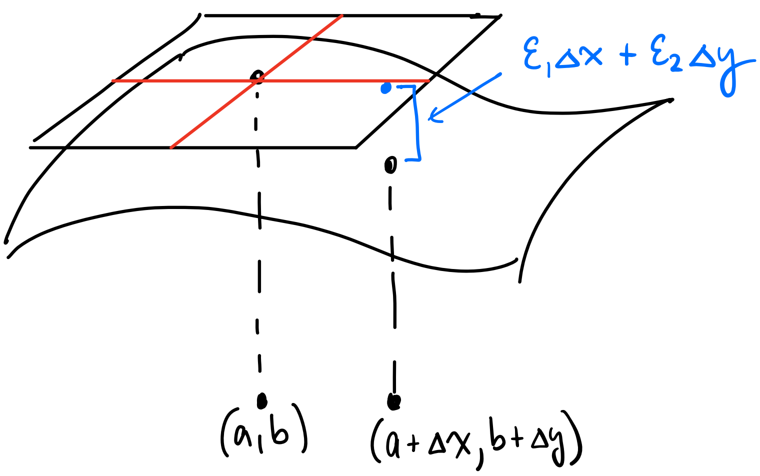
Fns of 2 variables: $\boxed{z = f(x,y)}$

Def: f is <u>differentiable</u> at (a,b) if we can express Δz as

$$\boxed{\Delta z = f_x(a,b)\cdot \Delta x + f_y(a,b)\,\Delta y + \underbrace{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}_{\text{ERROR}}}$$

where $\varepsilon_1 \to 0$ & $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

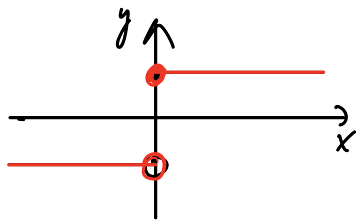
Think: Tangent plane gives a good approx. to f around (a,b)



Continuity vs Partial Derivatives vs Differentiability

T or F: If $f_x(a,b)$ and $f_y(a,b)$ exist, then f is differentiable.

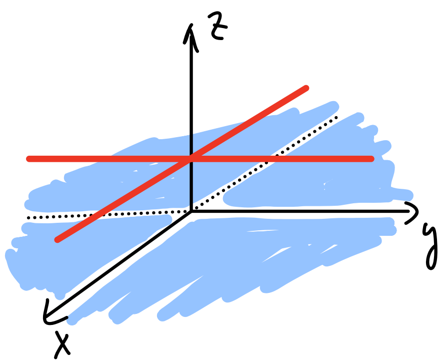
fns of 1 var: Differentiable $\Longrightarrow$ Continuous



- not continuous at 0

$\Rightarrow$ not differentiable

fns of 2 vars: Differentiable $\Longrightarrow$ Continuous



$$f(x,y) = \begin{cases} 1 & \text{if } xy = 0 \\ 0 & \text{if } xy \neq 0 \end{cases}$$

- not continuous at $(0,0)$

$\Longrightarrow$ not differentiable at $(0,0)$

- but partial derivatives exist! $f_x(0,0) = 0$

$f_y(0,0) = 0$

f_x & f_y exist $\not\Longrightarrow$ Differentiable

Which condition guarantees differentiability?

Thm: If f_x and f_y exist and are continuous at (a,b), then f is differentiable at (a,b).

Ex: Show that $f(x,y) = x^2y^3 + e^{2x+y}$ is differentiable on $\mathbb{R}^2$

$f_x = 2xy^3 + 2e^{2x+y}$

$f_y = 3x^2y^2 + e^{2x+y}$

continuous $\Rightarrow$ f differentiable