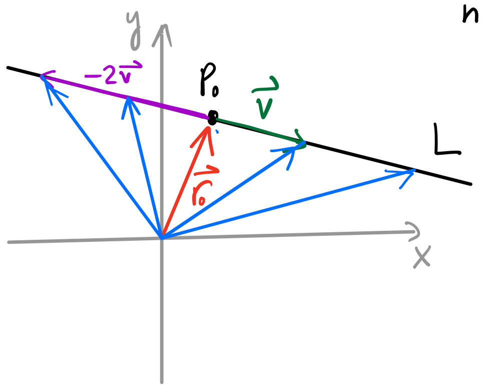
Lines in $\mathbb{R}^2$



need : a point P_0 with position vector $\vec{r_0}$ $\quad \vec{r_0} = \langle x_0, y_0 \rangle$

a direction vector $\vec{v}$ $\quad \vec{v} = \langle a, b \rangle$

The line is traced out by the tip of $\vec{r_0} + t\cdot\vec{v}$ as t varies

$\ldots, \vec{r_0} - 2\vec{v}, \vec{r_0} - \vec{v}, \vec{r_0}, \vec{r_0} + \vec{v}, \vec{r_0} + 2\vec{v}, \ldots$

vector equation

$$\boxed{\vec{r}(t) = \vec{r_0} + t\vec{v}}$$

$t\vec{v}$: scaled vector

t: parameter $t \in \mathbb{R}$, $-\infty < t < \infty$

parametric equations

$$\boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \end{aligned}}$$

slope-intercept form

$$y - y_0 = m(x - x_0)$$

$$\boxed{y = mx + b}$$

m: slope, b: y-interept

EX: $P_0 = (1,3) \Rightarrow \vec{r_0} = \langle 1,3 \rangle$

$\vec{v} = \langle 2,-1 \rangle$

vector equation

$$\vec{r}(t) = \langle 1,3 \rangle + t\langle 2,-1 \rangle = \langle \underbrace{1+2t}_{x}, \underbrace{3-t}_{y} \rangle$$

parametric equations

$$\left.\begin{aligned} x &= 1+2t \\ y &= 3-t \end{aligned}\right\} \xrightarrow{\text{combine}}$$

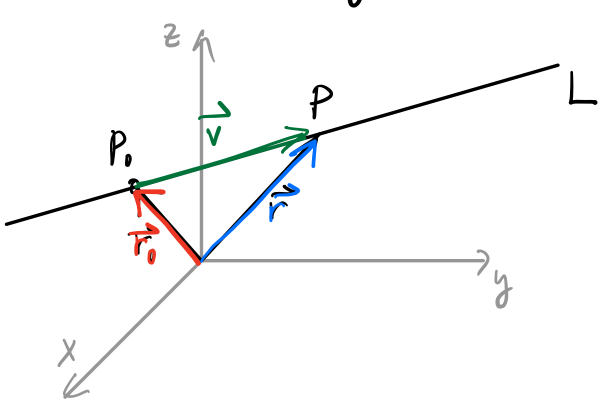
slope-intercept form

$$y = -\frac{1}{2}x + \frac{5}{2}$$

$$y = 3 - t = 3 - \left(\frac{x-1}{2}\right)$$

Lines in $\mathbb{R}^3$

Line determined by a point and direction (vector).



vector equation

$$\boxed{\vec{r}(t) = \vec{r_0} + t\vec{v}}$$

$\vec{r}(t)$: position vector of P

$\vec{r_0}$: position vector of P_0

$\vec{v}$: any vector parallel to L

Line through $P_0 = (x_0, y_0, z_0)$ w/ direction vector $\vec{v} = \langle a, b, c \rangle$

vector equation

$$\boxed{\vec{r}(t) = \vec{r_0} + t\vec{v}}$$

$t \in \mathbb{R}$

parametric equations

$$\boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}}$$

~~slope-intercept form~~

~~DNE~~

If $a, b, c \neq 0$, then

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned} \Longrightarrow \begin{aligned} t &= \frac{x - x_0}{a} \\ t &= \frac{y - y_0}{b} \\ t &= \frac{z - z_0}{c} \end{aligned} \Longrightarrow$$

symmetric equations

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

If $a, b \neq 0$, $c = 0$, for example, then write $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b}$, $z = z_0$

Ex: Find parametric equations and symmetric equations for the line through $P(1, -3, 5)$ and $Q(4, -7, 2)$

$\vec{r_0} = \langle 1, -3, 5 \rangle$

$\vec{v} = \overrightarrow{PQ} = \langle 3, -4, -3 \rangle$

vector equation

$$\vec{r} = \langle 1, -3, 5 \rangle + t \langle 3, -4, -3 \rangle$$

$$\vec{r} = \langle 1 + 3t, -3 - 4t, 5 - 3t \rangle$$

parametric equations

$$\begin{aligned} x &= 1 + 3t \\ y &= -3 - 4t \\ z &= 5 - 3t \end{aligned}$$

P $\quad \vec{v}$

many options, for example:

$$\begin{aligned} x &= 4 - 6t \\ y &= -7 + 8t \\ z &= 2 + 6t \end{aligned}$$

Q $\quad -2\vec{v}$

$$\begin{aligned} x &= 1 + 3t &&\Longrightarrow t = \frac{x - 1}{3} \\ y &= -3 - 4t &&\Longrightarrow t = \frac{y + 3}{-4} \\ z &= 5 - 3t &&\Longrightarrow t = \frac{z - 5}{-3} \end{aligned}$$

symmetric equations:

$$\frac{x - 1}{3} = \frac{y + 3}{-4} = \frac{z - 5}{-3}$$

2 lines in $\mathbb{R}^3$ can be:

- parallel $\longrightarrow$ direction vectors parallel $(\vec{v}_1 = k \cdot \vec{v}_2)$
- intersecting $\longrightarrow$ exist point (x,y,z) s.t. $x(t) = x(s)$, $y(t) = y(s)$, $z(t) = z(s)$
- skew $\longrightarrow$ not parallel and not intersecting

Ex: L_1:

$$x = 1 + 3t$$
$$y = -3 - 4t$$
$$z = 5 - 3t$$
$$t \in \mathbb{R}$$

L_2:

$$x = 2 - 3s$$
$$y = 2s$$
$$z = 1 + s$$
$$s \in \mathbb{R}$$

~~parallel~~, ~~intersecting~~, or skew?

$\langle 3, -4, -3 \rangle \neq k \langle -3, 2, 1 \rangle$

$\Rightarrow$ direction vectors not parallel

① $1 + 3t = 2 - 3s$

② $-3 - 4t = 2s$

③ $5 - 3t = 1 + s$

I will solve ①&③ first, then plug-in s and t into ②

①+③ $6 = 3 + 2s \Rightarrow s = -\frac{3}{2}$

① $1 + 3t = 2 - 3 \cdot (-\frac{3}{2})$

$3t = 2 + \frac{9}{2} - 1$

$t = \frac{11}{6}$

② $-3 - 4 \cdot \frac{11}{6} = 2 \cdot (-\frac{3}{2})$

$-\frac{31}{3} \neq -3$

$\Rightarrow$ no solution $\Rightarrow$ not intersecting

Line segment from $\vec{r}_0$ to $\vec{r}_1$

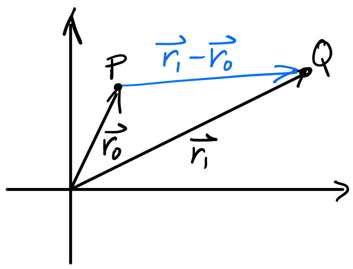
$$\boxed{\vec{r} = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)} \quad 0 \le t \le 1$$

same, just rearranged

$$\boxed{\vec{r} = (1-t)\,\vec{r}_0 + t\vec{r}_1}, \quad 0 \le t \le 1$$

Check: if $t = 0$, then $\vec{r} = \vec{r}_0$

$t = 1$ $\quad \vec{r} = \vec{r}_1$



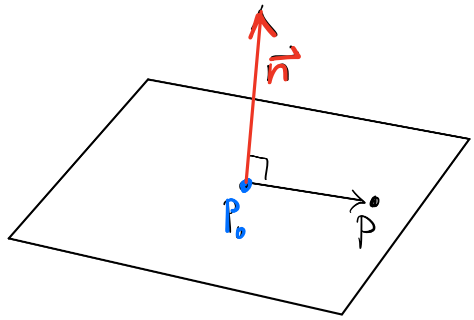
Ex: Find parametric equations for the line segment from $P(1, -3, 5)$ to $Q(4, -7, 2)$.

$$\vec{r}(t) = (1-t)\,\vec{r}_0 + t\vec{r}_1$$
$$= (1-t)\langle 1, -3, 5 \rangle + t\langle 4, -7, 2 \rangle$$
$$= \langle 1 + 3t, -3 - 4t, 5 - 3t \rangle$$

$0 \le t \le 1$

Planes in $\mathbb{R}^3$

How to specify plane?



- Specify a point $P_0 = (x_0, y_0, z_0)$ on plane
- and a normal vector $\vec{n} = \langle a, b, c \rangle$ which is perpendicular to plane

Then $P(x,y,z)$ is in plane $\iff$ $\vec{n}$ and $\overrightarrow{P_0P}$ are orthogonal (perpendicular)

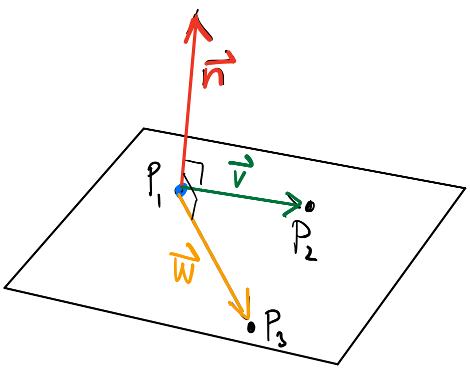
$$\iff \vec{n} \cdot \overrightarrow{P_0P} = 0$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Eqn. of a plane: $\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$

OR: $\boxed{ax + by + cz + d = 0}$, where $d = -(ax_0 + by_0 + cz_0)$

Ex: Find an eqn. of the plane throug the points $P_1(0,1,4)$, $P_2(2,4,1)$, $P_3(3,-1,5)$



$$\vec{v} = \overrightarrow{P_1P_2} = \langle 2, 3, -3 \rangle$$

$$\vec{w} = \overrightarrow{P_1P_3} = \langle 3, -2, 1 \rangle$$

$$\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -3 \\ 3 & -2 & 1 \end{vmatrix} = -3\hat{\imath} - 11\hat{\jmath} - 13\hat{k}$$

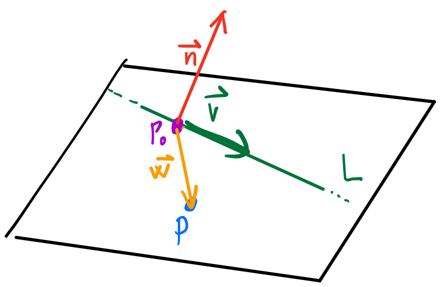
PLANE: $P_1(0,1,4)$, $\vec{n} = \langle -3, -11, -13 \rangle$

$$\boxed{-3(x-0) - 11(y-1) - 13(z-4) = 0}$$

$$+3(x-0) + 11(y-1) + 13(z-4) = 0$$

$$\boxed{3x + 11y + 13z - 63 = 0}$$

Ex: Find an eqn. of a plane that passes through the point $P(1,2,3)$ and contains the line L: $x = 1+t,\ y = 5-t,\ z = 2+3t,\ t\in\mathbb{R}$



Want: normal vector $\vec{n}$

2 non-parallel vectors determine a plane
their cross-product gives us $\vec{n}$

$\vec{v} = \langle 1,-1,3\rangle$ ... a vector parallel to L

$P_0 = (1,5,2)$ ... a point on the line L:

$P_0(1,5,2)$

$P(1,2,3)$

$\vec{w} = \overrightarrow{P_0P} = \langle 1-1, 2-5, 3-2\rangle = \langle 0,-3,1\rangle$

$$\vec{n} = \vec{v}\times\vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 3 \\ 0 & -3 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ -3 & 1 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 1 & -1 \\ 0 & -3 \end{vmatrix}\hat{k} = 8\hat{\imath} - \hat{\jmath} + (-3)\hat{k}$$

$\vec{n} = \langle 8,-1,-3\rangle$ & $P(1,2,3)$ $\Rightarrow$ $\boxed{8(x-1)-1(y-2)-3(z-3)=0}$

$$\boxed{8x - y - 3z + 3 = 0}$$

Two planes are parallel $\Longleftrightarrow$ their normal vectors are parallel
$(\vec{n}_1 = k\vec{n}_2)$

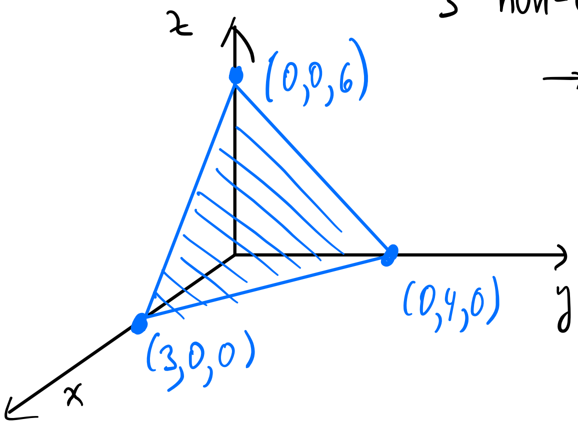
Ex: Find an eqn of a plane that is parallel to the plane $8x-y-3z+3=0$ and goes through the point $(5,6,7)$.

$$\boxed{8(x-5)-1(y-6)-3(z-7)=0}$$

Angle between two planes = Angle between their normal vectors

use $\cos\theta = \dfrac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

Ex: Sketch $4x + 3y + 2z = 12$



3 non-collinear points determine a plane

$\rightarrow$ Find intersections with x, y, z axes

x-intercept (set $y=0, z=0$) : $(3, 0, 0)$

$4x = 12 \Rightarrow x = 3$

y-intercept (set $x=0, z=0$) : $(0, 4, 0)$

z-intercept (set $x=0, y=0$) : $(0, 0, 6)$

Intersections

- **line & plane**

"substitute parametric eqns into eqn of plane"

Ex: Line: $x = 2 + 3t$
$y = 1 + 4t$
$z = -1 + 2t$

Plane: $8x - y - 3z + 3 = 0$

plug-in

then solve for t

$$8(2+3t) - (1+4t) - 3(-1+2t) + 3 = 0$$
$$16 + 24t - 1 - 4t + 3 - 6t + 3 = 0$$
$$21 + 14t = 0 \Longrightarrow t = -\frac{21}{14} = -\frac{3}{2}$$

plug-in $t = -\frac{3}{2}$ into Line eqn:

$$x = 2 + 3\left(-\tfrac{3}{2}\right) = 2 - \frac{9}{2} = -\frac{5}{2}$$
$$y = 1 + 4\left(-\tfrac{3}{2}\right) = 1 - 6 = -5$$
$$z = -1 + 2\left(-\tfrac{3}{2}\right) = -1 - 3 = -4$$

Point of intersection: $\left(-\frac{5}{2}, -5, -4\right)$

- plane & plane

$\langle 8,-1,-3\rangle \neq k\langle -4,1,2\rangle$

NOT PARALLEL

$\Rightarrow$ must intersect

Ex: Plane 1: $8x-y-3z+3=0$

Plane 2: $-4x+y+2z+5=0$

First method: Introduce a parameter. $z=t$.

① $8x-y-3t+3=0$

② $-4x+y+2t+5=0$

①+②: $4x - t + 8 = 0$

$4x = t-8$

$x = \frac{1}{4}t - 2$

$8\left(\frac{1}{4}t-2\right)-y-3t+3=0$

$2t-16-y-3t+3=0$

$-t-13=y$

$$L:\ \begin{aligned} x &= -2+\tfrac{1}{4}t \\ y &= -13 - t \\ z &= t, \quad t\in\mathbb{R} \end{aligned}$$

Second method:

① any point which belongs to both planes (i.e., satisfies both eqs)

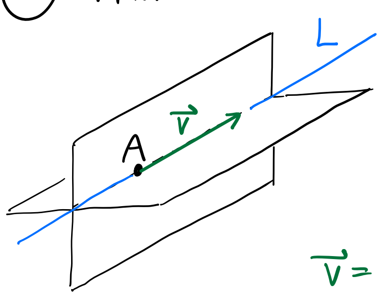
For example, set $z=0$ (or $x=0$, or $y=0$)

then
$$\begin{aligned} 8x-y+3&=0 \\ -4x+y+5&=0 \\ \hline 4x+8&=0 \\ x&=-2 \end{aligned}$$

$\longrightarrow y=-5+4x=-5-8=-13$

Point: $A(-2,-13,0)$

② Find the vector $\vec{v}=\vec{n}_1\times\vec{n}_2$, where $\vec{n}_1$ & $\vec{n}_2$ are the normal vectors of the given planes.



$\vec{v}$ will be orthogonal to both $\vec{n}_1$ and $\vec{n}_2$

$\vec{v}$ must lie in both planes

$$\vec{v}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & -1 & -3 \\ -4 & 1 & 2 \end{vmatrix} = \hat{i}\,(-2-(-3))-\hat{j}\,(16-12)+\hat{k}\,(8-4)$$

$$= \langle 1,-4,4\rangle$$

Line of intersection: $\langle x,y,z\rangle = \langle -2,-13,0\rangle + \langle 1,-4,4\rangle t$, $t\in\mathbb{R}$