Dot Product $\cdot$

- valid in any dimension
- result is a scalar
- Application: WORK

$\langle 1,-1,2\rangle \cdot \langle 0,3,-4\rangle = -11$

Thm: $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

versus

Cross Product $\times$

- valid only in $\mathbb{R}^3$
- result is a ~~number~~ vector!
- Application: torque

$\langle 1,-1,2\rangle \times \langle 0,3,-4\rangle = \langle ?,?,?\rangle$

Thm: $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

Determinant

determinant of order 2: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

determinant of order 3:

minus sign

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

EX: $$\begin{vmatrix} 3 & 4 & 1 \\ -2 & 5 & -1 \\ 0 & 7 & 6 \end{vmatrix} = 3\begin{vmatrix} 5 & -1 \\ 7 & 6 \end{vmatrix} - 4\begin{vmatrix} -2 & -1 \\ 0 & 6 \end{vmatrix} + 1\begin{vmatrix} -2 & 5 \\ 0 & 7 \end{vmatrix}$$

$$= 3(5\cdot 6 - (-1)\cdot 7) - 4((-2)6 - (-1)\cdot 0) + 1\cdot((-2)7 - 5\cdot 0)$$

$$= 3\cdot 37 - 4\cdot(-12) - 14 = 145$$

(Another way of computing 3x3 determinant

$$\left|\begin{array}{ccc|cc} 3 & 4 & 1 & 3 & 4 \\ -2 & 5 & -1 & -2 & 5 \\ 0 & 7 & 6 & 0 & 7 \end{array}\right| = 3\cdot 5\cdot 6 + 4\cdot(-1)\cdot 0 + 1\cdot(-2)\cdot 7$$
$$- 0\cdot 5\cdot 1 - 7\cdot(-1)\cdot 3 - 6\cdot(-2)\cdot 4$$
)

Cross Product

In general: $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

"symbolic determinant"

$$= \hat{i}\,(a_2 b_3 - a_3 b_2) - \hat{j}\,(a_1 b_3 - a_3 b_1) + \hat{k}\,(a_1 b_2 - a_2 b_1)$$

$$= \langle a_2 b_3 - a_3 b_2,\ -(a_1 b_3 - a_3 b_1),\ a_1 b_2 - a_2 b_1 \rangle$$

Ex: $\vec{a} = \langle 1, -1, 2 \rangle$

$\vec{b} = \langle 0, 3, -4 \rangle$

forgetting this minus is the #1 mistake!

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 0 & 3 & -4 \end{vmatrix} = \hat{i} \begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 2 \\ 0 & -4 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix}$$

"symbolic determinant"

$$= \hat{i}\big((-1)(-4) - 2 \cdot 3\big) - \hat{j}\big(1(-4) - 2 \cdot 0\big) + \hat{k}\big(1 \cdot 3 - (-1) \cdot 0\big)$$

$$= -2\,\hat{i} + 4\,\hat{j} + 3\,\hat{k}$$

$$= \langle -2, 4, 3 \rangle$$

Check orthogonality:

$(\vec{a} \times \vec{b}) \cdot \vec{a} = \langle -2, 4, 3 \rangle \cdot \langle 1, -1, 2 \rangle = -2 - 4 + 6 = 0$ ✓

$(\vec{a} \times \vec{b}) \cdot \vec{b} = \langle -2, 4, 3 \rangle \cdot \langle 0, 3, -4 \rangle = 0 + 12 - 12 = 0$ ✓

Alternative process:

$$\vec{a} \times \vec{b} = \left| \begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ 1 & -1 & 2 & 1 & -1 \\ 0 & 3 & -4 & 0 & 3 \end{array} \right|$$

$$= 4\,\hat{i} + 0\,\hat{j} + 3\,\hat{k}$$
$$- 0\,\hat{k} - 6\,\hat{i} - (-4) \cdot \hat{j}$$
$$= -2\,\hat{i} + 4\,\hat{j} + 3\,\hat{k}$$

Properties of the Cross Product

Remember: only works in $\mathbb{R}^3$!

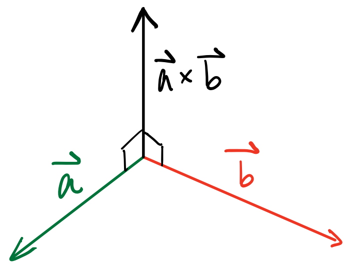
- It is a vector.
- $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$

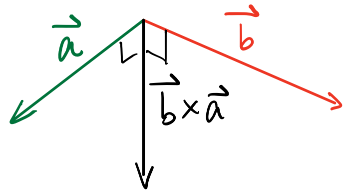
check: $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$

$(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$

- $\vec{b} \times \vec{a}$ is also orthogonal to both $\vec{a}$ and $\vec{b}$, but goes the opposite direction

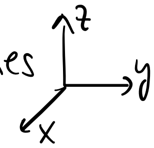
Right Hand Rule:
point your right hand in the direction of $\vec{a}$ and curl your fingers towards $\vec{b}$, then your thumb points in the dir. of $\vec{a} \times \vec{b}$





EX: $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, $\hat{k} \times \hat{\imath} = \hat{\jmath}$

$\hat{\jmath} \times \hat{\imath} = -\hat{k}$, $\hat{k} \times \hat{\jmath} = -\hat{\imath}$, $\hat{\imath} \times \hat{k} = -\hat{\jmath}$

(consequence of order of axes )

- $\boxed{\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}}$ NOT COMMUTATIVE!

How about the length of the vector $\vec{a} \times \vec{b}$?

EX: $\left.\begin{array}{l} \vec{a} = \langle a_1, a_2, a_3 \rangle \\ \vec{b} = \langle ca_1, ca_2, ca_3 \rangle \end{array}\right\}$ parallel $\Longrightarrow |\vec{a} \times \vec{b}| = ?$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ ca_1 & ca_2 & ca_3 \end{vmatrix} = \underbrace{(ca_2a_3 - ca_3a_2)}_{0}\hat{\imath} - \underbrace{(ca_1a_3 - ca_3a_2)}_{0}\hat{\jmath} + \underbrace{(ca_1a_2 - ca_2a_1)}_{0}\hat{k}$$

Ex: $\left.\begin{array}{l} \vec{a} = \langle 1, 0, 0 \rangle \\ \vec{b} = \langle 0, 1, 0 \rangle \end{array}\right\}$ orthogonal

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \langle 0, 0, 1 \rangle$$

Thm: $\boxed{|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta}$, where θ is the angle between $\vec{a}$ and $\vec{b}$. $0 \le \theta \le \pi$

Proof: Follows from $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, which relies on the Law of Cosines

OPTIONAL

$$\begin{aligned}
|\vec{a}\times\vec{b}|^2 &= |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 \quad \leftarrow \text{verified below}\\
&= |\vec{a}|^2|\vec{b}|^2 - (|\vec{a}||\vec{b}|\cos\theta)^2\\
&= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta\\
&= |\vec{a}|^2|\vec{b}|^2(\underbrace{1-\cos^2\theta}_{\sin^2\theta})
\end{aligned}$$

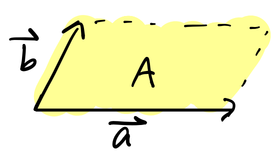
$$\begin{aligned}
|\vec{a}\times\vec{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (-1)^2(a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2\\
&= a_2^2b_3^2 \underline{- 2a_2a_3b_2b_3} + a_3^2b_2^2 + a_1^2b_3^2 \underline{- 2a_1a_3b_1b_3} + a_3^2b_1^2 + a_1^2b_2^2 \underline{- 2a_1a_2b_1b_2} + a_2^2b_1^2
\end{aligned}$$

$$\begin{aligned}
|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 &= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2\\
&= \cancel{a_1^2b_1^2} + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + \cancel{a_2^2b_2^2} + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + \cancel{a_3^2b_3^2}\\
&\quad - \Big(\cancel{a_1^2b_1^2} + \underline{a_1b_1a_2b_2} + \underline{a_1b_1a_3b_3} + \underline{a_2b_2a_1b_1} + \cancel{a_2^2b_2^2} + \underline{a_2b_2a_3b_3}\\
&\quad + \underline{a_3b_3a_1b_1} + \underline{a_3b_3a_2b_2} + \cancel{a_3^2b_3^2}\Big)
\end{aligned}$$

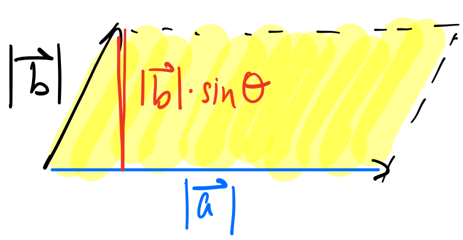
Geometric interpretation of $\vec{a}\times\vec{b}$

$\vec{a}\times\vec{b}$ = perpendicular vector to both $\vec{a}$ and $\vec{b}$

$|\vec{a}\times\vec{b}|$ = area of parallelogram



$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$



$A = |\vec{a}||\vec{b}|\sin\theta$

Properties of the Cross Product

If $\vec{a}, \vec{b}, \vec{c}$ are vectors in $\mathbb{R}^3$ and k is a scalar, then

1. $\boxed{\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}}$! NOT commutitive !

2. $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b})$

3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

$\Big\}$ "distributive over vector addition"

5. Scalar triple product:

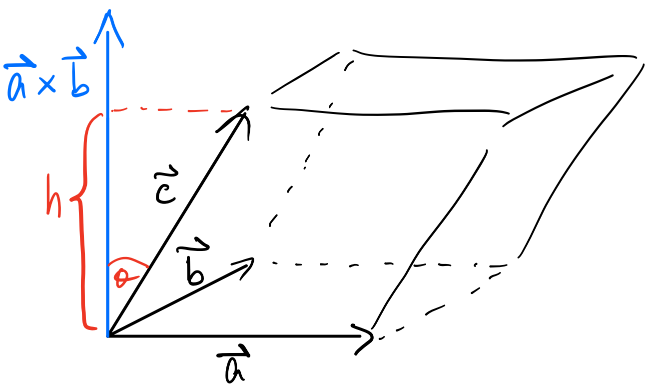
$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

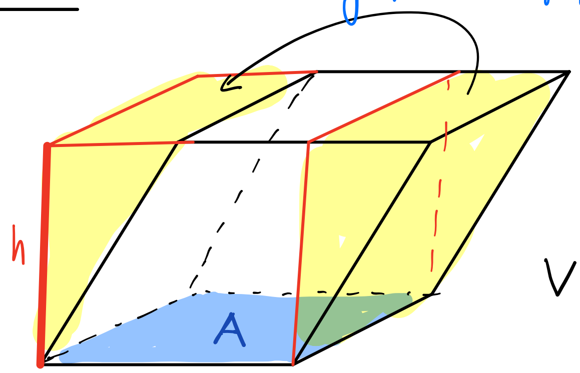
$$\cancel{\vec{a} \cdot (\vec{b} \times \vec{c}) \neq \underbrace{(\vec{a} \cdot \vec{b})}_{\text{number}} \times \underbrace{(\vec{a} \cdot \vec{c})}_{\text{number}}}$$

can be expressed as determinant:
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Geometric interpretation of $(\vec{a} \times \vec{b}) \cdot \vec{c}$

"Volume of parallelepiped"





$V = Ah$

$A = |\vec{a} \times \vec{b}|$, $h = |\vec{c}| \cdot |\cos\theta|$

Recall: $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$

So: $|\vec{u} \cdot \vec{v}| = |\vec{u}||\vec{v}||\cos\theta|$

$$V = |\vec{a} \times \vec{b}|\,|\vec{c}|\,|\cos\theta| = \Big|\underbrace{(\vec{a} \times \vec{b}) \cdot \vec{c}}_{\text{number}}\Big|$$

absolute value

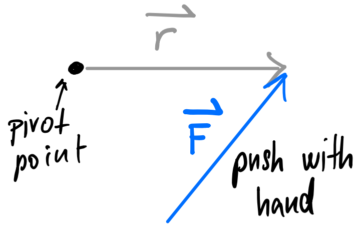
6. Vector triple product:

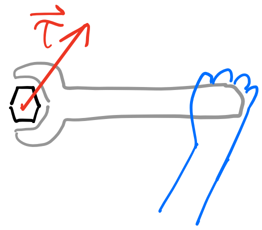
$$\vec{a} \times (\vec{b} \times \vec{c}) = \underbrace{(\vec{a} \cdot \vec{c})}_{\text{scalar}}\vec{b} - \underbrace{(\vec{a} \cdot \vec{b})}_{\text{scalar}}\vec{c}$$

$$\cancel{\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}}$$

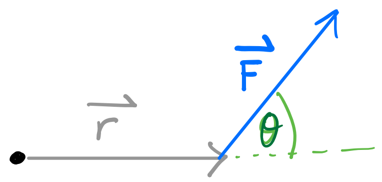
Torque $\vec{\tau}$ (optional)

= turning effect caused by a force at a point





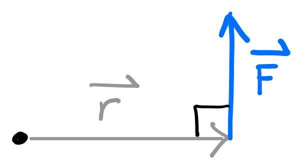
think: $\vec{r}$ represents a wrench



$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

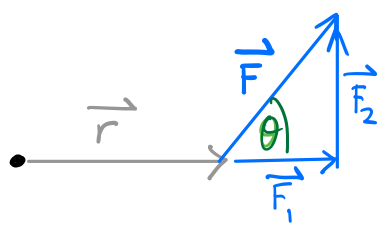
$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = \underbrace{|\vec{r}|}_{\substack{\text{length}\\ \text{of wrench}\\ [m]}}\underbrace{|\vec{F}|}_{\substack{\text{magnitude}\\ \text{of force}\\ [N]}} \sin\theta$$

EX: What if $\theta = \frac{\pi}{2}$?



$$|\vec{\tau}| = |\vec{r}||\vec{F}|\underbrace{\sin\frac{\pi}{2}}_{=1} = |\vec{r}||\vec{F}|$$

only the perpendicular component of $\vec{F}$ causes rotation



$$\vec{F} = \vec{F_1} + \vec{F_2}$$

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times (\vec{F_1} + \vec{F_2})$$

$$= \underbrace{\vec{r} \times \vec{F_1}}_{\substack{\vec{0}\\ \text{b/c parallel}}} + \vec{r} \times \vec{F_2}$$