Calc III Notes

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1 Multivariable calc goals

- vectors and geometry of R"
- functions of serval variables diff and int
- higher dimensional versions of FTC
- fund them for line integrals
- greens them
- stokes them
- divergence them

12 Chapter 12: vectors and geo of space

12.1

- $$R = line$$
  dist(x,y) = $|y - x|$
- $$R^2 = Plane$$
  Orders pairs of (x,y) of real numbers
- $$R^3 = space$$
  ordered triples of (x,y,z) of real numbers

right-hand rule: point fingers toward pos x-axis and curl towards pos y-axis, thumb should point to positive z

R^n = n-tuples $(x_1, x_2, ...x_n)$ n-dim space

Distance D from (0,0) to (x,y)

$$D^2 = x^2 + y^2$$

Distance D from (0,0,0) to (x,y,z)

$$D = \sqrt{x^2 + yy^2 + z^2}$$

Sphere with center X_0, Y_0, Z_0 and radius R has equation:

$$(X - X_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Find equation of the set of points equidistant from the points a1,4,2, and b 3,3,3

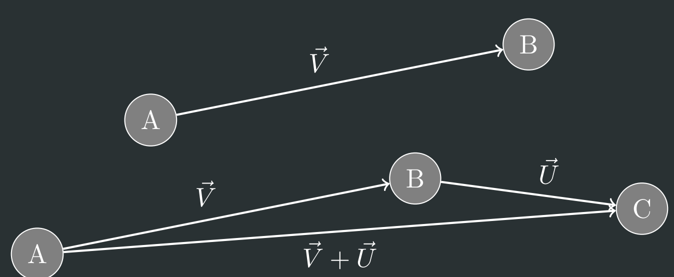
$$AX = \sqrt{(x-1)^2 + (y-4)^2 + (z-2)^2}$$
$$BX = \sqrt{(x-3)^2 + (y-3)^2 + (z-3)^2}$$

set AX = BX
Result is a **plane**

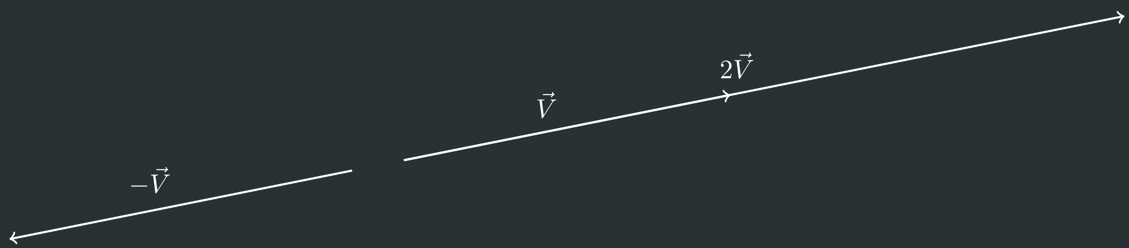
12.2 vectors and geo of space

scaler = magnitude ≤ 0
vector = magnitude + direction

$\vec{u} =< 5, 3 >$, $\vec{u} = 5\hat{i} + 3\hat{j}$



12.2.1 Scaler multiple



Changes the length of the vectors, ie, stretches them in a direction.
Length of k$\vec{v} = |K| * |V|$
direction of $k\vec{V} = sameas\vec{V}ifk > 0$
real numbers work like scaling factors

Points = (a,b)
Vector = $< a, b >$

12.2.2 Vectors in R^3

Extra cords for points and extra comps for vectors
point = (a,b,c)
vector = $< a, b, c >$

$$if\vec{V} = P_1\vec{P}_2, then|\vec{V}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$P(4, -1, -3)$
$Q(8, 0, -5)$
$R(0, -2, -1)$

$\vec{PQ} =< 8 - 4, 0 - -1, -5 - -3 >$

$$\vec{u} =< 3, 1, 3 >$$

$$\vec{v} =< 1, 3, 7 >$$

$$\vec{v} + \vec{v} =< 3 + 1, 1 + 3, 3 + 7 >$$

$$5 * \vec{v} =< 1 * 5, 3 * 5, 7 * 5 >$$

12.2.3 notation

$$V_n$$

set of all n-dim vectors $< V_1, V_2, ..., V_n >$
V_3 set of all vectors $< v_1, v_2, v_3 >$
$\hat{i} =< 1, 0, 0 >$

$\hat{j} =< 0, 1, 0 >$
$\hat{k} =< 0, 0, 1 >$

unit vector = vector of length 1
$\hat{i}, \hat{j}, \hat{k}$ are unit vectors
find the unit vector that has the same direction as $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

$$\sqrt{4^2(-2)^2 + 3^2}$$

$$\sqrt{16 + 4 + 9}$$

$$\sqrt{29}$$

$$\frac{\vec{b}}{|\vec{b}|} = \frac{1}{\sqrt{29}}(4\hat{i} - 2\hat{j} + 3\hat{k})$$

12.2.4 dot product

dot product $\rightarrow$ 2 vectors $\rightarrow$ 1 scaler
cross product $\rightarrow$ 2 vectors $\rightarrow$ 1 vector

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Properties

$$\vec{a} \cdot \vec{a} = \vec{a}^2$$

if θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$

follows from the law of cosines

$\vec{a}$ and $\vec{b}$ are orthogonal (or perp) if te angle between them is $\theta = \frac{\pi}{2}$

the zero vector $\vec{0}$ is considered orthogonal to all vectors

$\vec{a} \cdot \vec{b} = 0 <=> \vec{a} \perp \vec{b}$, orthogonal, form 90deg angle

accute

$$\vec{a} \cdot \vec{b} > 0$$

obtuse

$$\vec{a} \cdot \vec{b} > 0$$

find angle θ between $a =< 1, 1, 5 >$ and $b =< 0, 3, 4 >$

$$a = \sqrt{1^2 + 1^2 + 5^2}$$
$$b = \sqrt{0^2 + 3^2 + 4^2}$$
$$\cos\theta = \frac{\vec{a}\vec{b}}{|\vec{a}||\vec{b}|}$$
$$\cos\theta = \frac{23}{3\sqrt{3}\times 5}$$
$$\theta = \arccos(\frac{23}{15\sqrt{3}})$$

12.3 Projections

$proj_{\vec{a}}(\vec{b})$ = vector projection of $\vec{b}$ onto $\vec{a}$
Vector = A scaler-multiple of $\vec{a}$

$comp_{\vec{a}}(\vec{a})$ = Scaler projection of $\vec{b}$ onto $\vec{a}$
Scaler = Signed magnitude of the vector projection
$= |\vec{b} cos\theta|$

dot product

- valid in any dim
- result is a scaler
- application:work

cross product

- valid in only r^3
- result is a vector
- application: torque

$\hat{i} \times \hat{j} = \hat{k}$

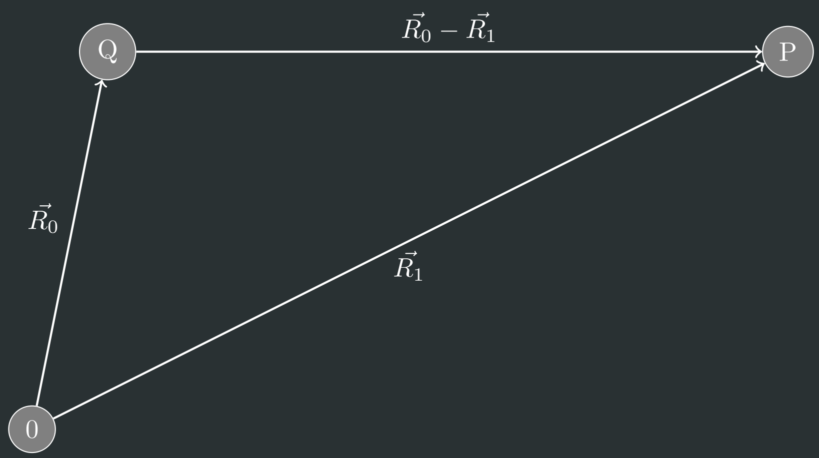
$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

12.4

12.5 Equations of lines and planes

line segment from $\vec{r_0}$ to $\vec{r_1}$

$$\vec{r} = \vec{_0} + t(\vec{r_1} - \vec{r_0}), 0 \leq t \leq 1$$



EX: Find parametric eq for the line seg from P(1,-3,5) to Q(4,-7,2)

12.5.1 Planes in R^3

How to specify plane?

- specify a point $P_0 \approx (x_0, y_0, z_0) onplane$
- and a normal vector $\vec{n} =< a, b, c >\neq \vec{0}$ which is perp to the plane

Then P (x,y,z) is in plane :$\Leftrightarrow$ $\vec{n}$ and $\vec{p_0 p}$ are orth

$$\vec{n} \cdot \vec{p_0 P} = 0$$

eqn of a plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$ax + by + cz + d = 0, \text{ where } d = -(ax_0 + by_0 + cz_0)$$

Ex. Find an eqn of the plane through points

$$P_1(0, 1, 4)$$

$$P_2(2, 4, 1)$$

$$P_3(3, -1, 5)$$

$$\vec{V} = \vec{P_1 P_2} =< 2, 3, -3 >= P_2 - P - 1$$

$$\vec{W} = \vec{P_1 P_3} =< 3, -2, 1 >= P_3 - P - 1$$

$$\vec{N} = \vec{V} \times \vec{W} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -3 \\ 3 & -2 & 1 \end{bmatrix} = -3\hat{i} - 11\hat{j} - 13\hat{k}$$

PLANE: $P_1(0,1,4), \vec{n} =< -3, -11, -13 >$

$$-3(x-0) + -11(y-1) + -13(z-1) = 0$$

$$+3(x-0) + 11(y-1) + 13(z-1) = 0$$

$$+3(x) + 11y + 13z - 63 = 0$$

EX: Find an eqn of a plane that is parallel to the plane $*x - y - 3z + 3 = 0$
and goes through the point (5,6,7)

$$8(x-5) - 1(y-6) - 3(z-7) = 0$$

Angle between two planes = Angle btween their normal vectors
use $\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$

Sketching a plane
3 noncollinear points determine a plane
find interactions with x,y,z axes
then draw plane between them

12.5.2 Intersections

ex: line and plane

Line=

$$X = 2 - 3t$$

$$y = 1 + 4t$$

$$z = -1 + 2t$$

Plane =

$$8x - y - 3z + 3 = 0$$

$$8(2+3t) - (1+4t) - 3(-1+2t) + 3 = 0$$

$$21 + 14t = 0$$

$$t = -\frac{3}{2}$$

ex: plane and plane

$$plane1 = 8x - y - 3z + 3 = 0$$

$$plane2 = -4x + y + 2z + 5 = 0$$

First method: introduce a parameter. z=t

$$plane1 = 8x - y - 3t + 3 = 0$$

$$plane2 = -4x + y + 2t + 5 = 0$$

$$P_1 + P_2 = 4x - t + 8 = 0$$

$$x = \frac{1}{4}t - 2$$

$$8(\frac{1}{4}t - 2) - y - 3t + 3 = 0$$

$$-t - 13 = y$$

$$x = 2 + 1/4t$$

$$y - 13 - t$$

$$z = t$$

13 Chapter 13: vector functions

14 Chapter 14: partial derivate

15 Chapter 15: multiple integrals

16 Chapter 16: vector calculus