

# Calc III Notes

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## 1 Multivariable calc goals

- vectors and geometry of  $\mathbb{R}^n$
- functions of several variables diff and int
- higher dimensional versions of FTC
- find them for line integrals
- greens them
- stokes them
- divergence them

## 12 Chapter 12: vectors and geo of space

### 12.1

- $$\mathbb{R} = \text{line}$$
  
$$\text{dist}(x,y) = |y - x|$$
- $$\mathbb{R}^2 = \text{Plane}$$
  
Orders pairs of  $(x,y)$  of real numbers
- $$\mathbb{R}^3 = \text{space}$$
  
ordered triples of  $(x,y,z)$  of real numbers

right-hand rule: point fingers toward pos x-axis and curl towards pos y-axis, thumb should point to positive z

$R^n$  = n-tuples  $(x_1, x_2, \dots, x_n)$  n-dim space

Distance D from (0,0) to (x,y)

$$D^2 = x^2 + y^2$$

Distance D from (0,0,0) to (x,y,z)

$$D = \sqrt{x^2 + y^2 + z^2}$$

Sphere with center  $X_0, Y_0, Z_0$  and radius R has equation:

$$(X - X_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Find equation of the set of points equidistant from the points a1,4,2, and b 3,3,3

$$AX = \sqrt{(x-1)^2 + (y-4)^2 + (z-2)^2}$$

$$BX = \sqrt{(x-3)^2 + (y-3)^2 + (z-3)^2}$$

set  $AX = BX$

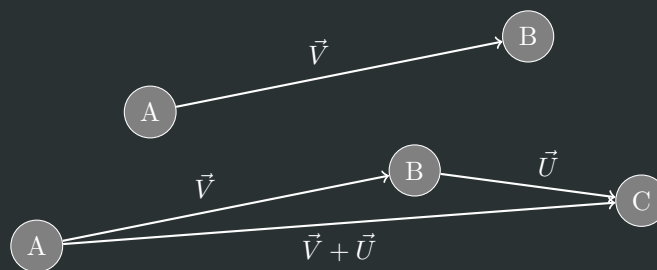
Result is a **plane**

## 12.2 vectors and geo of space

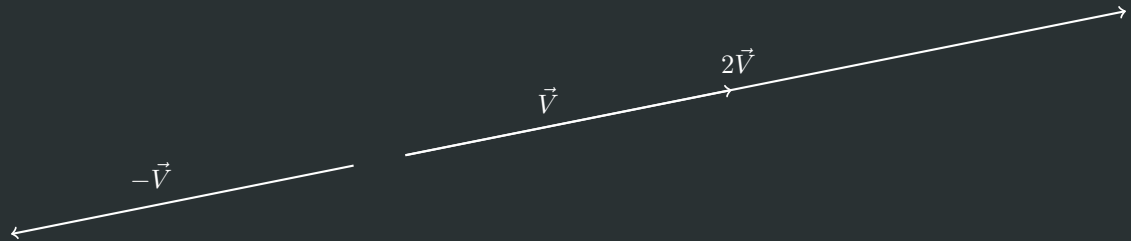
scaler = magnitude  $\leq 0$

vector = magnitude + direction

$$\vec{u} = \langle 5, 3 \rangle, \vec{u} = 5\hat{i} + 3\hat{j}$$



### 12.2.1 Scaler multiple



Changes the length of the vectors, ie, stretches them in a direction.  
 Length of  $k\vec{v} = |K| * |V|$   
 direction of  $k\vec{V} = \text{same as } \vec{V} \text{ if } k > 0$   
 real numbers work like scaling factors

Points = (a,b)  
 Vector =  $\langle a, b \rangle$

### 12.2.2 Vectors in $R^3$

Extra cords for points and extra comps for vectors

point = (a,b,c)  
 vector =  $\langle a, b, c \rangle$

$$\text{if } \vec{V} = P_1P_2, \text{ then } |\vec{V}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$P(4, -1, -3)$   
 $Q(8, 0, -5)$   
 $R(0, -2, -1)$

$$\vec{PQ} = \langle 8 - 4, 0 - (-1), -5 - (-3) \rangle$$

$$\vec{u} = \langle 3, 1, 3 \rangle$$

$$\vec{v} = \langle 1, 3, 7 \rangle$$

$$\vec{u} + \vec{v} = \langle 3 + 1, 1 + 3, 3 + 7 \rangle$$

$$5 * \vec{v} = \langle 1 * 5, 3 * 5, 7 * 5 \rangle$$

### 12.2.3 notation

$$V_n$$

set of all n-dim vectors  $\langle V_1, V_2, \dots, V_n \rangle$

$V_3$  set of all vectors  $\langle v_1, v_2, v_3 \rangle$

$$\hat{i} = \langle 1, 0, 0 \rangle$$

$$\hat{j} = \langle 0, 1, 0 \rangle$$

$$\hat{k} = \langle 0, 0, 1 \rangle$$

unit vector = vector of length 1

$\hat{i}, \hat{j}, \hat{k}$  are unit vectors

find the unit vector that has the same direction as  $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

$$\frac{\sqrt{4^2(-2)^2 + 3^2}}{\sqrt{16 + 4 + 9}}$$

$$\sqrt{29}$$

$$\frac{\vec{b}}{|\vec{b}|} = \frac{1}{\sqrt{29}}(4\hat{i} - 2\hat{j} + 3\hat{k})$$

#### 12.2.4 dot product

dot product  $\rightarrow$  2 vectors  $\rightarrow$  1 scalar

cross product  $\rightarrow$  2 vectors  $\rightarrow$  1 vector

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

##### Properties

$$\vec{a} \cdot \vec{a} = a^2$$

if  $\theta$  is the angle between the vectors  $\vec{a}$  and  $\vec{b}$ , then  $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$

follows from the law of cosines

$\vec{a}$  and  $\vec{b}$  are orthogonal (or perp) if the angle between them is  $\theta = \frac{\pi}{2}$

the zero vector  $\vec{0}$  is considered orthogonal to all vectors

$\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ , orthogonal, form 90deg angle

acute

$$\vec{a} \cdot \vec{b} > 0$$

obtuse

$$\vec{a} \cdot \vec{b} < 0$$

find angle  $\theta$  between  $a = \langle 1, 1, 5 \rangle$  and  $b = \langle 0, 3, 4 \rangle$

$$\begin{aligned}
a &= \sqrt{1^2 + 1^2 + 5^2} \\
b &= \sqrt{0^2 + 3^2 + 4^2} \\
\cos \theta &= \frac{\vec{a}\vec{b}}{|\vec{a}||\vec{b}|} \\
\cos \theta &= \frac{23}{3\sqrt{3} \times 5} \\
\theta &= \arccos\left(\frac{23}{15\sqrt{3}}\right)
\end{aligned}$$

## 12.3 Projections

$proj_{\vec{a}}(\vec{b})$  = vector projection of  $\vec{b}$  onto  $\vec{a}$   
 Vector = A scalar-multiple of  $\vec{a}$

$comp_{\vec{a}}(\vec{a})$  = Scalar projection of  $\vec{b}$  onto  $\vec{a}$   
 Scalar = Signed magnitude of the vector projection  
 $= |\vec{b}\cos\theta|$

### dot product

- valid in any dim
- result is a scalar
- application: work

### cross product

- valid in only  $r^3$
- result is a vector
- application: torque

$$\hat{i} \times \hat{j} = \hat{k}$$

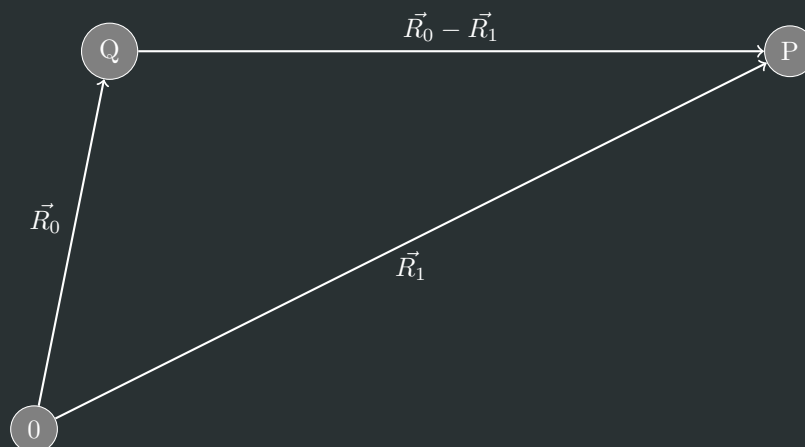
$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

## 12.4

## 12.5 Equations of lines and planes

line segment from  $\vec{r}_0$  to  $\vec{r}_1$

$$\vec{r} = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0), 0 \leq t \leq 1$$



EX: Find parametric eq for the line seg from P(1,-3,5) to Q(4,-7,2)

### 12.5.1 Planes in $R^3$

How to specify plane?

- specify a point  $P_0 \approx (x_0, y_0, z_0)$  on plane
- and a normal vector  $\vec{n} = \langle a, b, c \rangle \neq \vec{0}$  which is perp to the plane

Then P (x,y,z) is in plane  $\Leftrightarrow \vec{n}$  and  $\vec{p_0p}$  are orth

$$\vec{n} \cdot \vec{p_0p} = 0$$

#### eqn of a plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$ax + by + cz + d = 0, \text{ where } d = -(ax_0 + by_0 + cz_0)$$

Ex. Find an eqn of the plane through points

$$P_1(0, 1, 4)$$

$$P_2(2, 4, 1)$$

$$P_3(3, -1, 5)$$

$$\vec{V} = P_1\vec{P_2} = \langle 2, 3, -3 \rangle = P_2 - P_1$$

$$\vec{W} = P_1\vec{P_3} = \langle 3, -2, 1 \rangle = P_3 - P_1$$

$$\vec{N} = \vec{V} \times \vec{W} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -3 \\ 3 & -2 & 1 \end{bmatrix} = -3\hat{i} - 11\hat{j} - 13\hat{k}$$

PLANE:  $P_1(0, 1, 4), \vec{n} = \langle -3, -11, -13 \rangle$

$$-3(x - 0) + -11(y - 1) + -13(z - 1) = 0$$

$$+3(x - 0) + 11(y - 1) + 13(z - 1) = 0$$

$$+3(x) + 11y + 13z - 63 = 0$$

EX: Find an eqn of a plane that is parallel to the plane  $*x - y - 3z + 3 = 0$   
and goes through the point  $(5, 6, 7)$

$$8(x - 5) - 1(y - 6) - 3(z - 7) = 0$$

Angle between two planes = Angle between their normal vectors

$$\text{use } \cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}$$

Sketching a plane

3 noncollinear points determine a plane

find intersections with x,y,z axes

then draw plane between them

### 12.5.2 Intersections

ex: line and plane

**Line=**

$$X = 2 - 3t$$

$$y = 1 + 4t$$

$$z = -1 + 2t$$

**Plane =**

$$8x - y - 3z + 3 = 0$$

$$8(2 + 3t) - (1 + 4t) - 3(-1 + 2t) + 3 = 0$$

$$21 + 14t = 0$$

$$t = -\frac{3}{2}$$

ex: plane and plane

$$plane1 = 8x - y - 3z + 3 = 0$$

$$plane2 = -4x + y + 2z + 5 = 0$$

First method: introduce a parameter.  $z=t$

$$plane1 = 8x - y - 3t + 3 = 0$$

$$plane2 = -4x + y + 2t + 5 = 0$$

$$P_1 + P_2 = 4x - t + 8 = 0$$

$$x = \frac{1}{4}t - 2$$

$$8(\frac{1}{4}t - 2) - y - 3t + 3 = 0$$

$$-t - 13 = y$$

$$x = 2 + 1/4t$$

$$y = -13 - t$$

$$z = t$$

### 12.5.3 Parametric equations

Uses both a point and a vector. Pretty much a spot where the line starts, and then what direction it is going.

When parametrizing you then say it lies along this line at some time  $t$ , with  $-\infty < t < \infty$ .

For a line through  $P_0(x_0, y_0, z_0)$

$$\vec{V} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

$$x = x_0 + tv_1$$

$$y = y_0 + tv_2$$

$$z = z_0 + tv_3$$

**13 Chapter 13: vector functions**

**14 Chapter 14: partial derivate**

**15 Chapter 15: multiple integrals**

**16 Chapter 16: vector calculus**

**17 Examples**