

Calc III Notes

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1 Multivariable calc goals

- vectors and geometry of \mathbb{R}^n
- functions of several variables diff and int
- higher dimensional versions of FTC
- find them for line integrals
- greens them
- stokes them
- divergence them

12 Chapter 12: vectors and geo of space

12.1

- $R = \text{line}$

$$\text{dist}(x,y) = |y - x|$$

- $\mathbb{R}^2 = \text{Plane}$

Orders pairs of (x,y) of real numbers

- $\mathbb{R}^3 = \text{space}$

ordered triples of (x,y,z) of real numbers

right-hand rule: point fingers toward pos x-axis and curl towards pos y-axis,
thumb should point to positive z

$$R^n = \text{n-tuples } (x_1, x_2, \dots, x_n) \text{ n-dim space}$$

Distance D from (0,0) to (x,y)

$$D^2 = x^2 + y^2$$

Distance D from (0,0,0) to (x,y,z)

$$D = \sqrt{x^2 + y^2 + z^2}$$

Sphere with center X_0, Y_0, Z_0 and radius R has equation:

$$(X - X_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Find equation of the set of points equidistant from the points a1,4,2, and b
3,3,3

$$\begin{aligned} AX &= \sqrt{(x - 1)^2 + (y - 4)^2 + (z - 2)^2} \\ BX &= \sqrt{(x - 3)^2 + (y - 3)^2 + (z - 3)^2} \end{aligned}$$

set $AX = BX$

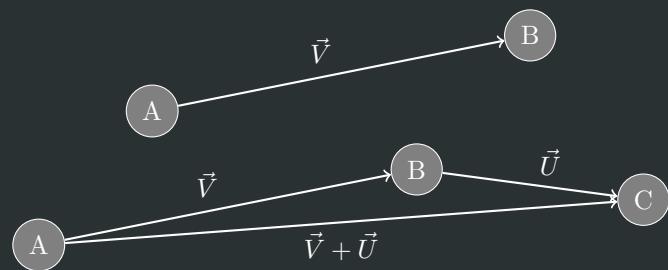
Result is a **plane**

12.2 vectors and geo of space

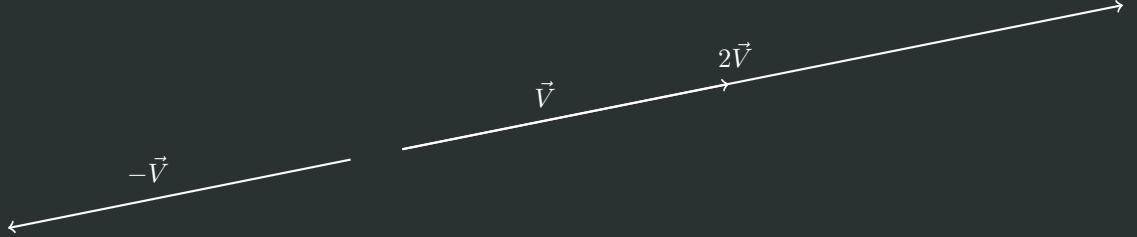
scalar = magnitude ≤ 0

vector = magnitude + direction

$$\vec{u} = \langle 5, 3 \rangle, \vec{u} = 5\hat{i} + 3\hat{j}$$



12.2.1 Scaler multiple



Changes the length of the vectors, ie, stretches them in a direction.

Length of $k\vec{v} = |K| * |\vec{v}|$

direction of $k\vec{V}$ = same as \vec{V} if $k > 0$
real numbers work like scaling factors

Points = (a,b)

Vector = $< a, b >$

12.2.2 Vectors in R^3

Extra cords for points and extra comps for vectors

point = (a,b,c)

vector = $< a, b, c >$

$$\text{if } \vec{V} = P_1 \vec{P}_2, \text{ then } |\vec{V}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$P(4, -1, -3)$$

$$Q(8, 0, -5)$$

$$R(0, -2, -1)$$

$$\vec{PQ} = < 8 - 4, 0 - -1, -5 - -3 >$$

$$\vec{u} = < 3, 1, 3 >$$

$$\vec{v} = < 1, 3, 7 >$$

$$\vec{v} + \vec{v} = < 3 + 1, 1 + 3, 3 + 7 >$$

$$5 * \vec{v} = < 1 * 5, 3 * 5, 7 * 5 >$$

12.2.3 notation

$$V_n$$

set of all n-dim vectors $< V_1, V_2, \dots, V_n >$

V_3 set of all vectors $< v_1, v_2, v_3 >$

$$\hat{i} = < 1, 0, 0 >$$

$$\begin{aligned}\hat{j} &= \langle 0, 1, 0 \rangle \\ \hat{k} &= \langle 0, 0, 1 \rangle\end{aligned}$$

unit vector = vector of length 1

$\hat{i}, \hat{j}, \hat{k}$ are unit vectors

find the unit vector that has the same direction as $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

$$\begin{aligned}&\sqrt{4^2(-2)^2 + 3^2} \\ &\sqrt{16 + 4 + 9} \\ &\sqrt{29} \\ \frac{\vec{b}}{|\vec{b}|} &= \frac{1}{\sqrt{29}}(4\hat{i} - 2\hat{j} + 3\hat{k})\end{aligned}$$

12.2.4 dot product

dot product \rightarrow 2 vectors \rightarrow 1 scalar

cross product \rightarrow 2 vectors \rightarrow 1 vector

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Properties

$$\vec{a} \cdot \vec{a} = \vec{a}^2$$

if θ is the angle between the vectors \vec{a} and \vec{b} , then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$

follows from the law of cosines

\vec{a} and \vec{b} are orthogonal (or perp) if the angle between them is $\theta = \frac{\pi}{2}$

the zero vector $\vec{0}$ is considered orthogonal to all vectors

$\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$, orthogonal, form 90deg angle

acute

$$\vec{a} \cdot \vec{b} > 0$$

obtuse

$$\vec{a} \cdot \vec{b} < 0$$

find angle θ between $a = \langle 1, 1, 5 \rangle$ and $b = \langle 0, 3, 4 \rangle$

$$\begin{aligned}
a &= \sqrt{1^2 + 1^2 + 5^2} \\
b &= \sqrt{0^2 + 3^2 + 4^2} \\
\cos \theta &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \\
\cos \theta &= \frac{23}{3\sqrt{3} \times 5} \\
\theta &= \arccos\left(\frac{23}{15\sqrt{3}}\right)
\end{aligned}$$

12.3 Projections

$\text{proj}_{\vec{a}}(\vec{b})$ = vector projection of \vec{b} onto \vec{a}

Vector = A scalar-multiple of \vec{a}

$\text{comp}_{\vec{a}}(\vec{b})$ = Scalar projection of \vec{b} onto \vec{a}
 Scalar = Signed magnitude of the vector projection
 $= |\vec{b} \cos \theta|$

dot product

- valid in any dim
- result is a scalar
- application: work

cross product

- valid in only r^3
- result is a vector
- application: torque

$$\hat{i} \times \hat{j} = \hat{k}$$

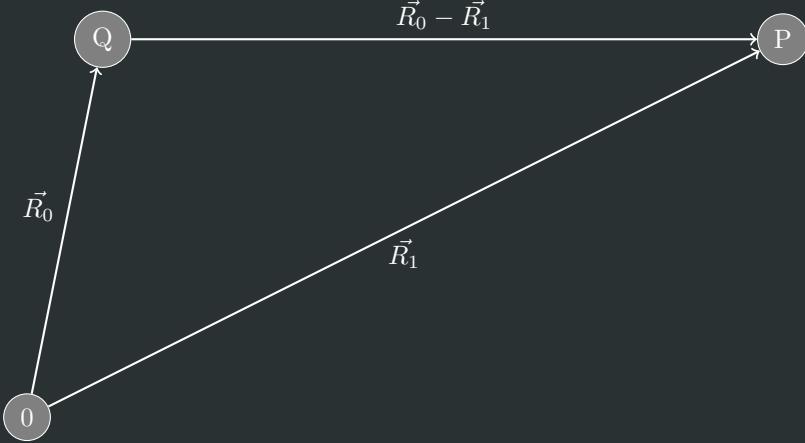
$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

12.4

12.5 Equations of lines and planes

line segment from \vec{r}_0 to \vec{r}_1

$$\vec{r} = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0), 0 \leq t \leq 1$$



EX: Find parametric eq for the line seg from P(1,-3,5) to Q(4,-7,2)

12.5.1 Planes in R^3

How to specify plane?

- specify a point $P_0 \approx (x_0, y_0, z_0)$ on plane
- and a normal vector $\vec{n} = \langle a, b, c \rangle \neq \vec{0}$ which is perp to the plane

Then P (x,y,z) is in plane $\Leftrightarrow \vec{n}$ and $\vec{p}_0\vec{p}$ are orth

$$\vec{n} \cdot \vec{p}_0\vec{P} = 0$$

eqn of a plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$ax + by + cz + d = 0, \text{ where } d = -(ax_0 + by_0 + cz_0)$$

Ex. Find an eqn of the plane through points

$$P_1(0, 1, 4)$$

$$P_2(2, 4, 1)$$

$$P_3(3, -1, 5)$$

$$\vec{V} = P_1\vec{P}_2 = \langle 2, 3, -3 \rangle = P_2 - P_1$$

$$\vec{W} = P_1\vec{P}_3 = \langle 3, -2, 1 \rangle = P_3 - P_1$$

$$\vec{N} = \vec{V} \times \vec{W} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -3 \\ 3 & -2 & 1 \end{bmatrix} = -3\hat{i} - 11\hat{j} - 13\hat{k}$$

PLANE: $P_1(0, 1, 4)$, $\vec{n} = \langle -3, -11, -13 \rangle$

$$-3(x - 0) + -11(y - 1) + -13(z - 1) = 0$$

$$+3(x - 0) + 11(y - 1) + 13(z - 1) = 0$$

$$+3(x) + 11y + 13z - 63 = 0$$

EX: Find an eqn of a plane that is parallel to the plane $x - y - 3z + 3 = 0$
and goes through the point $(5, 6, 7)$

$$8(x - 5) - 1(y - 6) - 3(z - 7) = 0$$

Angle between two planes = Angle btween their normal vectors
use $\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

Sketching a plane
3 noncollinear points determine a plane
find interactions with x,y,z axes
then draw plane between them

12.5.2 Intersections

ex: line and plane

Line=

$$X = 2 - 3t$$

$$y = 1 + 4t$$

$$z = -1 + 2t$$

Plane =

$$8x - y - 3z + 3 = 0$$

$$8(2 + 3t) - (1 + 4t) - 3(-1 + 2t) + 3 = 0$$

$$21 + 14t = 0$$

$$t = -\frac{3}{2}$$

ex: plane and plane

$$plane1 = 8x - y - 3z + 3 = 0$$

$$plane2 = -4x + y + 2z + 5 = 0$$

First method: introduce a parameter. $z=t$

$$plane1 = 8x - y - 3t + 3 = 0$$

$$plane2 = -4x + y + 2t + 5 = 0$$

$$P_1 + P_2 = 4x - t + 8 = 0$$

$$x = \frac{1}{4}t - 2$$

$$8(\frac{1}{4}t - 2) - y - 3t + 3 = 0$$

$$-t - 13 = y$$

$$x = 2 + 1/4t$$

$$y = -t - 13$$

$$z = t$$

12.5.3 Parametric equations

Uses both a point and a vector. Pretty much a spot where the line starts, and then what direction it is going.

When parametrizing you then say it lies along this line at some time t , with $-\infty < t < \infty$.

For a line through $P_0(x_0, y_0, z_0)$

$$\vec{V} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

$$x = x_0 + tv_1$$

$$y = y_0 + tv_2$$

$$z = z_0 + tv_3$$

13 Chapter 13: vector functions

14 Chapter 14: partial derivate

14.1

14.2

14.3

14.4 Tangent Planes and Linear Approximations

Tangent line to $f(x)$ at $(a, f(a))$
 $y = f(a) + f'(a)(x - a)$

Tangent plane to $f(x, y)$ at $(a, b, f(a, b))$
 $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

For linear Approximations

- Find Tangent plane
- Solve for $L(x, y)$. I.e. solve tangent plane at the point provided
 - if it asked linear approx at $(0, 0)$, solve for tangent plane, and then sub in $f(0, 0)$ and take that equation

14.4.1 Linear approx for $f(v, t)$

$$f(v, t) = f(v_0, t_0) + f_v(v_0, t_0)(v - v_0) + f_y(v_0, t_0)(t - t_0)$$

$$f(20, 40) = 28$$

$$f_v(20, 40) = \frac{f(50 - 20) - f(30, 20)}{50 - 30}$$

$$f(50, 20) = 36$$

$$f(30, 20) = 17$$

$$f_v(40, 20) \approx \frac{19}{20}$$

$$f_t(40, 20) = \frac{f(40 - 30) - f(40, 10)}{30 - 10}$$

$$f(40, 30) = 31$$

$$f(40, 10) = 21$$

$$f_t(40, 20) = \frac{10}{20}$$

15 Chapter 15: multiple integrals

16 Chapter 16: vector calculus

17 Examples