12.3 The Dot Product

Last time: $\langle 1,2,3\rangle + \langle 7,8,9\rangle = \langle 1+7, 2+8, 3+9\rangle$, $(-4)\cdot\langle 1,2,3\rangle = \langle -4,-8,-12\rangle$

Q: How do we multiply vectors?

Dot Product: 2 vectors $\longrightarrow$ 1 scalar — Application: Work

Cross Product: 2 vectors $\longrightarrow$ 1 vector — Application: Torque

Key concepts: dot product, angle between two vectors, projections

The Dot Product

$\vec{a} = \langle a_1, a_2, a_3\rangle$, $\vec{b} = \langle b_1, b_2, b_3\rangle$

~~bad notation: $\vec{a}\,\vec{b}$~~

Def: $\boxed{\vec{a} \bullet \vec{b} = a_1 b_1 + a_2 \cdot b_2 + a_3 \cdot b_3}$

never omit

Ex: $\vec{a} = \langle 1,2,3\rangle$, $\vec{b} = \langle 4,5,6\rangle$

$\vec{a}\bullet\vec{b} = 1\cdot 4 + 2\cdot 5 + 3\cdot 6 = 4\cdot 1 + 5\cdot 2 + 6\cdot 3 = \vec{b}\cdot\vec{a}$

$= 32$

- is comutative: $\boxed{\vec{a}\bullet\vec{b} = \vec{b}\bullet\vec{a}}$

Similarly, in any dimension:

EX: $\langle 1,2,3,4\rangle \cdot \langle 2,5,1,0\rangle = 1\cdot 2 + 2\cdot 5 + 3\cdot 1 + 4\cdot 0 = 15$

Ex: If $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\vec{b} = -\hat{i} + 4\hat{k}$, then $\vec{a}\cdot\vec{b} = 2\cdot(-1) + (-3)\cdot 0 + 1\cdot 4 = 2$

$\vec{a} = \langle 2,-3,1\rangle$ $\quad \vec{b} = \langle -1,0,4\rangle$

Properties:

- $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
- $\vec{a}\cdot\vec{a} = |\vec{a}|^2$ $\quad$ EX: $\vec{a} = \langle 1,2,3\rangle$ $\quad \vec{a}\cdot\vec{a} = 1^2 + 2^2 + 3^2$, $\quad |\vec{a}| = \sqrt{1^2+2^2+3^2}$
- $\vec{0}\cdot\vec{a} = \vec{a}\cdot\vec{0} = 0$
- $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$
- $(k\vec{a})\cdot\vec{b} = k(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(k\vec{b})$

scalar

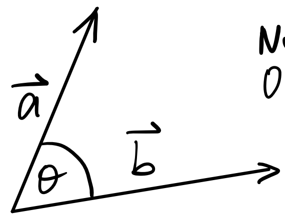
TRUE OR FALSE:

$|\vec{u}+\vec{v}|^2 = \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$

$$\begin{aligned}
|\vec{u}+\vec{v}|^2 &= (\vec{u}+\vec{v})\bullet(\vec{u}+\vec{v}) \\
&= (\vec{u}+\vec{v})\bullet\vec{u} + (\vec{u}+\vec{v})\bullet\vec{v} \\
&= \vec{u}\bullet\vec{u} + \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} \\
&= \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}
\end{aligned}$$

Thm: If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then

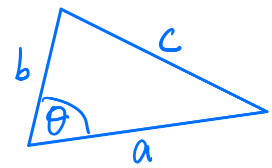
$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta}$$



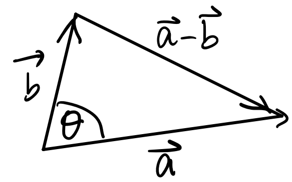
Note: $0 \le \theta \le \pi$

Proof: Follows from the Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



$$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}|\,|\vec{b}|\cos\theta$$



Now compute:

$$\begin{aligned}
|\vec{a}-\vec{b}|^2 &= (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) \\
&= (\vec{a}-\vec{b})\cdot\vec{a} - (\vec{a}-\vec{b})\cdot\vec{b} \\
&= \vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} \\
&= |\vec{a}|^2 - 2\,\vec{a}\cdot\vec{b} + |\vec{b}|^2
\end{aligned}$$

$$\cancel{|\vec{a}|^2} - 2\,\vec{a}\cdot\vec{b} + \cancel{|\vec{b}|^2} = \cancel{|\vec{a}|^2} + \cancel{|\vec{b}|^2} - 2|\vec{a}|\,|\vec{b}|\cos\theta$$

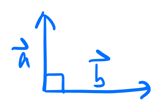
So $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$.

Def: $\vec{a}$ and $\vec{b}$ are orthogonal (or perpendicular) if the angle between them is $\theta = \frac{\pi}{2}$.

$\vec{a} \perp \vec{b}$

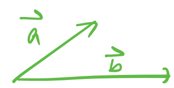
The zero vector $\vec{0}$ is considered orthogonal to all vectors.

$$\boxed{\vec{a}\cdot\vec{b} = 0 \iff \vec{a} \perp \vec{b}}$$



"orthogonal"

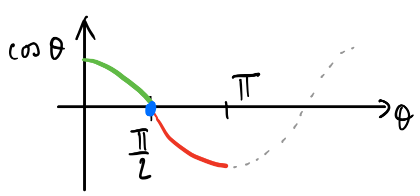
$$\vec{a}\cdot\vec{b} > 0 \iff \theta \in \left[0, \tfrac{\pi}{2}\right)$$



"accute"

$$\vec{a}\cdot\vec{b} < 0 \iff \theta \in \left(\tfrac{\pi}{2}, \pi\right]$$

"obtuse"



$\cos\theta > 0$ if $0 \le \theta < \frac{\pi}{2}$

$\cos\theta = 0$ if $\theta = \frac{\pi}{2}$

$\cos\theta < 0$ if $\frac{\pi}{2} < \theta \le \pi$

The dot product tells you what amount of one vector goes in the direction of another.

To find the <u>angle between two non-zero vectors</u>, use:

$$\boxed{\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}}$$

EX: Find the angle θ between $\vec{a} = \langle 1,1,5\rangle$ and $\vec{b} = \langle 0,3,4\rangle$

$$\vec{a}\cdot\vec{b} = 0+3+20 = 23$$

$$|\vec{a}| = \sqrt{1^2+1^2+5^2} = \sqrt{27} = 3\sqrt{3}$$

$$|\vec{b}| = \sqrt{0^2+3^2+4^2} = \sqrt{25} = 5$$

exact

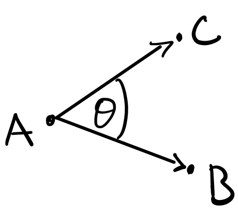
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{23}{3\sqrt{3}\cdot 5} = \frac{23}{15\sqrt{3}} \Rightarrow \theta = \arccos\left(\frac{23}{15\sqrt{3}}\right)$$

$$= 27.7152^\circ$$

calculator

approximate

EX: $A(0,3,4)$, $B(1,-2,3)$, $C(2,0,-1)$ $\quad \angle CAB = ?$



$$\cos\theta = \frac{\vec{AC}\cdot\vec{AB}}{|AC||\vec{AC}|}$$

$$\vec{AC} = C - A = \langle 2,-3,-5\rangle$$

$$\vec{AB} = B - A = \langle 1,-5,-1\rangle$$

$$\cos\theta = \frac{\langle 2,-3,-5\rangle\cdot\langle 1,-5,-1\rangle}{\sqrt{4+9+25}\,\sqrt{1+25+1}} = \frac{2+15+5}{\sqrt{38}\,\sqrt{27}} = \frac{22}{\sqrt{38}\,\sqrt{27}}$$

$$\Rightarrow \theta = \arccos\left(\frac{22}{\sqrt{38}\,\sqrt{27}}\right)$$

EX: Find a vector $\vec{u}$ which is orthogonal to $\vec{v} = \langle 1,1,5\rangle$

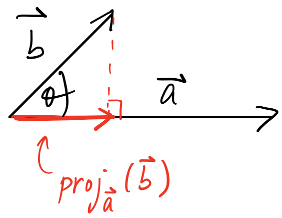
Need: $\vec{u}\cdot\vec{v} = 0$ $\quad$ Many such vectors $\vec{u}$!

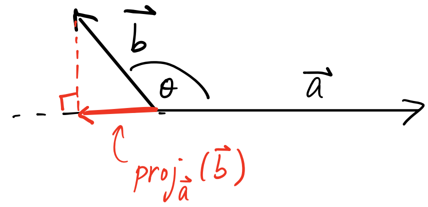
Example 1: $\vec{u} = \langle 1,-1,0\rangle$ Check: $\vec{u}\cdot\vec{v} = 1\cdot 1 + (-1)\cdot 1 + 0\cdot 5 = 0$ ✓

Example 2: $\vec{u} = \langle 0,0,0\rangle$

Example 3: $\vec{u} = \langle 2,3,-1\rangle$ Check: $\vec{u}\cdot\vec{v} = 2\cdot 1 + 3\cdot 1 + (-1)\cdot 5 = 0$ ✓

Projections





$\text{proj}_{\vec{a}}(\vec{b})$ = vector projection of $\vec{b}$ onto $\vec{a}$ "shadow of $\vec{b}$"

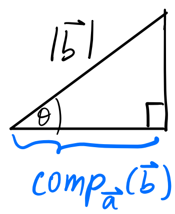
vector = a scalar-multiple of $\vec{a}$

OR: component

$\text{comp}_{\vec{a}}(\vec{b})$ = scalar projection of $\vec{b}$ onto $\vec{a}$ "$\pm$ length of the shadow"

scalar = signed magnitude of the vector projection

$= |\vec{b}| \cos\theta$



$$\cos\theta = \frac{adj}{hyp} = \frac{\text{comp}_{\vec{a}}(\vec{b})}{|\vec{b}|}$$

Recall: $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

So $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = |\vec{b}|\cdot\cos\theta$

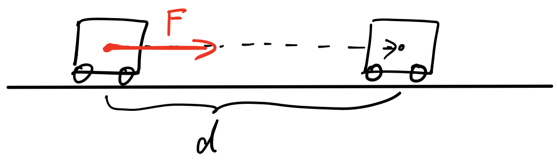
$$\boxed{\text{comp}_{\vec{a}}(\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$

$\text{proj}_{\vec{a}}(\vec{b}) = \text{comp}_{\vec{a}}(\vec{b}) \cdot \dfrac{\vec{a}}{|\vec{a}|}$ = unit vector in the direction of $\vec{a}$

$$\boxed{\text{proj}_{\vec{a}}(\vec{b}) = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|}}$$

Application in physics:

1 dimensional problem:



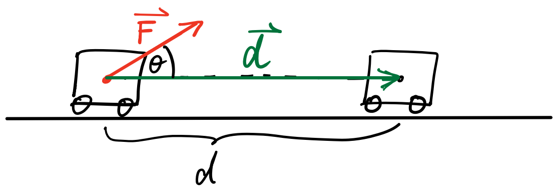
$$\underset{[J]}{\text{Work}} = \underset{[N]}{\text{Force}} \cdot \underset{[m]}{\text{distance}}$$

the amount of force in the same direction as the distance traveled

What if force is NOT applied in direction of movement?

$$\text{Work} = \Big(\underbrace{|\vec{F}| \cdot \cos\theta}\Big)|\vec{d}|$$

$= \text{comp}_{\vec{d}}(\vec{F})$



only the horizontal component of $\vec{F}$ matters.

$$\boxed{\text{Work} = \vec{F} \cdot \vec{d}}$$

scalar vector vector

EX: A wagon is pulled along a level path for 100 m.
The handle is held at a 30° angle from horizontal.
If Force pulling wagon is 80 N, how much work is done?

know: $|\vec{F}| = 80$
$|\vec{d}| = 100$
$\theta = \frac{\pi}{6}$

So: $W = \Big(|\vec{F}| \cdot \cos\theta\Big)|\vec{d}|$

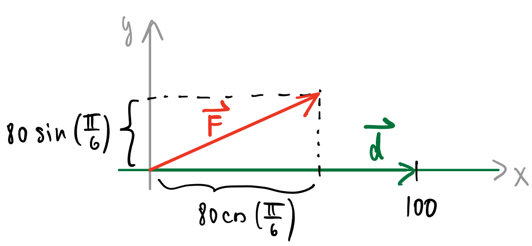
$= 80 \cdot \cos\left(\frac{\pi}{6}\right) \cdot 100 = 8000 \cdot \frac{\sqrt{3}}{2}$ $[\underbrace{N \cdot m}]$

or: Joules

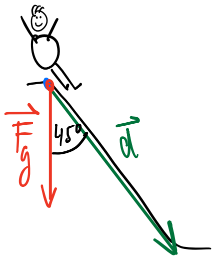
EX: If $\vec{F} = \langle 80\cos\left(\frac{\pi}{6}\right), 80\sin\left(\frac{\pi}{6}\right)\rangle$ and $\vec{d} = \langle 100, 0\rangle$, find W.

$W = \vec{F} \cdot \vec{d} = 80\cos\left(\frac{\pi}{6}\right) \cdot 100 + 80\sin\left(\frac{\pi}{6}\right) \cdot 0 = 8000 \cdot \frac{\sqrt{3}}{2}$ $[\underbrace{N \cdot m}]$

or Joules

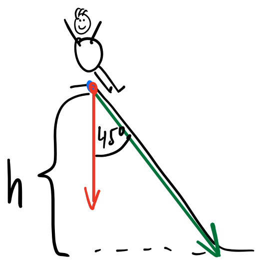


Ex. Child on a 5m long slide at angle 45°. Suppose that the weight force is 200 N. (i.e. $F_g = m \cdot g = 200N$)
Compute the work done by gravity.



$$W = |\vec{F_g}| \cdot \cos\left(\frac{\pi}{4}\right) \cdot |\vec{d}|$$
$$= 200 \cdot \frac{\sqrt{2}}{2} \cdot 5$$
$$= 500 \cdot \sqrt{2} \approx 707 \quad [J]$$

Note that this is equivalent to using the 1-dim. formula



$$W = \underbrace{\boxed{m \cdot g}}_{= |\vec{F_g}|} \cdot h$$

$$h = |\vec{d}| \cdot \cos\left(\frac{\pi}{4}\right)$$