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How does the structure of a group determine its Reidemeister spectrum?

Pieter Senden

Supervisor:
Prof. dr. Karel Dekimpe

Dissertation presented in partial fulfilment of the requirements for the degree of Doctor of Science (PhD): Mathematics

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Examination committee:
Prof. dr. David Dudal, chair
Prof. dr. Karel Dekimpe, supervisor
Prof. dr. Nero Budur
Prof. dr. Jonas Deré
Prof. dr. Stefaan Vaes
Prof. dr. Laura Ciobanu Radomirovic
(Heriot-Watt University)
Dr. Paula Macedo Lins de Araujo
(University of Lincoln)

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Uitgegeven in eigen beheer, Pieter Senden, Etienne Sabbelaan 53, B-8500 Kortrijk (Belgium)

Dankwoord

Feeling gratitude and not expressing it is like wrapping a present and not giving it.

Dankbaarheid voelen en haar niet uiten is als een cadeau inpakken en het niet geven.

William Arthur Ward

In de afgelopen drie en een half jaar heb ik op twee vlakken veel bereikt. Op academisch vlak ben ik gegroeid als wiskundige en heb ik gewerkt aan een doctoraat, waarvan je het eindresultaat nu in handen of op een scherm voor je hebt. Daarnaast heb ik ook op persoonlijk vlak een serieuze groei doorgemaakt. Beide vlakken zijn onlosmakelijk met elkaar verbonden en hebben invloed op elkaar gehad, en op beide vlakken had ik nooit zo veel kunnen bereiken zonder de mensen rondom mij. Daarom een woord van dank voor hen.

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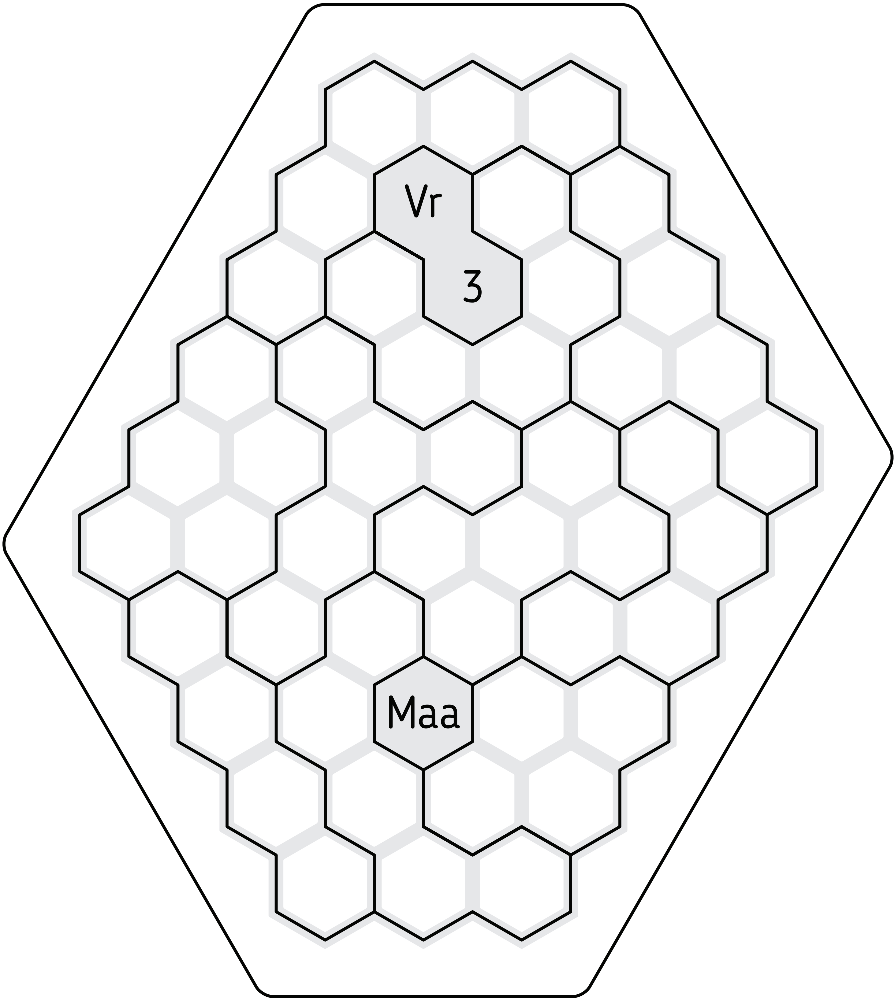
Pieter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Abstract

The concept of conjugacy in a group G has a natural generalisation to so-called *twisted conjugacy*: given an endomorphism $\varphi \in \mathrm{End}(G)$, we put an equivalence relation $\sim_\varphi$ on G by stating that, for all $x, y \in G$,

$$x \sim_\varphi y \iff \exists z \in G : x = zy\varphi(z)^{-1}.$$

The number of equivalence classes of $\sim_\varphi$ is called the *Reidemeister number* of φ and the set

$$\mathrm{Spec}_{\mathrm{R}}(G) := \{R(\psi) \mid \psi \in \mathrm{Aut}(G)\}$$

is called the *Reidemeister spectrum* of G. If $\mathrm{Spec}_{\mathrm{R}}(G) = \{\infty\}$, G is said to have the *R_∞-property.*

Twisted conjugacy arises naturally in Nielsen fixed-point theory, but there is also a strong algebraic interest in twisted conjugacy, in particular in the R_∞-property.

In this thesis, we investigate how structural properties of a group translate into information regarding the Reidemeister spectrum. By 'structural properties', we mean properties such as nilpotency, (residual) finiteness, splitting as a direct product and so on.

In the first part of this thesis, we study the behaviour of twisted conjugacy in group constructions. After recalling the known results about twisted conjugacy in general extensions, we focus on central extensions and derive an addition and product formula for Reidemeister numbers. Subsequently, we discuss the Reidemeister spectrum of free products, and we give a first family of groups for which the Reidemeister spectrum of a direct product of such groups is completely determined by the spectra of the individual factors.

The second part deals with nilpotent groups. We use the well-known product formula for Reidemeister numbers on finitely generated torsion-free nilpotent groups to prove relations among Reidemeister numbers of endomorphisms on both the group itself and its finite index subgroups. In addition, we revisit direct products and determine sufficient conditions for the Reidemeister spectrum of a direct product of nilpotent groups to be completely determined by the spectra of the individual factors. Finally, we prove an addition formula for finitely generated nilpotent groups with torsion and use it to calculate the Reidemeister spectra of several families of nilpotent groups.

The third and final part consists of results about finite groups. We first prove two alternative methods to compute the Reidemeister number of an endomorphism on a finite group. Afterwards, we completely compute the Reidemeister spectrum of two families of finite groups: the finite abelian groups and split metacyclic groups of the form $C_n \rtimes C_p$ with p a prime number.

Beknopte samenvatting

Het concept ‘conjugatie’ in een groep G heeft een natuurlijke veralgemening naar zogeheten *getwiste conjugatie*: gegeven een endomorfisme $\varphi \in \mathrm{End}(G)$ definiëren we een equivalentierelatie $\sim_\varphi$ op G door te eisen dat, voor alle $x, y \in G$,

$$x \sim_\varphi y \iff \exists z \in G : x = zy\varphi(z)^{-1}.$$

Het aantal equivalentieklassen van $\sim_\varphi$ noemen we het *Reidemeistergetal* van φ en de verzameling

$$\mathrm{Spec}_{\mathrm{R}}(G) := \{R(\psi) \mid \psi \in \mathrm{Aut}(G)\}$$

noemen we het *Reidemeisterspectrum* van G. Als $\mathrm{Spec}_{\mathrm{R}}(G) = \{\infty\}$, dan zeggen we dat G de *R_∞-eigenschap* heeft.

Getwiste conjugatie duikt op natuurlijke wijze op in Nielsen-vastepuntstheorie, maar er is ook een sterke algebraïsche interesse in getwiste conjugatie, in het bijzonder in de R_∞-eigenschap.

In deze thesis onderzoeken we hoe structurele eigenschappen van een groep zich vertalen in informatie over het Reidemeisterspectrum. Met ‘structurele eigenschappen’ bedoelen we eigenschappen zoals nilpotentie, (residuele) eindigheid, splitsen als een direct product en dergelijke.

In het eerste deel van de thesis bestuderen we het gedrag van getwiste conjugatie in groepsconstructies. Nadat we de gekende resultaten over getwiste conjugatie in algemene extensies herhaald hebben, focussen we op centrale extensies en bepalen we een som- en productformule voor Reidemeistergetallen. Vervolgens bespreken we het Reidemeisterspectrum van vrije producten en geven we een eerste familie van groepen waarvoor het Reidemeisterspectrum van een direct product van dergelijke groepen volledig vastligt door de spectra van de individuele factoren.

Het tweede deel gaat over nilpotente groepen. We gebruiken de gekende product-formule voor Reidemeistergetallen op eindig voortgebrachte torsievrije nilpotente groepen om verbanden tussen de Reidemeistergetallen van endomorfismen op zowel de groep zelf als zijn deelgroepen van eindige index te bewijzen. Daarna keren we terug naar directe producten en bepalen we voldoende voorwaarden opdat het Reidemeisterspectrum van een direct product van nilpotente groepen volledig vastligt door de spectra van de individuele factoren. Ten slotte bewijzen we een somformule voor eindig voortgebrachte nilpotente groepen met torsie en gebruiken die vervolgens om de Reidemeisterspectra van enkele families van nilpotente groepen te berekenen.

Het derde en laatste deel beslaat resultaten over eindige groepen. We bewijzen eerst twee alternatieve methodes om het Reidemeistergetal van een endormorfisme op een eindige groep te bepalen. Daarna berekenen we het Reidemeisterspectrum van twee families van eindige groepen: eindige abelse groepen en split-metacyclische groepen van de vorm $C_n \rtimes C_p$, waarbij p een priemgetal is.

Contents

Introduction

I.1 Historical positioning

Twisted conjugacy has its origins in topological fixed-point theory. In the 1920s, Jakob Nielsen used the theory of universal covering spaces to introduce a new approach to fixed-point theory [92, 93, 94]. In this approach, he studied the lifts of a continuous map $f : X \to X$ to the universal covering space E of X. He noted that the fixed-point sets of two lifts that are conjugate by an element of the group of covering transformations, $\mathcal{C}(E, p, X)$, project down to the same subset of fixed points of f under the covering map $p : E \to X$. Thus, with the equivalence relation given by

$$\tilde{f}_1 \sim \tilde{f}_2 \iff \exists \alpha \in \mathcal{C}(E, p, X) : \tilde{f}_1 = \alpha \circ \tilde{f}_2 \circ \alpha^{-1}$$

on the set of lifts of f, we obtain

$$\operatorname{Fix}(f) = \bigsqcup_{[\tilde{f}]} p(\operatorname{Fix}(\tilde{f})),$$

where $[\tilde{f}]$ is the equivalence class of the lift $\tilde{f}$.

Later, in the 1930s and 1940s, Kurt Reidemeister and one of his students, Franz Wecken, made this approach more algebraic [97, 122, 123, 124]. They noted that a (fixed) lift $\tilde{f}$ of f to the universal cover E of X induces an endomorphism $\tilde{f}_\pi$ on the group of covering transformations $\mathcal{C}(E, p, X)$. This map $\tilde{f}_\pi$ is (implicitly) given by

$$\tilde{f}_\pi(\alpha) \circ \tilde{f} = \tilde{f} \circ \alpha$$

for all $\alpha \in \mathcal{C}(E, p, X)$. Furthermore, using the fact that any lift of f is of the form $\alpha \circ \tilde{f}$, they argued that any two lifts $\alpha \circ \tilde{f}$ and $\beta \circ \tilde{f}$ are conjugate by an element of $\mathcal{C}(E, p, X)$ if and only if

$$\alpha = \gamma \circ \beta \circ \tilde{f}_\pi(\gamma)^{-1}$$

for some $\gamma \in \mathcal{C}(E,p,X)$. The relation

$$\alpha_1 \sim_{\tilde{f}_\pi} \alpha_2 \iff \exists \gamma \in \mathcal{C}(E,p,X) : \alpha_1 = \gamma \circ \alpha_2 \circ \tilde{f}_\pi(\gamma)^{-1}$$

on $\mathcal{C}(E,p,X)$ is called the $\tilde{f}_\pi$-conjugacy relation. Thus, the number of equivalence classes of lifts equals the number of $\tilde{f}_\pi$-conjugacy classes on $\mathcal{C}(E,p,X)$, and this number is called the Reidemeister number of f.

We can define the equivalence relation above on any group G. Given an arbitrary group G and an endomorphism φ of G, we define the φ-conjugacy relation $\sim_\varphi$ as follows: given $x, y \in G$,

$$x \sim_\varphi y \iff \exists z \in G : x = zy\varphi(z)^{-1}.$$

The number of equivalence classes under $\sim_\varphi$ is called the Reidemeister number of φ, denoted by $R(\varphi)$. The set

$$\operatorname{Spec_R}(G) := \{R(\psi) \mid \psi \in \operatorname{Aut}(G)\}$$

is called the Reidemeister spectrum of G. If $\operatorname{Spec_R}(G) = \{\infty\}$, G is said to have the R_∞-property, a term coined by Jenifer Taback and Peter Wong [115, 116].

The algebraic interest in twisted conjugacy became apparent at the end of the twentieth century, with an algebraic treatment of twisted conjugacy by Philip Heath in [56] and one of the first purely algebraic conjectures regarding twisted conjugacy due to Alexander Fel'shtyn and Richard Hill in 1994 [32]. They conjectured that every injective endomorphism on a finitely generated group of exponential growth has infinite Reidemeister number. This conjecture was confirmed for automorphisms on non-elementary hyperbolic groups by A. Fel'shtyn in the early 2000s [29], but the conjecture in general was disproven by Daciberg Gonçalves and P. Wong in 2003 [49]. In the same period, additional algebraic results about twisted conjugacy and Reidemeister numbers were published [43, 48].

The strong algebraic interest in twisted conjugacy, especially in the R_∞-property, arose in the second half of the 2000s and in the 2010s. In that time frame, the question whether a (family of) group(s) has the R_∞-property was answered for numerous groups. A non-exhaustive list comprises the following groups:

- Several saturated weakly branch groups, among which the Grigorchuk group and the Gupta-Sidki group [34];
- Non-abelian Baumslag-Solitar groups [31], their generalisations [72, 116], and groups quasi-isometric to them [115];

- Non-elementary Gromov hyperbolic groups [29, 73];
- Thompson's group F [9], its generalisations and their finite products [45];
- Some metabelian groups of the form $\mathbb{Q}^n \rtimes \mathbb{Z}$ and $\mathbb{Z}\left[\frac{1}{p}\right] \rtimes \mathbb{Z}$ [30];
- Certain Artin groups of large type [65];
- Free nilpotent groups $N_{r,c}$ with $r \geq 2$ [19]; see also [100];
- Non-abelian free solvable groups of finite rank [19]; see also [100];
- Infinitely generated free groups [19, Appendix];
- Extensions of $\mathrm{SL}(n,\mathbb{Z})$, $\mathrm{PSL}(n,\mathbb{Z})$, $\mathrm{GL}(n,\mathbb{Z})$, $\mathrm{PGL}(n,\mathbb{Z})$, $\mathrm{Sp}(2n,\mathbb{Z})$ and $\mathrm{PSp}(2n,\mathbb{Z})$ by a countable abelian group, where $n \geq 2$ [82];
- Lamplighter groups $(\mathbb{Z}/n\mathbb{Z}) \wr \mathbb{Z}$ [113];
- Certain Chevalley groups over fields [87];
- Houghton's groups [46, 63];
- Chevalley groups of types B_l, C_l, D_l over integral domains [86];
- Chevalley groups of types B_n, C_n, D_n over certain fields [85];
- Twin groups T_n for $n \geq 3$ [84];
- Several families of right-angled Artin groups [24];
- Pure Artin braid groups P_n for $n \geq 3$ [21].

The majority of these results use ad hoc methods to (dis)prove that the groups possess the R_∞-property, methods which do not (easily) lend themselves to generalisations. This leads us to the starting point of this PhD: we aim for general and structural results concerning twisted conjugacy, and especially the Reidemeister spectrum, in a group; hence the title 'How does the structure of a group determine its Reidemeister spectrum?'. The question then rises what we mean by 'structure' and 'determine'. The most essential structure of a group, its group operation, of course completely determines the Reidemeister spectrum of the group. Consequently, we are looking for broader forms of structure that still allow us to derive results for large classes of groups. Examples of such forms of structure include nilpotency, solvability, (residual) finiteness, but also notions as 'splits as a direct or free product'. A better[1] title would thus have been 'To what extent does a certain structural property of a group provide information about its Reidemeister spectrum?'.

Examples of such results already exist in the literature. The aforementioned papers by P. Heath [56] and D. Gonçalves [43] both deal with twisted conjugacy in group extensions, as does one by P. Wong [127]. D. Gonçalves and P. Wong have also determined which wreath products of abelian groups have the R_∞-

[1] but even less catchy

property [51] and have classified the virtually-$\mathbb{Z}$ groups with the R_∞-property [52]. In addition, they, together with Parameswaran Sankaran, have proven that any free product of finitely many groups has the R_∞-property under some mild conditions [47].

I.2 Overview of this thesis

As a consequence of the aim for general and/or structural results, the topics in this dissertation vary strongly. The common thread throughout the thesis is simply 'twisted conjugacy', and each chapter considers twisted conjugacy in groups where we impose one or more additional structural conditions such as nilpotency or splits as a direct product. To accentuate this common thread more clearly throughout the text, I found inspiration in music, for the approach of this thesis strongly resembles the compositional form of 'theme and variations'. A piece in this form starts with presenting a musical theme, which is often a simple and undecorated melody. This theme is then repeated several times with melodic, rhythmic, harmonic, or other variations. These variations can be of all sorts, but the theme is still recognisable within each of them. Examples of such works include the *Andante grazioso* from *Piano sonata no. 11 in A major* (KV 331 (300i)) by Wolfgang Amadeus Mozart and the *Andante* from *Symphony No. 94 in G major* (Hob. I:94) by Joseph Haydn.

I have incorporated this musical analogy in the names of the different parts of this thesis, which is structured as follows:

Prelude: A prelude is a composition that precedes the proper musical piece. In the Prelude, we elaborate further on the origin of twisted conjugacy in Nielsen fixed-point theory and we also discuss another branch of mathematics where twisted conjugacy occurs naturally, namely Galois cohomology.

Theme: In the Theme, we present the material, namely twisted conjugacy, on which the subsequent chapters, called 'Variations', are based. We introduce the necessary definitions and concepts, together with several results that hold in all groups.

The subsequent chapters are thus ten Variations on the Theme. These Variations are subdivided into three parts, or 'Collections', which contain three, four, and three Variations, respectively. All Variations in the same Collection deal with groups that share a common structural property.

Collection I: The first Collection concerns group constructions. We start in Variation 1 with studying the behaviour of twisted conjugacy on one of the most general constructions, namely extensions. In Variation 2, we provide a detailed proof of the aforementioned result by D. Gonçalves, P. Sankaran and P. Wong concerning the R_∞-property for free products of finitely many groups.

Variation 3 is the first of the two Variations that deal with direct products. We provide a first family of groups, the centreless directly indecomposable groups, for which the Reidemeister spectrum of a direct product of such groups is completely determined by the Reidemeister spectra of the individual factors.

Collection II: In this Collection, the common thread is nilpotency. In Variation 4, we recall the necessary definitions and properties of nilpotent groups, together with the powerful product formula for finitely generated torsion-free nilpotent groups. In addition, we present several general consequences of this product formula for Reidemeister numbers on finitely generated torsion-free nilpotent groups.

In Variation 5, we revisit direct products and provide a second family of groups for which the Reidemeister spectrum of a direct product is completely determined by the Reidemeister spectra of the individual factors. More concretely, we provide a complete description of the automorphism group of a direct product of finitely many finitely generated torsion-free non-abelian nilpotent groups whose rational Malcev completion is directly indecomposable. We then use this description to determine the Reidemeister spectrum of such a direct product.

Variation 6 deals with Reidemeister numbers on finite index subgroups, where the product formula again shows its power. Finally, in Variation 7, we prove an addition formula for Reidemeister numbers on nilpotent groups with torsion, which we subsequently use to completely determine the Reidemeister spectrum of three families of nilpotent groups.

Collection III: The last Collection deals with finite groups. In Variation 8, we discuss two alternative methods to compute Reidemeister numbers, as well as general relations among Reidemeister numbers on finite groups. Variations 9 and 10 deal with the complete computation of the Reidemeister spectrum of two classes of finite groups: Variation 9 deals with finite abelian groups, Variation 10 with split metacyclic groups of the form $C_n \rtimes C_p$, with p a prime number.

When relevant, we illustrate the results obtained in a Variation through examples. The end of each example is indicated by $\|$, which in music indicates the end of a piece.

Finally, there are two appendices[2], in musical terms ‘Codas’. A coda is a passage that brings a composition to an end.

[2] Originally, there were three, but one had to be removed.

Coda A: In the first Coda, we introduce and discuss the inverse Reidemeister problem. The key question is trying to determine which subsets of $\mathbb{N}_0 \cup \{\infty\}$ can occur as the Reidemeister spectrum of a group.

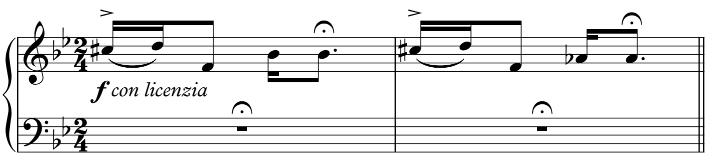
Coda B: The second Coda is a short elaboration on how we can view the map that associates to an automorphism its Reidemeister number as a class function on the automorphism group. This point of view yields an alternative proof of Menon's Identity.

To fully finish off the musical analogy, the beginning of the Prelude, the Theme, each Variation, and both Codas is accompanied by a short musical excerpt. The choice for the excerpts in the Prelude and the Codas is based on personal preference, but the choice for the excerpts in the Theme and the Variations is more substantiated. The theme chosen is a melody which in French is known as the melody of the children's song *Ah, vous dirai-je, Maman*, in English as the melody of the lullaby *Twinkle, Twinkle, Little Star*, and in Dutch as the melody of the children's song *Altijd is Kortjakje ziek*. The musical excerpt in each (mathematical) Variation is then the corresponding (musical) variation from the piece *Zwölf Variationen in C über das französische Lied 'Ah, vous dirai-je, Maman'* (KV 265 (300e)) by W. A. Mozart. Due to the international character of the melody, I deemed it a good choice to reach an audience as broad as possible. I invite every reader, whether they have a musical background or not, to try and recognise the theme in each of the variations, either visually or by hearing.

Prelude & Theme

Prelude

Twisted conjugacy in other branches of mathematics



Gershwin, George, *Prelude I* from *3 Preludes* (bars 1–2).[3]

In the introduction, we briefly sketched how twisted conjugacy arises in Nielsen fixed-point theory. Besides that, twisted conjugacy comes up in other branches of mathematics as well, such as Galois cohomology and the Arthur-Selberg trace formula. We discuss both Nielsen fixed-point theory and Galois cohomology here to some extent; the Arthur-Selberg trace formula would bring us too far. We refer the interested reader to [62] for more details on Nielsen fixed-point theory, to [6] for Galois cohomology, and to [111] for the Arthur-Selberg trace formula.

P.1 Nielsen fixed-point theory

Suppose that $f : X \to X$ is a continuous map on a topological space X. In topological fixed-point theory, one is interested in the (number of) fixed points of f. However, maps that are small continuous perturbations of each other can

[3]Excerpt adopted from [41].

differ vastly in the number of fixed points they have. For example, the identity map $\mathrm{Id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ has an infinite number of fixed points, whereas for $\varepsilon > 0$, the map

$$g : \mathbb{R} \to \mathbb{R} : x \mapsto x + \varepsilon$$

is a small homotopic perturbation of $\mathrm{Id}_{\mathbb{R}}$ without any fixed points.

Thus, a more rigid question is to try and determine $\min\{\mathrm{Fix}(g) \mid g \simeq f\}$, where $f \simeq g$ means that f and g are homotopic.

Throughout the remainder of this section, we assume that X is a path connected and locally path connected topological space that admits a universal cover E with associated covering map $p : E \to X$. Whenever we write that f is a map between topological spaces, we always mean a *continuous* map.

P.1.1 Topological Reidemeister number

Let $f : X \to X$ be a continuous map. In Nielsen fixed-point theory, to study fixed points of f, one looks at the fixed points of lifts of f to E.

Definition P.1.1. Let $f : X \to X$ be a continuous map. A *lift* of f to E is a map $\tilde{f} : E \to E$ such that the diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "p"] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

commutes.

A lift of the identity map Id_X is called a *covering transformation.* The set of all covering transformations forms a group. It is called the *group of covering transformations* and is denoted by $\mathcal{C}(E, p, X)$.

Whenever we speak of a lift of f, we mean a lift $\tilde{f} : E \to E$ to the universal covering space.

Another group associated to X is the fundamental group at a base point $x_0 \in X$, which we write as $\pi_1(X, x_0)$. In general, a continuous map $g : X \to Y$ between two topological spaces induces, for each $x \in X$, a group homomorphism

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x)) : [c] \mapsto [f \circ c].$$

Lemma P.1.2 (General lifting lemma; See e.g. [83, Lemma 79.1])**.** *Let $p : Y \to X$ be a covering map and let $y \in Y$ and $x \in X$ be such that $p(y) = x$. Let*

$f : Z \to X$ be a continuous map and $z \in Z$ with $f(z) = x$. Suppose that Z is path connected and locally path connected. Then there exists a map $\tilde{f} : Z \to Y$ such that $p \circ \tilde{f} = f$ and $\tilde{f}(z) = y$ if and only if

$$f_*(\pi_1(Z, z)) \subseteq p_*(\pi_1(Y, y)).$$

Moreover, if such a map $\tilde{f}$ exists, it is unique.

To study fixed points of lifts, we first need some preliminary results about lifts.

Proposition P.1.3 (See e.g. [62, Proposition 1.2]). *Let $f : X \to X$ be a continuous map.*

(1) Let $x_0 \in X$ and put $x_1 = f(x_0)$. For each $e_0 \in p^{-1}(x_0)$ and $e_1 \in p^{-1}(x_1)$, there is a unique lift $\tilde{f}$ of f satisfying $\tilde{f}(e_0) = e_1$.

(2) If $\tilde{f}$ is a lift of f and $\alpha, \beta \in \mathcal{C}(E, p, X)$, then $\alpha \circ \tilde{f} \circ \beta$ is again a lift of f.

(3) Let $\tilde{f}$ and $\tilde{f}'$ be two lifts of f. Then there exists a unique $\alpha \in \mathcal{C}(E, p, X)$ such that $\tilde{f}' = \alpha \circ \tilde{f}$.

Proof. We refer the reader to [83, Chapter 13] for the first item.

For the second item, let $\tilde{f}$ be a lift of f and $\alpha, \beta \in \mathcal{C}(E, p, X)$. Then

$$p \circ (\alpha \circ \tilde{f} \circ \beta) = p \circ \tilde{f} \circ \beta = f \circ p \circ \beta = f \circ p.$$

Therefore, $\alpha \circ \tilde{f} \circ \beta$ is a lift of f.

Finally, for the third item, let $\tilde{f}$ and $\tilde{f}'$ be two lifts of f. Fix an $x \in X$ and take $e \in p^{-1}(x)$. Since $p(\tilde{f}(e)) = f(p(e)) = p(\tilde{f}'(e))$, we find a unique covering transformation α such that $\alpha(\tilde{f}(e)) = \tilde{f}'(e)$ by applying the first item to the identity map. Then $\alpha \circ \tilde{f}$ and $\tilde{f}'$ are both lifts of f by the second item, and they agree on e. By the uniqueness of the first item, $\alpha \circ \tilde{f}$ and $\tilde{f}'$ are equal. □

Lemma P.1.4 (See e.g. [62, Chapter I, Lemma 1.3]). *Let $f : X \to X$ be a continuous map, let $\tilde{f}, \tilde{f}'$ be two lifts of f and let $x \in X$. Suppose that $e \in p^{-1}(x)$ is a fixed point of $\tilde{f}$. Let $\alpha \in \mathcal{C}(E, p, X)$. Then $\tilde{f}'$ has $\alpha(e)$ as a fixed point if and only if $\tilde{f}' = \alpha \circ \tilde{f} \circ \alpha^{-1}$.*

Proof. The 'if'-implication is immediate. For the converse, suppose that $\tilde{f}'(\alpha(e)) = \alpha(e)$. Then both $\tilde{f}'$ and $\alpha \circ \tilde{f} \circ \alpha^{-1}$ are lifts of f that map $\alpha(e)$ to $\alpha(e)$. By Proposition P.1.3(1), these lifts are equal. □

Using this conjugation relation, we can subdivide the set of all lifts of f based on their fixed points.

Definition P.1.5. Let $f: X \to X$ be a continuous map. Two lifts $\tilde{f}$ and $\tilde{f}'$ of f are called *conjugate* if $\tilde{f}' = \alpha \circ \tilde{f} \circ \alpha^{-1}$ for some $\alpha \in \mathcal{C}(E, p, X)$. We let $[\tilde{f}]$ denote the conjugacy class of $\tilde{f}$, and call it the *lifting class of* $\tilde{f}$.

We now link the fixed points of f to those of the lifts of f.

Theorem P.1.6 (See e.g. [62, Chapter I, Theorem 1.5])**.** *Let $f : X \to X$ be a continuous map and let $\tilde{f}, \tilde{f}'$ be two lifts of f. Then the following hold:*

(1) $\mathrm{Fix}(f) = \bigcup_{g \text{ is a lift of } f} p(\mathrm{Fix}(g))$.

(2) $p(\mathrm{Fix}(\tilde{f})) = p(\mathrm{Fix}(\tilde{f}'))$ *if* $[\tilde{f}] = [\tilde{f}']$.

(3) $p(\mathrm{Fix}(\tilde{f})) \cap p(\mathrm{Fix}(\tilde{f}')) = \emptyset$ *if* $[\tilde{f}] \neq [\tilde{f}']$.

Proof. For the first item, let g be a lift of f with fixed point e. Then

$$f(p(e)) = p(g(e)) = p(e).$$

Hence, $p(e) \in \mathrm{Fix}(f)$. Conversely, if $x \in \mathrm{Fix}(f)$, let $e \in p^{-1}(x)$. By Proposition P.1.3(1), there is a (unique) lift g of f such that $g(e) = e$. Consequently, $x \in p(\mathrm{Fix}(g))$.

For the second item, suppose that $\tilde{f}' = \alpha \circ \tilde{f} \circ \alpha^{-1}$ for some $\alpha \in \mathcal{C}(E, p, X)$. By Lemma P.1.4, $\alpha(\mathrm{Fix}(\tilde{f})) \subseteq \mathrm{Fix}(\tilde{f}')$ and $\alpha^{-1}(\mathrm{Fix}(\tilde{f}')) \subseteq \mathrm{Fix}(\tilde{f})$. Since α is bijective, $\mathrm{Fix}(\tilde{f}') = \alpha(\mathrm{Fix}(\tilde{f}))$. Therefore,

$$p(\mathrm{Fix}(\tilde{f}')) = p(\alpha(\mathrm{Fix}(\tilde{f}))) = p(\mathrm{Fix}(\tilde{f})).$$

Finally, for the third item, suppose that $x \in p(\mathrm{Fix}(\tilde{f})) \cap p(\mathrm{Fix}(\tilde{f}'))$. Then there are $e, e' \in p^{-1}(x)$ such that $\tilde{f}(e) = e$ and $\tilde{f}'(e') = e'$. As $p(e) = p(e')$, there is a covering transformation α such that $\alpha(e) = e'$, by Proposition P.1.3(1). Consequently, Lemma P.1.4 implies that $\tilde{f}' = \alpha \circ \tilde{f} \circ \alpha^{-1}$. Therefore, $[\tilde{f}] = [\tilde{f}']$. □

Given a lift $\tilde{f}$ of f, we call $p(\mathrm{Fix}(\tilde{f}))$ the *fixed-point class of* $[\tilde{f}]$. It follows from the previous theorem that $p(\mathrm{Fix}(\tilde{f}))$ is independent of the representative of $[\tilde{f}]$ and that $\mathrm{Fix}(f)$ can be written as the disjoint union of the fixed-point classes of all $[\tilde{f}]$.

Remark. Although a fixed-point class can be empty, we still regard two empty fixed-point classes as different if they originate from non-conjugate lifts.

Example P.1.7. An example of a map which has several empty fixed-point classes is the identity map on the circle S^1. The universal cover of S^1 is $\mathbb{R}$, and if we view S^1 as $\{z \in \mathbb{C} \mid |z| = 1\}$, then the covering map is given by $p : \mathbb{R} \to S^1 : t \mapsto \exp(2\pi i t)$. The group of covering transformations $\mathcal{C}(\mathbb{R}, p, S^1)$ is the set $\{t_n : \mathbb{R} \to \mathbb{R} : x \mapsto x + n \mid n \in \mathbb{Z}\}$, which is abelian. Hence, different lifts of Id_{S^1} are non-conjugate. Also,

$$p(\mathrm{Fix}(t_n)) = \begin{cases} S^1 & \text{if } n = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$ ‖

Definition P.1.8. Let $f : X \to X$ be a continuous map. The *(topological) Reidemeister number of* f, written as $R(f)$, is the number of distinct fixed-point classes. Equivalently, it is the number of lifting classes of f.

In general, the Reidemeister number does not provide any information regarding the number of fixed points of f, since fixed-point classes can either be empty or contain multiple fixed points. We later introduce a number that is an estimate for $|\mathrm{Fix}(f)|$ – and that coincides with the Reidemeister number in some cases –, but we first prove that the Reidemeister number is a homotopy invariant.

Theorem P.1.9 (See e.g. [62, Chapter I, Theorem 2.4])**.** *Let $f : X \to X$ be a continuous map. Then $R(f)$ is a homotopy invariant.*

Proof. Suppose that $g : X \to X$ is a continuous map homotopic to f and let $H : X \times [0, 1] \to X$ be a homotopy between f and g. Suppose that $\tilde{f}$ is a lift of f. Then there is a unique lifting of H, say $\tilde{H} : E \times [0, 1] \to E$, such that $\tilde{H}(\cdot, 0) = \tilde{f}$. The map $\tilde{g} : E \to E : e \mapsto \tilde{H}(e, 1)$ is then a lift of g. Indeed, since $\tilde{H}$ is a lift of H,

$$p(\tilde{g}(e)) = p(\tilde{H}(e, 1)) = H(p(e), 1) = g(p(e))$$

for all $e \in E$. By uniqueness of lifts, H induces a bijective correspondence between the lifts of f and those of g.

Moreover, this correspondence preservers conjugacy of lifts. Indeed, let $\tilde{f}$ be a lift of f and $\alpha \in \mathcal{C}(E, p, X)$. Let $\tilde{H}$ be the unique lift of H such that $\tilde{H}(\cdot, 0) = \tilde{f}$, and write $\tilde{g} = \tilde{H}(\cdot, 1)$. Then the map

$$E \times [0, 1] \to E : (e, t) \mapsto \alpha(H(\alpha^{-1}(e), t))$$

is a homotopy between $\alpha \circ \tilde{f} \circ \alpha^{-1}$ and $\alpha \circ \tilde{g} \circ \alpha^{-1}$. Therefore, the bijective correspondence between lifts of f and lifts of g induces a correspondence between lifting classes of f, which maps $[\tilde{f}]$ to $[\tilde{g}]$ if $\tilde{f}$ and $\tilde{g}$ are homotopic through a lift of H.

By the preceding paragraph, this correspondence is both independent of the chosen representative and bijective itself. Hence, $R(f) = R(g)$. $\square$

Example P.1.10. A non-empty fixed-point class can vanish under homotopy. Consider again S^1 with universal cover $p : \mathbb{R} \to S^1 : t \mapsto \exp(2\pi i t)$. Let $\varepsilon > 0$ be arbitrary but small. Consider the map

$$H : S^1 \times [0,1] \to S^1 : (z,t) \mapsto z \exp(2\pi i t\varepsilon).$$

This is a homotopy between the identity map on S^1 and the map $g : S^1 \to S^1 : z \mapsto z \exp(2\pi i \varepsilon)$. The lift

$$\tilde{H} : \mathbb{R} \times [0,1] \to \mathbb{R} : (x,t) \mapsto x + t\varepsilon,$$

of H is a homotopy between $\tilde{f} := \mathrm{Id}_{\mathbb{R}}$ and $\tilde{g} : \mathbb{R} \to \mathbb{R} : x \mapsto x + \varepsilon$. However, $p(\mathrm{Fix}(\tilde{f})) = S^1$ whereas $p(\mathrm{Fix}(\tilde{g})) = \emptyset$. ‖

P.1.2 Nielsen number

Thus far, there are two issues with the fixed-point classes to use their number as an estimate for the number of fixed points: (1) they can vanish under homotopy and (2) one fixed-point class can contain multiple fixed points. To address this issue, we use the notion of the *index* of a fixed-point class. From now on, we assume that X is a compact polyhedron.

The index of a fixed-point class is an integer. There are several ways to introduce this index: geometrically and homologically (see e.g. [62, Chapter 1, § 3], [11]), or axiomatically (see e.g. [39]). We do not introduce it formally here, as this would take us too far.

Definition P.1.11. Let $f : X \to X$ be continuous map and let $\mathbb{F}$ be a fixed-point class of f. If the index of $\mathbb{F}$ is non-zero, we call $\mathbb{F}$ an *essential* fixed-point class; otherwise, we call it *inessential*.

The number of essential fixed-point classes is called the *Nielsen number* of f and is denoted by $N(f)$.

The main idea behind this terminology is that an essential fixed-point class is never empty and that it cannot vanish under homotopy, whereas an inessential fixed-point class can vanish under a homotopy.

Theorem P.1.12 (See e.g. [62, Chapter II, § 4])**.** *Suppose that X is compact and let $f : X \to X$ be a continuous map. Then the following hold:*

(1) $N(f)$ is a finite non-negative integer.

(2) $N(f) \leq R(f)$.

(3) $N(f) \leq |\mathrm{Fix}(f)|$.

(4) $N(f)$ is a homotopy invariant.

In particular, $N(f) \leq \min\{|\mathrm{Fix}(g)| \mid g \simeq f\}$.

Thus, the Nielsen number provides us with information regarding the number of fixed points of f. In certain situations, the bound is sharp; F. Wecken proved that $N(f) = \min\{|\mathrm{Fix}(g)| \mid g \simeq f\}$ if f is a continuous map on a connected compact manifold of dimension at least 3 [122, 123, 124].

Generally, the Nielsen number is hard to compute. However, in several situations, the Reidemeister number and the Nielsen number are equal. For instance, on infra-solvmanifolds of type (R), if $R(f) < \infty$, then $R(f) = N(f)$ (see e.g. [33]). For the subfamily of (compact) nilmanifolds, the Reidemeister number yields complete information (see e.g. [57]): if $R(f) = \infty$, then $N(f) = 0$; if $R(f) < \infty$, then $R(f) = N(f)$.

P.1.3 Linking topology with group theory

Let $f : X \to X$ be a continuous map and let $\tilde{f}_0$ be a fixed lift. By Proposition P.1.3, $\tilde{f}_0 \circ \alpha$ is a lift of f for each $\alpha \in \mathcal{C}(E, p, X)$. By the same proposition, there exists for each $\alpha \in \mathcal{C}(E, p, X)$ a unique $\alpha' \in \mathcal{C}(E, p, X)$ such that $\alpha' \circ \tilde{f}_0 = \tilde{f}_0 \circ \alpha$. Thus, we obtain a map

$$\tilde{f}_\pi : \mathcal{C}(E, p, X) \to \mathcal{C}(E, p, X) : \alpha \mapsto \tilde{f}_\pi(\alpha) := \alpha',$$

which is in addition a homomorphism. Indeed, let $\alpha, \beta \in \mathcal{C}(E, p, X)$. Then

$$\begin{aligned}(\tilde{f}_0 \circ \alpha) \circ \beta &= \tilde{f}_\pi(\alpha) \circ \tilde{f}_0 \circ \beta \\ &= \tilde{f}_\pi(\alpha) \circ \tilde{f}_\pi(\beta) \circ \tilde{f}_0,\end{aligned}$$

so $\tilde{f}_\pi(\alpha \circ \beta) = \tilde{f}_\pi(\alpha) \circ \tilde{f}_\pi(\beta)$.

Lemma P.1.13 (See e.g. [62, Chapter II, Lemma 1.4])**.** *Let $\alpha, \beta \in \mathcal{C}(E, p, X)$. Then $\alpha \circ \tilde{f}_0$ and $\beta \circ \tilde{f}_0$ lie in the same lifting class if and only if $\beta = \gamma \circ \alpha \circ \tilde{f}_\pi(\gamma)^{-1}$ for some $\gamma \in \mathcal{C}(E, p, X)$.*

Proof. Note that

$$\gamma \circ \alpha \circ \tilde{f}_\pi(\gamma)^{-1} \circ \tilde{f}_0 = \gamma \circ \alpha \circ \tilde{f}_0 \circ \gamma^{-1}$$

for all $\gamma \in \mathcal{C}(E, p, X)$. Therefore, $[\beta \circ \tilde{f}_0] = [\alpha \circ \tilde{f}_0]$ if and only if $\beta \circ \tilde{f}_0 = \gamma \circ \alpha \circ \tilde{f}_\pi(\gamma)^{-1} \circ \tilde{f}_0$ for some $\gamma \in \mathcal{C}(E, p, X)$, thus if and only if $\beta = \gamma \circ \alpha \circ \tilde{f}_\pi(\gamma)^{-1}$ for some $\gamma \in \mathcal{C}(E, p, X)$, by Proposition P.1.3(3). □

Thus, we see how the notion of twisted conjugacy arises. By the definition of the (topological) Reidemeister number of f, this is consequently also the same as the (algebraic) Reidemeister number of $\tilde{f}_\pi$, i.e. $R(f) = R(\tilde{f}_\pi)$.

There is another group homomorphisms induced naturally by f, namely the map f_* on the fundamental group of X. We show that $R(f_*) = R(f)$.

Fix a base point $x_0 \in X$ and a point $e_0 \in p^{-1}(x_0)$. We identify E with path classes in X starting at x_0 using the following map:

$$\Phi : E \to \{\text{path classes in } X \text{ starting at } x_0\} : e \mapsto [p \circ c],$$

where $c : [0, 1] \to E$ is a path from e_0 to e. Since E is simply connected, $[p \circ c]$ is independent of the chosen path c. The map Φ is surjective, for if c is a path starting at x_0, we can lift this path to a path $\tilde{c}$ in E starting at e_0 by [83, Lemma 54.1]. By construction, $[p \circ \tilde{c}] = [c]$. To prove that Φ is injective, suppose that $\Phi(e) = \Phi(e')$ and let c and c' be paths from e_0 to e and e', respectively. Then $[p \circ c] = [p \circ c']$. Let $H : [0, 1] \times [0, 1] \to X$ be a homotopy between $p \circ c$ and $p \circ c'$. We can lift this homotopy to a homotopy $\tilde{H} : [0, 1] \times [0, 1] \to E$ such that $\tilde{H}(\cdot, 0) = c$. Since

$$p(\tilde{H}(t, 1)) = H(t, 1) = (p \circ c')(t)$$

for all $t \in [0, 1]$, $\tilde{H}(\cdot, 1)$ is a lift of $p \circ c'$, which moreover starts at e_0. By the uniqueness of such lifts, $\tilde{H}(\cdot, 1) = c'$. Consequently, c and c' end in the same point, which means that $e = e'$.

We now identify $\mathcal{C}(E, p, X)$ and $\pi_1(X, x_0)$ using Φ. If $e \in p^{-1}(x_0)$, then $\Phi(e)$ is a loop at x_0. Therefore, $\Phi(e) \in \pi_1(X, x_0)$. Conversely, every lift of a loop at x_0 to E ends in a point of $p^{-1}(x_0)$. Therefore, Φ restricted to $p^{-1}(x_0)$ is a bijection with $\pi_1(X, x_0)$. Moreover, each $e \in p^{-1}(x_0)$ defines a unique element of $\mathcal{C}(E,p,X)$ by Proposition P.1.3(1), and vice versa. Therefore, we get a bijection

$$\Psi : \mathcal{C}(E,p,X) \to \pi_1(X,x_0) : \alpha \mapsto \Phi(\alpha(e_0)).$$

This map yields a homomorphism f_π on $\pi_1(X,x_0)$ given by $f_\pi := \Psi \circ \tilde{f}_\pi \circ \Psi^{-1}$. Thus, $R(f_\pi) = R(\tilde{f}_\pi)$.

Let $\tilde{w} : [0,1] \to E$ be a path from e_0 to $\tilde{f}_0(e_0)$ and let $w := p \circ \tilde{w}$ be its projection to X. Then w is a path from $p(e_0) = x_0$ to $p(\tilde{f}_0(e_0) = f(p(e_0)) = f(x_0)$. With w, we define a map $\hat{w} : \pi_1(X, f(x_0)) \to \pi_1(X,x_0) : [c] \mapsto [w] * [c] * [\bar{w}]$, where $\bar{w}$ is the path w traversed in opposite direction.

Lemma P.1.14 (See e.g. [62, Chapter II, Lemma 1.3])**.** *With the notation as above, the following diagram commutes:*

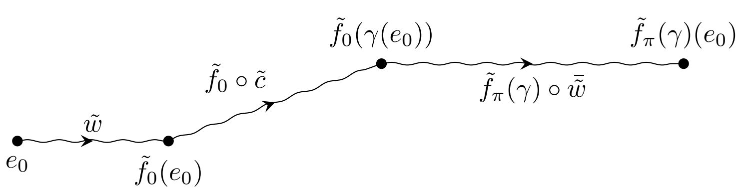
$$\begin{tikzcd}
& \pi_1(X,f(x_0)) \arrow[dr, "\hat{w}"] & \\
\pi_1(X,x_0) \arrow[ur, "f_*"] \arrow[rr, "f_\pi"] & & \pi_1(X,x_0)
\end{tikzcd}$$

Proof. Let $[c] \in \pi_1(X,x_0)$. Following the top two arrows we get

$$\hat{w}(f_*([c])) = [w] * [f \circ c] * [\bar{w}].$$

For $f_\pi([c])$, put $\gamma := \Psi^{-1}([c])$. Then $f_\pi([c]) = \Psi(\tilde{f}_\pi(\gamma)) = \Phi(\tilde{f}_\pi(\gamma)(e_0))$, by definition. Thus, we have to find a path in E that starts at e and ends in $\tilde{f}_\pi(\gamma)(e_0)$.

The path $\tilde{w}$ starts at e_0 and ends at $\tilde{f}_0(e_0)$, by construction. Let $\tilde{c}$ be the unique lift of c to E that starts at e_0. Then $\tilde{c}$ ends in $\gamma(e_0)$, since $\gamma = \Psi^{-1}([c])$. Consequently, $\tilde{f}_0 \circ \tilde{c}$ is a path from $\tilde{f}_0(e_0)$ to $\tilde{f}_0(\gamma(e_0))$. By definition is $\tilde{f}_0(\gamma(e_0)) = \tilde{f}_\pi(\gamma)(\tilde{f}_0(e_0))$. Finally, since $\bar{\tilde{w}}$ is a path from $\tilde{f}_0(e_0)$ to e_0, $\tilde{f}_\pi(\gamma) \circ \bar{\tilde{w}}$ is a path from $\tilde{f}_\pi(\gamma)(\tilde{f}_0(e_0))$ to $\tilde{f}_\pi(\gamma)(e_0)$.



Combining these three paths, we get the path

$$\tilde{w} * (\tilde{f}_0 \circ \tilde{c}) * (\tilde{f}_\pi(\gamma) \circ \bar{\tilde{w}})$$

from e_0 to $\tilde{f}_\pi(\gamma)(e_0)$. With this path, we compute $\Phi(\tilde{f}_\pi(\gamma)(e_0))$:

$$\begin{aligned}\Phi(\tilde{f}_\pi(\gamma)(e_0)) &= [p \circ (\tilde{w} * (\tilde{f}_0 \circ \tilde{c}) * (\tilde{f}_\pi(\gamma) \circ \bar{\tilde{w}}))]\\ &= [p \circ \tilde{w}] * [p \circ \tilde{f}_0 \circ \tilde{c}] * [p \circ \tilde{f}_\pi(\gamma) \circ \bar{\tilde{w}}]\\ &= [w] * [f \circ p \circ \tilde{c}] * [p \circ \bar{\tilde{w}}]\\ &= [w] * [f \circ c] * [\bar{w}]\\ &= \hat{w}(f_*([c])).\end{aligned}$$

Therefore, the diagram commutes. □

If x_0 is a fixed point of f, then $f_* = f_\pi$, so also $R(f_*) = R(f_\pi)$. Since $R(f_\pi) = R(\tilde{f}_\pi) = R(f)$, we conclude that $R(f_*) = R(f)$.

P.2 Galois cohomology

We now discuss another branch of mathematics where twisted conjugacy arises naturally. The exposition below is based on the introduction of [6].

Let L/K be a field extension and let $A, B \in K^{n\times n}$ be two matrices. It is a classical result in linear algebra that, if $A = QBQ^{-1}$ for some $Q \in \mathrm{GL}(n, L)$, then there also exists a $P \in \mathrm{GL}(n, K)$ such that $A = PBP^{-1}$ (see e.g. [101, Corollary 9.36]). For instance, if $L = \mathbb{C}$ and $K = \mathbb{R}$, then the matrices

$$A = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}$$

are similar over $\mathbb{C}$, as they both have eigenvalues $\pm i\sqrt{3}$. In fact,

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} B \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^{-1}.$$

In this case, it also easy to find a matrix $P \in \mathrm{GL}(n, \mathbb{R})$ such that $A = PBP^{-1}$, namely

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So, being conjugate by an element of $\mathrm{GL}(n, L)$ is equivalent with being conjugate by an element of $\mathrm{GL}(n, K)$. However, we can ask the same question over smaller groups of matrices, for instance, over $\mathrm{SL}(n, K)$. Going back to the example, we see that there exists a matrix $Q \in \mathrm{SL}(2, \mathbb{C})$ such that $A = QBQ^{-1}$, namely

$$Q = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

However, there does not such a matrix in $\mathrm{SL}(2, \mathbb{R})$. Indeed, one can verify that each $P \in \mathrm{GL}(2, \mathbb{R})$ that satisfies $A = PBP^{-1}$ is of the form

$$P = \begin{pmatrix} a & 3c \\ c & -a \end{pmatrix},$$

where $a, c \in \mathbb{R}$ are not both zero. The determinant of such matrix equals $-a^2 - 3c^2 < 0$. Consequently, there does not exist a matrix $P \in \mathrm{SL}(2, \mathbb{R})$ such that $A = PBP^{-1}$.

To explain the difference between the two cases, we can use Galois cohomology. For ease of notation, we write $G(K)$ to denote either $\mathrm{GL}(2, K)$ or $\mathrm{SL}(2, K)$, where K is a field.

So, suppose that $A = QBQ^{-1}$ for some $Q \in G(\mathbb{C})$. The goal is to decide whether there exists a $P \in G(\mathbb{R})$ such that $A = PBP^{-1}$. We try and construct such a matrix using Q. Note that $M \in G(\mathbb{C})$ belonging to $G(\mathbb{R})$ is equivalent with $M = \overline{M}$, where $\bar{\cdot}$ denotes complex conjugation. Suppose first that we have a $P \in G(\mathbb{R})$ such that $A = PBP^{-1}$. Then $C := QP^{-1}$ belongs to

$$C_{G(\mathbb{C})}(A) := \{M \in G(\mathbb{C}) \mid MA = AM\}.$$

Indeed,

$$CAC^{-1} = QP^{-1}APQ^{-1} = QBQ^{-1} = A.$$

Moreover, since $P = \overline{P}$, we have the equality

$$Q\overline{Q}^{-1} = CP\big(\overline{CP}\big)^{-1} = CPP^{-1}\overline{C}^{-1} = C\overline{C}^{-1}.$$

Conversely, suppose that $C \in C_{G(\mathbb{C})}(A)$ is such that $Q\overline{Q}^{-1} = C\overline{C}^{-1}$. Put $P := C^{-1}Q$. Then

$$PBP^{-1} = C^{-1}QBQ^{-1}C = C^{-1}AC = A$$

and

$$\overline{P} = \overline{C}^{-1}\overline{Q} = C^{-1}Q\overline{Q}^{-1}\overline{Q} = C^{-1}Q = P.$$

Thus, we conclude that

$$\exists P \in G(\mathbb{R}) : A = PBP^{-1} \iff \exists C \in C_{G(\mathbb{C})}(A) : Q\overline{Q}^{-1} = C\overline{C}^{-1}.$$

In our example, $Q = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, so $Q\overline{Q}^{-1} = -I$. A straightforward computation yields

$$C_{G(\mathbb{C})}(A) = \left\{ C \in G(\mathbb{C}) \;\middle|\; C = \begin{pmatrix} z & -3w \\ w & z \end{pmatrix} \text{ for some } w, z \in \mathbb{C} \right\}.$$

Now, for $C \in C_{G(\mathbb{C})}(A)$ is $C\overline{C}^{-1} = -I$ equivalent with

$$C = \begin{pmatrix} ic & -3id \\ id & ic \end{pmatrix}$$

for some $c, d \in \mathbb{R}$, not both zero. The determinant of C is thus equal to $-(c^2 + 3d^2) < 0$. Thus, the equation $C\overline{C}^{-1} = -I$ has no solutions with $C \in C_{\mathrm{SL}(2,\mathbb{C})}(A)$, but does with $C \in C_{\mathrm{GL}(2,\mathbb{C})}(A)$. The matrix $Q\overline{Q}^{-1}$ therefore measures to some extent how far A and B are from being conjugate over $\mathbb{R}$.

We can extend this discussion to a more general setting. Let L/K be a finite Galois extension, $n \geq 2$ an integer and $A, B \in K^{n \times n}$ two matrices. For a field F, let $G(F)$ denote a fixed matrix group (e.g. $\mathrm{GL}(n, F), \mathrm{SL}(n, F), \mathrm{O}(n, F), \mathrm{Sp}(2n, F) \ldots$). Suppose that $A = QBQ^{-1}$ for some $Q \in G(L)$. We want to decide whether there exists a $P \in G(K)$ such that $A = PBP^{-1}$. First, recall that the Galois group $\mathrm{Gal}(L/K)$ acts on $L^{n \times n}$ by applying $\sigma \in \mathrm{Gal}(L/K)$ to each entry of a matrix $M \in L^{n \times n}$. We write this as $\sigma \cdot M$. In this notation, if $M \in L^{n \times n}$, then

$$M \in K^{n \times n} \iff \forall \sigma \in \mathrm{Gal}(L/K) : \sigma \cdot M = M,$$

since L/K is a finite Galois extension. If we apply a given $\sigma \in \mathrm{Gal}(L/K)$ to the equality $A = QBQ^{-1}$, we derive that

$$Q(\sigma \cdot Q)^{-1} A (\sigma \cdot Q) Q^{-1} = QBQ^{-1} = A,$$

i.e. $Q(\sigma \cdot Q)^{-1} \in C_{G(L)}(A)$.

Suppose that $P \in G(K)$ satisfies $A = PBP^{-1}$. Then $C := QP^{-1}$ belongs to $C_{G(L)}(A)$, as before. Also, for each $\sigma \in \mathrm{Gal}(L/K)$,

$$Q(\sigma \cdot Q)^{-1} = CP(\sigma \cdot (CP))^{-1} = CPP^{-1}(\sigma \cdot C)^{-1} = C(\sigma \cdot C)^{-1}.$$

Conversely, a similar argument as before shows that, for each $C \in C_{G(L)}(A)$ satisfying $Q(\sigma \cdot Q)^{-1} = C(\sigma \cdot C)^{-1}$ for each $\sigma \in \mathrm{Gal}(L/K)$, $P := C^{-1}Q$ is a matrix in $G(K)$ such that $A = PBP^{-1}$. Thus, if we define for each matrix $D \in G(L)$ the map

$$\alpha^D : \mathrm{Gal}(L/K) \to G(L) : \sigma \mapsto D(\sigma \cdot D)^{-1},$$

we see that a $P \in G(K)$ such that $A = PBP^{-1}$ exists if and only if $\alpha^Q = \alpha^C$ for some $C \in C_{G(L)}(A)$.

Summarised, each $A' \in K^{n\times n}$ conjugate to B by an element of $G(L)$ induces a map α^Q that measures to some extent how far this conjugation is from being by an element of $G(K)$.

On the other hand, if $\alpha : \mathrm{Gal}(L/K) \to C_{G(L)}(A) : \alpha \mapsto \alpha_\sigma$ is a map such that $\alpha = \alpha^Q$ for some $Q \in G(L)$, then $A_\alpha := Q^{-1}AQ$ is an element of $G(K)$ conjugate to A. Indeed, for all $\sigma \in \mathrm{Gal}(L/K)$, we have

$$\begin{aligned}
\sigma \cdot A_\alpha &= \sigma \cdot (Q^{-1}AQ) \\
&= (\sigma \cdot Q)^{-1}(\sigma \cdot A)(\sigma \cdot Q) \\
&= Q^{-1}Q(\sigma \cdot Q)^{-1}A(\sigma \cdot Q) \\
&= Q^{-1}AQ(\sigma \cdot Q)^{-1}(\sigma \cdot Q) \\
&= Q^{-1}AQ \\
&= A_\alpha.
\end{aligned}$$

In the third equality, we use that $A \in G(K)$, and in the fourth, we use that $Q(\sigma \cdot Q)^{-1} = \alpha_\sigma$ lies in $C_{G(L)}(A)$.

Not all maps α can be written as α^Q for some $Q \in G(L)$. A necessary condition is that

$$\alpha_{\sigma\tau} = \alpha_\sigma \sigma \cdot \alpha_\tau \tag{P.2.1}$$

for all $\sigma, \tau \in \mathrm{Gal}(L/K)$. Indeed, if $\alpha = \alpha^Q$ for $Q \in G(L)$, then

$$\begin{aligned}
\alpha^Q_\sigma \sigma \cdot \alpha^Q_\tau &= Q(\sigma \cdot Q)^{-1}\sigma \cdot \left(Q(\tau \cdot Q)^{-1}\right) \\
&= Q(\sigma \cdot Q)^{-1}(\sigma \cdot Q)\sigma \cdot (\tau \cdot Q)^{-1} \\
&= Q((\sigma\tau) \cdot Q)^{-1} \\
&= \alpha^Q_{\sigma\tau}
\end{aligned}$$

for all $\sigma, \tau \in \mathrm{Gal}(L/K)$.

A map that satisfies (P.2.1) is called a *cocycle*. Although in general not every cocycle is of the form α^Q, it is for $\mathrm{GL}(n, L)$ or $\mathrm{SL}(n, L)$; this follows from Hilbert's Theorem 90 (see e.g. [6, §III.9]).

Thus far, we have worked with one specific choice for $Q \in G(L)$. If $Q' \in G(L)$ is another matrix satisfying $A = Q'B(Q')^{-1}$, then we can ask ourself the question whether α^Q and $\alpha^{Q'}$ are related. Since both Q and Q' conjugate B to A, $C := Q'Q^{-1} \in C_{G(L)}(A)$. Thus,

$$\begin{aligned}\alpha_\sigma^{Q'} &= Q'(\sigma \cdot Q')^{-1}\\ &= CQ(\sigma \cdot CQ)^{-1}\\ &= C\Big(Q(\sigma \cdot Q)^{-1}\Big)(\sigma \cdot C)^{-1}\\ &= C\alpha_\sigma^Q(\sigma \cdot C)^{-1}\end{aligned}$$

for all $\sigma \in \mathrm{Gal}(L/K)$. In general, if two cocycles $\alpha, \alpha' : \mathrm{Gal}(L/K) \to C_{G(L)}(A)$ and a matrix $C \in C_{G(L)}(A)$ are such that

$$\alpha'_\sigma = C\alpha_\sigma(\sigma \cdot C)^{-1}$$

for all $\sigma \in \mathrm{Gal}(L/K)$, α and α' are said to be *cohomologous*. This defines an equivalence relation on the set of all cocycles, and we write the equivalence class of α as $[\alpha]$. We let $H^1(\mathrm{Gal}(L/K), C_{G(L)}(A))$ denote the set of all equivalences classes. We see here how twisted conjugacy pops up, although the cohomology equivalence relation is much stronger than twisted conjugacy: for each $\sigma \in \mathrm{Gal}(L/K)$, let

$$\varphi_\sigma : C_{G(L)}(A) \to C_{G(L)}(A) : M \mapsto \sigma \cdot M$$

denote the induced automorphism on $C_{G(L)}(A)$. We then have two sets of elements, $\{\alpha_\sigma\}_{\sigma \in \mathrm{Gal}(L/K)}$ and $\{\alpha'_\sigma\}_{\sigma \in \mathrm{Gal}(L/K)}$, and we require (1) that α_σ is φ_σ-conjugate to α'_σ for each $\sigma \in \mathrm{Gal}(L/K)$, and (2) that there exists a *single* $C \in C_{G(L)}(A)$ realising these twisted conjugacy relations.

Thus, if we fix $A \in G(L)$, we can associate to each $B \in G(L)$ that is conjugate to A a cohomology class $[\alpha^Q]$, where $Q \in G(L)$ is any matrix such that $A = QBQ^{-1}$. It is, however, not the case that $[\alpha^Q]$ fully determines B. Indeed, if $P \in G(K)$, then $\sigma \cdot P = P$ for all $\sigma \in \mathrm{Gal}(L/K)$. Hence,

$$\alpha_\sigma^{QP} = (QP)(\sigma \cdot (QP))^{-1} = QPP^{-1}(\sigma \cdot Q)^{-1} = Q(\sigma \cdot Q)^{-1} = \alpha_\sigma^Q,$$

for all $\sigma \in \mathrm{Gal}(L/K)$. Consequently, $\alpha^{QP} = \alpha^Q$, although $Q^{-1}AQ$ and $(QP)^{-1}AQP$ are not (necessarily) equal. They are, however, $G(K)$-conjugate.

Even more general, suppose that α^Q and $\alpha^{Q'}$ are cohomologous, say, $\alpha_\sigma^{Q'} = C\alpha_\sigma^Q(\sigma \cdot C)^{-1}$ for some $C \in C_{G(L)}(A)$. Put $P := Q^{-1}C^{-1}Q'$ and let $\sigma \in$

$\mathrm{Gal}(L/K)$. Then

$$\begin{aligned}\sigma \cdot P &= (\sigma \cdot Q)^{-1}(\sigma \cdot C)^{-1}(\sigma \cdot Q')\\ &= Q^{-1}\alpha_\sigma^Q(\sigma \cdot C)^{-1}(\sigma \cdot Q')\\ &= Q^{-1}C^{-1}\alpha_\sigma^{Q'}(\sigma \cdot Q')\\ &= Q^{-1}C^{-1}Q'\\ &= P.\end{aligned}$$

As σ is arbitrary, it follows that $P \in G(K)$. Furthermore,

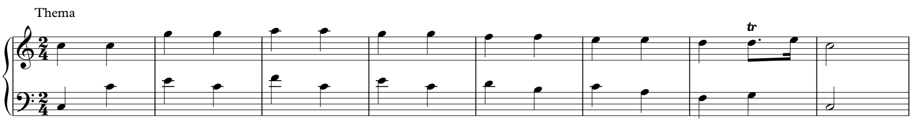
$$PA_{\alpha^{Q'}}P^{-1} = P(Q')^{-1}AQ'P^{-1} = Q^{-1}C^{-1}ACQ = Q^{-1}AQ = A_{\alpha^Q}.$$

Thus, the associated matrices $A_{\alpha^{Q'}}$ and A_{α^Q} are $G(K)$-conjugate.

Combining all of the previous considerations, we conclude that if all cocycles $\mathrm{Gal}(L/K) \to C_{G(L)}(A)$ are of the form α^Q, then the map $[\alpha] \mapsto [A_\alpha]$ is a bijection between $H^1(\mathrm{Gal}(L/K), C_{G(L)}(A)$ and the set of $G(K)$-conjugacy classes of matrices in $K^{n\times n}$ that are conjugate to A by some element in $G(L)$.

Theme

Twisted conjugacy



Mozart, Wolfgang Amadeus, *Zwölf Variationen in C über das franzözische Lied 'Ah, vous dirai-je, Maman'*, (KV 265 (300e)), Theme (bars 1–8).[4]

As mentioned in the introduction, the theme of this thesis is twisted conjugacy. We therefore start with formally defining all the notions regarding twisted conjugacy and introducing the necessary notation.

Definition T.1.1. Let G be a group and $\varphi \in \mathrm{End}(G)$. We say that $x, y \in G$ are *φ-conjugate* if $x = zy\varphi(z)^{-1}$ for some $z \in G$. We write this as $x \sim_\varphi y$.

This defines an equivalence relation on G. We let $[x]_\varphi$ denote the equivalence class of $x \in G$. We refer to this as the φ-conjugacy class of x. When no φ is specified, we also speak of *twisted conjugacy classes* or *Reidemeister classes*. We let $\mathcal{R}[\varphi]$ denote the set of equivalence classes and $R(\varphi)$ the cardinality of $\mathcal{R}[\varphi]$. We call the latter the *Reidemeister number* of φ. If $R(\varphi)$ is infinite, we write $R(\varphi) = \infty$.

Definition T.1.2. Let G be a group. The *Reidemeister spectrum* of G is the set

$$\mathrm{Spec}_{\mathrm{R}}(G) := \{R(\varphi) \mid \varphi \in \mathrm{Aut}(G)\},$$

and the *extended Reidemeister spectrum* of G is the set

$$\mathrm{ESpec}_{\mathrm{R}}(G) := \{R(\varphi) \mid \varphi \in \mathrm{End}(G)\}.$$

[4]This and all subsequent excerpts from Mozart's work are adopted from [80, 81].

Clearly, $\mathrm{Spec}_{\mathrm{R}}(G) \subseteq \mathrm{ESpec}_{\mathrm{R}}(G)$. We say that G has the *R_∞-property* if $\mathrm{Spec}_{\mathrm{R}}(G) = \{\infty\}$. We say that G has *full (extended) Reidemeister spectrum* if $\mathrm{Spec}_{\mathrm{R}}(G) = \mathbb{N}_0 \cup \{\infty\}$ (resp. $\mathrm{ESpec}_{\mathrm{R}}(G) = \mathbb{N}_0 \cup \{\infty\}$). Here, $\mathbb{N}_0$ is the set of all positive integers, not including 0.

To stay within the philosophy of the structure of this thesis, in this section, we almost exclusively discuss results that hold for every group, with no additional structural properties imposed. These mainly consist of equalities of Reidemeister numbers of related endomorphisms. At the end, however, we make a small deviation to abelian groups. They occur in many Variations, which justifies their position in the theme of the thesis.

Lemma T.1.3 (See e.g. [29, Lemma 1])**.** *Let G be a group and $\varphi \in \mathrm{End}(G)$. For every $g \in G$, we have $g \sim_\varphi \varphi(g)$.*

Proof. This is immediate from the observation that $g = g\varphi(g)\varphi(g)^{-1}$ holds for all $g \in G$. ☐

Proposition T.1.4. *Let G be a group. Then $1 \in \mathrm{ESpec}_{\mathrm{R}}(G)$.*

Proof. Let $\varphi : G \to G : g \mapsto 1$ be the trivial endomorphism. By Lemma T.1.3, $g \sim_\varphi 1$ for all $g \in G$. Therefore, $[1]_\varphi = G$, which implies that $R(\varphi) = 1$. ☐

Lemma T.1.5. *Let G and H be two groups, and φ and ψ endomorphisms on G and H, respectively. Suppose that $f : G \to H$ is a homomorphism such that the following square commutes:*

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "f"] & G \arrow[d, "f"] \\ H \arrow[r, "\psi"] & H \end{tikzcd}$$

Then f induces a map $\hat{f} : \mathcal{R}[\varphi] \to \mathcal{R}[\psi] : [g]_\varphi \mapsto [f(g)]_\psi$.

Moreover, if f is surjective, then so is $\hat{f}$. In particular, $R(\varphi) \geq R(\psi)$ in this situation.

If f is bijective, then $R(\varphi) = R(\psi)$.

Proof. Suppose that $x, y \in G$ are φ-conjugate. Write $x = gy\varphi(g)^{-1}$. Then

$$\begin{aligned} f(x) &= f(g)f(y)f\Big(\varphi(g)^{-1}\Big) \\ &= f(g)f(y)\psi(f(g))^{-1} \end{aligned}$$

by commutativity of the diagram. This shows that $f(x) \sim_\psi f(y)$. Hence, $\hat{f}$ is well defined.

Next, suppose that f is surjective. Given $h \in H$, write $h = f(g)$ for some $g \in G$. Then $[h]_\psi = \hat{f}([g]_\varphi)$, which shows that $\hat{f}$ is surjective.

If f is bijective, then we see that both $R(\varphi) \geq R(\psi)$ and $R(\psi) \geq R(\varphi)$ hold, by applying the diagram to f and f^{-1}. This yields the result. □

Definition T.1.6. Let G be a group and $g \in G$. We define $\tau_g : G \to G : x \mapsto gxg^{-1}$, the inner automorphism given by conjugation by g.

Given a group G and two elements $g, h \in G$, we write $g^h = h^{-1}gh$. We use the same notation for two elements g, h in a monoid if h has an inverse.

Lemma T.1.7 (See e.g. [37, Corollary 2.5])**.** *Let G be a group and $\varphi \in \mathrm{End}(G)$.*

(1) For all $\psi \in \mathrm{Aut}(G)$, $R(\varphi^\psi) = R(\varphi)$.

(2) For all $g \in G$, $R(\varphi \circ \tau_g) = R(\tau_g \circ \varphi) = R(\varphi)$.

Proof. Let $\psi \in \mathrm{Aut}(G)$ be arbitrary. Consider the following diagram:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\psi"] & G \arrow[d, "\psi"] \\ G \arrow[r, "\varphi^\psi"] & G \end{tikzcd}$$

By Lemma T.1.5, $R(\varphi) \geq R(\varphi^\psi)$, as ψ is surjective. Conversely, the diagram

$$\begin{tikzcd} G \arrow[r, "\varphi^\psi"] \arrow[d, "\psi^{-1}"] & G \arrow[d, "\psi^{-1}"] \\ G \arrow[r, "\varphi"] & G \end{tikzcd}$$

yields $R(\varphi^\psi) \geq R(\varphi)$. Combining both yields equality.

Next, let $g \in G$. Since τ_g is an automorphism of G and $(\tau_g \circ \varphi)^{\tau_g} = \varphi \circ \tau_g$, we only have to argue that $R(\tau_g \circ \varphi) = R(\varphi)$, by the first item. Define the map

$$F : \mathcal{R}[\varphi] \to \mathcal{R}[\tau_g \circ \varphi] : [x]_\varphi \mapsto \left[xg^{-1}\right]_{\tau_g \circ \varphi}.$$

We prove that F is a well-defined bijection. Suppose that $x \sim_\varphi y$. Write $x = zy\varphi(z)^{-1}$. Then

$$xg^{-1} = zy\varphi(z)^{-1}g^{-1} = z(yg^{-1})(g\varphi(z)^{-1}g^{-1}) = z(yg^{-1})(\tau_g \circ \varphi)(z)^{-1},$$

which proves that $F([x]_\varphi)$ is independent of the representative. The same argument in reverse order proves that $xg^{-1} \sim_{\tau_g \circ \varphi} yg^{-1}$ implies $x \sim_\varphi y$. Thus, F is injective.

Finally, since $y = (yg^{-1})g$ for all $y \in G$, the map F is also surjective, which finishes the proof. □

Lemma T.1.8. *Let G be a group and $\psi \in \mathrm{Aut}(G)$. Then $R(\psi) = R(\psi^{-1})$.*

Proof. We again provide an explicit bijection, this time between $\mathcal{R}[\psi]$ and $\mathcal{R}[\psi^{-1}]$. Consider

$$F : \mathcal{R}[\psi] \to \mathcal{R}[\psi^{-1}] : [x]_\psi \mapsto \left[x^{-1}\right]_{\psi^{-1}}.$$

Suppose that $x = gy\psi(g)^{-1}$. Then

$$x^{-1} = \psi(g)y^{-1}g^{-1} = \psi(g)y^{-1}\left(\psi^{-1}(\psi(g))\right)^{-1},$$

which proves that $x^{-1} \sim_{\psi^{-1}} y^{-1}$. Thus, F is well defined. Similarly, the map

$$F' : \mathcal{R}[\psi^{-1}] \to \mathcal{R}[\psi] : [x]_{\psi^{-1}} \mapsto \left[x^{-1}\right]_{\psi}$$

is well defined as well. Since $F' \circ F$ and $F \circ F'$ are the identity maps on $\mathcal{R}[\psi]$ and $\mathcal{R}[\psi^{-1}]$, respectively, F and F' are bijections. □

We can also view twisted conjugacy as a group action. Let G be a group and $\varphi \in \mathrm{End}(G)$. Then G acts on itself from the left via φ by

$$g \cdot x := gx\varphi(g)^{-1}.$$

We note that $[x]_\varphi$ is the orbit of x under this action.

Definition T.1.9. Let G be a group, $\varphi \in \mathrm{End}(G)$ and $g \in G$. The *φ-stabiliser* of g, written as $\mathrm{Stab}_\varphi(g)$, is the stabiliser of g under the φ-conjugacy action, i.e.

$$\mathrm{Stab}_\varphi(g) := \{x \in G \mid xg\varphi(x)^{-1} = g\}.$$

Being a stabiliser of a group action, $\mathrm{Stab}_\varphi(x)$ is a subgroup of G for all $x \in G$. In general, we also speak of *twisted stabilisers.*

Given a group G and an endomorphism φ of G, we write $\mathrm{Fix}(\varphi)$ for the set of fixed points of φ, i.e.

$$\mathrm{Fix}(\varphi) := \{g \in G \mid \varphi(g) = g\}.$$

Lemma T.1.10. *Let G be a group, $\varphi \in \operatorname{End}(G)$ and $g \in G$. Then* $\operatorname{Stab}_\varphi(g) = \operatorname{Fix}(\tau_g \circ \varphi)$.

Proof. We have the following chain of equalities:

$$\begin{aligned}\operatorname{Stab}_\varphi(g) &= \{x \in G \mid xg\varphi(x)^{-1} = g\}\\ &= \{x \in G \mid x = g\varphi(x)g^{-1}\}\\ &= \operatorname{Fix}(\tau_g \circ \varphi).\end{aligned}$$ □

As a first example of how to compute Reidemeister numbers, we consider the free abelian groups of finite rank. For ease of notation, we introduce the following map:

$$|\cdot|_\infty : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\} : n \mapsto \begin{cases} |n| & \text{if } n \neq 0,\\ \infty & \text{otherwise.}\end{cases}$$

Lemma T.1.11 (See also [62, Chapter II, Theorem 2.5])**.** *Let A be an abelian group and $\varphi \in \operatorname{End}(A)$. Then*

$$\mathcal{R}[\varphi] = \operatorname{coker}(\varphi - \mathrm{Id}) = \frac{A}{\operatorname{Im}(\varphi - \mathrm{Id})}$$

and thus $R(\varphi) = [A : \operatorname{Im}(\varphi - \mathrm{Id})]$.

Proof. Let $x, y \in A$ be arbitrary. Then the following equivalences hold:

$$x \sim_\varphi y \iff \exists a \in A : x = a + y - \varphi(a) \iff x - y \in \operatorname{Im}(\varphi - \mathrm{Id}).$$

Consequently, $\mathcal{R}[\varphi]$ is the set of all cosets of $\operatorname{Im}(\varphi - \mathrm{Id})$, i.e. $\mathcal{R}[\varphi] = A/\operatorname{Im}(\varphi - \mathrm{Id}) = \operatorname{coker}(\varphi - \mathrm{Id})$. □

Proposition T.1.12 (See e.g. [100, §3])**.** *Let $A \cong \mathbb{Z}^r$ be a free abelian group of finite rank $r \geq 1$. Let $\varphi \in \operatorname{End}(A)$. Then*

$$R(\varphi) = |\det(\mathrm{Id}_A - \varphi)|_\infty = |\chi_\varphi(1)|_\infty.$$

Here, given $\psi \in \operatorname{End}(A)$, $\det(\psi)$ and χ_ψ are the determinant and characteristic polynomial, respectively, of any matrix representation of ψ w.r.t. a $\mathbb{Z}$-basis of A.

In particular, $R(\varphi) = \infty$ if and only if φ has a non-trivial fixed point.

Proof. By Lemma T.1.11, $R(\varphi) = [A : \operatorname{Im}(\varphi - \mathrm{Id})]$. Therefore, it is sufficient to prove that $|\operatorname{coker}(\psi)| = |\det(\psi)|_\infty$ for $\psi \in \operatorname{End}(A)$.

Let $\psi \in \mathrm{End}(A)$ be arbitrary. Let M be the matrix representation of ψ with respect to a (fixed) basis of A. As A is a free $\mathbb{Z}$-module, there exist $\mathbb{Z}$-invertible matrices P and Q such that $A = P\Sigma Q$, where

$$\Sigma = \begin{pmatrix} s_1 & 0 & 0 & \dots & 0 \\ 0 & s_2 & 0 & \dots & 0 \\ 0 & 0 & s_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & s_r \end{pmatrix},$$

and where $s_1 \mid s_2 \mid \dots \mid s_r$ are non-negative integers. This means there are $\mathbb{Z}$-bases $\{x_1, \dots, x_r\}$ and $\{y_1, \dots, y_r\}$ of A such that $\psi(x_i) = s_i y_i$ for each $i \in \{1, \dots, r\}$. From this, it follows that

$$\mathrm{coker}(\psi) \cong \bigoplus_{i=1}^{r} \frac{\mathbb{Z}}{s_i\mathbb{Z}}.$$

We can now prove the desired equality. Suppose that $s_r \neq 0$. Then $|\mathrm{coker}(\psi)| = \prod_{i=1}^{r} s_i = |\det(A)|$, as $|\det(P)| = |\det(Q)| = 1$.

If $s_r = 0$, then $|\mathrm{coker}(\psi)| = \infty = |0|_\infty = |\det(A)|_\infty$. □

Theorem T.1.13. *Let $A \cong \mathbb{Z}^r$ be a free abelian group of finite rank $r \geq 1$. Then*

$$\mathrm{Spec}_{\mathrm{R}}(A) = \begin{cases} \{2, \infty\} & \textit{if } r = 1, \\ \mathbb{N}_0 \cup \{\infty\} & \textit{otherwise}, \end{cases}$$

and $\mathrm{ESpec}_{\mathrm{R}}(A) = \mathbb{N}_0 \cup \{\infty\}$.

Proof. The only automorphisms of $\mathbb{Z}$ are the identity map and the inversion map, and their Reidemeister numbers are $|1-1|_\infty = \infty$ and $|-1-1|_\infty = 2$, respectively. Therefore, $\mathrm{Spec}_{\mathrm{R}}(\mathbb{Z}) = \{2, \infty\}$. On the other hand, given $n \in \mathbb{Z}$, the map $\varphi_n : \mathbb{Z} \to \mathbb{Z} : x \mapsto (n+1)x$ is a well-defined homomorphism with Reidemeister number equal to $|n+1-1|_\infty = |n|_\infty$. Consequently, as n was arbitrary, $\mathbb{Z}$ has full extended Reidemeister spectrum.

Next, suppose that $r \geq 2$. Let $m \in \mathbb{Z}$ be arbitrary. Let M be the companion matrix of the polynomial $p_m(x) := x^r + mx^{r-1} - 1$. Then $\det(M) = p_m(0) = -1$, so M defines an automorphism of $\mathbb{Z}^r$. By Proposition T.1.12, $R(M) = |p_m(1)|_\infty = |m|_\infty$. As m was arbitrary, it follows that $\mathrm{Spec}_{\mathrm{R}}(\mathbb{Z}^r) = \mathrm{ESpec}_{\mathrm{R}}(\mathbb{Z}^r) = \mathbb{N}_0 \cup \{\infty\}$.

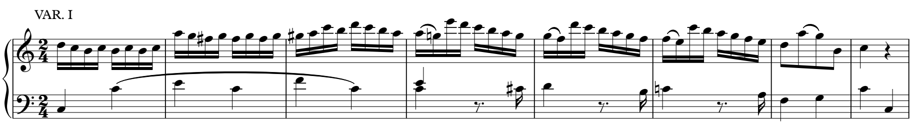
□

Collection I

Group constructions

Variation 1

Extensions



Mozart, Wolfgang Amadeus, *Zwölf Variationen in C über das franzözische Lied 'Ah, vous dirai-je, Maman'*, (KV 265 (300e)), Variation I (bars 1–8).

Extensions are a ubiquitous tool to study groups. It is therefore natural to begin with studying the behaviour of twisted conjugacy and Reidemeister numbers in group extensions.

Due to the broad nature of extensions, the results in this Variation are quite technical. On the other hand, we use the results obtained here frequently in later Variations, when we impose additional structure to derive more specific results.

1.1 General extensions

Proposition 1.1.1 (See e.g. [56, Theorem 1.8],[50, Lemma 1.1],[70, §2])**.** *Let G be a group and N a normal subgroup. Suppose that $\varphi \in \mathrm{End}(G)$ satisfies $\varphi(N) \leq N$. Let $\varphi|_N$ and $\bar{\varphi}$ denote the induced endomorphisms on N and G/N,* *respectively. We then have the following commuting diagram with exact rows:*

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r, "i"] \arrow[d, "\varphi|_N"] & G \arrow[r, "\pi"] \arrow[d, "\varphi"] & G/N \arrow[r] \arrow[d, "\bar{\varphi}"] & 1 \\
1 \arrow[r] & N \arrow[r, "i"] & G \arrow[r, "\pi"] & G/N \arrow[r] & 1
\end{tikzcd}$$

This diagram induces an exact sequence of pointed sets

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Fix}(\varphi|_N) \arrow[r, "i_{Fix}"] & \mathrm{Fix}(\varphi) \arrow[r, "\pi_{Fix}"] & \mathrm{Fix}(\bar{\varphi}) \arrow[dll, "\delta"] \\
 & \mathcal{R}[\varphi|_N] \arrow[r, "\hat{\imath}"] & \mathcal{R}[\varphi] \arrow[r, "\hat{\pi}"] & \mathcal{R}[\bar{\varphi}] \arrow[r] & 1,
\end{tikzcd}$$

where i_{Fix} and π_{Fix} are the restrictions of i and π to $\mathrm{Fix}(\varphi|_N)$ and $\mathrm{Fix}(\varphi)$, respectively, $\hat{\imath}$ and $\hat{\pi}$ are the induced maps on the respective sets of Reidemeister classes, and δ is given by $\delta(gN) = \left[g\varphi(g)^{-1}\right]_{\varphi|_N}$. The base point for the sets in the top row is 1, the base point for the sets in the bottom row is the respective twisted conjugacy class of 1.

Proof. The commutativity of the diagram follows immediately from the definitions of $\varphi|_N$ and $\bar{\varphi}$.

We prove the exactness of the induced sequence step by step, starting at $\mathrm{Fix}(\varphi|_N)$. Since $\varphi|_N$ is the restriction of φ to N, i_{Fix} is well defined and injective.

Next, let $g \in \mathrm{Fix}(\varphi)$. Then $\varphi(g) = g$, hence $\bar{\varphi}(gN) = gN$ as well, which proves that $\pi_{Fix}(g) = gN \in \mathrm{Fix}(\bar{\varphi})$.

Since $\pi \circ i$ is the trivial map, so is $\pi_{Fix} \circ i_{Fix}$. Hence, $\mathrm{Im}\, i_{Fix} \subseteq \ker \pi_{Fix}$. Conversely, suppose that $g \in \mathrm{Fix}(\varphi)$ is such that $\pi_{Fix}(g) = N$. Then $g \in N$ and since $\varphi(g) = g$, we conclude that $g \in \mathrm{Fix}(\varphi|_N)$.

The definition of δ is the only non-trivial one. It comes from the natural action of $\mathrm{Fix}(\bar{\varphi})$ on $\mathcal{R}[\varphi|_N]$ given by

$$gN \cdot [n]_{\varphi|_N} = \left[gn\varphi(g)^{-1}\right]_{\varphi|_N},$$

for all $gN \in \mathrm{Fix}(\bar{\varphi})$ and $[n]_{\varphi|_N} \in \mathcal{R}[\varphi|_N]$. We first argue that this is a well-defined action. Let $gN \in \mathrm{Fix}(\bar{\varphi})$ and $[n]_{\varphi|_N} \in \mathcal{R}[\varphi|_N]$. Then

$$\pi(gn\varphi(g)^{-1}) = g\varphi(g)^{-1}N = gN\bar{\varphi}(gN)^{-1} = gNgN^{-1} = N,$$

since $gN \in \mathrm{Fix}(\bar{\varphi})$. Thus, $gn\varphi(g)^{-1} \in N$. We also have to argue that the action is independent from the coset representative and the $\varphi|_N$-representative. So, suppose that $gN = hN \in \mathrm{Fix}(\bar{\varphi})$ and $[n]_{\varphi|_N} = [m]_{\varphi|_N}$. Then $g = n_0 h$ for some $n_0 \in N$, as N is normal, and $n = xm\varphi|_N(x)^{-1}$ for some $x \in N$. Then

$$\begin{aligned}\left[gn\varphi(g)^{-1}\right]_{\varphi|_N} &= \left[n_0hxm\varphi|_N(x)^{-1}\varphi(h)^{-1}\varphi(n_0)^{-1}\right]_{\varphi|_N}\\ &= \left[hxm\varphi|_N(x)^{-1}\varphi(h)^{-1}\right]_{\varphi|_N}\\ &= \left[(hxh^{-1})hm\varphi(h)^{-1}\varphi|_N(hxh^{-1})^{-1}\right]_{\varphi|_N}\\ &= \left[hm\varphi(h)^{-1}\right]_{\varphi|_N}.\end{aligned}$$

Thus, this action is well defined, as is the map δ.

Suppose that $g \in \mathrm{Fix}(\varphi)$. Then $\delta(gN) = \left[g\varphi(g)^{-1}\right]_{\varphi|_N} = [1]_{\varphi|_N}$. Hence, $\mathrm{Im}\,\pi_{Fix} \subseteq \ker\delta$. Conversely, suppose that $\left[g\varphi(g)^{-1}\right]_{\varphi|_N} = [1]_{\varphi|_N}$ for some $gN \in \mathrm{Fix}(\bar{\varphi})$. Then $g\varphi(g)^{-1} = n\varphi(n)^{-1}$ for some $n \in N$. Thus, $\varphi(gn^{-1}) = gn^{-1}$, which implies that $gn^{-1} \in \mathrm{Fix}(\varphi)$. Then $gN = \pi_{Fix}(gn^{-1})$, which shows that $gN \in \mathrm{Im}\,\pi_{Fix}$.

For $\hat{\imath}$, note that $\hat{\imath}(\delta(gN)) = \left[g\varphi(g)^{-1}\right]_{\varphi} = [1]_\varphi$ for all $g \in \mathrm{Fix}(\bar{\varphi})$, which proves that $\mathrm{Im}\,\delta \subseteq \ker\hat{\imath}$. Conversely, suppose that $[n]_\varphi = [1]_\varphi$ for some $n \in N$. Write $n = g\varphi(g)^{-1}$ for some $g \in G$. Then $\bar{\varphi}(gN) = gN$, so $gN \in \mathrm{Fix}(\bar{\varphi})$. Consequently,

$$[n]_{\varphi|_N} = \left[g\varphi(g)^{-1}\right]_{\varphi|_N} = \delta(gN),$$

which proves that $[n]_{\varphi|_N} \in \mathrm{Im}\,\delta$.

Finally, if $n \in N$, then

$$\hat{\pi}(\hat{\imath}([n]_{\varphi|_N})) = \hat{\pi}([n]_\varphi) = [N]_{\bar{\varphi}},$$

so $\mathrm{Im}\,\hat{\imath} \subseteq \ker\hat{\pi}$. Conversely, if $[gN]_{\bar{\varphi}} = [N]_{\bar{\varphi}}$, for some $g \in G$, then $gN = xN\bar{\varphi}(xN)^{-1}$ for some $x \in G$. So, $g = xn\varphi(x)^{-1}$ for some $n \in N$. Therefore, $[g]_\varphi = [n]_\varphi$, which proves that $[g]_\varphi = \hat{\imath}([n]_{\varphi|_N})$. To end the proof, Lemma T.1.5 applied to the square

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"] & G \arrow[d, "\pi"] \\ G/N \arrow[r, "\bar{\varphi}"] & G/N \end{tikzcd}$$

implies that $\hat{\pi}$ is surjective. $\square$

Corollary 1.1.2. *Let G, N and φ be as in the previous proposition. Then*

(1) $R(\varphi) \geq R(\bar{\varphi})$;

(2) if $R(\varphi|_N) = \infty$ and $|\mathrm{Fix}(\bar{\varphi})| < \infty$, then $R(\varphi) = \infty$.

In particular, if N is characteristic, then G has the R_∞-property in the following cases:

(1') G/N has the R_∞-property;

(2') N has the R_∞-property and G/N is finite.

Proof. The first item follows from the surjectivity of $\hat{\pi}$.

For the second item, the aforementioned action of $\mathrm{Fix}(\bar{\varphi})$ on $\mathcal{R}[\varphi|_N]$ has infinitely many orbits of finite size, since $\mathrm{Fix}(\bar{\varphi})$ is finite and $R(\varphi|_N)$ is infinite. For $n, m \in N$, we have that $[n]_\varphi = [m]_\varphi$ if and only if $n = gm\varphi(g)^{-1}$ for some $g \in G$. For such g, we automatically have $N = gN\bar{\varphi}(gN)^{-1}$, i.e. $\bar{\varphi}(gN) = gN$. Hence, $[n]_\varphi = [m]_\varphi$ if and only if $[n]_{\varphi|_N}$ and $[m]_{\varphi|_N}$ lie in the same orbit under the action of $\mathrm{Fix}(\bar{\varphi})$. Consequently, distinct orbits in $\mathcal{R}[\varphi|_N]$ are mapped to distinct elements of $\mathcal{R}[\varphi]$. As there are infinitely many orbits, $R(\varphi) = \infty$. $\square$

The next result was first proven by S. Kim, J. Lee and K. Lee in a topological context [70], and later in an algebraic context by S. Tertooy [117]. We provide here the algebraic proof.

Proposition 1.1.3 (See e.g. [70, Lemma 2.1], [117, Proposition 2.5.13]). *Consider the same situation as in Proposition 1.1.1. Let $g \in G$ and let τ_g be the corresponding inner automorphism. The endomorphism $\tau_g \circ \varphi$ induces the exact sequence*

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Fix}((\tau_g \circ \varphi)|_N) \arrow[r, "i_g"] & \mathrm{Fix}(\tau_g \circ \varphi) \arrow[r, "\pi_g"] & \mathrm{Fix}(\tau_{gN} \circ \bar{\varphi}) \arrow[dlll, "\delta_g"'] \\ & \mathcal{R}[(\tau_g \circ \varphi)|_N] \arrow[r, "\hat{\imath}_g"] & \mathcal{R}[\tau_g \circ \varphi] \arrow[r, "\hat{\pi}_g"] & \mathcal{R}[\tau_{gN} \circ \bar{\varphi}] \arrow[r] & 1. \end{tikzcd}$$

Then the following hold:

(1) $\left|\hat{\pi}^{-1}([gN]_{\bar{\varphi}})\right| = \left|\hat{\pi}_g^{-1}([N]_{\tau_{gN} \circ \bar{\varphi}})\right| = |\mathrm{Im}\, \hat{\imath}_g|$;

(2) $R(\varphi) = \sum_{[hN]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} |\operatorname{Im} \hat{\imath}_h|$*;*

(3) For all $n \in N$, $\left|\hat{\imath}_g^{-1}([n]_{\tau_g \circ \varphi})\right| = [\operatorname{Fix}(\tau_{gN} \circ \bar{\varphi}) : \pi_{ng}(\operatorname{Fix}(\tau_{ng} \circ \varphi))]$*;*

(4) $[G : N] = |[gN]_{\bar{\varphi}}| \cdot |\operatorname{Fix}(\tau_{gN} \circ \bar{\varphi})|$.

In the second item, an infinite sum or a sum with one of its terms equal to ∞ *is to be interpreted as* ∞*. In the forth item, a product of* ∞ *with a positive integer or* ∞ *is to be interpreted as* ∞ *as well.*

Proof. We prove the items one by one.

For the first equality in the first item, we know from Lemma T.1.7(2) that

$$F : \mathcal{R}[\varphi] \to \mathcal{R}[\tau_g \circ \varphi] : [h]_\varphi \mapsto \left[hg^{-1}\right]_{\tau_g \circ \varphi} \tag{1.1.1}$$

is a well-defined bijection. We thus have to argue that F restricts to a bijection between $\hat{\pi}^{-1}([gN]_{\bar{\varphi}})$ and $\hat{\pi}_g^{-1}([N]_{\tau_{gN} \circ \bar{\varphi}})$. This follows, however, from the following chain of equivalences, which hold for all $h \in G$:

$$\begin{aligned}
[h]_\varphi \in \hat{\pi}^{-1}([gN]_{\bar{\varphi}}) &\iff \exists x \in G : hN = xg\varphi^{-1}(x)N \\
&\iff \exists x \in G : hg^{-1}N = xg\varphi^{-1}(x)g^{-1}N \\
&\iff [hg^{-1}N]_{\tau_{gN} \circ \bar{\varphi}} = [N]_{\tau_{gN} \circ \bar{\varphi}} \\
&\iff [hg^{-1}]_{\tau_g \circ \varphi} \in \hat{\pi}_g^{-1}([N]_{\tau_{gN} \circ \bar{\varphi}}).
\end{aligned}$$

The equality $\left|\hat{\pi}_g^{-1}([N]_{\tau_{gN} \circ \bar{\varphi}})\right| = |\operatorname{Im} \hat{\imath}_g|$ follows from the exactness of the sequence.

For the second item, we know that

$$\mathcal{R}[\varphi] = \bigsqcup_{[hN]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} \hat{\pi}^{-1}([hN]_{\bar{\varphi}}).$$

Indeed, since $[g]_\varphi \in \hat{\pi}^{-1}([gN]_{\bar{\varphi}})$ for each $g \in G$, the union fully covers $\mathcal{R}[\varphi]$. The union is disjoint, for $\hat{\pi}([g]_\varphi) = \hat{\pi}([h]_\varphi)$ implies that $[gN]_{\bar{\varphi}} = [hN]_{\bar{\varphi}}$. Thus, applying the first item to each set in the disjoint union, we get

$$R(\varphi) = \sum_{[hN]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} \left|\hat{\pi}^{-1}([hN]_{\bar{\varphi}})\right| = \sum_{[hN]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} |\operatorname{Im} \hat{\imath}_h|.$$

For the third item, let $n \in N$ be arbitrary. First, we show that $\left|\hat{\imath}_g^{-1}([n]_{\tau_g \circ \varphi})\right| = \left|\hat{\imath}_{ng}^{-1}([1]_{\tau_{ng} \circ \varphi})\right|$. Let $h \in N$ be arbitrary. Then the following equivalences hold:

$$\begin{aligned}
[h]_{(\tau_g \circ \varphi)|_N} \in \hat{\imath}_g^{-1}([n]_{\tau_g \circ \varphi}) &\iff \exists x \in G : h = xng\varphi(x)^{-1}g^{-1} \\
&\iff \exists x \in G : hn^{-1} = xng\varphi(x)^{-1}g^{-1}n^{-1} \\
&\iff \left[hn^{-1}\right]_{\tau_{ng} \circ \varphi} = [1]_{\tau_{ng} \circ \varphi} \\
&\iff \left[hn^{-1}\right]_{(\tau_{ng} \circ \varphi)|_N} \in \hat{\imath}_{ng}^{-1}([1]_{\tau_{ng} \circ \varphi})
\end{aligned}$$

Consequently, the map

$$\hat{\imath}_g^{-1}([n]_{\tau_g \circ \varphi}) \to \hat{\imath}_{ng}^{-1}([1]_{\tau_{ng} \circ \varphi}) : [h]_{(\tau_g \circ \varphi)|_N} \mapsto \left[hn^{-1}\right]_{(\tau_{ng} \circ \varphi)|_N}$$

is a bijection. Next, by exactness, $\hat{\imath}_{ng}^{-1}([1]_{\tau_{ng} \circ \varphi}) = \mathrm{Im}(\delta_{ng})$. To compute $|\mathrm{Im}(\delta_{ng})|$, note that, for all $xN, yN \in \mathrm{Fix}(\tau_{gN} \circ \bar{\varphi})$, the following equivalences hold:

$$\begin{aligned}
&\delta_{ng}(xN) = \delta_{ng}(yN) \\
\iff& \left[x(\tau_{ng} \circ \varphi)(x)^{-1}\right]_{(\tau_{ng} \circ \varphi)|_N} = \left[y(\tau_{ng} \circ \varphi)(y)^{-1}\right]_{(\tau_{ng} \circ \varphi)|_N} \\
\iff& \exists z \in N : x(\tau_{ng} \circ \varphi)(x)^{-1} = zy(\tau_{ng} \circ \varphi)(y)^{-1}(\tau_{ng} \circ \varphi)(z)^{-1} \\
\iff& \exists z \in N : y^{-1}z^{-1}x = (\tau_{ng} \circ \varphi)(y)^{-1}(\tau_{ng} \circ \varphi)(z)^{-1}(\tau_{ng} \circ \varphi)(x) \\
\iff& \exists z \in N : y^{-1}z^{-1}x \in \mathrm{Fix}(\tau_{ng} \circ \varphi) \\
\iff& y^{-1}xN \in \pi_{ng}(\mathrm{Fix}(\tau_{ng} \circ \varphi))
\end{aligned}$$

Consequently, $\delta_{ng}(xN) = \delta_{ng}(yN)$ if and only if xN and yN represent the same coset of $\pi_{ng}(\mathrm{Fix}(\tau_{ng} \circ \varphi))$ in $\mathrm{Fix}(\tau_{gN} \circ \bar{\varphi})$, which implies that $|\mathrm{Im}\,\delta_{ng}| = [\mathrm{Fix}(\tau_{gN} \circ \bar{\varphi}) : \pi_{ng}(\mathrm{Fix}(\tau_{ng} \circ \varphi))]$.

Finally, for the fourth item, recall that $\bar{\varphi}$-conjugacy can be seen as an action of G/N on itself. The orbit-stabiliser theorem then yields

$$[G : N] = |[gN]_{\bar{\varphi}}| \cdot |\mathrm{Stab}_{\bar{\varphi}}(gN)|.$$

As $\mathrm{Stab}_{\bar{\varphi}}(gN) = \mathrm{Fix}(\tau_{gN} \circ \bar{\varphi})$ by Lemma T.1.10, the desired equality follows. $\square$

Proposition 1.1.4 (See e.g. [117, Proposition 2.5.14]). *Let G be a group and N a finite index normal subgroup. Suppose that $\varphi \in \mathrm{End}(G)$ satisfies $\varphi(N) \leq N$. Let $\varphi|_N$ and $\bar{\varphi}$ denote the induced endomorphisms on N and G/N, respectively. Then*

$$\frac{1}{[G:N]} \sum_{gN \in G/N} R((\tau_g \circ \varphi)|_N) \leq R(\varphi) \leq \sum_{gN \in G/N} R((\tau_g \circ \varphi)|_N).$$

Proof. We start from Proposition 1.1.3(2) and rewrite the summation:

$$\begin{aligned}
R(\varphi) &= \sum_{[gN]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} |\mathrm{Im}\, \hat{\imath}_g| \\
&= \sum_{[gN]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} \sum_{[n]_{\tau_g \circ \varphi} \in \mathrm{Im}\, \hat{\imath}_g} 1 \\
&= \sum_{gN \in G/N} \frac{1}{|[gN]_{\bar{\varphi}}|} \sum_{[n]_{(\tau_g \circ \varphi)|_N} \in \mathcal{R}[(\tau_g \circ \varphi)|_N]} \frac{1}{|\hat{\imath}_g^{-1}([n]_{\tau_g \circ \varphi})|} \\
&= \sum_{gN \in G/N} \frac{|\mathrm{Fix}(\tau_{gN} \circ \bar{\varphi})|}{[G:N]} \sum_{[n]_{(\tau_g \circ \varphi)|_N} \in \mathcal{R}[(\tau_g \circ \varphi)|_N]} \frac{|\pi_{ng}(\mathrm{Fix}(\tau_{ng} \circ \varphi))|}{|\mathrm{Fix}(\tau_{gN} \circ \bar{\varphi})|} \\
&= \frac{1}{[G:N]} \sum_{gN \in G/N} \sum_{[n]_{(\tau_g \circ \varphi)|_N} \in \mathcal{R}[(\tau_g \circ \varphi)|_N]} |\pi_{ng}(\mathrm{Fix}(\tau_{ng} \circ \varphi))|.
\end{aligned}$$

Note that both $|\pi_{ng}(\mathrm{Fix}(\tau_{ng} \circ \varphi))|$ and $|\mathrm{Fix}(\tau_{gN} \circ \bar{\varphi})|$ are finite by Proposition 1.1.3(3) and Proposition 1.1.3(4)

As clearly $|\pi_{ng}(\mathrm{Fix}(\tau_{ng} \circ \varphi))| \geq 1$ for all $g \in G$ and $n \in N$, we get

$$R(\varphi) \geq \frac{1}{[G:N]} \sum_{gN \in G/N} \sum_{[n]_{(\tau_g \circ \varphi)|_N} \in \mathcal{R}[(\tau_g \circ \varphi)|_N} 1 = \frac{1}{[G:N]} \sum_{gN \in G/N} R((\tau_g \circ \varphi)|_N).$$

On the other hand, $|\pi_{ng}(\mathrm{Fix}(\tau_{ng} \circ \varphi))| \leq |\mathrm{Fix}(\tau_{gN} \circ \bar{\varphi})| \leq [G:N]$ by Proposition 1.1.3(4). Therefore,

$$R(\varphi) \leq \frac{1}{[G:N]} \sum_{gN \in G/N} \sum_{[n]_{(\tau_g \circ \varphi)|_N} \in \mathcal{R}[(\tau_g \circ \varphi)|_N} [G:N] = \sum_{gN \in G/N} R((\tau_g \circ \varphi)|_N),$$

which proves the second inequality. $\square$

Corollary 1.1.5 (See e.g. [117, Corollary 2.5.15])**.** *Let G be a group and N a finite index normal subgroup. Suppose that $\varphi \in \mathrm{End}(G)$ satisfies $\varphi(N) \leq N$. Let $\varphi|_N$ and $\bar{\varphi}$ denote the induced endomorphisms on N and G/N, respectively.*

Then $R(\varphi) = \infty$ if and only if $R((\tau_g \circ \varphi)|_N) = \infty$ for some $gN \in G/N$.

Example 1.1.6 (See e.g. [50, Proposition 2.3])**.** Consider the infinite dihedral group $D_\infty := \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ acts via inversion. We claim that $\mathbb{Z}$ is characteristic. Let t be a generator of $\mathbb{Z}/2\mathbb{Z}$ and let $z \in \mathbb{Z}$. Then

$$(z, t)^2 = (z, t)(z, t) = (z + t \cdot z, t^2) = (z - z, 1) = (0, 1).$$

Therefore, any element outside of $\mathbb{Z}$ has finite order. Consequently, $\mathbb{Z}$ is characteristic.

Now, let $\varphi \in \mathrm{Aut}(D_\infty)$. Let $\varphi|_\mathbb{Z}$ be the induced automorphism on $\mathbb{Z}$. Note that

$$\tau_t(z) = z^t = -z$$

for all $z \in \mathbb{Z}$. Therefore, $\tau_t|_\mathbb{Z}$ is the inversion map on $\mathbb{Z}$. Since $\varphi|_\mathbb{Z}$ is either the identity map or the inversion map, $\varphi|_\mathbb{Z}$ or $\tau_t|_\mathbb{Z} \circ \varphi|_\mathbb{Z}$ is the identity map. Thus, one of the two has infinite Reidemeister number. By Corollary 1.1.5, φ has infinite Reidemeister number. We conclude that D_∞ has the R_∞-property. ‖

Besides the general formulae from Proposition 1.1.3, there are two instances where we can express the Reidemeister number on G in terms of Reidemeister numbers on the subgroup N. The first one is often called the averaging formula.

Proposition 1.1.7 (See e.g. [117, Proposition 2.5.16])**.** *Let G be a torsion-free group and N a finite index normal subgroup. Let $\varphi \in \mathrm{End}(G)$ be such that $\varphi(N) \leq N$. Suppose that $\mathrm{Fix}((\tau_g \circ \varphi)|_N) = 1$ for all $g \in G$. Then*

$$R(\varphi) = \frac{1}{[G : N]} \sum_{gN \in G/N} R((\tau_g \circ \varphi)|_N).$$

Proof. We derived the lower bound using the inequality $|\pi_{ng}(\mathrm{Fix}(\tau_{ng} \circ \varphi))| \geq 1$ for all $g \in G$ and $n \in N$. We prove that equality holds in this case, by proving that $\mathrm{Fix}(\tau_g \circ \varphi) = 1$ for all $g \in G$.

So, suppose that $h \in \mathrm{Fix}(\tau_g \circ \varphi)$. Since G/N is finite, there is an $m \in \mathbb{N}_0$ such that $h^m \in N$. Then $(\tau_g \circ \varphi)|_N(h^m) = (\tau_g \circ \varphi)(h)^m = h^m$. This implies that $h^m \in \mathrm{Fix}((\tau_g \circ \varphi)|_N)$. Consequently, $h^m = 1$, and thus $h = 1$, as G is torsion-free. □

The second one is often called the addition formula.

Proposition 1.1.8 (See e.g. [127, Proposition 1])**.** *Let G be a group and N a normal subgroup. Let $\varphi \in \mathrm{End}(G)$ be such that $\varphi(N) \leq N$. Suppose that* $\mathrm{Fix}(\tau_{gN} \circ \bar{\varphi}) = 1$ *for all $g \in G$. Then*

$$R(\varphi) = \sum_{[gN]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} R((\tau_g \circ \varphi)|_N).$$

Proof. Let $g \in G$ be arbitrary. As $\mathrm{Fix}(\tau_{gN} \circ \bar{\varphi}) = 1$, Proposition 1.1.3(3) implies that $\left|\hat{\imath}_g^{-1}([n]_{\tau_g \circ \varphi})\right| = 1$ for all $n \in N$. Therefore, $\hat{\imath}_g$ is injective, which implies that $|\mathrm{Im}\, \hat{\imath}_g| = R((\tau_g \circ \varphi)|_N)$. Combining this with Proposition 1.1.3(2), we get the desired equality. □

While there are not many groups with full Reidemeister spectrum known, there is a relatively mild sufficient condition for a group to have a full extended Reidemeister spectrum. The proof is essentially the same as the one K. Dekimpe, S. Tertooy and A. Vargas give for finitely generated torsion-free nilpotent groups [26, § 6] (see also [117, Theorem 5.2.1]).

Theorem 1.1.9. *Let G be a group. Suppose that G admits a surjective homomorphism to $\mathbb{Z}$. Then G has full extended Reidemeister spectrum.*

Proof. As G admits a surjective homomorphism to $\mathbb{Z}$, G fits in an exact sequence

$$1 \to K \to G \to \mathbb{Z} \to 1$$

for some (normal) subgroup $K \leq G$. As $\mathbb{Z}$ is a free group, this sequence splits, which means that $G \cong K \rtimes \mathbb{Z}$. Without loss of generality, we can assume that G is in fact equal to $K \rtimes \mathbb{Z}$.

Let

$$\pi : K \rtimes \mathbb{Z} \to \mathbb{Z} : (k, z) \mapsto z$$

and

$$\iota : \mathbb{Z} \to K \rtimes \mathbb{Z} : z \mapsto (1, z)$$

be the canonical projection and inclusion map, respectively. Let $n \in \mathbb{Z}$. Define $\varphi_n : \mathbb{Z} \to \mathbb{Z} : x \mapsto nx$, and, finally, put $\psi_n := \iota \circ \varphi_n \circ \pi$. This yields an endomorphism of G, for which we claim that $R(\psi_n) = |n - 1|_\infty$ holds.

Let $(k_1, z_1), (k_2, z_2) \in K \rtimes \mathbb{Z}$ be arbitrary. The following equivalences hold:

$$\begin{aligned}
(k_1, z_1) \sim_{\psi_n} (k_2, z_2) &\iff \exists (k, z) \in K \rtimes \mathbb{Z} : (k_1, z_1) = (k, z)(k_2, z_2)\varphi_n(k, z)^{-1} \\
&\iff \exists (k, z) \in K \rtimes \mathbb{Z} : (k_1, z_1) = (k, z)(k_2, z_2)(1, nz)^{-1} \\
&\iff \exists (k, z) \in K \rtimes \mathbb{Z} : k_1 = kzk_2z^{-1}, z_1 = z + z_2 - nz.
\end{aligned}$$

Given $z \in \mathbb{Z}$, we can always find a $k \in K$ such that $k_1 = kzk_2z^{-1}$. Therefore, the last statement is equivalent with

$$\exists z \in \mathbb{Z} : z_1 = z + z_2 - nz.$$

In other words, $(k_1, z_1) \sim_{\psi_n} (k_2, z_2)$ if and only if $z_1 \sim_{\varphi_n} z_2$. Therefore, $R(\psi_n) = |n-1|_\infty$. Since n is arbitrary, this proves that $\mathbb{N}_0 \cup \{\infty\} \subseteq \mathrm{ESpec_R}(G)$. $\square$

1.2 Central extensions

For central extensions, we can obtain additional information regarding the Reidemeister number, ranging from an upper bound to an exact product formula in specific cases.

We fix some notation for this section. Let G be a group and $C \leq Z(G)$ a central subgroup. Let $\varphi \in \mathrm{End}(G)$ be such that $\varphi(C) \leq C$. Let $\varphi|_C$ and $\bar{\varphi}$ denote the induced endomorphisms on C and G/C, respectively. By Proposition 1.1.3(2),

$$R(\varphi) = \sum_{[gC]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} |\mathrm{Im}\, \hat{\imath}_g|,$$

where $\hat{\imath}_g : \mathcal{R}[(\tau_g \circ \varphi)|_C] \to \mathcal{R}[\tau_g \circ \varphi]$. Now, since C is central, τ_g restricted to C is the identity map for each $g \in G$. Therefore, the domain of $\hat{\imath}_g$ is $\mathcal{R}[\varphi|_C]$ for each $g \in G$.

Let $g \in G$ and $z_1, z_2 \in C$ be arbitrary. Then we have the following chain of equivalences:

$$\begin{aligned} \hat{\imath}_g([z_1]_{\varphi|_C}) = \hat{\imath}_g([z_2]_{\varphi|_C}) &\iff \exists x \in G : z_1 = xz_2g\varphi(x)^{-1}g^{-1} \\ &\iff \exists x \in G : z_1 = z_2g^{-1}xg\varphi(x)^{-1}, \end{aligned}$$

where we conjugate by g in the last equivalence and simultaneously use that C is central. Furthermore, if we project the last equality down to G/C, we find $C = g^{-1}xg\varphi(x)^{-1}C$, or, equivalently, $gC = xg\varphi(x)^{-1}C$. Thus, $xC \in \mathrm{Stab}_{\bar{\varphi}}(gC)$. Therefore,

$$\begin{aligned} &\exists x \in G : z_1 = z_2g^{-1}xg\varphi(x)^{-1} \\ \iff &\exists x \in \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC)) : z_1 = z_2g^{-1}xg\varphi(x)^{-1}, \end{aligned}$$

where $\pi : G \to G/C$ is the canonical projection. Thus, if we define the *φ-twisted commutator of $g, x \in G$* as $[g, x]^\varphi := g^{-1}x^{-1}g\varphi(x)$, we see that $\hat{\imath}_g([z_1]_{\varphi|_C}) = \hat{\imath}_g([z_2]_{\varphi|_C})$ if and only if $z_1 = [g, x]^\varphi z_2$ for some $x \in \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))$.

Lemma 1.2.1. *For each $g \in G$, the set*

$$[g, \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))]^{\varphi} := \{[g, x]^{\varphi} \mid x \in \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))\}$$

is a subgroup of C.

Proof. First, if $x \in \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))$, then $gC = x^{-1}g\varphi(x)C$ and thus $C = g^{-1}x^{-1}g\varphi(x)C = [g,x]^{\varphi}C$, which shows that $[g,x]^{\varphi} \in C \leq Z(G)$. It follows that $h[g,x]^{\varphi}h^{-1} = [g,x]^{\varphi}$ for all $h \in G$ and $x \in \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))$. Consequently,

$$\begin{aligned}
([g,x]^{\varphi})^{-1} &= \varphi(x)([g,x]^{\varphi})^{-1}\varphi(x)^{-1} \\
&= \varphi(x)\varphi(x)^{-1}g^{-1}xg\varphi(x)^{-1} \\
&= g^{-1}xg\varphi(x)^{-1} \\
&= \left[g, x^{-1}\right]^{\varphi}.
\end{aligned}$$

If $x_1, x_2 \in \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))$, note first that $x_1x_2 \in \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))$, as $\mathrm{Stab}_{\bar{\varphi}}(gC)$ is a subgroup. Now,

$$\begin{aligned}
[g,x_1]^{\varphi}[g,x_2]^{\varphi} &= g^{-1}x_1^{-1}g\varphi(x_1)g^{-1}x_2^{-1}g\varphi(x_2) \\
&= g^{-1}\left(x_1^{-1}g\varphi(x_1)g^{-1}\right)x_2^{-1}g\varphi(x_2) \\
&= g^{-1}x_2^{-1}\left(x_1^{-1}g\varphi(x_1)g^{-1}\right)g\varphi(x_2) \\
&= g^{-1}x_2^{-1}x_1^{-1}g\varphi(x_1)\varphi(x_2) \\
&= g^{-1}(x_1x_2)^{-1}g\varphi(x_1x_2) \\
&= [g, x_1x_2]^{\varphi},
\end{aligned}$$

since $x_1^{-1}g\varphi(x_1)g^{-1} = g([g,x_1]^{\varphi})g^{-1} \in C \leq Z(G)$. Therefore, the set $[g, \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))]^{\varphi}$ is closed under taking products and inverses. Hence, it is a subgroup. $\square$

Corollary 1.2.2. *For each $g \in G$, the map*

$$\lambda_g : [g, \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))]^{\varphi} \times \mathcal{R}[\varphi|_C] \to \mathcal{R}[\varphi|_C] : (x, [z]_{\varphi|_C}) \mapsto [xz]_{\varphi|_C}$$

is a well-defined left action.

Proof. Since $[g, \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))]^{\varphi}$ is a subgroup of C, the product xz again lies in C for $x \in [g, \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))]^{\varphi}$ and $[z]_{\varphi|_C} \in \mathcal{R}[\varphi|_C]$.

Next, suppose that $[z]_{\varphi|_C} = [y]_{\varphi|_C}$. Then $z = wy\varphi|_C(w)^{-1}$ for some $w \in C$. Let $x \in [g, \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))]^{\varphi}$. Then

$$xz = xwy\varphi|_C(w)^{-1} = wxy\varphi|_C(w)^{-1},$$

which shows that $[xz]_{\varphi|_C} = [xy]_{\varphi|_C}$.

Finally, if $x_1, x_2 \in [g, \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))]^{\varphi}$ and $[z]_{\varphi|_C} \in \mathcal{R}[\varphi|_C]$, it is immediate that

$$\lambda_g(x_1, \lambda_g(x_2, [z]_{\varphi|_C})) = \lambda_g(x_1x_2, [z]_{\varphi|_C}),$$

which proves that λ_g is a left action. □

Theorem 1.2.3. *With the notation as above,*

$$R(\varphi) = \sum_{[gC]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} \#\text{Orbits of } \lambda_g.$$

In particular, $R(\varphi) \leq R(\varphi|_C)R(\bar{\varphi})$.

Proof. We combine everything discussed so far, starting from Proposition 1.1.3(2):

$$R(\varphi) = \sum_{[gC]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} |\mathrm{Im}\,\hat{\imath}_g|.$$

We have argued that, for $g \in G$ and $z_1, z_2 \in C$, $\hat{\imath}_g([z_1]_{\varphi|_C}) = \hat{\imath}_g([z_2]_{\varphi|_C})$ if and only if $z_1 = xz_2$ for some $x \in [g, \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))]^{\varphi}$. We now argue that this is equivalent with $[z_1]_{\varphi|_C} = [yz_2]_{\varphi|_C}$ for some $y \in [g, \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))]^{\varphi}$.

One implication is immediate. For the other, suppose that $z_1 = cyz_2\varphi|_C(c)^{-1}$ for some $c \in C$ and $y \in [g, \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))]^{\varphi}$. Now note that

$$cyz_2\varphi|_C(c)^{-1} = yc\varphi(c)^{-1}z_2 = y[g, c^{-1}]^{\varphi}z_2,$$

since $g^{-1}cg\varphi(c)^{-1} = c\varphi(c)^{-1}$ as C is central. As $[g, \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))]^{\varphi}$ is a subgroup, $z_1 = xz_2$ for some $x \in [g, \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))]^{\varphi}$.

Thus, we have proven that $\hat{\imath}_g([z_1]_{\varphi|_C}) = \hat{\imath}_g([z_2]_{\varphi|_C})$ if and only if $[z_1]_{\varphi|_C} = [xz_2]_{\varphi|_C}$ for some $x \in [g, \pi^{-1}(\mathrm{Stab}_{\bar{\varphi}}(gC))]^{\varphi}$; in other words, if and only if $[z_1]_{\varphi|_C}$ and $[z_2]_{\varphi|_C}$ lie in the same orbit of λ_g. Thus, $|\mathrm{Im}\,\hat{\imath}_g| = \#\text{Orbits of } \lambda_g$, which proves the sum formula.

Since λ_g acts on $\mathcal{R}[\varphi|_C]$ for each $g \in G$, the number of orbits of each λ_g is at most $R(\varphi|_C)$. This implies that $R(\varphi) \leq R(\varphi|_C)R(\bar{\varphi})$. □

Remark. This theorem can also be proven by starting from (fixed) representatives $\{z_i\}_{i\in\mathcal{I}}$ of $\mathcal{R}[\varphi|_C]$ and $\{g_jC\}_{j\in\mathcal{J}}$ of $\mathcal{R}[\bar{\varphi}]$, showing that $\mathcal{R}[\varphi] = \{[z_ig_j]_\varphi \mid i \in \mathcal{I}, j \in \mathcal{J}\}$ and then determining when $z_ig_j \sim_\varphi z_kg_l$ for $i,k \in \mathcal{I}, j,l \in \mathcal{J}$. This approach yields the same action λ_g.

Example 1.2.4. In contrast to what is stated in [50, Lemma 1.1(3)], equality in $R(\varphi) \leq R(\varphi|_C)R(\bar{\varphi})$ does not always hold. Consider the group $\mathbb{Z}/4\mathbb{Z}$ with subgroup C generated by $\bar{2}$. These fit in the exact sequence

$$0 \to C \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

As $\mathbb{Z}/4\mathbb{Z}$ is abelian, C is central. Also, C is characteristic in $\mathbb{Z}/4\mathbb{Z}$. Let $\varphi : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} : x \mapsto -x$ be the inversion map. As $\mathbb{Z}/2\mathbb{Z}$ has only one automorphism, both $\varphi|_C$ and $\bar{\varphi}$ are the identity map, so $R(\varphi|_C) = R(\bar{\varphi}) = 2$.

On the other hand, we prove that $R(\varphi)$ equals 2 as well. Let $x, y \in \mathbb{Z}/4\mathbb{Z}$ be arbitrary. Then $x \sim_\varphi y$ if and only if $x = z + y - (-z) = y + 2z$ for some $z \in \mathbb{Z}/4\mathbb{Z}$. In other words, $x \sim_\varphi y$ if and only if $x - y \in C$. Therefore, $R(\varphi) = [\mathbb{Z}/4\mathbb{Z} : C] = 2$. Thus, the inequality reads

$$R(\varphi) = 2 < 4 = R(\varphi|_C)R(\bar{\varphi}).$$

The formula in Theorem 1.2.3 can theoretically be used to compute Reidemeister numbers, but doing so is generally work-intensive. However, for finitely generated torsion-free residually finite groups, the formula simplifies to equality in the stated inequality.

Proposition 1.2.5. *Let G be a group and C a central subgroup such that G/C is finitely generated torsion-free residually finite. Let $\varphi \in \mathrm{End}(G)$ be such that $\varphi(C) \leq C$. Let $\varphi|_C$ and $\bar{\varphi}$ denote the induced endomorphisms on C and G/C, respectively. Then*

$$R(\varphi) = R(\varphi|_C)R(\bar{\varphi}).$$

Before proving this result, we first study the behaviour of twisted conjugacy in finitely generated residually finite groups.

Lemma 1.2.6 (See e.g. [59, Lemma 2.2])**.** *Let G be a finitely generated group and H a subgroup of finite index. Then H contains a subgroup K that is fully characteristic in G and has finite index in G. In particular, if H is a proper subgroup of G, then so is K.*

Proof. Put $n = [G : H]$. Since G is finitely generated, G only has finitely many subgroups of index k for a fixed $k \in \mathbb{N}_0$. Therefore, the set

$$X := \{L \leq G \mid [G : L] \leq n\}$$

is finite. Define
$$K := \bigcap_{L \in X} L.$$
Since $H \in X$, we immediately have $K \leq H$. Being an intersection of finitely many groups of finite index, K has finite index in G. Next, for arbitrary $\varphi \in \operatorname{End}(G)$ and $L \in X$,

$$\begin{aligned}
[G : L] &\geq [\operatorname{Im}\varphi : \operatorname{Im}\varphi \cap L] \\
&= \left[\frac{G}{\ker\varphi} : \frac{\varphi^{-1}(\operatorname{Im}\varphi \cap L)}{\ker\varphi}\right] \\
&= \left[\frac{G}{\ker\varphi} : \frac{\varphi^{-1}(L)}{\ker\varphi}\right] \\
&= [G : \varphi^{-1}(L)],
\end{aligned}$$

which proves that $\varphi^{-1}(L) \in X$ as well (the map $X \to X : L \mapsto \varphi^{-1}(L)$ does not need to be surjective, however). Therefore,
$$\varphi^{-1}(K) = \bigcap_{L \in X} \varphi^{-1}(L) \geq \bigcap_{L \in X} L = K.$$
Applying φ to both sides, we get $\varphi(\varphi^{-1}(K)) \geq \varphi(K)$, and since $\varphi(\varphi^{-1}(K)) \leq K$ in general, we obtain $\varphi(K) \leq K$. As φ was arbitrary, K is fully characteristic. $\square$

Put $\zeta_1 := 1$ and define $\zeta_{i+1} := \zeta_i(\zeta_i + 1)$ for $i \geq 1$. The following is mentioned in [91, p. 109], and is proven in [16]:

Proposition 1.2.7 (See e.g. [16])**.** *Let $n \geq 1$ be an integer and suppose that $a_1 \leq \ldots \leq a_n$ are positive integers such that*
$$\sum_{i=1}^{n} \frac{1}{a_i} = 1.$$
Then $a_n \leq \zeta_n$.

Lemma 1.2.8. *Let $n \geq 1$. Then $\zeta_n \leq 2^{2^{n-1}}$.*

Proof. For $n = 1$, this is immediate. So, assume that $n \geq 2$. We first prove by induction on n that
$$\zeta_n = \prod_{i=1}^{n-1} (\zeta_i + 1).$$

For $n = 2$ this is trivial. Suppose the equality holds for n. Then

$$\zeta_{n+1} = \zeta_n(\zeta_n + 1) = \left(\prod_{i=1}^{n-1}(\zeta_i + 1)\right)(\zeta_n + 1) = \prod_{i=1}^{n}(\zeta_i + 1).$$

Now, we prove the lemma, again by induction on n, the case $n = 2$ being clear. Suppose it holds for 1 up to $n-1$. Then

$$\begin{aligned}\zeta_n &= \prod_{i=1}^{n-1}(\zeta_i + 1)\\ &\leq \prod_{i=1}^{n-1}\left(2^{2^{i-1}} + 1\right).\end{aligned}$$

We finish the proof by showing, by induction on n yet again, that

$$\prod_{i=1}^{n-1}\left(2^{2^{i-1}} + 1\right) = 2^{2^{n-1}} - 1.$$

For $n = 2$, this is immediate. Suppose it holds for n. Then

$$\prod_{i=1}^{n}\left(2^{2^{i-1}} + 1\right) = \left(2^{2^{n-1}} - 1\right)\left(2^{2^{n-1}} + 1\right) = \left(2^{2^{n-1}}\right)^2 - 1 = 2^{2^n} - 1.$$

This concludes the proof. $\square$

Combining this lemma with Proposition 1.2.7 and the trivial inequality $2^{2^{n-1}} \leq 2^{2^n}$, we obtain the following:

Corollary 1.2.9. *Let $n \geq 1$ be an integer and suppose that $a_1 \leq \ldots \leq a_n$ are positive integers such that*

$$\sum_{i=1}^{n} \frac{1}{a_i} = 1.$$

Then $a_n \leq 2^{2^n}$.

Lemma 1.2.10. *Let G be a finite group and $\varphi \in \mathrm{End}(G)$. Put $R(\varphi) = r$. Then $|\mathrm{Fix}(\varphi)| \leq 2^{2^r}$.*

Proof. Let $1 = g_1, \ldots, g_r$ represent the r distinct φ-conjugacy classes. The orbit-stabiliser theorem then yields

$$|G| = \sum_{i=1}^{r} \frac{|G|}{|\mathrm{Stab}_\varphi(g_i)|}.$$

This simplifies to

$$1 = \sum_{i=1}^{r} \frac{1}{|\mathrm{Stab}_\varphi(g_i)|}.$$

Put $m = \max\{|\mathrm{Stab}_\varphi(g_i)| \mid i \in \{1, \ldots, r\}\}$. Corollary 1.2.9 then yields

$$m \leq 2^{2^r}.$$

In particular,

$$|\mathrm{Fix}(\varphi)| = |\mathrm{Stab}_\varphi(1)| \leq m \leq 2^{2^r}. \qquad \square$$

The next result can implicitly be found in [60] for automorphisms. Using Lemma 1.2.6, we can generalise this to endomorphisms as well.

Proposition 1.2.11. *Let G be a finitely generated residually finite group. Let $\varphi \in \mathrm{End}(G)$. If $\mathrm{Stab}_\varphi(g)$ is infinite for some $g \in G$, then $R(\varphi) = \infty$.*

Proof. First, recall that $\mathrm{Stab}_\varphi(g) = \mathrm{Fix}(\tau_g \circ \varphi)$ by Lemma T.1.10. By the second item of Lemma T.1.7(2), $R(\tau_g \circ \varphi) = R(\varphi)$. Therefore, by switching to $\tau_g \circ \varphi$, we may assume that $\mathrm{Fix}(\varphi)$ is infinite.

Fix $n \geq 1$ and let $g_1, \ldots, g_n$ be (distinct) elements in $\mathrm{Fix}(\varphi)$. Since G is residually finite, we can find a finite index normal subgroup N such that $x_1N, \ldots, x_nN$ are all distinct. Since G is finitely generated, we can find, using Lemma 1.2.6, a fully characteristic subgroup K of finite index contained in N such that $x_1K, \ldots, x_nK$ are all distinct. Let $\bar{\varphi}$ denote the induced endomorphism on G/K. Then $\{x_1K, \ldots, x_nK\} \subseteq \mathrm{Fix}(\bar{\varphi})$, which implies that

$$|\mathrm{Fix}(\bar{\varphi})| \geq n.$$

By Lemma 1.2.10, $|\mathrm{Fix}(\bar{\varphi})| \leq 2^{2^{R(\bar{\varphi})}}$, which implies that

$$R(\bar{\varphi}) \geq \log_2(\log_2(|\mathrm{Fix}(\bar{\varphi})|)) \geq \log_2(\log_2(n)).$$

By Corollary 1.1.2, $R(\varphi) \geq R(\bar{\varphi})$. Combining everything, we get

$$R(\varphi) \geq \log_2(\log_2(n)).$$

As n was arbitrary and $\log_2(\log_2(n))$ tends to infinity as n does, we obtain the equality $R(\varphi) = \infty$. $\square$

Corollary 1.2.12. *Let G be a finitely generated residually finite group. Suppose that G is infinite. Then $\infty \in \mathrm{Spec}_\mathrm{R}(G)$.*

Proof. The identity map has an infinite number of fixed points, so $\mathrm{Stab}_{\mathrm{Id}}(1) = \mathrm{Fix}(\varphi)$ is infinite. Therefore, $R(\mathrm{Id}) = \infty$. $\square$

Corollary 1.2.13. *Let G be a finitely generated torsion-free residually finite group and let $\varphi \in \mathrm{End}(G)$. If $R(\varphi) < \infty$, then $\mathrm{Stab}_\varphi(g)$ is trivial for all $g \in G$.*

Proof. Proposition 1.2.11 implies that $\mathrm{Stab}_\varphi(g)$ is finite for all $g \in G$. As twisted stabilisers are subgroups, they have to be trivial, since G is torsion-free. □

The condition that G is finitely generated cannot be dropped. In fact, there already exists a non-finitely generated torsion-free residually finite abelian group admitting an automorphism with finite Reidemeister number and non-trivial twisted stabilisers.

Example 1.2.14 (Based on [18, Proposition 3.6])**.** Consider the direct sum A of countably many copies of $\mathbb{Z}$, indexed by the positive integers, i.e.

$$A = \bigoplus_{n=1}^{\infty} \mathbb{Z},$$

and define $\varphi : A \to A$ as

$$(a_1, a_2, a_3, a_4, \ldots) \mapsto (a_1 + a_2 + a_3, a_2 + a_3, a_3 + a_4 + a_5, a_4 + a_5, \ldots).$$

In other words, the $(2k-1)$th component is given by $a_{2k-1} + a_{2k} + a_{2k+1}$ and the $2k$th by $a_{2k} + a_{2k+1}$. This map is an endomorphism of A, and even an automorphism since the map $\psi : A \to A$ defined as

$$(a_1, a_2, a_3, a_4, \ldots) \mapsto (a_1 - a_2, a_2 - a_3 + a_4, a_3 - a_4, a_4 - a_5 + a_6, \ldots)$$

is the inverse of φ. The map φ has non-trivial fixed points, for instance $(1, 0, 0, \ldots)$. However, $R(\varphi) = 1$, since

$$[(0, 0, \ldots)]_\varphi = \{a - \varphi(a) \mid a \in A\} = \mathrm{Im}(\varphi - \mathrm{Id})$$

and $\varphi - \mathrm{Id} : A \to A$ is given by

$$(a_1, a_2, a_3, a_4, \ldots) \mapsto (a_2 + a_3, a_3, a_4 + a_5, a_5, \ldots),$$

which is clearly surjective. ‖

Proof of Proposition 1.2.5. If $R(\bar{\varphi}) = \infty$, then $R(\varphi) = \infty$ as well, so the product formula holds. Therefore, assume that $R(\bar{\varphi}) < \infty$. Then by Proposition 1.2.11, the twisted stabilisers of $\bar{\varphi}$ are trivial. Consequently, the action λ_g in Theorem 1.2.3 is trivial for each $g \in G$, which means that each λ_g has $R(\varphi|_C)$ orbits. We conclude that $R(\varphi) = R(\varphi|_C)R(\bar{\varphi})$.

Alternatively, we can apply Proposition 1.1.8 if $R(\bar{\varphi}) < \infty$. In that case, the twisted stabilisers of $\bar{\varphi}$ are trivial, so by Lemma T.1.10, $\operatorname{Fix}(\tau_{gC} \circ \bar{\varphi}) = 1$ for all $g \in G$. Consequently, by Proposition 1.1.8,

$$R(\varphi) = \sum_{[gC]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} R((\tau_g \circ \varphi)|_C).$$

Since C is central, τ_g restricted to C is the identity map for each $g \in G$. Consequently, the summation above simplifies to

$$R(\varphi) = \sum_{[gC]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} R(\varphi|_C) = R(\varphi|_C) R(\bar{\varphi}). \qquad \square$$

Corollary 1.2.15. *Let G be a finitely generated residually finite group such that $G/Z(G)$ is torsion-free. Let $\varphi \in \operatorname{Aut}(G)$. Write $\varphi|_{Z(G)}$ and $\bar{\varphi}$ for the induced automorphisms on $Z(G)$ and $G/Z(G)$, respectively. Then $R(\varphi) = R(\varphi|_{Z(G)}) R(\bar{\varphi})$.*

Proof. Since G is finitely generated residually finite, $\operatorname{Aut}(G)$ is residually finite as well (see e.g. [12, Theorem 2.5.1]). As subgroups of residually finite groups are residually finite, $G/Z(G) \cong \operatorname{Inn}(G) \leq \operatorname{Aut}(G)$ is residually finite. By assumption, $G/Z(G)$ is torsion-free, and it is also finitely generated, being a quotient of G. Therefore, Proposition 1.2.5 applies and we derive $R(\varphi) = R(\varphi|_{Z(G)}) R(\bar{\varphi})$. $\square$

Example 1.2.16. Consider the semi-direct product $G := \mathbb{Z}^3 \rtimes_A \mathbb{Z}$, where

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 5 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

We write the elements in G as (v, l), where $v \in \mathbb{Z}^3$ is a column vector and $l \in \mathbb{Z}$. We first determine $Z(G)$. Suppose that $(v, l) \in Z(G)$. Then $(v, l) = (0, 1)(v, l)(0, -1)$, which yields

$$(v, l) = (Av, l).$$

Consequently, $Av = v$. Since $\ker(A - I)$ is generated by $e_1 := (1, 0, 0)^{\mathsf{T}}$, $v = ie_1$ for some $i \in \mathbb{Z}$. Next, with $e_2 := (0, 1, 0)^{\mathsf{T}}$, also $(e_2, 0)(v, l)(-e_2, 0) = (v, l)$. Hence,

$$(v, l) = (e_2 + v - A^l e_2, l).$$

This implies that $(I - A^l) e_2 = 0$. As $\operatorname{Spec}(A) = \left\{1, \frac{6 \pm \sqrt{32}}{2}\right\}$, each non-zero power of A has three distinct eigenvalues, one of which is 1. Moreover, $\ker(A^k - I)$

is generated by e_1 for all non-zero k. Hence, if $A^l e_2 = e_2$, then $l = 0$. Hence, $(v, l) = (ie_1, 0)$. Conversely, it is readily verified that $(e_1, 0) \in Z(G)$. Therefore, $Z(G) = \langle (e_1, 0) \rangle \cong \mathbb{Z}$.

The quotient group $G/Z(G)$ is isomorphic with $H := \mathbb{Z}^2 \rtimes_B \mathbb{Z}$, where

$$B = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

Corollary 1.2.15 then yields

$$\operatorname{Spec}_{\mathrm{R}}(G) \subseteq \operatorname{Spec}_{\mathrm{R}}(\mathbb{Z}) \cdot \operatorname{Spec}_{\mathrm{R}}(H) \subseteq \{2, \infty\} \cdot \{4, \infty\} = \{8, \infty\},$$

where the second inclusion follows from Theorem T.1.13 and [120, Proposition 4.5.4].

Clearly, $R(\mathrm{Id}_G) = \infty$. To prove that 8 lies in $\operatorname{Spec}_{\mathrm{R}}(G)$, consider the map φ given by

$$\varphi : G \to G : (v, l) \mapsto (Mv, -l),$$

where

$$M = \begin{pmatrix} -1 & -5 & -1 \\ 0 & 2 & 1 \\ 0 & -5 & -2 \end{pmatrix}.$$

It is readily verified that φ defines an automorphism of G. Let $\varphi|_{Z(G)}$ and $\bar{\varphi}$ denote the induced automorphisms on $Z(G)$ and $G/Z(G)$, respectively. Then Corollary 1.2.15 states $R(\varphi) = R(\varphi|_{Z(G)}) R(\bar{\varphi})$. Now, $\varphi|_{Z(G)}(e_1) = -e_1$, so $R(\varphi|_{Z(G)}) = 2$. For $R(\bar{\varphi})$, note that $\bar{\varphi}$ is, under the isomorphism $G/Z(G) \cong \mathbb{Z}^2 \rtimes_B \mathbb{Z} = H$, given by

$$\bar{\varphi} : H \to H : (v, l) \mapsto (M'v, -l),$$

where

$$M' = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix}.$$

The induced automorphism on $H/\mathbb{Z}^2$ is given by $-\mathrm{Id}_{\mathbb{Z}}$, so we can apply Proposition 1.1.8 to obtain

$$R(\bar{\varphi}) = R(M') + R(BM') = \left\| \begin{vmatrix} 1 & 1 \\ -5 & -3 \end{vmatrix} \right\|_\infty + \left\| \begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} \right\|_\infty = 2 + 2 = 4.$$

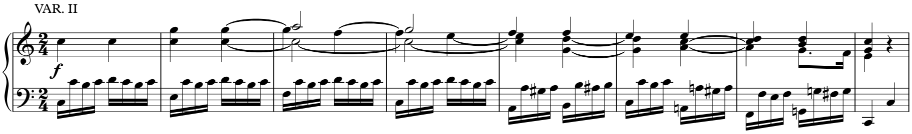
We conclude that

$$R(\varphi) = R(\varphi|_{Z(G)}) R(\bar{\varphi}) = 2 \cdot 4 = 8.$$

Therefore, $\operatorname{Spec}_{\mathrm{R}}(G) = \{8, \infty\}$. ‖

Variation 2

Free products



Mozart, Wolfgang Amadeus, *Zwölf Variationen in C über das französische Lied 'Ah, vous dirai-je, Maman'*, (KV 265 (300e)), Variation II (bars 1–8).

In this Variation and the next one, we study twisted conjugacy in two product constructions, starting with free products. In [47], D. Gonçalves, P. Sankaran and P. Wong prove that, under two mild conditions, a free product of finitely many groups has the R_∞-property. As this is a strong result that illustrates how the structure of a group completely determines its Reidemeister spectrum, we include it here, together with a detailed proof.

2.1 Free groups

Given $n \geq 1$, we let F_n denote the free group of rank n.

Theorem 2.1.1. *Let $n \geq 2$ be arbitrary. Then F_n has the R_∞-property.*

One way of proving this result uses a notion related to Reidemeister numbers called *isogredience numbers*: given a group G and an element $\Phi \in \mathrm{Out}(G)$, we say that $\varphi, \psi \in \Phi$ are *isogredient* if there exists a $\tau \in \mathrm{Inn}(G)$ such that

$$\varphi = \tau \circ \psi \circ \tau^{-1}.$$

This defines an equivalence relation on Φ and we call the corresponding equivalence classes *isogredience classes*. The number of isogredience classes is called the *isogredience number* of Φ and is denoted by $S(\Phi)$. Similar to the Reidemeister spectrum, we define the *isogrendience spectrum* of G to be

$$\mathrm{Spec}_{\mathrm{S}}(G) := \{S(\Phi) \mid \Phi \in \mathrm{Out}(G)\}.$$

Finally, a group G has the *S_∞-property* if $\mathrm{Spec}_{\mathrm{S}}(G) = \{\infty\}$.

The concept of isogredience has a topological motivation as well, see e.g. [73, §0] and [92, §18].

A. Fel'shtyn and E. Troitsky have studied the relation between isogredience and Reidemeister numbers. In particular, they prove in [36, Lemma 3.3] that for a group G, an element $\Phi \in \mathrm{Out}(G)$ and an automorphism $\varphi \in \Phi$, the equality $S(\Phi) = R(\bar{\varphi})$ holds, where $\bar{\varphi}$ is the induced automorphism on $G/Z(G)$. Together with Corollary 1.1.2(1), it then follows $R(\varphi) \geq S(\Phi)$. Thus, if a group has the S_∞-property, it also has the R_∞-property.

The S_∞-property has been studied, although not as extensively as the R_∞-property. G. Levitt and M. Lustig show in [73] that non-elementary (Gromov) hyperbolic groups (i.e. hyperbolic groups that are neither finite nor virtually-$\mathbb{Z}$) have the S_∞-property. Thus, the combination of this result with the one by Fel'shtyn and Troitsky yields the following:

Theorem 2.1.2 (See e.g. [29, Theorem 3])**.** *Let G be a non-elementary hyperbolic group. Then G has the R_∞-property.*

Theorem 2.1.1 now follows from this theorem and the fact that non-abelian free groups of finite rank are non-elementary hyperbolic (see e.g. [74, Example 7.3.3]).

We come back to hyperbolic groups in Variation 3, and in Variation 4 we discuss a second way of proving Theorem 2.1.1 using nilpotent groups.

We now return to free groups and discuss virtually free groups and free groups of infinite rank.

Definition 2.1.3. Let G be a group. We say that G is *non-elementary* virtually free if there is a non-abelian free subgroup F of finite index in G.

Proposition 2.1.4. *Let G be a finitely generated non-elementary virtually free group. Then G has the R_∞-property.*

Proof. Let F be a non-abelian free subgroup of finite index in G. Define

$$F_0 := \bigcap_{\varphi \in \mathrm{Aut}(G)} \varphi(F).$$

Then F_0 is characteristic by construction. Moreover, since G is a finitely generated, it has only finitely many subgroups of index $[G : F]$. Therefore, F_0 is an intersection of finitely many subgroups of finite index, so F_0 has finite index as well. As F_0 is a subgroup of finite index of F, it is non-abelian free as well, so it has the R_∞-property by Theorem 2.1.1. Finally, Corollary 1.1.2(2') implies that G has the R_∞-property as well, as G/F_0 is finite. ☐

In contrast, the free group of countably infinite rank, denoted by F_∞, has full Reidemeister spectrum.

Theorem 2.1.5 (See e.g. [19, §5])**.** *The Reidemeister spectrum of F_∞ is $\mathbb{N}_0 \cup \{\infty\}$.*

Proof. Without loss of generality, we can assume that F_∞ is the free group on $X = \{x_0, x_1, x_2, \ldots\}$. We first construct a surjective map $\theta : X \to F_\infty$ such that, for each $i \geq 0$, $\theta(x_i)$ is a possibly trivial element of F_∞ that only involves the generators $x_0, \ldots, x_{i-1}$. To do so, let $f_0 = 1, f_1, f_2, \ldots$ be an enumeration of F_∞, which is possible, as F_∞ is a countable group. We define θ inductively. We put $\theta(x_0) := 1 = f_0$. Suppose that, for $i \geq 1$, we have defined $\theta(x_0)$ up to $\theta(x_{i-1})$. Let j be the minimal index such that $f_j \notin \{\theta(x_0), \ldots, \theta(x_{i-1})\}$. If f_j only involves the generators $x_0, \ldots, x_{i-1}$, we put $\theta(x_i) := f_j$; otherwise, we put $\theta(x_i) := 1$.

To prove that θ is surjective, we argue that the index j always increases after finitely many steps. For ease of notation, we put $j(l)$ to be the value of the index j when $\theta(x_0)$ up to $\theta(x_{l-1})$ have been defined. Suppose we are at index i. If we define $\theta(x_i) = f_{j(i)}$, then $j(i+1) = j(i) + 1$. Otherwise, $f_{j(i)}$ involves the generators $x_0, \ldots, x_k$ for some $k \geq i$. Without loss of generality, we may assume that x_k does occur in $f_{j(i)}$. In that case, $\theta(x_i)$ up to $\theta(x_k)$ will all be defined to be 1, so $j(i) = j(i+1) = \ldots = j(k) = j(k+1)$. At step $k+1$, however, $f_{j(k+1)}$ only involves the generators $x_0, \ldots, x_k$, so $\theta(x_{k+1}) = f_{j(k+1)} = f_{j(i)}$. Consequently, $j(k+2) = j(k+1) + 1$.

Now, fix a positive integer n. For $i \in \{0, \ldots, n-1\}$, let $J_i \subseteq X$ be the set of elements x_j such that the exponent sum of $\theta(x_j)$ is congruent to i modulo n. Putting $W_i := \theta(J_i)$, we obtain that X is the disjoint union of $J_0, \ldots, J_{n-1}$ and that F_∞ is the disjoint union of $W_0, \ldots, W_{n-1}$.

Next, let $\varphi_n : F_\infty \to F_\infty$ be the homomorphism given by $\varphi_n(x_k) = \theta(x_k)^{-1} x_k x_0^i$, where $i \in \{0, \ldots, n-1\}$ is such that $x_k \in J_i$. We argue that φ_n is an automorphism. We construct an inverse ψ_n of φ_n by defining the image of x_i under ψ_n for each $i \geq 0$. Define $\psi_n(x_0) := x_0$. Let $k \geq 1$ and suppose that $\psi_n(x_j)$ has been defined for $j \in \{0, \ldots, k-1\}$. For ψ_n to be an inverse of φ_n, it must hold that $\psi_n\Big(\theta(x_k)^{-1}x_k x_0^i\Big) = x_k$. Therefore, $\psi_n(x_k)$ must be equal to

$$\psi_n(x_k) := \psi_n(\theta(x_k))x_k\psi_n(x_0)^{-i}.$$

Since, by construction, $\theta(x_k)$ only involves the generators $x_0, \dots, x_{k-1}$, we can compute $\psi_n(\theta(x_k))$ by applying ψ_n to each of the generators in the word representing $\theta(x_k)$. This gives us a homomorphism $\psi_n : F_\infty \to F_\infty$, which is, by construction, a left inverse of φ_n.

To prove that it is a right inverse as well, we again proceed by induction. Clearly, $\varphi_n(\psi_n(x_0)) = \varphi_n(x_0) = x_0$. Next, let $k \geq 1$ and suppose that $\varphi_n(\psi_n(x_j)) = x_j$ for $j \in \{0, \dots, k-1\}$. Then

$$\begin{aligned}
\varphi_n(\psi_n(x_k)) &= \varphi_n\big(\psi_n(\theta(x_k))x_k\psi_n(x_0)^{-i}\big) \\
&= \varphi_n(\psi_n(\theta(x_k)))\varphi_n(x_k)\varphi_n(\psi_n(x_0))^{-i} \\
&= \varphi_n(\psi_n(\theta(x_k)))\theta(x_k)^{-1}x_k x_0^i\varphi_n(\psi_n(x_0))^{-i}
\end{aligned}$$

Since $\theta(x_k)$ only involves $x_0, \dots, x_{k-1}$, $\varphi_n(\psi_n(\theta(x_k))) = \theta(x_k)$ by the induction hypothesis. Similarly, $\varphi_n(\psi_n(x_0))^{-i} = x_0^{-i}$. Consequently,

$$\begin{aligned}
\varphi_n(\psi_n(x_k)) &= \varphi_n(\psi_n(\theta(x_k)))\theta(x_k)^{-1}x_k x_0^i\varphi_n(\psi_n(x_0))^{-i} \\
&= \theta(x_k)\theta(x_k)^{-1}x_k x_0^i x_0^{-i} \\
&= x_k.
\end{aligned}$$

Thus, ψ_n is a right inverse of φ_n as well, which proves that φ_n is an automorphism.

Finally, we show that $R(\varphi_n) = n$. First, we prove that any element $x \in F_\infty$ is φ_n-conjugate to x_0^i for some $i \in \{0, \dots, n-1\}$. Since θ is surjective, there is an $x_k \in X$ such that $\theta(x_k) = x$. Thus, as $\varphi_n(x_k) = \theta(x_k)^{-1}x_k x_0^i$ where $i \in \{0, \dots, n-1\}$ is such that $x_k \in J_i$, we obtain

$$x = \theta(x_k) = x_k x_0^i \varphi_n(x_k)^{-1}.$$

Secondly, we argue that no two elements in $\{1, x_0, \dots, x_0^{n-1}\}$ are φ_n-conjugate. Note that, by construction, $\varphi_n(x_k)$ has the same exponent sum modulo n as x_k. Thus, inductively, for any $x \in F_\infty$, $\varphi_n(x)$ and x have the same exponent sum modulo n as well. Thus, if $x_0^j = x x_0^k \varphi_n(x)^{-1}$ for some $x \in F_\infty$ and $j, k \in \{0, \dots, n-1\}$, then the exponent sum modulo n on the left-hand side equals j, and the one on the right-hand side equals k. Thus, $j \equiv k \bmod n$, which implies that $j = k$. Consequently, $1, x_0, \dots, x_0^{n-1}$ represent n different φ_n-conjugacy classes. Since any element in x is φ_n-conjugate to one of these n powers of x_0, we conclude that $R(\varphi_n) = n$.

To finish the proof, we have to provide an automorphism with infinitely many twisted conjugacy classes. We claim that identity map Id_{F_∞} suffices. Indeed, since $\mathbb{Z}$ is a quotient of F_∞ and $R(\mathrm{Id}_\mathbb{Z}) = \infty$, Corollary 1.1.2(1) implies that $R(\mathrm{Id}_{F_\infty}) = \infty$ as well. $\square$

2.2 Free products

In what follows, we work out in more detail the main algebraic result by D. Gonçalves, P. Sankaran and P. Wong [47] regarding the R_∞-property of free products.

2.2.1 External and internal free product

When speaking of a free product, we often mean the external free product. To prove the result concerning the R_∞-property, we also need the notion of internal free product.

Definition 2.2.1. Let G be a group and $\{A_i\}_{i\in\mathcal{I}}$ a collection of subgroups. Then G is called the *internal free product of* $\{A_i\}_{i\in\mathcal{I}}$ if G is generated by $\{A_i\}_{i\in\mathcal{I}}$ and if each $1 \neq g \in G$ has a *unique* representation of the form $a_1 \dots a_n$, where $1 \neq a_j \in A_{i_j}$ for all $j \in \{1, \dots, n\}$ and $i_j \neq i_{j-1}$ for all $j \in \{2, \dots, n\}$.

We refer to $a_1 \dots a_n$ as the *reduced form* of g, call n the *length* of $g \in G$ and write this as $n = \ell(g)$.

We reserve the notation $A * B$ for the external free product of two groups A and B, and, given a collection of groups $\{G_i\}_{i\in\mathcal{I}}$, we also write $\ast_{i\in\mathcal{I}} G_i$ for their (external) free product. A. Kurosh proves in [71] that the internal free product satisfies essentially the same universal property as the external one:

Theorem 2.2.2 (See e.g. [71, p. 15])**.** *Let G be a group and $\{A_i\}_{i\in\mathcal{I}}$ a collection of subgroups. Suppose that G is generated by $\{A_i\}_{i\in\mathcal{I}}$. Then G is the internal free product of $\{A_i\}_{i\in\mathcal{I}}$ if and only if the following universal property holds: given an arbitrary group H and a collection of homomorphisms $\{\varphi_i : A_i \to H\}_{i\in\mathcal{I}}$, there exists a (unique) homomorphism $\varphi : G \to H$ such that $\varphi|_{A_i} = \varphi_i$ for all $i \in \mathcal{I}$.*

Remark. The uniqueness is not stated in [71]. However, it follows from the fact that G is generated by $\{A_i\}_{i\in\mathcal{I}}$ that φ is uniquely determined by its image on each A_i.

Thus, when defining a map from an internal or external free product, we only have to specify what it does on the factors. We do so without further mention of this universal property.

A. Kurosh also argues that, if G is the external free product of a collection of groups $\{G_i\}_{i\in\mathcal{I}}$, G is also the internal free product of the canonical images of $\{G_i\}_{i\in\mathcal{I}}$ in G ([71, p. 14]).

Corollary 2.2.3. *Let F be a free group and let $\{G_i\}_{i\in\mathcal{I}}$ be a collection of groups . Put $G = F * \mathop{\ast}_{i\in\mathcal{I}} G_i$ and let $\varphi \in \operatorname{Aut}(G)$. Then G is the internal free product of $\varphi(F)$ and $\{\varphi(G_i)\}_{i\in\mathcal{I}}$.*

Proof. We use Theorem 2.2.2. Since G is generated by F and $\{G_i\}_{i\in\mathcal{I}}$ and as φ is a bijection, G is also generated by $\varphi(F)$ and $\{\varphi(G_i)\}_{i\in\mathcal{I}}$. Next, let H be an arbitrary group. Suppose that we are given homomorphisms $\psi_F : \varphi(F) \to H$ and $\psi_i : \varphi(G_i) \to H$, for each $i \in \mathcal{I}$. Then $\psi_F \circ \varphi$ is a homomorphism from F to H, and for each $i \in \mathcal{I}$ is $\psi_i \circ \varphi$ a homomorphism from $G_i \to H$, so by Theorem 2.2.2, there is a unique map $\theta : G \to H$ such that $\theta|_F = \psi_F \circ \varphi$ and $\theta|_{G_i} = \psi_i \circ \varphi$ for each $i \in \mathcal{I}$. Consequently, $\theta' := \theta \circ \varphi^{-1} : G \to H$ is the desired map. Indeed, let $g \in \varphi(F)$. Then $g = \varphi(f)$ for some $f \in F$. Computing $\theta'(g)$, we obtain

$$\theta'(g) = \theta(\varphi^{-1}(\varphi(f))) = \theta(f) = \psi_F(\varphi(f)) = \psi_F(g).$$

Similarly, for $i \in \mathcal{I}$ and $g = \varphi(g_i) \in \varphi(G_i)$, we see that

$$\theta'(g) = \theta(\varphi^{-1}(\varphi(g_i))) = \theta(g_i) = \psi_i(\varphi(g_i)) = \psi_i(g).$$

Therefore, Theorem 2.2.2 implies that G is the internal free product of $\varphi(F)$ and $\{\varphi(G_i)\}_{i\in\mathcal{I}}$. $\square$

Definition 2.2.4. Let G be a group. We say that G is *freely indecomposable* if $G \cong A * B$ for two groups A and B implies that either $A = 1$ or $B = 1$.

Theorem 2.2.5 (See e.g. [71, p. 27])**.** *Let G be a group. Suppose that G is the internal free product of both $\{F_1\} \cup \{A_i\}_{i\in\mathcal{I}}$ and $\{F_2\} \cup \{B_j\}_{j\in\mathcal{J}}$, where F_1 and F_2 are free groups and where A_i and B_j are both freely indecomposable and non-isomorphic to $\mathbb{Z}$ for all $i \in \mathcal{I}, j \in \mathcal{J}$. Then F_1 and F_2 are isomorphic, and there exists a bijection $f : \mathcal{I} \to \mathcal{J}$ such that A_i and $B_{f(i)}$ are conjugate in G.*

Corollary 2.2.6. *Let F be a free group and let $\{G_i\}_{i\in\mathcal{I}}$ be a collection of freely indecomposable groups such that no G_i is isomorphic to $\mathbb{Z}$. Put $G = F * \mathop{\ast}_{i\in\mathcal{I}} G_i$ and let $\varphi \in \mathrm{Aut}(G)$. Then there exist a bijection $f : \mathcal{I} \to \mathcal{I}$ and, for each $i \in \mathcal{I}$, an element $g_i \in G$ such that $\varphi(G_i) = g_i G_{f(i)} g_i^{-1}$.*

Proof. By Corollary 2.2.3, G is the internal free product of $\varphi(F)$ and $\{\varphi(G_i)\}_{i\in\mathcal{I}}$. By Theorem 2.2.5, there exists a bijection $f : \mathcal{I} \to \mathcal{I}$ such that $\varphi(G_i)$ and $G_{f(i)}$ are conjugate in G. This yields the elements $\{g_i\}_{i\in\mathcal{I}}$. □

To end this section, we state the so-called Kurosh subgroup theorem, which describes the subgroups of a free product.

Theorem 2.2.7 (See e.g. [71, p. 17])**.** *Let G be a group. Suppose that G is the internal free product of $\{A_i\}_{i\in\mathcal{I}}$. Let H be a subgroup of G. Then H is the internal free product of F and $\{B_j\}_{j\in\mathcal{J}}$, where F is a free group and where each B_j is conjugate (in G) to a subgroup of one of the A_i.*

2.2.2 The R_∞-property for free products

We now head to the main result of this Variation. The key idea is to find suitable characteristic subgroups of the given free products.

Lemma 2.2.8. *Let G and H be isomorphic groups and let C be a characteristic subgroup of G. Let $\alpha : G \to H$ and $\beta : G \to H$ be isomorphisms between G and H. Then $\alpha(C) = \beta(C)$.*

Proof. Since α and β are isomorphisms, $\beta^{-1} \circ \alpha : G \to G$ is an automorphism of G. Consequently, $(\beta^{-1} \circ \alpha)(C) = C$, which implies that $\alpha(C) = \beta(C)$. □

Lemma 2.2.9 (See also [47, Lemma 3])**.** *Let $G_1, \ldots, G_n$ be freely indecomposable groups that are not isomorphic to $\mathbb{Z}$, and let $k \geq 1$ be an integer. For each $i \in \{1, \ldots, n\}$, let C_i be a characteristic subgroup of G_i. For any pair of indices i and j such that G_i and G_j are isomorphic, fix an isomorphism $\alpha_{ij} : G_j \to G_i$. Put $G = F_k * G_1 * \ldots * G_n$. Put*

$$\begin{aligned}\mathcal{C} := &\{gC_ig^{-1} \mid g \in G, i \in \{1, \ldots, n\}\}\\ &\cup \{g\alpha_{ij}(C_j)g^{-1} \mid g \in G, i, j \in \{1, \ldots, n\} \text{ such that } \alpha_{ij} \text{ exists}\}.\end{aligned}$$

Let K be the subgroup of G generated by $\mathcal{C}$. Then K is characteristic in G.

Proof. Let $\varphi \in \operatorname{Aut}(G)$ be arbitrary. By Corollary 2.2.6, there exist a bijection $f : \{1, \dots, n\} \to \{1, \dots, n\}$ and elements $g_1, \dots, g_n$ such that $\varphi(G_i) = g_i G_{f(i)} g_i^{-1}$. Fix $i \in \{1, \dots, n\}$. Then $\kappa_i := \tau_{g_i^{-1}}|_{\varphi(G_i)} \circ \varphi|_{G_i}$ is an isomorphism between G_i and $G_{f(i)}$. By Lemma 2.2.8, $\kappa_i(C_i) = \alpha_{f(i)i}(C_i)$, or, in other words, $\varphi(C_i) = g_i \alpha_{f(i)i}(C_i) {g_i}^{-1}$. Thus, if we take $g \in G$ arbitrary, we see that

$$\varphi(gC_i g^{-1}) = \varphi(g) g_i \alpha_{f(i)i}(C_i) {g_i}^{-1} \varphi(g)^{-1}$$

again lies in $\mathcal{C}$. Similarly, if $j \in \{1, \dots, n\}$ is such that α_{ji} exists, then $\alpha_{ji}(C_i)$ is characteristic in G_j. Therefore, for arbitrary $g \in G$, we obtain that

$$\varphi(g\alpha_{ji}(C_i)g^{-1}) = \varphi(g)\varphi(\alpha_{ji}(C_i))\varphi(g)^{-1} = \varphi(g) g_j \alpha_{f(j)j}(\alpha_{ji}(C_i)) g_j^{-1} \varphi(g)^{-1}.$$

Now note that $\alpha_{f(j)i}$ also exists, as $\alpha_{f(j)j} \circ \alpha_{ji}$ is an isomorphism between G_i and $G_{f(j)}$, and that $\alpha_{f(j)j}(\alpha_{ji}(C_i)) = \alpha_{f(j)i}(C_i)$ by Lemma 2.2.8. Therefore, $\varphi(g\alpha_{ji}(C_i)g^{-1}) \in \mathcal{C}$. We conclude that $\mathcal{C}$ is closed under φ, and as φ was arbitrary, $\mathcal{C}$ is closed under all automorphisms of G. This implies that K is characteristic in G. □

Lemma 2.2.10. *Let G be the internal free product of $\{A_i\}_{i \in \mathcal{I}}$. For each $i \in \mathcal{I}$, let N_i be a normal subgroup of A_i and put $B_i := A_i / N_i$. Define $B := \ast_{i \in \mathcal{I}} B_i$. For each $i \in \mathcal{I}$, let $\pi_i : A_i \to B$ be the composition of the canonical projection map $A_i \to B_i$ and the inclusion map $B_i \to B$. Let $\varphi : G \to B$ the induced map on G. Then*

$$\ker \pi = \big\langle \{ g N_i g^{-1} \mid g \in G, i \in \mathcal{I} \} \big\rangle.$$

Proof. Put $L = \big\langle \{ g N_i g^{-1} \mid g \in G \} \big\rangle$. Clearly, $\pi(x) = 1$ for each $x \in L$, as $\pi(N_i) = 1$ for each $i \in \mathcal{I}$. Conversely, let $1 \neq g = a_1 \dots a_n$ be an element in $\ker \pi$ in its reduced form. We prove by induction on n that $g \in L$. Suppose that $n = 1$. Let $i \in \mathcal{I}$ be the index such that $a_1 \in A_i$. As $\pi(g) = 1$, this implies that $1 = \pi(g) = \pi_i(a_1)$. Therefore, $a_1 \in \ker \pi_i = N_i$. We conclude that $g = a_1 \in L$.

Next, suppose that the result holds for all $g \in \ker \pi$ of length at most n. Let $g = a_1 \dots a_{n+1}$ be an element in $\ker \pi$ in reduced form of length $n + 1$. If $\pi(a_i) \neq 1$ for all $i \in \{1, \dots, n+1\}$, then $\pi(a_1) \dots \pi(a_{n+1})$ is the reduced form of $\pi(g)$, which contradicts the fact that $\pi(g) = 1$. Hence, there is an $i \in \{1, \dots, n+1\}$ such that $\pi(a_i) = 1$. Similarly as before, this implies that $a_i \in N_i \leq L$. Now, define $y := a_1 \dots a_{i-1} a_{i+1} \dots a_{n+1}$. Then $\ell(y) < \ell(g) = n+1$ and $\pi(y) = \pi(g) = 1$, so the induction hypothesis yields $y \in L$. Finally, remark that

$$g = y (a_{i+1} \dots a_{n+1})^{-1} a_i a_{i+1} \dots a_{n+1} = y a_i^z,$$

where $z = a_{i+1} \dots a_{n+1}$. Since L is normal and $y, a_i \in L$, we conclude that $g \in L$ as well. As $g \in \ker \pi$ was arbitrary, this proves that $\ker \pi = L$. □

Theorem 2.2.11 (See e.g. [75, Theorem 2]). *Let $G_1, \ldots, G_n$ be groups, and let K be the kernel of the canonical map $G_1 * \ldots * G_n \to G_1 \times \ldots \times G_n$. Then K is a free group with basis given by all non-trivial elements of the form*

$$a_1 a_2 \ldots a_{i-1} a_{i+1} a_{i+2} \ldots a_n a_i (a_1 \ldots a_n)^{-1},$$

where $i \in \{1, \ldots, n\}$ and $a_j \in G_j$ for all $j \in \{1, \ldots, n\}$.

Proposition 2.2.12. *Let $E_1, \ldots, E_n$ be (non-trivial) finite groups and put $E = E_1 * \ldots * E_n$. Then E is virtually free.*

Moreover, if $n \geq 2$, then E has the R_∞-property.

Proof. By Theorem 2.2.11, the kernel K of the canonical map from E to $E_1 \times \ldots \times E_n$ is free. As E/K is finite, K has finite index in E. Thus, E is virtually free.

Now, suppose that $n \geq 2$. If $n = 2$ and $E_1 \cong E_2 \cong \mathbb{Z}/2\mathbb{Z}$, then E is the infinite dihedral group. By Example 1.1.6, this group has the R_∞-property. So, suppose that E is not the infinite dihedral group. Let a, b, c be three distinct non-trivial elements in E_1 and E_2, or in E_1, E_2 and E_3, such that neither a and b nor a and c lie in the same E_i. Then $ba(ab)^{-1}$ and $ca(ac)^{-1}$ are distinct non-trivial elements in K, and they are part of a free basis of K, again by Theorem 2.2.11. Thus, K is non-abelian free in this case. Hence, E is a finitely generated non-elementary virtually free group. Consequently, it has the R_∞-property by Proposition 2.1.4. $\square$

We can now state and prove the main result.

Theorem 2.2.13 (See e.g. [47, Theorem 1]). *Let $n \geq 2$ be an integer and let $G_1, \ldots, G_n$ be freely indecomposable groups. Suppose that at least two of the G_i have a proper finite index characteristic subgroup. Put $G = G_1 * \ldots * G_n$. Then G has the R_∞-property.*

Proof. We may assume after reordering that there exists an $m \geq 2$ such that

(i) G_1 up to G_m have a proper finite index characteristic subgroup, and

(ii) G_{m+1} up to G_n do not.

Note that, with this numbering, every (if any) G_i isomorphic to $\mathbb{Z}$ satisfies $i \in \{1, \ldots, m\}$. Indeed, $\mathbb{Z}$ contains a proper finite index characteristic subgroup (e.g. $2\mathbb{Z}$).

For each $i \in \{1, \ldots, m\}$, let C_i a proper finite index characteristic subgroup of G_i. Furthermore, for each pair G_i and G_j of isomorphic groups with $i, j \in \{1, \ldots, m\}$, we choose C_i and C_j such that they correspond under any isomorphism between G_i and G_j. By Lemma 2.2.8, this C_i is well defined.

First, we argue that it is sufficient to prove that $G_0 := G_1 * \ldots * G_m$ has the R_∞-property. Let $\pi : G \to G_0$ be the homomorphism that is the identity map on G_1 up to G_m, and the trivial map on G_{m+1} up to G_n. Let K be its kernel. By Lemma 2.2.10, K is equal to $\left\langle \{gG_ig^{-1} \mid g \in G, i \in \{m+1, \ldots, n\}\} \right\rangle$. By Lemma 2.2.9, K is characteristic in G. Therefore, by Corollary 1.1.2, if $G/K \cong G_0$ has the R_∞-property, so does G.

Now, we prove that G_0 has the R_∞-property. Let r be the number of indices $i \in \{1, \ldots, m\}$ such that G_i is isomorphic to $\mathbb{Z}$. We distinguish three cases: $r = 0, r = 1$ and $r \geq 2$.

Case 1: $r = 0$. For $i \in \{1, \ldots, m\}$, define $E_i := G_i/C_i$. Then each E_i is a non-trivial finite group. Put $E := E_1 * \ldots * E_n$ and let $\pi_E : G \to E$ be the homomorphism associated to the canonical projections $G_i \to E_i$. By Lemma 2.2.10, $K_E := \ker \pi_E$ is generated by $\{gC_ig^{-1} \mid g \in G, i \in \{1, \ldots, m\}\}$. By the assumptions on the C_i, Lemma 2.2.9 then implies that K_E is characteristic in G_0. As E has the R_∞-property by Proposition 2.2.12, so does G_0 by Corollary 1.1.2.

Case 2: $r = 1$. We may assume, after renumbering, that $G_1 \cong \mathbb{Z}$. As before, for $i \in \{2, \ldots, m\}$, define $E_i := G_i/C_i$. Put $D := G_1 * E_2 * \ldots * E_m$ and let $\pi_D : G_0 \to D$ be the homomorphism associated to the identity map $G_1 \to G_1$ and the canonical projections $G_i \to E_i$. Again, $K_D := \ker \pi_D$ is generated by $\{gC_ig^{-1} \mid g \in G, i \in \{2, \ldots, m\}\}$ and is characteristic in G by Lemma 2.2.10 and Lemma 2.2.9, respectively. We claim that D is non-elementary virtually free. To that end, we consider the map $\psi : D \to E_2 \times \ldots \times E_m$, which is induced by the trivial map $G_1 \to E_2 \times \ldots \times E_m$ and, for each $i \in \{2, \ldots, m\}$, the composition of the identity map $E_i \to E_i$ and the canonical inclusion $E_i \to E_2 \times \ldots \times E_m$. By Theorem 2.2.7, $\ker \psi$ is a free product of the form

$$F * \mathop{\Large\ast}_{i \in \mathcal{I}} b_i H_i b_i^{-1},$$

where F is a free group, $\mathcal{I}$ is some (possibly empty) index set, and each H_i is a subgroup of one of the E_j. We argue that $\mathcal{I}$ is empty, or, equivalently, that each H_i is trivial. Suppose that $h \in H_i$. Then $1 = \psi(b_i h b_i^{-1})$, so $\psi(h) = 1$ as well. As $h \in H_i \leq E_j$ for some $j \in \{2, \ldots, m\}$, this implies that $h = 1$. Therefore, $\ker \psi$ is free. As $D/\ker \psi$ is finite, $\ker \psi$ has finite index and is consequently finitely generated, since D is finitely generated as well. To argue that $\ker \psi$ is non-abelian, let $t \in G_1$ and $e \in E_2$ both be non-trivial elements (recall that

$m \geq 2$). By definition of ψ, both t and t^e lie in $\ker \psi$. Suppose that t and t^e commute. Then $e^{-1}tete^{-1}t^{-1}et^{-1} = 1$. However, as $e \neq 1 \neq t$, the element on the left-hand side is in reduced form. Therefore, it cannot be equal to 1, which is a contradiction. Thus, t and t^e do not commute, which proves that $\ker \psi$ contains the non-abelian group $\langle t, t^e \rangle$. It follows that $\ker \psi$ is a non-abelian free group of finite rank.

Thus, D is non-elementary virtually free, so Proposition 2.1.4 implies that D has the R_∞-property. As $D \cong G_0/K_D$ is a characteristic quotient of G_0, Corollary 1.1.2 implies that G_0 has the R_∞-property as well.

Case 3: $r \geq 2$. Again, after renumbering, we may assume that $G_1 \cong \ldots \cong G_r \cong \mathbb{Z}$. Put $A = G_1 * \ldots * G_r$. Then A is a non-abelian free group of rank r. Let $\pi_A : G_0 \to A$ be the homomorphism associated to the identity maps on G_1 up to G_r, and the trivial maps on G_{r+1} up to G_m. Let K_A be its kernel. Then K_A is generated by $\{gG_ig^{-1} \mid g \in G, i \in \{r+1, \ldots, m\}\}$ and is characteristic in G by Lemma 2.2.10 and Lemma 2.2.9, respectively. Thus, $G_0/K_A \cong A$, and as the latter has the R_∞-property by Theorem 2.1.1, so does G_0. □

2.2.3 Examples

As announced in the introduction, we provide (families of) groups that satisfy the conditions of Theorem 2.2.13.

First of all, every finitely generated group having a proper finite index subgroup has a proper finite index (fully) characteristic subgroup by Lemma 1.2.6. Therefore, Theorem 2.2.13 applies in particular to all finitely generated residually finite and freely indecomposable groups.

To find other families of examples, we turn to direct products for inspiration.

Lemma 2.2.14 (See e.g. [76, p. 177]). *No group splits as both a non-trivial free product and a non-trivial direct product.*

Proposition 2.2.15 (See e.g. [47, p. 3917]). *Let A be a non-trivial finite centreless group and let B be a torsion-free group. Then B is a proper finite index characteristic subgroup of $G := A \times B$.*

Proof. Since B is torsion-free, A is characteristic in G. Next, let $b \in B$ and $\varphi \in \operatorname{Aut}(G)$. Note that $A \leq C_G(b)$. Write $\varphi(b) = (a, b')$. Suppose that $a \neq 1$. From $\varphi(A) = A$ and $C_G(\varphi(b)) = \varphi(C_G(b))$, we derive that $A \leq C_G(a, b')$. However, since $a \neq 1$ and A is centreless, there is an $a' \in A$ such that $[a, a'] \neq 1$. Therefore, $a' \notin C_G(a, b')$, which yields a contradiction. Thus, $a = 1$, which proves that $\varphi(b) \in B$. Consequently, B is characteristic in G.

Since $G/B \cong A$ is finite and non-trivial, B is a proper finite index subgroup. $\square$

Lemma 2.2.14 ensures that every direct product is freely indecomposable, so this proposition yields a method for constructing groups that satisfy the conditions of Theorem 2.2.13.

Definition 2.2.16. Let G be a group. We call G *divisible* if for each $g \in G$ and $n \in \mathbb{N}_0$ there exists an $x \in G$ such that $g = x^n$.

Examples of abelian divisible groups include the (additive) groups $\mathbb{Q}^r, \mathbb{R}^r$ and $\mathbb{C}^r$ for any integer $r \geq 1$.

Proposition 2.2.17. *Let G be a divisible group and E a finite group. Then each homomorphism from G to E is trivial.*

Proof. Let $\varphi : G \to E$ be a homomorphism. Let $g \in G$. Since G is divisible, there is an $h \in G$ such that $g = h^{|E|}$. Consequently,

$$\varphi(g) = \varphi(h^{|E|}) = \varphi(h)^{|E|} = 1.$$

As g is arbitrary, we conclude that $\varphi(G) = 1$, i.e. φ is trivial. $\square$

Corollary 2.2.18. *Let G be a divisible group. Then G has no non-trivial finite index subgroups.*

Proof. Let H be a finite index subgroup of G. By intersecting all conjugates of H, we obtain a finite index normal subgroup N contained in H. Since N gives rise to a surjective homomorphism $\pi : G \to G/N$ to the finite group G/N, the group G/N must be trivial, by Proposition 2.2.17. Consequently, N equals G. As N is contained in H, also $H = G$. $\square$

Proposition 2.2.19. *Let G be a divisible group and E a finite group. Then G is characteristic in $G \times E$.*

In particular, if E is non-trivial, then G is a proper finite index characteristic subgroup of $G \times E$.

Proof. Let $\varphi \in \mathrm{Aut}(G \times E)$. Let $\iota : G \to G \times E$ and $\pi : G \times E \to E$ be the canonical inclusion and projection, respectively. Then $\pi \circ \varphi \circ \iota$ is a homomorphism from G to E. By Proposition 2.2.17, this map is trivial. Therefore, for every $g \in G$, $\varphi(g) = (g', 1)$ for some $g' \in G$. Consequently, $\varphi(G) \leq G$, i.e. G is characteristic in $G \times E$.

If E is non-trivial, then $[G \times E : G] = |E| > 1$. Thus, G is a proper finite index subgroup of $G \times E$. $\square$

Thus, all groups of the form $G \times E$ with G divisible and E non-trivial finite satisfy the conditions of Theorem 2.2.13, since they are also freely indecomposable by Lemma 2.2.14.

It is interesting to note that Theorem 2.2.13 does *not* cover free products of the form $\mathbb{Q}^r * \mathbb{Q}^s$ for integers $r, s \geq 1$, since $\mathbb{Q}^r$ has no proper finite index subgroups by Corollary 2.2.18.

Example 2.2.20 (See e.g. [47, Remark 6(ii)])**.** Let $\mathcal{P}$ be the set of all prime numbers. Let $P \subseteq \mathcal{P}$ be a proper non-empty subset. Let $\mathbb{Z}[P] \subseteq \mathbb{Q}$ be the subring generated by $\mathbb{Z}$ and $\{1/q \mid q \in P\}$. Then every element in $\mathbb{Z}[P]$ is of the form $\frac{a}{b}$, where the prime factors of b all lie in P.

Let $p \in \mathcal{P} \setminus P$ and let $\pi : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ be the natural projection between the rings $\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$. We claim that

$$\pi' : \mathbb{Z}[P] \to \mathbb{Z}/p\mathbb{Z} : x = \frac{a}{b} \mapsto \pi(a)\pi(b)^{-1} \tag{2.2.1}$$

is a well-defined ring homomorphism as well; in particular, it is a group homomorphism between the additive groups $\mathbb{Z}[P]$ and $\mathbb{Z}/p\mathbb{Z}$. Let $\frac{a}{b} \in \mathbb{Z}[P]$ be arbitrary. First, since b only contains prime factors in P, $\pi(b) \neq 0$. Second, if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$, so $\pi(ad) = \pi(bc)$, which implies that $\pi(a)\pi(b)^{-1} = \pi(c)\pi(d)^{-1}$. This shows that $\pi'(x)$ is independent of the representation of x as a fraction.

Third, let $x = \frac{a}{b}, y = \frac{c}{d} \in \mathbb{Z}[P]$ be arbitrary. It is clear that $\pi'(xy) = \pi'(x)\pi(y)$. For addition, note that

$$\begin{aligned}
\pi'(x+y) &= \pi\left(\frac{ad+bc}{bd}\right) \\
&= \pi(ad+bc)\pi(bd)^{-1} \\
&= \pi(ad)\pi(bd)^{-1} + \pi(bc)\pi(bd)^{-1} \\
&= \pi(a)\pi(b)^{-1} + \pi(c)\pi(d)^{-1} \\
&= \pi'(x) + \pi'(y).
\end{aligned}$$

We conclude that π' is a well-defined ring homomorphism. It is surjective, hence its kernel K_p is a proper finite index subgroup.

Note that $K_p = p\mathbb{Z}[p] := \{px \mid x \in \mathbb{Z}[P]\}$. Indeed, it is clear that $\pi'(p\mathbb{Z}[P]) = 0$. Conversely, if $\pi'(x) = 0$ with $x = \frac{a}{b}$, then $a \equiv 0 \bmod p$. Writing $a = pA$ for some $A \in \mathbb{Z}$, we see that $x = p \cdot (A/b) \in p\mathbb{Z}[P]$, which proves that $K_p = p\mathbb{Z}[P]$. Finally, let $\varphi : \mathbb{Z}[P] \to \mathbb{Z}[P]$ be an arbitrary group homomorphism on $\mathbb{Z}[P]$ as additive group. Since $\varphi(px) = p\varphi(x)$ for all $x \in \mathbb{Z}[P]$, we see that $\varphi(p\mathbb{Z}[P]) \leq p\mathbb{Z}[P]$. As φ is arbitrary, we conclude that $p\mathbb{Z}[P]$ is fully characteristic. Therefore, $\mathbb{Z}[P]$ satisfies the condition of Theorem 2.2.13.

Furthermore, if $P \neq P'$ are both proper non-empty subsets of $\mathcal{P}$, then $\mathbb{Z}[P]$ and $\mathbb{Z}[P']$ are non-isomorphic groups. Indeed, without loss of generality, there exists a prime $p \in P \setminus P'$. Let $x \in \mathbb{Z}[P]$. Then $x = p \cdot (x/p)$, which proves that each element in $\mathbb{Z}[P]$ has a pth root (additively). On the other hand, as $p \notin P'$, we have the map $\pi' : \mathbb{Z}[P'] \to \mathbb{Z}/p\mathbb{Z}$ from (2.2.1). If $1 \in \mathbb{Z}[P']$ were to have a pth root y, then

$$1 = \pi'(1) = \pi'(py) = 0,$$

which is a contradiction. Therefore, $\mathbb{Z}[P]$ and $\mathbb{Z}[P']$ are non-isomorphic groups.

Combining everything, we thus obtain an uncountable family of non-isomorphic groups, each of which satisfies the conditions of Theorem 2.2.13. ∎

Variation 3

Direct products

Mozart, Wolfgang Amadeus, *Zwölf Variationen in C über das französische Lied 'Ah, vous dirai-je, Maman'*, (KV 265 (300e)), Variation III (bars 1–8).

In this Variation, we study another product construction, namely direct products. To study Reidemeister spectra of direct products, we adopt two approaches On the one hand, we study twisted conjugacy itself and how it behaves under the construction of a direct product. On the other hand, we investigate the automorphism group (or more generally, the endomorphism monoid) of a direct product and see how it relates to the automorphism groups of the individual factors.

The contents of this Variation are largely the same as in [109].

3.1 Matrix description of endomorphism monoid of direct product

Given a group G, the set $\mathrm{End}(G)$ of all endomorphisms on G forms a monoid under composition, with the identity map as neutral element. For endomorphisms of direct products of groups, there exists an alternative way to represent this monoid by means of matrices of group homomorphisms, as described by e.g. F. Johnson in [64, §1], J. Bidwell, M. Curran and D. McCaughan in [8, Theorem 1.1] and by J. Bidwell in [7, Lemma 2.1].

If $\varphi, \psi : G \to H$ are two homomorphisms with commuting images, then it is easily seen that $\varphi{+}\psi : G \to H : g \mapsto (\varphi{+}\psi)(g) := \varphi(g)\psi(g)$ is a homomorphism as well.

Now, let $G_1, \ldots, G_n$ be groups. Define $\mathcal{M}$ to be the set

$$\left\{ \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & \ddots & \vdots \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{pmatrix} \;\middle|\; \begin{array}{c} \forall 1 \leq i, j \leq n : \varphi_{ij} \in \operatorname{Hom}(G_j, G_i) \\ \forall 1 \leq i, k, l \leq n : k \neq l \implies [\operatorname{Im}\varphi_{ik}, \operatorname{Im}\varphi_{il}] = 1 \end{array} \right\}$$

and equip it with matrix multiplication, where the addition of two homomorphisms $\varphi, \psi \in \operatorname{Hom}(G_j, G_i)$ with commuting images is defined as above and the multiplication of $\psi \in \operatorname{Hom}(G_i, G_k)$ and $\varphi \in \operatorname{Hom}(G_j, G_i)$ is defined as $\psi \circ \varphi$. It is readily verified that this puts a monoid structure on $\mathcal{M}$, where the diagonal matrix with the respective identity maps on the diagonal is the neutral element.

Lemma 3.1.1. *For* $G = \bigtimes_{i=1}^{n} G_i$*,* $\operatorname{End}(G) \cong \mathcal{M}$ *as monoids.*

F. Johnson proves this result for $G_1 = \ldots = G_n$ and although J. Bidwell, M. Curran and D. McCaughan state in both aforementioned papers that they only consider finite groups, their proof holds up for infinite groups as well. For the sake of completeness, we give a proof here as well.

Proof. For $1 \leq i \leq n$, let $\pi_i : G \to G_i$ denote the canonical projection and $e_i : G_i \to G$ the canonical inclusion. Given $\varphi \in \operatorname{End}(G)$, put $\varphi_{ij} := \pi_i \circ \varphi \circ e_j \in \operatorname{Hom}(G_j, G_i)$.

Fix $i, k, l \in \{1, \ldots, n\}$ with $k \neq l$. If $g_k \in G_k$ and $g_l \in G_l$, then $e_k(g_k)$ and $e_l(g_l)$ commute in G. Hence, $\varphi(e_k(g_k))$ and $\varphi(e_l(g_l))$ commute as well, which implies that $\varphi_{ik}(g_k)$ and $\varphi_{il}(g_l)$ commute too. Therefore, $[\operatorname{Im}\varphi_{ik}, \operatorname{Im}\varphi_{il}] = 1$.

Since the commuting condition is satisfied, we can define $F : \operatorname{End}(G) \to \mathcal{M}$ by putting $F(\varphi) := (\varphi_{ij})_{ij}$. If $\varphi, \psi \in \operatorname{End}(G)$, then we need to prove for all $1 \leq i, j \leq n$ that

$$\pi_i \circ \varphi \circ \psi \circ e_j = (\varphi \circ \psi)_{ij} = \sum_{k=1}^{n} \varphi_{ik}\psi_{kj}.$$

For $1 \leq i, j \leq n$ and $g \in G_j$, we see that

$$\begin{aligned}
(\pi_i \circ \varphi \circ \psi \circ e_j)(g) &= (\pi_i \circ \varphi \circ \psi)(1, \ldots, 1, g, 1, \ldots, 1) \\
&= (\pi_i \circ \varphi)(\psi_{1j}(g), \ldots, \psi_{nj}(g)) \\
&= (\pi_i \circ \varphi)(e_1(\psi_{1j}(g)) \ldots e_n(\psi_{nj}(g))) \\
&= \prod_{k=1}^{n} (\pi_i \circ \varphi \circ e_k)(\psi_{kj}(g)) \\
&= \prod_{k=1}^{n} (\varphi_{ik} \circ \psi_{kj})(g) \\
&= \left(\sum_{k=1}^{n} \varphi_{ik} \psi_{kj} \right)(g),
\end{aligned}$$

hence the equality holds. Therefore, F is a monoid homomorphism. It is also clear that F is injective.

To prove that F is surjective, let $(\varphi_{ij})_{ij} \in \mathcal{M}$ and define

$$\varphi : G \to G : (g_1, \ldots, g_n) \mapsto \left(\prod_{k=1}^{n} \varphi_{1k}(g_k), \ldots, \prod_{k=1}^{n} \varphi_{nk}(g_k) \right).$$

Due to the commuting conditions and the fact that all φ_{ij} are group homomorphisms, the map φ is a well-defined endomorphism of G and it is clear that $F(\varphi) = (\varphi_{ij})_{ij}$. □

We often identify an endomorphism of G with is image under F and write $\varphi = (\varphi_{ij})_{ij}$. In a matrix, we let 1 denote the identity map and 0 the trivial homomorphism.

Lemma 3.1.2. *With the notations as above, let $\varphi \in \operatorname{Aut}(G)$. Then*

(1) for all $1 \leq i \leq n$, G_i is generated by $\{\operatorname{Im} \varphi_{ij} \mid 1 \leq j \leq n\}$;

(2) $\operatorname{Im} \varphi_{ij}$ is normal in G_i for all $1 \leq i, j \leq n$.

Proof. Suppose φ is an automorphism. Then φ is surjective. Let $1 \leq i \leq n$. Then $G_i = \pi_i(\varphi(G))$. Since G is generated by $\{e_j(G_j) \mid 1 \leq j \leq n\}$, we see that G_i is generated by $\{(\pi_i \circ \varphi \circ e_j)(G_j) \mid 1 \leq j \leq n\} = \{\operatorname{Im} \varphi_{ij} \mid 1 \leq j \leq n\}$. This proves the first item.

For the second, fix $1 \leq i, j \leq n$. Let $g \in \operatorname{Im} \varphi_{ij}$ and $h \in G_i$ be arbitrary. We can write $h = xy$ for some $x \in \operatorname{Im} \varphi_{ij}$ and $y \in \langle \{\operatorname{Im} \varphi_{ik} \mid k \neq j\} \rangle$, as G_i is generated by $\operatorname{Im} \varphi_{i1}, \ldots, \operatorname{Im} \varphi_{in}$ and the images of φ_{ij} and φ_{ik} commute if $j \neq k$. Then

$$[g, h] = g^{-1}(xy)^{-1}gxy = g^{-1}y^{-1}x^{-1}gxy = g^{-1}x^{-1}gx = [g, x] \in \operatorname{Im} \varphi_{ij}.$$

Consequently, $[\operatorname{Im} \varphi_{ij}, G_i] \leq \operatorname{Im} \varphi_{ij}$, which implies that $\operatorname{Im} \varphi_{ij}$ is normal in G_i. □

Lemma 3.1.3. *With the notations as above, suppose that all automorphisms of G have a matrix representation that is upper triangular, or all of them are lower triangular. Let $\varphi \in \operatorname{Aut}(G)$. Then $\varphi_{ii} \in \operatorname{Aut}(G_i)$ for each $i \in \{1, \ldots, n\}$ and $\varphi_{ij} \in \operatorname{Hom}(G_j, Z(G_i))$ for all $1 \leq i \neq j \leq n$.*

Proof. We prove the result for upper triangular matrices, the proof for lower triangular is similar.

Let $\varphi \in \operatorname{Aut}(G)$ and write $\varphi^{-1} = (\psi_{ij})_{ij}$. Fix $i \in \{1, \ldots, n\}$. Since both φ and φ^{-1} are upper triangular, we find that

$$\operatorname{Id}_{G_i} = (\varphi \circ \varphi^{-1})_{ii} = \varphi_{ii} \circ \psi_{ii}$$

and

$$\operatorname{Id}_{G_i} = (\varphi^{-1} \circ \varphi)_{ii} = \psi_{ii} \circ \varphi_{ii},$$

which shows that φ_{ii} must be an automorphism of G_i.

Now, let $1 \leq i, j \leq n$ be indices with $i \neq j$. Since $\operatorname{Im} \varphi_{ij}$ and $\operatorname{Im} \varphi_{ii} = G_i$ commute, we conclude that $\operatorname{Im} \varphi_{ij} \leq Z(G_i)$. □

Using this alternative description of the endomorphism monoid, we can deduce some general results regarding Reidemeister numbers of specific endomorphisms on direct products.

We define the diagonal endomorphisms to be all endomorphisms of $\operatorname{End}(G)$ of the form

$$\operatorname{Diag}(\varphi_1, \ldots, \varphi_n) : G \to G : (g_1, \ldots, g_n) \mapsto (\varphi_1(g_1), \ldots, \varphi_n(g_n)),$$

where each $\varphi_i \in \operatorname{End}(G_i)$. We let $\operatorname{Diag}(G)$ denote the submonoid of all diagonal endomorphisms. Note that it is isomorphic with $\operatorname{End}(G_1) \times \ldots \times \operatorname{End}(G_n)$.

The following is then quite straightforward:

Proposition 3.1.4. *Let $G_1, \ldots, G_n$ be groups and put $G = \bigtimes_{i=1}^{n} G_i$. Let φ be an element of $\operatorname{Diag}(G)$ and write $\varphi = \operatorname{Diag}(\varphi_1, \ldots, \varphi_n)$. Then $R(\varphi) = \prod_{i=1}^{n} R(\varphi_i)$.*

Proof. It is clear that, for $(g_1, \ldots, g_n), (h_1, \ldots, h_n) \in G_1 \times \ldots \times G_n$, we have

$$(g_1, \ldots, g_n) \sim_\varphi (h_1, \ldots, h_n) \iff \forall i \in \{1, \ldots, n\} : g_i \sim_{\varphi_i} h_i.$$

Thus, the map

$$\mathcal{R}[\varphi] \to \mathcal{R}[\varphi_1] \times \ldots \times \mathcal{R}[\varphi_n] : [(g_1, \ldots, g_n)]_\varphi \mapsto ([g_1]_{\varphi_1}, \ldots, [g_n]_{\varphi_n})$$

is a well-defined bijection, which implies that

$$R(\varphi) = \prod_{i=1}^{n} R(\varphi_i). \qquad \square$$

Definition 3.1.5. For $a \in \mathbb{N}_0 \cup \{\infty\}$, we define the product $a \cdot \infty$ to be equal to ∞. Let $A_1, \ldots, A_n$ be subsets of $\mathbb{N}_0 \cup \{\infty\}$. We then define

$$A_1 \cdot \ldots \cdot A_n := \prod_{i=1}^{n} A_i := \{a_1 \ldots a_n \mid \forall i \in \{1, \ldots, n\} : a_i \in A_i\}.$$

If $A_1 = \ldots = A_n =: A$, we also write $A^{(n)}$ for the n-fold product of A with itself.

Corollary 3.1.6. *Let $G_1, \ldots, G_n$ be groups and put $G = \bigtimes_{i=1}^{n} G_i$. Then*

$$\prod_{i=1}^{n} \mathrm{Spec}_{\mathrm{R}}(G_i) \subseteq \mathrm{Spec}_{\mathrm{R}}(G).$$

Equality holds if $\mathrm{Aut}(G) = \bigtimes_{i=1}^{n} \mathrm{Aut}(G_i)$.

More generally,

$$\prod_{i=1}^{n} \mathrm{ESpec}_{\mathrm{R}}(G_i) \subseteq \mathrm{ESpec}_{\mathrm{R}}(G).$$

We now specify to the case of an n-fold direct product of a group with itself, i.e. G^n for some group G and integer $n \geq 1$. First of all, the symmetric group S_n embeds in $\mathrm{End}(G^n)$ in the following way:

$$S_n \to \mathrm{End}\, G^n : \sigma \mapsto (P_{\sigma^{-1}} : G^n \to G^n : (g_1, \ldots, g_n) \mapsto (g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)})),$$

where the matrix representation of $P_{\sigma^{-1}}$ is given by

$$P_{\sigma^{-1}} = \begin{pmatrix} \varepsilon_{\sigma^{-1}(1)} \\ \vdots \\ \varepsilon_{\sigma^{-1}(n)} \end{pmatrix}.$$

Here, ε_i is a row with a 1 on the i-th spot and zeroes elsewhere.

To prove that is indeed a monoid morphism, let $\sigma, \tau \in S_n$ and $(g_1, \ldots, g_n) \in G^n$ be arbitrary. Put $h_i := g_{\sigma^{-1}(i)}$, then

$$P_{\tau^{-1}}(h_1, \ldots, h_n) = (h_{\tau^{-1}(1)}, \ldots, h_{\tau^{-1}(n)}) = (g_{\sigma^{-1}(\tau^{-1}(1))}, \ldots, g_{\sigma^{-1}(\tau^{-1}(n))})$$

and

$$P_{(\tau\sigma)^{-1}}(g_1, \ldots, g_n) = (g_{\sigma^{-1}(\tau^{-1}(1))}, \ldots, g_{\sigma^{-1}(\tau^{-1}(n))}).$$

Therefore, $P_{(\tau\sigma)^{-1}} = P_{\tau^{-1}} P_{\sigma^{-1}}$.

Now, we define $\mathrm{End}_w(G^n)$ to be the submonoid of $\mathrm{End}(G^n)$ generated by S_n and $\mathrm{Diag}(G^n)$.

Lemma 3.1.7. *Let G be a group and $n \geq 1$ an integer. Then each endomorphism φ in $\mathrm{End}_w(G^n)$ can be written as*

$$\varphi = \mathrm{Diag}(\varphi_1, \ldots, \varphi_n) P_{\sigma^{-1}}$$

for some $\varphi_i \in \mathrm{End}(G)$ and $\sigma \in S_n$. Moreover, $\varphi \in \mathrm{Aut}(G^n)$ if and only if $\varphi_i \in \mathrm{Aut}(G)$ for each $i \in \{1, \ldots, n\}$.

Proof. The existence of such a decomposition relies on the following equality, which we claim holds for all $\sigma \in S_n$ and $\varphi_i \in \mathrm{End}(G)$:

$$P_{\sigma^{-1}} \mathrm{Diag}(\varphi_1, \ldots, \varphi_n) = \mathrm{Diag}(\varphi_{\sigma^{-1}(1)}, \ldots, \varphi_{\sigma^{-1}(n)}) P_{\sigma^{-1}}. \tag{3.1.1}$$

Indeed, evaluating the left-hand side in $(g_1, \ldots, g_n)$ yields

$$P_{\sigma^{-1}}(\varphi_1(g_1), \ldots, \varphi_n(g_n)) = (\varphi_{\sigma^{-1}(1)}(g_{\sigma^{-1}(1)}), \ldots, \varphi_{\sigma^{-1}(n)}(g_{\sigma^{-1}(n)}))$$

whereas the right-hand side yields

$$\mathrm{Diag}(\varphi_{\sigma^{-1}(1)}, \ldots, \varphi_{\sigma^{-1}(n)})(g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)}),$$

hence we see they are equal.

Thus, given an element in $\mathrm{End}_w(G^n)$, we can apply the equality above several times in order to gather all diagonal endomorphisms and all elements of the form $P_{\sigma^{-1}}$ together, which yields the desired representation.

The claim regarding the automorphism is immediate, as each $P_{\sigma^{-1}}$ is an automorphism. □

The representation is not necessarily unique, as the trivial endomorphism equals $\mathrm{Diag}(0, \ldots, 0) P_{\sigma^{-1}}$ for all $\sigma \in S_n$. If we restrict ourselves to automorphisms, however, we have an injective group homomorphism

$$\Psi : \mathrm{Aut}(G) \wr S_n \to \mathrm{Aut}(G^n) : (\varphi, \sigma) = (\varphi_1, \ldots, \varphi_n, \sigma) \mapsto \mathrm{Diag}(\varphi_1, \ldots, \varphi_n) P_{\sigma^{-1}}.$$

Here, $\mathrm{Aut}(G) \wr S_n$ is the wreath product, i.e. the semi-direct product $\mathrm{Aut}(G)^n \rtimes S_n$, where the action is given by

$$\sigma \cdot (\varphi_1, \ldots, \varphi_n) = (\varphi_{\sigma^{-1}(1)}, \ldots, \varphi_{\sigma^{-1}(n)}).$$

To see that is indeed homomorphism, note that, for $(\varphi, \sigma), (\psi, \tau) \in \mathrm{Aut}(G) \wr S_n$,

$$\Psi((\varphi, \sigma)(\psi, \tau)) = \Psi(\varphi \circ (\sigma \cdot \psi), \sigma\tau) = \mathrm{Diag}(\varphi_1 \psi_{\sigma^{-1}(1)}, \ldots, \varphi_n \psi_{\sigma^{-1}(n)}) P_{(\sigma\tau)^{-1}}$$

and

$$\begin{aligned}
\Psi(\varphi, \sigma)\Psi(\psi, \tau) &= \mathrm{Diag}(\varphi_1, \ldots, \varphi_n) P_{\sigma^{-1}} \mathrm{Diag}(\psi_1, \ldots, \psi_n) P_{\tau^{-1}} \\
&= \mathrm{Diag}(\varphi_1, \ldots, \varphi_n) \mathrm{Diag}(\psi_{\sigma^{-1}(1)}, \ldots, \psi_{\sigma^{-1}(n)}) P_{\sigma^{-1}} P_{\tau^{-1}} \\
&= \mathrm{Diag}(\varphi_1 \psi_{\sigma^{-1}(1)}, \ldots, \varphi_n \psi_{\sigma^{-1}(n)}) P_{(\sigma\tau)^{-1}}
\end{aligned}$$

where we used (3.1.1). We identify $\mathrm{Aut}(G) \wr S_n$ with its image under Ψ and thus regard it as a subgroup of $\mathrm{Aut}(G^n)$.

We now determine the Reidemeister number of an arbitrary element of $\mathrm{End}_w(G^n)$, which also generalises [24, Proposition 5.1.2].

Proposition 3.1.8. *Let G be a group and let $\varphi := \mathrm{Diag}(\varphi_1, \ldots, \varphi_n) P_\sigma \in \mathrm{End}_w(G^n)$. Let*

$$\sigma = (c_1 \ldots c_{n_1})(c_{n_1+1} \ldots c_{n_2}) \ldots (c_{n_{k-1}+1} \ldots c_{n_k})$$

be the disjoint cycle decomposition of σ, where $n_0 := 0 < n_1 < n_2 < \ldots < n_k = n$. Put $\tilde{\varphi}_j := \varphi_{c_{n_{j-1}+1}} \circ \ldots \circ \varphi_{c_{n_j}}$. Then

$$R(\varphi) = \prod_{j=1}^{k} R(\tilde{\varphi}_j).$$

Proof. Let $\tau \in S_n$ be the permutation given by $\tau(i) = c_i$ for all $i \in \{1, \ldots, n\}$. Then

$$\sigma^\tau = \tau^{-1}\sigma\tau = (1 \ldots n_1)(n_1 + 1 \ \ldots n_2) \ldots (n_{k-1} + 1 \ldots n_k).$$

By Lemma T.1.7(1), conjugate endomorphisms have the same Reidemeister number. Since, by (3.1.1),

$$P_\tau \mathrm{Diag}(\varphi_1, \ldots, \varphi_n) P_\sigma P_{\tau^{-1}} = \mathrm{Diag}(\varphi_{\tau(1)}, \ldots, \varphi_{\tau(n)}) P_{\sigma^\tau},$$

it is sufficient to determine the Reidemeister number of the latter endomorphism. Note that P_{σ^τ} is a block matrix consisting of k square blocks of the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

on the diagonal and zero matrices elsewhere. This implies that

$$\mathrm{Diag}(\varphi_{\tau(1)}, \dots, \varphi_{\tau(n)})P_{\sigma^\tau} \in \bigtimes_{i=1}^{k} \mathrm{End}(G^{n_i - n_{i-1}}).$$

Therefore, it is sufficient to determine Reidemeister numbers of endomorphisms of the form $\psi := \mathrm{Diag}(\psi_1, \dots, \psi_n)P_\alpha$, where $\alpha = (1\ 2 \dots\ n)$. Indeed, if we know that $R(\psi) = R(\psi_1 \circ \dots \circ \psi_n)$, then

$$\begin{aligned} R(\varphi) &= \prod_{i=1}^{k} R\big(\varphi_{\tau(n_{i-1}+1)} \circ \dots \circ \varphi_{\tau(n_i)}\big) \\ &= \prod_{i=1}^{k} R(\varphi_{c_{n_{i-1}+1}} \circ \dots \circ \varphi_{c_{n_i}}) \\ &= \prod_{j=1}^{k} R(\tilde{\varphi}_j), \end{aligned}$$

by construction of τ.

So, let $(g_1, \dots, g_n) \in G^n$. We claim there exists a $g \in G$ such that

$$(g_1, \dots, g_n) \sim_\psi (g, 1, \dots, 1).$$

Note that, for $(x_1, \dots, x_n) \in G^n$,

$$(x_1, \dots, x_n)(g_1, \dots, g_n)\psi(x_1, \dots, x_n)^{-1} = \Big(x_1 g_1 \psi_1(x_2)^{-1}, \dots, x_n g_n \psi_n(x_1)^{-1}\Big).$$

Put $x_1 := 1, x_n := g_n^{-1}$ and $x_i := \psi_i(x_{i+1})g_i^{-1}$ for $i \in \{2, \dots, n-1\}$, starting with $i = n-1$. Finally, put $g := x_1 g_1 \psi_1(x_2)^{-1}$. Then

$$\begin{cases} g = x_1 g_1 \psi_1(x_2)^{-1} \\ 1 = x_i g_i \psi_i(x_{i+1})^{-1} & \text{for } i \geq 2, \end{cases}$$

where $x_{n+1} := x_1$. This implies that $(g_1, \ldots, g_n) \sim_\psi (g, 1, \ldots, 1)$.

Next, put $\tilde{\psi} := \psi_1 \circ \ldots \circ \psi_n$ and suppose $(g, 1, \ldots, 1) \sim_\psi (h, 1, \ldots, 1)$ for some $g, h \in G$. Then there exist $x_1, \ldots, x_n \in G$ such that

$$\begin{cases} g = x_1 h \psi_1(x_2)^{-1} \\ 1 = x_i \psi_i(x_{i+1})^{-1} & \text{for } i \geq 2, \end{cases}$$

where again $x_{n+1} := x_1$. Consequently, $x_i = \psi_i(x_{i+1})$ for $i \geq 2$. This implies that

$$g = x_1 h \psi_1(x_2)^{-1} = x_1 h \psi_1(\psi_2(x_3))^{-1} = \ldots = x_1 h \tilde{\psi}(x_1)^{-1},$$

i.e. $g \sim_{\tilde{\psi}} h$. Conversely, if $g \sim_{\tilde{\psi}} h$, then there exists an $x \in G$ such that $g = xh\tilde{\psi}(x)^{-1}$. Put $x_1 := x$ and $x_i := \psi_i(x_{i+1})$ for $i \geq 2$, starting with $i = n$ and where, again, $x_{n+1} := x_1$. Then

$$\begin{cases} g = x_1 h \psi_1(x_2)^{-1} \\ 1 = x_i \psi_i(x_{i+1})^{-1} & \text{for } i \geq 2, \end{cases}$$

hence $(g, 1, \ldots, 1) \sim_\psi (h, 1, \ldots, 1)$.

Combining all results, we find that there is a bijection between the ψ-conjugacy classes and the $\tilde{\psi}$-conjugacy classes. Consequently, $R(\psi) = R(\tilde{\psi})$. □

Corollary 3.1.9. *Let G be a group and $n \geq 1$ a natural number. Then*

$$\{R(\varphi) \mid \varphi \in \operatorname{Aut}(G) \wr S_n\} = \bigcup_{i=1}^{n} \operatorname{Spec}_{\mathrm{R}}(G)^{(i)} \subseteq \operatorname{Spec}_{\mathrm{R}}(G^n).$$

Equality holds if $\operatorname{Aut}(G^n) = \operatorname{Aut}(G) \wr S_n$.

More generally,

$$\{R(\varphi) \mid \varphi \in \operatorname{End}_w(G^n)\} = \bigcup_{i=1}^{n} \operatorname{ESpec}_{\mathrm{R}}(G)^{(i)} \subseteq \operatorname{ESpec}_{\mathrm{R}}(G^n).$$

Proof. Let $1 \leq k \leq n$. Let $\varphi_1, \ldots, \varphi_k$ be automorphisms of G. We prove that $R(\varphi_1) \ldots R(\varphi_k) \in \operatorname{Spec}_{\mathrm{R}}(G^n)$. Consider the automorphism

$$\varphi := \operatorname{Diag}(\varphi_1, \ldots, \varphi_k, \operatorname{Id}_G, \ldots, \operatorname{Id}_G) P_\sigma$$

where $\sigma = (1)(2)(3) \ldots (k-1)(k \; k+1 \; \ldots \; n-1 \; n)$. By the previous proposition $R(\varphi) = R(\varphi_1) \ldots R(\varphi_k)$. The same argument holds if $\varphi_1, \ldots, \varphi_k$ are endomorphisms of G.

This combined with the previous proposition also proves the first equality, from which the additional claim follows immediately. □

Corollary 3.1.10. *Let G be a group and $n \geq 1$ an integer. Suppose that $G \in R_\infty$ and that $\operatorname{Aut}(G^n) = \operatorname{Aut}(G) \wr S_n$. Then $G^n \in R_\infty$.*

Proof. Since $\operatorname{Aut}(G) \wr S_n = \operatorname{Aut}(G^n)$, the previous corollary shows that

$$\operatorname{Spec}_{\mathrm{R}}(G^n) = \bigcup_{i=1}^{n} \operatorname{Spec}_{\mathrm{R}}(G)^{(i)} = \bigcup_{i=1}^{n} \{\infty\}^{(i)} = \{\infty\}. \qquad \square$$

Example 3.1.11. Although Corollary 3.1.9 yields some information regarding $\operatorname{Spec}_{\mathrm{R}}(G^n)$, this information can be limited. Consider the case $G = \mathbb{Z}$. Recall from Theorem T.1.13 that, for $r \geq 1$,

$$\operatorname{Spec}_{\mathrm{R}}(\mathbb{Z}^r) = \begin{cases} \{2, \infty\} & \text{if } r = 1 \\ \mathbb{N}_0 \cup \{\infty\} & \text{otherwise.} \end{cases}$$

However, using only the result for $r = 1$, Corollary 3.1.9 merely yields $\{2^i \mid 1 \leq i \leq n\} \cup \{\infty\} \subseteq \operatorname{Spec}_{\mathrm{R}}(\mathbb{Z}^n)$. Nonetheless, we will later provide conditions under which Corollaries 3.1.9 and 3.1.10 yield full information on $\operatorname{Spec}_{\mathrm{R}}(G^n)$. ‖

We can also generalise Lemma T.1.7(1).

Corollary 3.1.12. *Let G be a group, $n \geq 1$ an integer and $\varphi_1, \ldots, \varphi_n \in \operatorname{End}(G)$. Then*

$$R(\varphi_1 \circ \varphi_2 \circ \ldots \circ \varphi_n) = R(\varphi_2 \circ \varphi_3 \ldots \circ \varphi_n \circ \varphi_1).$$

Proof. Consider the endomorphism $\varphi := \operatorname{Diag}(\varphi_1, \ldots, \varphi_n) P_{(1\ 2\ \ldots\ n)}$ of G^n. Since $(1\ 2\ \ldots\ n)$ and $(2\ \ldots\ n\ 1)$ are both cycle representations of the same permutation, Proposition 3.1.8 yields

$$R(\varphi_1 \circ \varphi_2 \circ \ldots \circ \varphi_n) = R(\varphi) = R(\varphi_2 \circ \varphi_3 \circ \ldots \circ \varphi_n \circ \varphi_1). \qquad \square$$

Remark. For $G = \mathbb{Z}^m$, Corollary 3.1.12 takes on a familiar form when considering two endomorphisms. Let A and B be two $(m \times m)$-matrices with integer entries. Combined with Proposition T.1.12, Corollary 3.1.12 then states that

$$|\det(I - AB)|_\infty = |\det(I - BA)|_\infty.$$

The equality also holds without $|\cdot|_\infty$ and is often attributed to J. Sylvester, who stated a more general result in [114], albeit without proof.

3.2 Direct products of two groups

We now restrict to the case of direct products of two groups. Let H and K be two groups and put $G := H \times K$. Instead of using indices to represent endomorphisms of G as matrices, we use the notation

$$\mathcal{M} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;\middle|\; \begin{matrix} \alpha \in \mathrm{End}(H), & \beta \in \mathrm{Hom}(K, H), & [\mathrm{Im}\,\alpha, \mathrm{Im}\,\beta] = 1 \\ \gamma \in \mathrm{Hom}(H, K), & \delta \in \mathrm{End}(K), & [\mathrm{Im}\,\gamma, \mathrm{Im}\,\delta] = 1 \end{matrix} \right\}.$$

Let $\varphi \in \mathrm{End}(G)$ be an endomorphism leaving H invariant. Then, in the notation above, γ is the trivial homomorphism, which sends everything to 1. The induced endomorphisms on H and $K \cong \frac{H \times K}{H}$ are α and δ, respectively. By Proposition 1.1.3(2),

$$R(\varphi) = \sum_{[k]_\delta \in \mathcal{R}[\delta]} |\mathrm{Im}\,\hat{\imath}_k|,$$

where $\hat{\imath}_k : \mathcal{R}[(\tau_k \circ \varphi)|_H] \to \mathcal{R}[\tau_k \circ \varphi]$ for each $k \in K$. Since K and H commute in G, $(\tau_k \circ \varphi)|_H = \alpha$ for all $k \in K$.

Let $k \in K$ and $h, h' \in H$ be arbitrary. Then the following chain of equivalences holds:

$$\begin{aligned} &\hat{\imath}_k([h]_\alpha) = \hat{\imath}_k([h']_\alpha) \\ &\iff \exists (x, y) \in H \times K : (h, 1) = (x, y)(h', 1)(1, k)(\alpha(x)\beta(y), \delta(y))^{-1}(1, k^{-1}) \\ &\iff \exists (x, y) \in H \times K : h = xh'(\alpha(x)\beta(y))^{-1}, k = yk\delta(y)^{-1} \\ &\iff \exists (x, y) \in H \times \mathrm{Stab}_\delta(k) : h = xh'\alpha(x)^{-1}\beta(y)^{-1}. \end{aligned}$$

We claim that the last statement is equivalent with

$$\exists y \in \mathrm{Stab}_\delta(k) : [h]_\alpha = [h'\beta(y)]_\alpha. \tag{3.2.1}$$

It is clear that the last statement implies (3.2.1). For the converse, suppose that $[h]_\alpha = [h'\beta(y)]_\alpha$ for some $y \in \mathrm{Stab}_\delta(k)$. Then $h = xh'\beta(y)\alpha(x)^{-1}$ for some $x \in H$. Since $\mathrm{Im}\,\alpha$ and $\mathrm{Im}\,\beta$ commute, we can rewrite this as $h = xh'\alpha(x)^{-1}\beta(y)$. Combined with the fact that $\mathrm{Stab}_\delta(k)$ is a subgroup, this proves that the two statements are equivalent.

Based on (3.2.1), we define, for each $k \in K$, an action of $\mathrm{Stab}_\delta(k)$ on $\mathcal{R}[\alpha]$ in the following way:

$$\rho_k : \mathcal{R}[\alpha] \times \mathrm{Stab}_\delta(k) \to \mathcal{R}[\alpha] : ([h]_\alpha, y) \mapsto [h\beta(y)]_\alpha.$$

The action is independent of the representative, for if $h' = xh\alpha(x)^{-1}$, then

$$h'\beta(y) = xh\alpha(x)^{-1}\beta(y) = xh\beta(y)\alpha(x)^{-1},$$

since $[\operatorname{Im}\alpha, \operatorname{Im}\beta] = 1$. Moreover, if $y, y' \in \operatorname{Stab}_\delta(k)$, then

$$\rho_k([h\beta(y')]_\alpha, y) = [h\beta(y')\beta(y)]_\alpha = [h\beta(y'y)]_\alpha = \rho_k([h]_\alpha, y'y).$$

Thus, it is indeed an action. Therefore, from (3.2.1) it follows that

$$\hat{\imath}_k([h]_\alpha) = \hat{\imath}_k([h']_\alpha) \iff [h]_\alpha \text{ and } [h']_\alpha \text{ lie in the same } \rho_k\text{-orbit.}$$

Combining all of the previous, we obtain the following result:

Theorem 3.2.1. *With the notations as above,*

$$R(\varphi) = \sum_{[k]_\delta \in \mathcal{R}[\delta]} \#\mathit{Orbits\ of\ } \rho_k.$$

In particular, $R(\varphi) \leq R(\alpha)R(\delta)$.

Of course, the analogous result for φ leaving K invariant holds as well.

Remark. Similarly as for Theorem 1.2.3, we can also derive Theorem 3.2.1 by picking representatives $\{h_i\}_{i\in\mathcal{I}}$ of $\mathcal{R}[\alpha]$ and $\{k_j\}_{j\in\mathcal{J}}$ of $\mathcal{R}[\delta]$, showing that $\mathcal{R}[\varphi] = \{[(h_i, k_j)]_\varphi \mid i \in \mathcal{I}, j \in \mathcal{J}\}$, and determining when $(h_i, k_j) \sim_\varphi (h_l, k_m)$ for $i, l \in \mathcal{I}, j, m \in \mathcal{J}$. This approach yields the same action ρ_k.

We now use this theorem to determine the Reidemeister spectrum of direct products of the form $G \times F$, where F is a group generated by torsion elements and G is a finitely generated torsion-free residually finite group, generalising a result due to A. Fel'shtyn (see [28, Proposition 3]).

Theorem 3.2.2. *Let G and H be groups with G finitely generated torsion-free residually finite. Suppose that H is characteristic in $G \times H$. Then*

$$\operatorname{Spec}_{\mathrm{R}}(G \times H) = \operatorname{Spec}_{\mathrm{R}}(G) \cdot \operatorname{Spec}_{\mathrm{R}}(H).$$

If H is fully characteristic, the same equality holds for the extended Reidemeister spectra.

Proof. Since H is characteristic in $G \times H$, each automorphism of $G \times H$ is of the form

$$\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix},$$

and $\alpha \in \operatorname{Aut}(G)$, $\delta \in \operatorname{Aut}(H)$ and $\gamma \in \operatorname{Hom}(G, Z(H))$, by Lemma 3.1.3. Now, fix $\varphi \in \operatorname{Aut}(G \times H)$. We claim that $R(\varphi) = R(\alpha)R(\delta)$. We know by Corollary 1.1.2(1) that $R(\varphi) \geq R(\alpha)$. Thus, $R(\varphi) = \infty$ if $R(\alpha) = \infty$. In that case, $R(\varphi) = R(\alpha)R(\delta)$. So suppose that $R(\alpha) < \infty$. Then, by Corollary 1.2.13, we know that $\operatorname{Stab}_\alpha(g) = 1$ for all $g \in G$. This means that the action of $\operatorname{Stab}_\alpha(g)$ on $\mathcal{R}[\delta]$ is trivial for all $g \in G$, which implies that the number of orbits for each action is equal to $R(\delta)$. Applying Theorem 3.2.1 then yields

$$R(\varphi) = \sum_{[g]_\alpha \in \mathcal{R}[\alpha]} R(\delta) = R(\alpha)R(\delta).$$

This proves that $\operatorname{Spec}_{\mathrm{R}}(G \times H) \subseteq \operatorname{Spec}_{\mathrm{R}}(G) \cdot \operatorname{Spec}_{\mathrm{R}}(H)$. The converse inclusion follows directly from Corollary 3.1.6.

If H is fully characteristic, we can use almost the same argument as above, as Corollary 1.2.13 and Theorem 3.2.1 hold for arbitrary endomorphisms. The only difference is that α and δ are mere endomorphisms of G and H, respectively, and that $\gamma : G \to H$ is a homomorphism such that $\operatorname{Im}\delta$ and $\operatorname{Im}\gamma$ commute. $\square$

Corollary 3.2.3. *Let G be a finitely generated torsion-free residually finite group and F a group which is generated by torsion elements. Then*

$$\operatorname{Spec}_{\mathrm{R}}(G \times F) = \operatorname{Spec}_{\mathrm{R}}(G) \cdot \operatorname{Spec}_{\mathrm{R}}(F)$$

and

$$\operatorname{ESpec}_{\mathrm{R}}(G \times F) = \operatorname{ESpec}_{\mathrm{R}}(G) \cdot \operatorname{ESpec}_{\mathrm{R}}(F).$$

Proof. Since G is torsion-free, F is fully characteristic in $G \times F$ as it is the subgroup generated by the torsion elements. Therefore, we can apply Theorem 3.2.2. $\square$

Using the fact that finitely generated linear groups are residually finite (see e.g. [77]) and virtually torsion-free if the underlying field has characteristic zero (see e.g. [105]), one can easily generate examples of groups to which the theorem and its corollary apply. In particular, they apply to finitely generated torsion-free nilpotent groups and for several families of these groups their Reidemeister spectrum is known, see e.g. [19, 26, 117] and Collection II.

Corollary 3.2.4. *Let G be a finitely generated torsion-free centreless residually finite group and let $r \geq 1$ be an integer. Then*

$$\operatorname{Spec}_{\mathrm{R}}(G \times \mathbb{Z}^r) = \operatorname{Spec}_{\mathrm{R}}(G) \cdot \operatorname{Spec}_{\mathrm{R}}(\mathbb{Z}^r).$$

Proof. Since G is centreless, $Z(G \times \mathbb{Z}^r) = \mathbb{Z}^r$. Therefore, $\mathbb{Z}^r$ is characteristic in $G \times \mathbb{Z}^r$, so the result follows from Theorem 3.2.2. $\square$

3.3 Direct products of centreless groups

In this section, we use the matrix description of the endomorphism monoid to describe the automorphism group under certain conditions. The first result is a generalisation of one by F. Johnson [64, Corollary 2.2].

Definition 3.3.1. Let G be a group. We say that G is *directly indecomposable* if $G \cong H \times K$ for some groups H, K implies that $H = 1$ or $K = 1$.

Theorem 3.3.2. *Let $G_1, \ldots, G_n$ be non-isomorphic non-trivial, centreless, directly indecomposable groups. Let $r_1, \ldots, r_n$ be positive integers. Put $G := \bigtimes_{i=1}^{n} G_i^{r_i}$. Then*

$$\mathrm{Aut}(G) = \bigtimes_{i=1}^{n} \left(\mathrm{Aut}(G_i) \wr S_{r_i}\right)$$

In [64], there is the extra condition that there do not exist non-trivial homomorphisms $\varphi : G_i \to G_j$ with normal image for $1 \leq i < j \leq n$, but it is redundant. In fact, the proof we give is nearly identical to the one F. Johnson gave for the case $n = 1$, see [64, Theorem 1.1].

Lemma 3.3.3. *Let G be a group and let $N_1, \ldots, N_n$ be commuting normal subgroups such that $N_1 \ldots N_n = G$. If*

$$\left(\prod_{j \neq i} N_j\right) \cap N_i = 1$$

for all $i \in \{1, \ldots, n\}$, then $G \cong N_1 \times \ldots \times N_n$.

Proof. Since G is generated by N_1 up to N_n, we know that each $g \in G$ can be written as $g_1 \ldots g_n$ for some $g_i \in N_i$. This yields a surjective map

$$F : \bigtimes_{i=1}^{n} N_i \to G : (g_1, \ldots, g_n) \mapsto g_1 \ldots g_n.$$

Since the N_i commute, F is also a group homomorphism. Therefore, the only thing we need to prove is that F is injective. So, suppose that $g_1 \ldots g_n = 1$ for some $g_i \in N_i$. We can rewrite this as

$$g_1 = (g_2 \cdots g_n)^{-1}.$$

The left-hand side is an element of N_1, the right-hand side one of $N_2 \ldots N_n$. Therefore, both are trivial, which shows that $g_1 = 1$. By induction, we find that $g_i = h_i$ for all $i \in \{1, \ldots, n\}$, which shows that F is injective. $\square$

To make the notation easier, we first prove the following (equivalent) result:

Theorem 3.3.4. *Let $G_1, \ldots, G_n$ be non-trivial, centreless, directly indecomposable groups. Put $G = \bigtimes_{i=1}^{n} G_i$. Under the monoid isomorphism $\operatorname{End}(G) \cong \mathcal{M}$, $\operatorname{Aut}(G)$ corresponds to those matrices $(\varphi_{ij})_{ij}$ satisfying the following conditions:*

(1) Each row and column contains exactly one non-trivial homomorphism.

(2) Each non-trivial homomorphism is an isomorphism.

Proof. Let $\varphi = (\varphi_{ij})_{ij} \in \operatorname{Aut}(G)$ and fix $i \in \{1, \ldots, n\}$. By Lemma 3.1.2, we know that G_i is generated by $\operatorname{Im}\varphi_{i1}$ up to $\operatorname{Im}\varphi_{in}$ and that each of these images is normal in G_i. Let $k \in \{1, \ldots, n\}$ be arbitrary. Suppose that g is an element of

$$\left(\prod_{j \neq k} \operatorname{Im}\varphi_{ij}\right) \cap \operatorname{Im}\varphi_{ik} \leq G_i.$$

For $l \neq k$, we find that

$$[g, \operatorname{Im}\varphi_{il}] \subseteq [\operatorname{Im}\varphi_{ik}, \operatorname{Im}\varphi_{il}] = 1,$$

and for $l = k$, we find that

$$[g, \operatorname{Im}\varphi_{ik}] \subseteq \left[\prod_{j \neq k} \operatorname{Im}\varphi_{ij}, \operatorname{Im}\varphi_{ik}\right] = 1.$$

Therefore, $g \in Z(G_i) = 1$. We derive from this that G_i is isomorphic to the direct product of $\operatorname{Im}\varphi_{i1}$ up to $\operatorname{Im}\varphi_{in}$ by Lemma 3.3.3. As G_i is directly indecomposable, exactly one of these images is non-trivial. Since i was arbitrary, we thus find a map $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $\operatorname{Im}\varphi_{ij} \neq 1$ if and only if $j = \sigma(i)$.

If σ were not surjective, say, $m \notin \operatorname{Im}\sigma$, we find that $\operatorname{Im}\varphi_{im} = 1$ for all $i \in \{1, \ldots, n\}$. Recall that $e_i : G_i \to G$ and $\pi_i : G \to G_i$ are the canonical inclusion and projection, respectively. Then $e_m(G_m) \leq \ker(\pi_i \circ \varphi)$ for all $i \in \{1, \ldots, n\}$. We thus find that

$$e_m(G_m) \leq \bigcap_{i=1}^{n} \ker(\pi_i \circ \varphi) = \bigcap_{i=1}^{n} \varphi^{-1}(\ker(\pi_i)) = \varphi^{-1}\left(\bigcap_{i=1}^{n} \ker(\pi_i)\right) = 1,$$

which contradicts the non-triviality of G_m. Thus, σ is surjective, and therefore bijective. We thus find that the matrix representation of φ contains exactly one non-trivial homomorphism on each row and column. As φ is an automorphism, each of these non-trivial homomorphisms must be both injective and surjective. Consequently, they must all be isomorphisms. $\square$

Proof of Theorem 3.3.2. Let $\varphi \in \operatorname{Aut}(G)$. By Theorem 3.3.4, the matrix representation $(\varphi_{ij})_{ij}$ contains exactly one non-trivial homomorphism per row and column, and each of those homomorphisms is in fact an isomorphism. Due to the fact that G_i and G_j are not isomorphic for $i \neq j$, $(\varphi_{ij})_{ij}$ is of the form

$$\begin{pmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & A_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_n, \end{pmatrix}$$

where each A_i is an $(r_i \times r_i)$-matrix containing exactly one non-trivial homomorphism per row and column, and each of those homomorphisms is an automorphism of G_i. These block matrices correspond to automorphisms lying in $\operatorname{Aut}(G_i) \wr S_{r_i}$. Therefore, φ lies in $\bigtimes_{i=1}^{n} (\operatorname{Aut}(G_i) \wr S_{r_i})$. $\square$

Corollary 3.3.5. *Let $G_1, \dots, G_n$ be non-trivial, non-isomorphic, centreless, directly indecomposable groups. Let $r_1, \dots, r_n$ be positive integers and put $G = \bigtimes_{i=1}^{n} G_i^{r_i}$. Then*

$$\operatorname{Spec}_{\mathrm{R}}(G) = \prod_{i=1}^{n} \left(\bigcup_{j=1}^{r_i} \operatorname{Spec}_{\mathrm{R}}(G_i)^{(j)} \right).$$

In particular, G has the R_∞-property if and only if G_i has the R_∞-property for some $i \in \{1, \dots, n\}$.

Proof. This follows by combining Theorem 3.3.2 with Corollaries 3.1.6 and 3.1.9. $\square$

Remark. In Variation 5, we prove that the same equality holds for direct products of certain nilpotent groups. As nilpotent groups have non-trivial centre, they do not satisfy the conditions of Corollary 3.3.5. This shows that those conditions are sufficient, but not necessary.

Example 3.3.6. Recall Theorem 2.2.13 regarding the R_∞-property for free products. Since free products are centreless (see e.g. [68, Corollary 4.5]) and directly indecomposable by Lemma 2.2.14, Corollary 3.3.5 then implies that any direct product of such free products has the R_∞-property. This includes, for instance, direct products of groups isomorphic to non-abelian free groups of finite rank, the modular group $\mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, or the infinite dihedral group $D_\infty \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. ‖

Example 3.3.7. Several known results regarding automorphism groups and the R_∞-property follow from Theorem 3.3.2 and Corollary 3.3.5. For instance, A. Fel'shtyn and T. Nasybullov prove in [35, Theorem 3] that certain reductive linear algebraic groups G have the R_∞-property by proving it for the quotient group $G/R(G)$, which splits as a direct product of Chevalley groups. The latter can also be proved by combining the results from [85, 86, 87, 88, 89] with Corollary 3.3.5.

Another example concerns right-angled Artin groups (RAAGs). Each RAAG admits a unique maximal decomposition as a direct product of RAAGs (see e.g. [40, Proposition 3.1]) and N. Fullarton [38] and G. Gandini and N. Wahl [40] described the automorphism group of the RAAG in terms of the automorphism groups of the direct factors. For centreless RAAGs, this description also follows from Theorem 3.3.2. ‖

Example 3.3.8. For each $n \geq 2$, let $H_n := \mathbb{Z}^n \rtimes_{-I} \mathbb{Z}/2\mathbb{Z}$, where the action is given by inversion. K. Dekimpe, T. Kaiser and S. Tertooy determined the Reidemeister spectra of these groups [22, Proposition 5.7]:

$$\mathrm{Spec}_{\mathrm{R}}(H_n) = \begin{cases} 2\mathbb{N}_0 \cup \{3, \infty\} & \text{if } n = 2 \\ \mathbb{N}_0 \setminus \{1\} \cup \{\infty\} & \text{if } n \geq 3. \end{cases}$$

Moreover, we argue that each H_n is directly indecomposable and centreless as well. We write the elements of H_n as (x, t^i), where $x \in \mathbb{Z}^n$ and $i \in \{0, 1\}$. Consider an element of the form (x, t) with $x \in \mathbb{Z}^n$. For $y \in \mathbb{Z}^n$ and $i \in \{0, 1\}$, note that

$$\begin{aligned} (y, t^i)(x, t)(y, t^i)^{-1} &= (y + (-1)^i x, t^{i+1})((-1)^{i+1} y, t^i) \\ &= (y + (-1)^i x + y, t) = ((-1)^i x + 2y, t). \end{aligned}$$

For this to be equal to (x, t), we must have $y = 0$ and $i = 0$, or $y = x$ and $i = 1$. Consequently, the centraliser $C_{H_n}(x, t)$ of (x, t) equals $\langle (x, t) \rangle \cong \mathbb{Z}/2\mathbb{Z}$. If we take two distinct $x, y \in \mathbb{Z}^n$, then $C_{H_n}(x, t) \cap C_{H_n}(y, t) = 1$. Since the centre of H_n is contained in this intersection, the centre is trivial.

Now, suppose that $H_n \cong A \times B$ via an isomorphism $\xi : H_n \to A \times B$. Under ξ, A or B must contain $\xi(x, t)$ for some $x \in \mathbb{Z}^n$. Indeed, let $y \in \mathbb{Z}^n$ be a fixed element. Then, $\xi(y, t) = (a, b)$ for some $a \in A$ and $b \in B$. In turn,

$(a,1) = \xi(x_a, t^\alpha)$ and $(1,b) = \xi(x_b, t^\beta)$ for some $x_a, x_b \in \mathbb{Z}^n$ and $\alpha, \beta \in \{0,1\}$. As $(a,b) = (a,1)(1,b)$, also

$$(y,t) = (x_a, t^\alpha)(x_b, t^\beta) = (x_a + (-1)^\alpha x_b, t^{\alpha+\beta}).$$

In particular, $\alpha + \beta = 1$. Consequently, $\alpha = 1$ or $\beta = 1$, which implies that $\xi(x_a, t) \in A$ or $\xi(x_b, t) \in B$, respectively.

So, suppose that A contains an element $g = \xi(x,t)$ for some $x \in \mathbb{Z}^n$. The centraliser of g is then isomorphic to $\mathbb{Z}/2\mathbb{Z}$, but it also equals $C_A(g) \times B$. Since $C_A(g)$ is non-trivial and $\mathbb{Z}/2\mathbb{Z}$ is directly indecomposable, it must hold that $B = 1$. Therefore, H_n is directly indecomposable.

Consequently, we can apply Corollary 3.3.5 to compute $\mathrm{Spec_R}(H_{n_1}^{r_1} \times \ldots \times H_{n_k}^{r_k})$ for all $2 \leq n_1 < \ldots < n_k$ and $r_1, \ldots, r_k \geq 1$. For instance,

$$\mathrm{Spec_R}(H_2 \times H_3) = \big((2\mathbb{N}_0 \cup 3\mathbb{N}_0) \setminus \{2,3\}\big) \cup \{\infty\}.$$ ‖

As a last example, we prove that finite wreath products of groups with the R_∞-property have it as well.

Proposition 3.3.9. *Let G be a centreless directly indecomposable group and let F be a finite group. If G has the R_∞-property, then so does $G \wr F$.*

Proof. In [90, Theorem 9.1], it is proven that the base group K of $G \wr F$ is characteristic in $G \wr F$. This base group is isomorphic with $G^{|F|}$, thus, it has the R_∞-property by Corollary 3.3.5. Since F is finite, we can use Corollary 1.1.2(2) to conclude that $G \wr F$ has the R_∞-property. □

The next result provides us with sufficient conditions for a direct product to have the R_∞-property when one of the factors has it.

Theorem 3.3.10. *Let $G_1, \ldots, G_n, H$ be non-trivial groups such that each G_i is centreless and directly indecomposable and such that, for each $i \in \{1, \ldots, n\}$, H has no direct factor isomorphic with G_i. Let $G = \bigtimes_{i=1}^{n} G_i$ Then, under the monoid isomorphism $\mathrm{End}(G \times H) \cong \mathcal{M}$, $\mathrm{Aut}(G \times H)$ corresponds to*

$$\left\{ \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \;\middle|\; \begin{matrix} \alpha \in \mathrm{Aut}(G), & \delta \in \mathrm{Aut}(H), \\ \gamma \in \mathrm{Hom}(G, Z(H)) & \end{matrix} \right\}.$$

In other words, H is characteristic in $G \times H$.

Proof. Let $\varphi \in \mathrm{Aut}(G \times H)$ and write

$$\varphi = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1n} & \beta_1 \\ \vdots & \ddots & \vdots & \vdots \\ \varphi_{n1} & \cdots & \varphi_{nn} & \beta_n \\ \gamma_1 & \cdots & \gamma_n & \delta \end{pmatrix}$$

with, for all $1 \leq i, j \leq n$, $\varphi_{ij} \in \mathrm{Hom}(G_j, G_i), \beta_i \in \mathrm{Hom}(H, G_i), \gamma_j \in \mathrm{Hom}(G_j, H)$ and $\delta \in \mathrm{End}(H)$. By a similar argument as in the beginning of Theorem 3.3.4, we find that each of the first n rows of φ contains precisely one non-trivial homomorphism. The inverse automorphism φ^{-1} has a similar matrix form as φ. Hence, each of the first n rows of φ^{-1} also contains precisely one non-trivial homomorphism, and we let $\psi_{ij}, \beta'_i, \gamma'_j$ and δ' denote the corresponding homomorphisms of φ^{-1}.

The goal is to prove that $\beta_i = 0$ for all $1 \leq i \leq n$. First, we make some observations.

Suppose that $\varphi_{ij} \neq 0$ for some $i, j \in \{1, \ldots, n\}$. Then $\beta_i = 0$ and $\varphi_{ik} = 0$ for $k \in \{1, \ldots, n\}$ different from j. Consequently, $\mathrm{Id}_{G_i} = (\varphi \circ \varphi^{-1})_{ii} = \varphi_{ij}\psi_{ji}$. This implies that $\psi_{ji} \neq 0$, so by symmetry, $\mathrm{Id}_{G_j} = (\varphi^{-1} \circ \varphi)_{jj} = \psi_{ji}\varphi_{ij}$. Moreover, this also shows that $\varphi_{kj} = 0$ for $k \neq i$. Indeed, if $\varphi_{kj} \neq 0$, the same arguments as before show that $\psi_{jk} \neq 0$. This would imply that the jth row of φ^{-1} contains two non-trivial homomorphisms, which is a contradiction. Hence, the index i is unique, i.e. the jth column of φ contains only φ_{ij} and γ_j as (potentially) non-trivial homomorphisms. By symmetry, the ith column of φ^{-1} only contains γ'_i and ψ_{ji} as (potentially) non-trivial homomorphisms.

Next, suppose that β_l is non-trivial for some $l \in \{1, \ldots, n\}$. Since each of the first n rows of φ contains at most one non-trivial homomorphism, there is a column of φ, say the jth one, containing only γ_j as (potentially) non-trivial homomorphism. So, let $\mathcal{J}$ be the set of indices of the columns of φ of this form and $\mathcal{I}$ be the set of indices i with $\beta_i \neq 0$.

Suppose that $\mathcal{I}$, and thus also $\mathcal{J}$, is non-empty. Note that, for each $j \in \mathcal{J}$, γ_j is injective, since

$$\mathrm{Id}_{G_j} = (\varphi^{-1} \circ \varphi)_{jj} = \beta_j \gamma_j.$$

From this it follows that $\mathrm{Im}\,\gamma_j \cap \langle \mathrm{Im}\,\gamma_i \mid i \in \mathcal{J} \setminus \{j\} \rangle = 1$ for $j \in \mathcal{J}$. Indeed, suppose that h lies in the intersection. Since the images of the γ_k pairwise commute, we can write

$$h = \prod_{i \in \mathcal{J} \setminus \{j\}} \gamma_i(g_i) = \gamma_j(g_j)$$

for some $g_j \in G_j$ and $g_i \in G_i$. For $g \in G_j$, we then find that

$$\gamma_j(g)\gamma_j(g_j) = \gamma_j(g) \prod_{i \in \mathcal{J}\setminus\{j\}} \gamma_i(g_i) = \left(\prod_{i \in \mathcal{J}\setminus\{j\}} \gamma_i(g_i) \right) \gamma_j(g) = \gamma_j(g_j)\gamma_j(g),$$

again by the commuting condition. Injectivity of γ_j implies that $gg_j = g_j g$, and as this has to hold for all $g \in G_j$, we conclude that $g_j \in Z(G_j) = 1$. Thus, $h = 1$, which proves the claim. This shows that $\langle \operatorname{Im} \gamma_j \mid j \in \mathcal{J} \rangle$ is isomorphic to $\bigtimes_{j \in \mathcal{J}} G_j$, by Lemma 3.3.3. We thus have a subgroup of H isomorphic to $\bigtimes_{j \in \mathcal{J}} G_j$. We now proceed to prove that this subgroup is in fact a direct factor to obtain a contradiction.

First, we claim that $\delta\gamma_i' = 0$ for $i \in \mathcal{I}$. To do so, we compute $(\varphi \circ \varphi^{-1})_{n+1,i}$ and obtain

$$0 = \delta\gamma_i' + \sum_{j=1}^{n} \gamma_j \psi_{ji}.$$

If ψ_{ji} is non-trivial, then also φ_{ij} would be non-trivial, which would yield two non-trivial homomorphisms on the ith row of φ. Hence, $0 = \delta\gamma_i'$, which proves the claim.

With this equality, we can prove that $\delta\delta'\delta = \delta$, by noting that

$$\mathrm{Id}_H = (\varphi^{-1} \circ \varphi)_{n+1,n+1} = \delta'\delta + \sum_{i \in \mathcal{I}} \gamma_i' \beta_i,$$

as $\mathcal{I}$ contains all indices i for which β_i is non-trivial. Composing with δ on the left and using $\delta\gamma_i' = 0$, we obtain $\delta = \delta\delta'\delta$ as desired.

Next, we prove the following equality for all $j \in \{1, \dots, n\}$:

$$\delta\delta'\gamma_j = \begin{cases} 0 & \text{if } j \in \mathcal{J} \\ \gamma_j & \text{if } j \notin \mathcal{J} \end{cases} \tag{3.3.1}$$

Fix $j \in \{1, \dots, n\}$. Suppose that $j \in \mathcal{J}$. Then $\varphi_{ij} = 0$ for all $i \in \{1, \dots, n\}$. Consequently, $0 = (\varphi^{-1} \circ \varphi)_{n+1,j} = \delta'\gamma_j$, hence $\delta\delta'\gamma_j = 0$ as well.

If $j \notin \mathcal{J}$, then there is a unique $i \in \{1, \dots, n\}$ such that $\varphi_{ij} \neq 0$. Then also $\psi_{ji} \neq 0$, and this will be the only non-trivial homomorphism in the first n rows of the ith column of φ^{-1}. Using this, we find that

$$0 = (\varphi^{-1} \circ \varphi)_{n+1,j} = \gamma_i' \varphi_{ij} + \delta'\gamma_j$$

and

$$0 = (\varphi \circ \varphi^{-1})_{n+1,i} = \gamma_j \psi_{ji} + \delta\gamma_i'.$$

Composing the first equality on the left with δ yields

$$\begin{aligned} 0 &= \delta\gamma_i'\varphi_{ij} + \delta\delta'\gamma_j \\ &= -\gamma_j\psi_{ji}\varphi_{ij} + \delta\delta'\gamma_j \\ &= -\gamma_j + \delta\delta'\gamma_j, \end{aligned}$$

where we used that $\mathrm{Id}_{G_j} = \psi_{ji}\varphi_{ij}$. Rearranging terms proves that $\delta\delta'\gamma_j = \gamma_j$, which finishes the proof of (3.3.1).

Finally, we prove that

$$\langle \{\operatorname{Im}\gamma_j \mid j \in \mathcal{J}\}\rangle \cap \langle \operatorname{Im}\delta, \{\operatorname{Im}\gamma_j \mid j \notin \mathcal{J}\}\rangle = 1. \tag{3.3.2}$$

Let h be an element of the intersection. Since all the images of the γ_i and δ commute pairwise, we can write

$$h = \prod_{j\in\mathcal{J}} \gamma_j(g_j) = \delta(h') \prod_{j\notin\mathcal{J}} \gamma_j(g_j)$$

where $h' \in H$ and $g_j \in G_j$ for all indices j. Applying $\delta\delta'$ to this equality and using (3.3.1) yields on the one hand

$$\delta\delta'(h) = \prod_{j\in\mathcal{J}} \delta\delta'\gamma_j(g_j) = 1,$$

and on the other hand, again by using (3.3.1) and $\delta = \delta\delta'\delta$,

$$\delta\delta'(h) = \delta\delta'\delta(h') \prod_{j\notin\mathcal{J}} \delta\delta'\gamma_j(g_j) = \delta(h') \prod_{j\notin\mathcal{J}} \gamma_j(g_j) = h.$$

Hence, $h = 1$.

Now, both subgroups in (3.3.2) are normal, being a product of normal subgroups by Lemma 3.1.2, they generate H, also by Lemma 3.1.2, and they intersect trivially. Therefore, we conclude that H equals the internal direct product of the groups $\operatorname{Im}\varphi_j$ with $j \in \mathcal{J}$ and the group $\langle \operatorname{Im}\delta, \{\operatorname{Im}\gamma_j \mid j \notin \mathcal{J}\}\rangle$, i.e.

$$H \cong \left(\bigtimes_{j\in\mathcal{J}} G_j \right) \times N$$

for some normal subgroup N of H. As $|\mathcal{J}| \geq 1$, we obtain a contradiction, since we assume that H has no direct factors isomorphic with G_j for $j \in \{1, \ldots, n\}$.

Therefore, $\mathcal{J}$ is empty. Hence, $\mathcal{I}$ is empty as well, which means that all β_i are trivial. Thus, the matrix form of φ is

$$\varphi = \begin{pmatrix} \varphi_{11} & \dots & \varphi_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \varphi_{n1} & \dots & \varphi_{nn} & 0 \\ \gamma_1 & \dots & \gamma_n & \delta \end{pmatrix}.$$

Rewrite this as

$$\varphi = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$$

with $\alpha \in \operatorname{End}(G), \gamma \in \operatorname{Hom}(G, H), \delta \in \operatorname{End}(H)$. As φ was arbitrary, each automorphism of $G \times H$ is of this form. Lemma 3.1.3 then implies that $\alpha \in \operatorname{Aut}(G)$, $\delta \in \operatorname{Aut}(H)$ and $\gamma \in \operatorname{Hom}(G, Z(H))$. □

Remark. By Theorem 3.3.4, we also know what $\operatorname{Aut}(G)$ looks like. In fact, we could have merged Theorems 3.3.4 and 3.3.10 into one theorem, since the start of the proof of Theorem 3.3.10 is the same as that of Theorem 3.3.4. However, for the sake of clarity, we have split the results: Theorem 3.3.4 deals with the internal structure of $\operatorname{Aut}(G)$ and Theorem 3.3.10 determines the influence of G on an additional factor H on which we impose less strict conditions.

Corollary 3.3.11. *Let $G_1, \dots, G_n, H$ and G be as in the previous theorem.*

(1) If $G_i \in R_\infty$ for some $i \in \{1, \dots, n\}$, then $G \times H \in R_\infty$ as well.

(2) If G_i is finitely generated residually finite for each $i \in \{1, \dots, n\}$ and $H \in R_\infty$, then $G \times H \in R_\infty$ as well.

(3) If G_i is finitely generated torsion-free residually finite for each $i \in \{1, \dots, n\}$, then

$$\operatorname{Spec}_{\mathrm{R}}(G \times H) = \operatorname{Spec}_{\mathrm{R}}(G) \cdot \operatorname{Spec}_{\mathrm{R}}(H).$$

Proof. By the previous theorem, H is characteristic in $G \times H$. If $G_i \in R_\infty$ for some $i \in \{1, \dots, n\}$, then so does G by Corollary 3.3.5. Therefore, $G \times H$ has the R_∞-property as well by Corollary 1.1.2(1'), since $G \cong \frac{G \times H}{H}$.

Now, suppose that $H \in R_\infty$ and that each G_i is finitely generated residually finite. Then also G is finitely generated residually finite. For an automorphism φ of $G \times H$, let $\varphi|_H$ denote the induced automorphism on H and $\bar{\varphi}$ the induced automorphism on $G \cong \frac{G \times H}{H}$. If $R(\bar{\varphi}) = \infty$, then Corollary 1.1.2(1) yields $R(\varphi) = \infty$. If $R(\bar{\varphi}) < \infty$, then Proposition 1.2.11 implies that $\operatorname{Fix}(\bar{\varphi})$ is finite.

Since $H \in R_\infty$, we know that $R(\varphi|_H) = \infty$ and then Corollary 1.1.2 (2) yields that also $R(\varphi) = \infty$.

The last item follows from Theorem 3.2.2. $\square$

Example 3.3.12. We continue Example 3.3.8. For each $k \geq 2$, the group H_k contains torsion and since H_k fits in the exact sequence $1 \to \mathbb{Z}^k \to H_k \to \mathbb{Z}/2\mathbb{Z} \to 1$, it is a finitely generated residually finite group. On the other hand, the non-abelian Baumslag-Solitar groups $BS(m,n)$ have the R_∞-property [31, Theorem 4.4], and are torsion-free, so they cannot contain H_k as a direct factor. Therefore, if we consider direct products of the form $\left(\bigtimes_{i=1}^{k} H_{n_i}^{r_i}\right) \times BS(m,n)$ for some $2 \leq n_1 < \ldots < n_k$, $r_1, \ldots, r_k \geq 1$ and $BS(m,n)$ non-abelian, we can apply the second item of Corollary 3.3.11 to prove that these groups have the R_∞-property. ‖

Example 3.3.13. This example is based on the results from [49]. Consider the semi-direct product $G = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$, where $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. As in Example 3.3.8, we write the elements in the group as (x, t^i), where $x \in \mathbb{Z}^2$ and $i \in \mathbb{Z}$. It is a polycyclic group of Hirsch length 3, hence finitely generated and residually finite, and clearly torsion-free. Also note that G is solvable.

To prove that its centre is trivial, let (x, t^i) be an element in the centre. Conjugating with $(0, t)$ yields the equality $(x, t^i) = (Ax, t^i)$. Since the eigenvalues of A are $\frac{3\pm\sqrt{5}}{2}$, A does not have any non-trivial fixed points. Therefore, $x = 0$. If we then conjugate $(0, t^i)$ with $(y, 1)$, where $y \in \mathbb{Z}^2$ is a non-trivial element, we find that $(0, t^i) = (y - A^i y, t^i)$. Since neither of the eigenvalues of A is a root of unity, A^i has no eigenvalue 1 if $i \neq 0$. Therefore, $i = 0$ and we find that $(x, t^i) = (0, 1)$, which proves that the centre of G is trivial.

To prove that G is directly indecomposable, note that, for $x \in \mathbb{Z}^2$, we have $[(x, 1), (0, t^{-1})] = (Ax - x, 1)$. Since $\det(A - I) = -1$, $A - I$ defines a surjective map on $\mathbb{Z}^2$, thus $[G, G]$ contains $\mathbb{Z}^2$. The inclusion $[G, G] \subseteq \mathbb{Z}^2$ follows immediately from the definition of G. So, $[G, G] = \mathbb{Z}^2$ and $G/[G, G] \cong \mathbb{Z}$. Next, suppose that $G \cong H \times K$. Factoring out the commutator subgroup, we get the isomorphism $\mathbb{Z} \cong \frac{H}{[H,H]} \times \frac{K}{[K,K]}$. As $\mathbb{Z}$ is directly indecomposable, one of the factors, say $H/[H, H]$, is trivial. This implies $[H, H] = H$. But H is isomorphic to a subgroup of G, which is a solvable group, hence H is solvable itself. Combined with the equality $H = [H, H]$, this yields $H = 1$, which shows that G is directly indecomposable.

Finally, in [49, Theorem 4.1, Example 4.3], it is proven that G admits automorphisms with finite Reidemeister number. A concrete example of an automorphism with finite Reidemeister number is

$$\varphi : G \to G : (x, t^i) \mapsto (Mx, t^{-i}),$$

where $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. One can verify that $R(\varphi) = 4$. Thus, if we consider the direct product $G \times P$ where P is any polycyclic group of Hirsch length at most 2, we can apply the third item of Corollary 3.3.11 to conclude that $\mathrm{Spec}_{\mathrm{R}}(G \times P) = \mathrm{Spec}_{\mathrm{R}}(G) \cdot \mathrm{Spec}_{\mathrm{R}}(P)$, since G cannot be a direct factor (not even a subgroup) of P. ‖

3.4 Application: direct products of hyperbolic groups

In addition to the examples given in the previous section, we now use the results obtained to study the R_∞-property for direct products of hyperbolic groups. We do not need the precise definition of a hyperbolic group, only some of the (algebraic) properties they satisfy; for more information, we refer the interested reader to [55, 74].

By definition, hyperbolic groups are finitely generated. The simplest examples are finite groups and groups containing $\mathbb{Z}$ as a finite index subgroup. These two families are often called *elementary hyperbolic*; all other hyperbolic groups are called *non-elementary hyperbolic*. Examples of the latter include groups containing a non-abelian free subgroup of finite index.

Recall from Theorem 2.1.2 that non-elementary hyperbolic groups have the R_∞-property. We now use this and the results obtained in the previous section to prove that direct products of non-elementary hyperbolic groups have the R_∞-property.

Lemma 3.4.1 (See e.g. [74, Theorem 7.5.10, Corollary 7.5.15])**.** *Let G be a torsion-free hyperbolic group and let $x \in G$. Then $C_G(x)$ is cyclic.*

If G is, moreover, non-elementary hyperbolic, then

(1) $C_G(x) \cap C_G(y) = 1$ for all $x, y \in G$ that do not commute;

(2) G contains two elements that do not commute;

(3) G is centreless.

Proposition 3.4.2. *Let G be a torsion-free hyperbolic group. Then G is directly indecomposable.*

Proof. Suppose that $G \cong A \times B$ and that both A and B are non-trivial. Let $1 \neq a \in A$. Then $C_G(a) = C_A(A) \times B$ is a non-trivial direct product. However, $C_G(a)$ is cyclic by Lemma 3.4.1. As G is torsion-free, $C_G(a)$ is thus isomorphic to $\mathbb{Z}$. However, $\mathbb{Z}$ is directly indecomposable, which contradicts $C_G(a) = C_A(a) \times B$. Therefore, either A or B is trivial, which proves that G is directly indecomposable. □

Corollary 3.4.3. *Let $G_1, \dots, G_n$ be torsion-free non-elementary hyperbolic groups. Then $G_1 \times \dots G_n$ has the R_∞-property.*

Proof. Let $i \in \{1, \dots, n\}$ be arbitrary. By Theorem 2.1.2, G_i has the R_∞-property. By Proposition 3.4.2, G_i is directly indecomposable, and by Lemma 3.4.1(3), G_i is centreless. Therefore, Corollary 3.3.5 applies to $G_1, \dots, G_n$ and implies that $G_1 \times \dots G_n$ has the R_∞-property. □

The aim of this section is to prove the following generalisation of the previous corollary:

Theorem 3.4.4. *Let $G_1, \dots, G_n$ be virtually torsion-free non-elementary hyperbolic groups. Put $G := G_1 \times \dots \times G_n$ and let $H \leq G$ be a finite index subgroup. Then H has the R_∞-property.*

In order to do so, we need a way of constructing characteristic subgroups. The work of M. Bridson and C. Miller [10, §3] on full subgroups provides us with the necessary tools. Let $G_1, \dots, G_n$ be torsion-free non-elementary hyperbolic groups and put $G := G_1 \times \dots \times G_n$. We call a subgroup $H \leq G$ *full* if, for each $i \in \{1, \dots, n\}$, the group $H \cap G_i$ contains two non-commuting elements.

Suppose that $H \leq G$ is a full subgroup and let, for each $i \in \{1, \dots, n\}$, x_i and y_i be two non-commuting elements in $H \cap G_i$. Consider the set

$$\mathcal{C} := \{C_H(a) \cap C_H(b) \mid a, b \in H, a \text{ and } b \text{ do not commute}\}.$$

In particular, $\mathcal{C}$ contains, for each $i \in \{1, \dots, n\}$, the subgroup $M_i := C_H(x_i) \cap C_H(y_i)$, which is moreover equal to

$$H \cap (G_1 \times \dots \times G_{i-1} \times 1 \times G_{i+1} \times \dots \times G_n).$$

Indeed, it is clear that M_i contains the subgroup above. Conversely, if $z = (z_1, \dots, z_n)$ is an element that commutes with both x_i and y_i, then $z_i = 1$ by Lemma 3.4.1(1).

Let $i \in \{1, \dots, n\}$ and suppose that $M_i \subseteq C_H(a) \cap C_H(b)$ for some non-commuting elements a and b in H. Then there exists a $j \in \{1, \dots, n\}$ such that

a_j and b_j do not commute. Consequently,

$$M_i \leq C_H(a) \cap C_H(b) \leq C_H(a_j) \cap C_H(b_j) = M_j.$$

Hence, $i = j$ and $M_i = C_H(a) \cap C_H(b)$. This proves that M_i is a maximal element of $\mathcal{C}$. Conversely, if $C_H(a) \cap C_H(b) \in \mathcal{C}$ is a maximal element for some non-commuting elements $a, b \in H$, then the previous argument also shows that $C_H(a) \cap C_H(b)$ must equal M_j for some $j \in \{1, \dots, n\}$.

Summarised, the set $\mathcal{M} := \{M_1, \dots, M_n\}$ is exactly the set of maximal elements of $\mathcal{C}$. In addition, as automorphisms map centralisers to centralisers and preserve inclusions, each $\varphi \in \mathrm{Aut}(H)$ permutes the elements in $\mathcal{M}$.

Now, for each $i \in \{1, \dots, n\}$, the subgroup $H \cap G_i$ is exactly the intersection of all M_j except for M_i. Therefore, each $\varphi \in \mathrm{Aut}(H)$ also permutes the elements in the set $\{H \cap G_1, \dots, H \cap G_n\}$.

Putting all of this together, we obtain the following result:

Proposition 3.4.5 (See e.g. [10, Proposition 7])**.** *Let $G_1, \dots, G_n$ be torsion-free non-elementary hyperbolic groups and put $G := G_1 \times \ldots \times G_n$. Suppose that H is a full subgroup of G. Then*

$$(H \cap G_1) \times \ldots \times (H \cap G_n)$$

is a characteristic subgroup of H.

We can now prove Theorem 3.4.4.

Proof of Theorem 3.4.4. We are given a finite index subgroup H of a direct product of virtually torsion-free non-elementary hyperbolic groups $G = G_1 \times \ldots \times G_n$. The goal is to construct a finite index characteristic subgroup of H that is a direct product of non-elementary hyperbolic groups. By Corollary 3.4.3, this subgroup has the R_∞-property, and then Corollary 1.1.2(2') implies that H has the R_∞-property as well.

Firstly, for $i \in \{1, \dots, n\}$, let F_i be a finite index torsion-free subgroup of G_i and put $H_i := H \cap F_i$. Secondly, define $H_0 := H_1 \times \ldots \times H_n$. We claim that

$[H : H_0]$ is finite. To that end, note that

$$\begin{aligned}[H : H_0] &\leq [G : H_0] \\ &= \prod_{i=1}^{n} [G_i : H_i] \\ &= \prod_{i=1}^{n} [G_i : H \cap F_i] \\ &= \prod_{i=1}^{n} [G_i : F_i][F_i : H \cap F_i] \\ &\leq \prod_{i=1}^{n} [G_i : F_i][G : H].\end{aligned}$$

By assumption, $[G_i : F_i]$ is finite for each $i \in \{1, \ldots, n\}$, as is $[G : H]$. Therefore, $[H : H_0]$ is finite.

Lastly, putting

$$K_0 := \bigcap_{\varphi \in \mathrm{Aut}(H)} \varphi(H_0),$$

we obtain a characteristic subgroup of H that is also contained in H_0. Furthermore, K_0 has finite index in H, as H_0 has finite index in H and H is finitely generated.

Fix $i \in \{1, \ldots, n\}$. Then $K_i := K_0 \cap H_i$ is a torsion-free hyperbolic finite index subgroup of H_i. As

$$[G_i : K_i] = [G_i : H_i][H_i : K_i]$$

is finite, K_i is non-elementary hyperbolic as well. Therefore, K_i contains two non-commuting elements. As i was arbitrary, this proves that K_0 is a full subgroup of H_0. Finally, let K be the characteristic subgroup of K_0 given by Proposition 3.4.5, i.e.

$$K = K_1 \times \ldots \times K_n$$

As each K_i is non-elementary hyperbolic, Corollary 3.4.3 implies that K has the R_∞-property.

We prove that K is the desired subgroup of H. In other words, we prove that K is characteristic in H and has finite index in H. We have constructed the following chain of subgroups:

$$G \geq_f H \geq_f H_0 \geq_f K_0 \geq K,$$

where a subscript f indicates finite index. Recall that K_0 is characteristic in H and that K is characteristic in K_0. Consequently, K is characteristic in H.

To end the proof, we prove that $[H : K]$ is finite. To do so, it is sufficient to prove that $[H_0 : K]$ is finite. Since

$$[H_0 : K] = \prod_{i=1}^{n} [H_i : K_0 \cap H_i]$$

and $[H_i : K_0 \cap H_i]$ is finite for each $i \in \{1, \ldots, n\}$, $[H_0 : K]$ is finite as well. □

Corollary 3.4.6. *Let G be a group and H a finite index subgroup. Suppose that H is isomorphic to a finite index subgroup of a finite direct product of virtually torsion-free non-elementary hyperbolic groups. Then G has the R_∞-property.*

Proof. Since hyperbolic groups are finitely generated, a finite direct product of such groups is finitely generated as well. Therefore, H is finitely generated too, being a finite index subgroup of a finitely generated group. This in turn implies that G is finitely generated, as H has finite index in G.

Thus, using H and Lemma 1.2.6, we obtain a (fully) characteristic finite index subgroup K of G that is contained in H. As $[G : K]$ is finite, so is $[H : K]$. Hence, K is also a finite index subgroup of a finite direct product of virtually torsion-free non-elementary hyperbolic groups. Theorem 3.4.4 then implies that K has the R_∞-property, and using Corollary 1.1.2(2'), we conclude that G has the R_∞-property as well. □

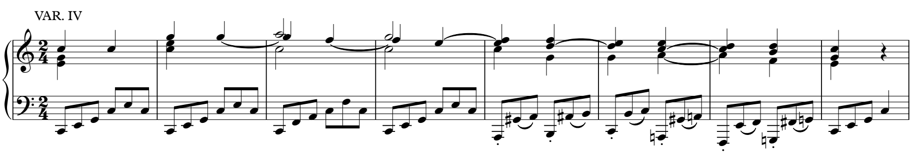
In particular, every direct product of finitely many non-abelian free groups has the R_∞-property, as do their finite index subgroups and their finite extensions.

Collection II

Nilpotent groups

Variation 4

Finitely generated torsion-free nilpotent groups



Mozart, Wolfgang Amadeus, *Zwölf Variationen in C über das französische Lied 'Ah, vous dirai-je, Maman'*, (KV 265 (300e)), Variation IV (bars 1–8).

In each Variation of this Collection, we require that our groups are nilpotent. Nilpotency is a strong property that yields several powerful and elegant results concerning Reidemeister numbers and spectra.

In this Variation, we recall the necessary definitions and properties of nilpotent groups and study the well-known product formula for Reidemeister numbers on finitely generated torsion-free nilpotent groups.

4.1 Nilpotent groups

4.1.1 Preliminaries

We recall the definition and several properties of nilpotent groups, without the restriction that they are finitely generated or torsion-free.

Definition 4.1.1. Let G be a group. We call G *nilpotent* if G admits a central series, that is, a series

$$1 = H_0 \leq H_1 \leq \ldots \leq H_c = G$$

such that $[H_i, G] \leq H_{i-1}$ for all $i \in \{1, \ldots, c\}$. We call c the *length* of the central series, and the minimal c for which G admits a central series of length c is the *nilpotency class* of G.

We refer to the quotient groups H_{i+1}/H_i for $i \in \{0, \ldots, c-1\}$ as the *factors of the series.*

Two series that are often used are the lower and upper central series, which can be defined for any group G. The *lower central series*

$$\gamma_1(G) \geq \gamma_2(G) \geq \ldots$$

is defined as $\gamma_1(G) := G$ and $\gamma_{i+1}(G) := [\gamma_i, G]$ for all $i \geq 1$. The *upper central series*

$$Z_1(G) \leq Z_2(G) \leq \ldots$$

is defined as $Z_1(G) := Z(G)$ and $\frac{Z_{i+1}(G)}{Z_i(G)} := Z\left(\frac{G}{Z_i(G)}\right)$ for all $i \geq 1$. For notational purposes, we put $Z_0(G) := 1$.

The group G is then nilpotent if and only if $Z_i(G) = G$ for some i or, equivalently, $\gamma_{i+1}(G) = 1$ for some i. If G is nilpotent of class c, then c is the minimal index i such that $Z_i(G) = G$ or $\gamma_{i+1}(G) = 1$.

Lemma 4.1.2 (See e.g. [13, Theorem 2.1])**.** *Let N be a nilpotent group of class c. Suppose that*

$$1 = H_0 \leq H_1 \leq \ldots \leq H_k = G$$

is a central series of G. Then $\gamma_{k-i+1}(G) \leq H_i \leq Z_i(G)$ for all $i \in \{1, \ldots, k\}$.

If G is torsion-free and nilpotent, the factors of the upper central series are torsion-free (see e.g. [99, p. 5.2.19]). The factors of the lower central series, on the other hand, do not need to be.

Example 4.1.3. Let $k \geq 2$ be an integer and let N be the group given by the presentation

$$N = \langle a, b, c \mid [a, b] = c^k, [a, c] = [b, c] = 1 \rangle.$$

The upper central series is given by $1 \leq \langle c \rangle \leq N$, with respective factors

$$Z_1(N) = \langle c \rangle \cong \mathbb{Z},$$

$$\frac{Z_2(N)}{Z_1(N)} \cong \langle a, b, c \mid [a, b] = c^k, [a, c] = [b, c] = c = 1 \rangle \cong \mathbb{Z}^2.$$

The lower central series, on the other hand, is given by $N \geq \gamma_2(N) = \langle c^k \rangle \geq 1$. However, the factors are not torsion-free, since

$$\frac{N}{\gamma_2(N)} \cong \langle a, b, c \mid [a,b] = c^k, [a,c] = [b,c] = 1 = c^k \rangle \cong \mathbb{Z}^2 \oplus \mathbb{Z}/k\mathbb{Z}. \qquad \|$$

Given a group G, a normal subgroup N and two elements $g, h \in G$, we write $g \equiv h \bmod N$ if $gN = hN$.

Lemma 4.1.4 (See e.g. [68, Theorem 5.3])**.** *Let G be a group and let $a, b, c \in G$ be arbitrary. Then the following identities hold:*

(1) $[ab, c] = [a,c]^b [b,c]$;

(2) $[a, bc] = [a,c][a,b]^c$.

Suppose moreover that $a \in \gamma_i(G)$, $b \in \gamma_j(G)$ and $c \in \gamma_k(G)$ for some integers $i, j, k \geq 1$. Then the following congruences hold:

(3) $[ab, c] \equiv [a,c][b,c] \bmod \gamma_{i+j+k}(G)$;

(4) $[a, bc] \equiv [a,b][a,c] \bmod \gamma_{i+j+k}(G)$.

Definition 4.1.5. Let G be a group and H a subgroup of G. We define the *isolator* of H to be the set

$$\sqrt[G]{H} := \{ g \in G \mid \exists n \in \mathbb{N}_0 : g^n \in H \}.$$

In general, $\sqrt[G]{H}$ is not a subgroup. For instance, if $H = 1$, then $\sqrt[G]{H}$ is the set of torsion elements in G, which do not form a subgroup in general.

Lemma 4.1.6 (See e.g. [17, Lemmata 1.1.2 and 1.1.4], [95, p. 472-473])**.** *Let G be a group. Then the following hold:*

(1) $\forall i \in \mathbb{N}_0 : \sqrt[G]{\gamma_i(G)}$ *is a fully characteristic subgroup of G;*

(2) $\forall i \in \mathbb{N}_0 : G/\sqrt[G]{\gamma_i(G)}$ *is torsion-free;*

(3) $\forall i, j \in \mathbb{N}_0 : \left[\sqrt[G]{\gamma_i(G)}, \sqrt[G]{\gamma_j(G)} \right] \leq \sqrt[G]{\gamma_{i+j}(G)}$.

(4) $\forall i, j \in \mathbb{N}_0$ *with $i \leq j$: if $N := \sqrt[G]{\gamma_j(G)}$, then*

$$\sqrt[G/N]{\gamma_{i+1}(G/N)} = \frac{\sqrt[G]{\gamma_{i+1}(G)}}{N}.$$

This leads us to a new series of a group.

Definition 4.1.7. Let G be a group. The *adapted lower central series* is given by

$$G = \sqrt[G]{\gamma_1(G)} \geq \sqrt[G]{\gamma_2(G)} \geq \dots$$

It follows from Lemma 4.1.6(2) that the factors of the adapted lower central series are torsion-free.

For instance, the adapted lower central series of the group introduced in Example 4.1.3 is given by $N \geq \langle c \rangle \geq 1$.

Lemma 4.1.8. *Let G be a group. Then G is torsion-free nilpotent if and only if $\sqrt[G]{\gamma_{i+1}(G)} = 1$ for some $i \geq 0$.*

Moreover, if G is torsion-free nilpotent of class c, then $\sqrt[G]{\gamma_{c+1}(G)} = 1$.

Proof. Suppose that $\sqrt[G]{\gamma_{i+1}(G)} = 1$. As $\gamma_j(G) \leq \sqrt[G]{\gamma_j(G)}$ for all $j \geq 1$, it follows that $\gamma_{i+1}(G) = 1$ as well. Therefore, G is nilpotent. By Lemma 4.1.6(2), $G \cong G/\sqrt[G]{\gamma_{i+1}(G)}$ is torsion-free.

Conversely, suppose that G is torsion-free nilpotent of class c. Then $\gamma_{c+1}(G) = 1$. Let $g \in \sqrt[G]{\gamma_{c+1}(G)}$. Then there exists a $k \geq 1$ such that $g^k \in \gamma_{c+1}(G) = 1$. Therefore, $g^k = 1$, and as G is torsion-free, this implies that $g = 1$. Therefore, $\sqrt[G]{\gamma_{c+1}(G)} = 1$. $\square$

We mentioned polycyclic groups already in Example 3.3.13. As we make more heavy use of them in later Variations, we provide the definition for the sake of completeness.

Definition 4.1.9. Let G be a group. We call G *polycyclic* if there exists a series

$$1 = H_0 \lhd H_1 \lhd \dots \lhd H_n = G$$

such that H_i/H_{i-1} is cyclic for each $i \in \{1, \dots, n\}$.

The number of infinite cyclic factors is called the *Hirsch length* of G and is written as $h(G)$. It is independent of the series chosen.

Theorem 4.1.10 (See e.g. [66, Theorem 17.2.2])**.** *Let N be a finitely generated nilpotent group. Then N is polycyclic, thus also residually finite.*

In particular, if N is in addition torsion-free, then N is poly-$\mathbb{Z}$, which means that N is of the form

$$(((\mathbb{Z} \rtimes \mathbb{Z}) \rtimes \mathbb{Z}) \dots) \rtimes \mathbb{Z}.$$

Corollary 4.1.11. *Let N be a finitely generated and infinite nilpotent group. Then $\infty \in \mathrm{Spec}_{\mathrm{R}}(N)$.*

Proof. This follows from Theorem 4.1.10 and Corollary 1.2.12. □

4.1.2 Extended Reidemeister spectrum

Finitely generated nilpotent groups that are infinite are sufficiently well-structured to allow for a complete description of their extended Reidemeister spectrum.

Definition 4.1.12. Let N be a nilpotent group. The set of torsion elements of N is written as $\tau(N)$.

Proposition 4.1.13. *Let N be a nilpotent group.*

(1) The set $\tau(N)$ is a fully characteristic subgroup of N;

(2) The quotient group $N/\tau(N)$ is torsion-free;

(3) If $H \leq N$ is a normal subgroup such that N/H is torsion-free, then $\tau(N) \leq H$.

Proof. It is well known (see e.g. [66, Theorem 16.2.7]) that $\tau(N)$ is a subgroup of N. To prove that it is fully characteristic, let $n \in \tau(N)$ and suppose it has order k. Then, for arbitrary $\varphi \in \mathrm{End}(N)$, $\varphi(n)^k = \varphi(n^k) = 1$, which proves that $\varphi(n) \in \tau(N)$.

Next, $N/\tau(N)$ is torsion-free. For if $(n\tau(N))^k = \tau(N)$ for some $k \in \mathbb{Z}$, then $n^k \in \tau(N)$ and thus $n^{kl} = 1$ for some $l \in \mathbb{Z}$. Hence, $n \in \tau(N)$.

Finally, suppose now that N/H is torsion-free. Let $n \in \tau(N)$. Then $(nH)^k = H$ for some $k \geq 1$. As N/H is torsion-free, $nH = H$. Therefore, $n \in H$. □

Corollary 4.1.14. *Let N be an infinite finitely generated nilpotent group. Then N has full extended Reidemeister spectrum.*

Proof. By Theorem 1.1.9, it is sufficient to prove that N admits a surjective homomorphism to $\mathbb{Z}$.

Put $M := N/\tau(N)$. Then M is a (non-trivial) finitely generated torsion-free nilpotent group, since $\tau(N)$ is finite, as N is finitely generated. So, by Theorem 4.1.10, M is isomorphic to $K \rtimes \mathbb{Z}$ for some group K. Thus, if we let $\pi : N \to N/\tau(N)$ and $\varphi : M \to M/K \cong \mathbb{Z}$ be the canonical projections, then $\varphi \circ \pi$ is a surjective homomorphism from N to $\mathbb{Z}$. □

4.2 Product formula

The structure of finitely generated torsion-free nilpotent groups is strong enough to admit an elegant product formula for Reidemeister numbers.

4.2.1 Formulation, proof and examples

Theorem 4.2.1 (Product formula; See e.g. [100, Theorem 2.6])**.** *Let N be a finitely generated torsion-free nilpotent group and $\varphi \in \mathrm{End}(N)$. Suppose that*

$$1 = N_0 \lhd N_1 \lhd \ldots \lhd N_{c-1} \lhd N_c = N$$

is a central series with the following properties:

(1) For each $i \in \{1, \ldots, c\}$, N_i/N_{i-1} is torsion-free;

(2) For each $i \in \{1, \ldots, c\}$, $\varphi(N_i) \leq N_i$.

For $i \in \{1, \ldots, c\}$, let φ_i denote the induced endomorphism on N_i/N_{i-1}. Then

$$R(\varphi) = \prod_{i=1}^{c} R(\varphi_i).$$

In particular, $R(\varphi) = \infty$ if and only if one of the following equivalent conditions holds:

- *There exists an $i \in \{1, \ldots, c\}$ such that $R(\varphi_i) = \infty$;*
- *There exists an $i \in \{1, \ldots, c\}$ such that φ_i has a non-trivial fixed point.*

Proof. We proceed by induction on c. The case $c = 1$ is immediate, as $\varphi_1 = \varphi$ in that case. So, suppose the result holds for all finitely generated torsion-free nilpotent groups and central series of length $c - 1$. Let $1 = N_0 \lhd N_1 \lhd \ldots \lhd N_{c-1} \lhd N_c = N$ be a central series as in the statement. Consider $\frac{N}{N_1}$ with induced morphism $\bar{\varphi}$. As a quotient of a finitely generated torsion-free nilpotent group, it is finitely generated and nilpotent. Now,

$$1 = \frac{N_1}{N_1} \lhd \frac{N_2}{N_1} \lhd \ldots \lhd \frac{N_{c-1}}{N_1} \lhd \frac{N_c}{N_1} = \frac{N}{N_1}$$

is a central series of N/N_1 satisfying the conditions of the theorem. Thus, if we let $\bar{\varphi}_i$ denote the restriction of $\bar{\varphi}$ to $\frac{N_i}{N_1}$ for $i \in \{2, \ldots, c\}$, then

$$R(\bar{\varphi}) = \prod_{i=2}^{c} R(\bar{\varphi}_i),$$

by the induction hypothesis.

Now, from the commuting diagram

$$\begin{tikzcd}
\frac{N_i/N_1}{N_{i-1}/N_1} \arrow[r, "\bar{\varphi}_i"] \arrow[d, "\cong"] & \frac{N_i/N_1}{N_{i-1}/N_1} \arrow[d, "\cong"] \\
\frac{N_i}{N_{i-1}} \arrow[r, "\varphi_i"] & \frac{N_i}{N_{i-1}}
\end{tikzcd}$$

where the vertical arrows are the canonical isomorphisms, it follows that $R(\varphi_i) = R(\overline{\varphi}_i)$ for all $i \in \{2, \ldots, c\}$ by Lemma T.1.5. Therefore, it is sufficient to prove that

$$R(\varphi) = R(\varphi|_{N_1})R(\overline{\varphi}),$$

as $\varphi|_{N_1} = \varphi_1$. This, however, immediately follows from Proposition 1.2.5, which finishes the proof of the product formula.

Since each N_i/N_{i-1} is a free abelian group of finite rank, the equivalent conditions for $R(\varphi)$ to be infinite follow from the product formula and Proposition T.1.12. □

For a finitely generated torsion-free nilpotent group, there does exists at least one central series where each subgroup is fully characteristic and such that its factors are torsion-free, namely the adapted lower central series, by Lemmata 4.1.6 and 4.1.8(1).

The following result shows how we can construct other such central series based on a given normal subgroup:

Proposition 4.2.2. *Let N be a finitely generated torsion-free nilpotent group. Suppose that H is a normal subgroup such that N/H is torsion-free and that $\varphi \in \operatorname{End}(N)$ is such that $\varphi(H) \leq H$. Then there exists a central series*

$$1 = N_0 \lhd N_1 \lhd \ldots \lhd N_{k-1} \lhd N_k = N$$

such that the following hold:

(1) $H = N_i$ for some $i \in \{0, \ldots, k\}$;

(2) For each $i \in \{1, \ldots, k\}$, N_i/N_{i-1} is torsion-free;

(3) For each $i \in \{1, \ldots, k\}$, $\varphi(N_i) \leq N_i$.

Proof. Since N is torsion-free nilpotent, there is a c such that $\sqrt[N]{\gamma_{c+1}(N)} = 1$. Let $j \geq 0$ be the smallest index such that $H \leq \sqrt[N]{\gamma_{c-j+1}(N)}$. For $i \in \{0, \ldots, j\}$, put

$$N_i := \sqrt[N]{\gamma_{c-i+1}(N)} \cap H.$$

Next, let $p : N \to N/H$ be the natural projection. Since N/H is torsion-free nilpotent, there is a d such that $\sqrt[N/H]{\gamma_{d+1}(N/H)} = 1$ by Lemma 4.1.8. For $i \in \{0, \ldots, d\}$, put

$$N_{i+j} := p^{-1}\left(\sqrt[N/H]{\gamma_{d-i+1}\left(\frac{N}{H}\right)}\right).$$

We prove that the N_i form the desired central series. First, note that we have defined N_j in two ways. However, they coincide, as

$$H \cap \sqrt[N]{\gamma_{c-j+1}(N)} = H$$

by definition of j and

$$p^{-1}\left(\sqrt[N/H]{\gamma_{d-0+1}\left(\frac{N}{H}\right)}\right) = p^{-1}(1) = H$$

as well.

Put $k := j + d$. As the adapted lower central series is descending, the N_i are ascending. To prove that they are central (and normal), we compute $[N_i, N]$ for each $i \in \{0, \ldots, k\}$. If $i \leq j$, then $[N_i, N] \leq [H, N] \leq H$, as H is normal, and

$$[N_i, N] \leq \left[\sqrt[N]{\gamma_{c-i+1}(N)}, N\right] \leq \sqrt[N]{\gamma_{c-i+2}(N)}.$$

Thus, $[N_i, N] \leq N_{i-1}$. If $i \geq j+1$, then write $i = j+l$ for some $l \in \{1, \dots, d\}$. With this notation, we find

$$\begin{aligned}
[N_i, N] = [N_{j+l}, N] &= \left[p^{-1}\left(\sqrt[N/H]{\gamma_{d-l+1}\left(\frac{N}{H}\right)}\right), N\right] \\
&= \left[p^{-1}\left(\sqrt[N/H]{\gamma_{d-l+1}\left(\frac{N}{H}\right)}\right), p^{-1}\left(\frac{N}{H}\right)\right] \\
&= p^{-1}\left(\left[\left(\sqrt[N/H]{\gamma_{d-l+1}\left(\frac{N}{H}\right)}\right), \frac{N}{H}\right]\right) \\
&\leq p^{-1}\left(\sqrt[N/H]{\gamma_{d-l+2}\left(\frac{N}{H}\right)}\right) \\
&= N_{j+l-1} = N_{i-1}.
\end{aligned}$$

Next, we argue that the factors N_i/N_{i-1} are torsion-free for all $i \in \{1, \dots, k\}$. Let $i \in \{1, \dots, k\}$ and suppose that $i \leq j$. Then

$$\frac{N_i}{N_{i-1}} = \frac{N_i}{\sqrt[N]{\gamma_{c-i}(N)} \cap N_i} \cong \frac{N_i \sqrt[N]{\gamma_{c-i}(N)}}{\sqrt[N]{\gamma_{c-i}(N)}} \leq \frac{N}{\sqrt[N]{\gamma_{c-i}(N)}},$$

where the latter is torsion-free by Lemma 4.1.6(2). Now, suppose that $i \geq j+1$ and write $i = j+l$ for some $l \in \{1, \dots, d\}$. Then by the third isomorphism theorem,

$$\frac{N}{N_{i-1}} \cong \frac{N/H}{\sqrt[N/H]{\gamma_{d-l+2}(N/H)}}.$$

The latter group is torsion-free, again by Lemma 4.1.6(2), and as N_i/N_{i-1} is a subgroup of N/N_{i-1}, it is torsion-free as well.

Finally, we prove that $\varphi(N_i) \leq N_i$ for each $i \in \{0, \dots, k\}$. Let $i \in \{0, \dots, k\}$ be arbitrary. If $i \leq j$, then

$$\begin{aligned}
\varphi(N_i) &= \varphi\left(\sqrt[N]{\gamma_{c-i+1}(N)} \cap H\right) \\
&\leq \varphi\left(\sqrt[N]{\gamma_{c-i+1}(N)}\right) \cap \varphi(H) \\
&\leq \sqrt[N]{\gamma_{c-i+1}(N)} \cap H \\
&= N_i,
\end{aligned}$$

by assumption on H and by the fact that the adapted lower central series consists of fully characteristic subgroups by Lemma 4.1.6(1).

If $i \geq j+1$, write $i = j+l$ for some $l \in \{1, \dots, d\}$. Let $\bar{\varphi}$ denote the induced endomorphism on N/H. Let $n \in N_i$ be arbitrary. Then $p(n) \in \sqrt[N/H]{\gamma_{d-l+1}(N/H)}$. Consequently,

$$p(\varphi(n)) = \bar{\varphi}(p(n)) \in \sqrt[N/H]{\gamma_{d-l+1}(N/H)}.$$

This implies that $\varphi(n) \in \sqrt[N/H]{\gamma_{d-l+1}(N/H)}$ as well. Thus, as n is arbitrary, $\varphi(N_i) \leq N_i$. □

Corollary 4.2.3. *Let N be a finitely generated torsion-free nilpotent group. Suppose that H is a normal subgroup such that N/H is torsion-free. Let $\varphi \in \mathrm{End}(N)$ such that $\varphi(H) \leq H$. Let $\varphi|_H$ and $\bar{\varphi}$ denote the induced endomorphisms on H and N/H, respectively. Then $R(\varphi) = R(\varphi|_H)R(\bar{\varphi})$.*

Proof. By Proposition 4.2.2, there is a central series $1 = H_0 \lhd H_1 \lhd \dots \lhd H_{c-1} \lhd H_c = N$ containing H as one of its groups, say, $H = H_j$. The product formula yields

$$R(\varphi) = \prod_{i=1}^{j} R(\varphi_i) \prod_{i=j+1}^{c} R(\varphi_i), \tag{4.2.1}$$

where φ_i is the induced endomorphism on H_i/H_{i-1}. As $1 \lhd H_1 \lhd \dots H_{i-1} \lhd H_i = H$ is a central series of H, the first product in (4.2.1) equals $R(\varphi|_H)$ by Theorem 4.2.1. Analogously as in the proof of Theorem 4.2.1, one can argue that

$$\prod_{i=j+1}^{c} R(\varphi_i) = R(\bar{\varphi}).$$

Note that in Theorem 4.2.1 the case $j = 1$ is proven. Therefore, the result follows. □

The product formula and the criterion for infinite Reidemeister number have been extensively used in the literature. One notable result classifies the free nilpotent groups that have the R_∞-property.

Definition 4.2.4. Let $r \geq 2$ and $c \geq 1$ be integers. The *free nilpotent group $N_{r,c}$ of rank r and class c* is the group

$$N_{r,c} := \frac{F_r}{\gamma_{c+1}(F_r)},$$

where F_r is the free group of rank r.

Theorem 4.2.5 (See e.g. [100, Theorem 4.3], [19, Theorem 2.1]). *Let $r \geq 2$ and $c \geq 1$ be integers. Then the free nilpotent group $N_{r,c}$ has the R_∞-property if and only if $c \geq 2r$.*

This result provides us with a second way of proving that non-abelian free groups of finite rank have the R_∞-property (Theorem 2.1.1): for each integer $n \geq 2$, $N_{n,2n}$ is a quotient of F_n by a characteristic subgroup, namely $\gamma_{2n+1}(F_n)$. By Theorem 4.2.5, $N_{n,2n}$ has the R_∞-property and thus by Corollary 1.1.2(1'), so does F_n.

For free 2-step nilpotent groups, there is also a classification of which of those have full Reidemeister spectrum.

Theorem 4.2.6 (See e.g. [26]). *Let $r \geq 2$ be an integer. Then the free nilpotent group $N_{r,2}$ has full Reidemeister spectrum if and only if $r \geq 4$.*

Other groups for which the product formula has been used include 2-step nilpotent groups associated to graphs [23], nilpotent quotients of surface groups [20] and of Baumslag-Solitar groups [19].

We end by computing the Reidemeister spectrum of some specific finitely generated torsion-free nilpotent groups.

Example 4.2.7. We continue Example 4.1.3 and compute the Reidemeister spectrum. Recall that the adapted lower central series is given by $N \geq \langle c \rangle \geq 1$. Let $\varphi \in \mathrm{Aut}(N)$ and write φ_1 and φ_2 for the induced automorphisms on $N/\langle c \rangle$ and $\langle c \rangle$, respectively. As the latter is isomorphic to $\mathbb{Z}$, $R(\varphi_2) \in \{2, \infty\}$. By Theorem 4.2.1, $R(\varphi) = R(\varphi_1)R(\varphi_2)$. Therefore, $R(\varphi) \in 2\mathbb{N}_0 \cup \{\infty\}$.

Conversely, let $m \in \mathbb{N}_0$ be arbitrary. Let φ_m be the map on N defined by

$$\varphi(a) = b, \varphi(b) = ab^m, \varphi(c) = c^{-1}.$$

It is readily verified that φ respects the relations of N and that it defines an automorphism on N.

The induced automorphism $\varphi_{m,1}$ on $N/\langle c \rangle$ has matrix representation

$$\varphi_{m,1} = \begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix}.$$

Using Proposition T.1.12, we find $R(\varphi_{m,1}) = m$. As $\varphi_{m,2}$ is the inversion map on $\langle c \rangle$, $R(\varphi_{m,2}) = 2$. Consequently, $R(\varphi) = R(\varphi_{m,1})R(\varphi_{m,2}) = 2m$. We conclude that $\mathrm{Spec}_{\mathrm{R}}(N) = 2\mathbb{N}_0 \cup \{\infty\}$, as $\infty \in \mathrm{Spec}_{\mathrm{R}}(N)$ by Corollary 4.1.11. ∥

Example 4.2.8. Let $n \geq 3$ be an integer. Let G_n be the group

$$G_n := \left\langle a_1, \ldots, a_n, t \;\middle|\; \begin{array}{c} \forall i \in \{1, \ldots, n-1\} : [a_i, t] = a_{i+1}; \\ \forall i, j \in \{1, \ldots, n\} : [a_i, a_j] = 1 \\ [a_n, t] = 1 \end{array} \right\rangle$$

It is readily verified that $\gamma_i(G_n) = \langle a_i, \ldots, a_n \rangle$ for all $i \in \{2, \ldots, n\}$. As a_n commutes with all generators of G_n, by construction, $\gamma_{n+1}(G_n) = 1$. Therefore, G_n is nilpotent of class n.

We prove that G_n has the R_∞-property. First, remark that $G_n/\gamma_4(G_n) \cong G_3$. Since $\gamma_4(G_n)$ is characteristic, it is therefore sufficient to prove that G_3 has the R_∞-property by Corollary 1.1.2(1').

Thus, from now on, we work in G_3. Let $\varphi \in \operatorname{Aut}(G_3)$. The factors of the lower central series are torsion-free, so we can use them to find non-trivial fixed points as stated in Theorem 4.2.1. Recall that $\gamma_2(G_3) = \langle a_2, a_3 \rangle$ and $\gamma_3(G_3) = \langle a_3 \rangle$ are characteristic. For $i \in \{1, \ldots, 3\}$, let φ_i denote the induced automorphism on $\gamma_i(G_3)/\gamma_{i+1}(G_3)$. Then $\varphi_2(a_2\gamma_3(G_3)) = a_2^{\varepsilon_2}\gamma_3(G_3)$ for some $\varepsilon_2 \in \{-1, 1\}$, as $\gamma_2(G_3)/\gamma_3(G_3) \cong \mathbb{Z}$. In other words, $\varphi(a_2) = a_2^{\varepsilon_2}a_3^k$ for some $k \in \mathbb{Z}$.

Next, we argue that the centraliser C of $\langle a_2, a_3 \rangle$ equals $H := \langle a_1, a_2, a_3 \rangle$. Clearly, $H \leq C$. Conversely, let $g \in G_n \setminus H$. Then $g = at^l$ for some $a \in H$ and $l \in \mathbb{Z}$ with $l \neq 0$. If $g \in C$, then also $t^l \in C$. However, using the presentation of G, we find that

$$[a_2, t^l] = a_3^l \neq 1.$$

Consequently, t^l does not commute with a_2. Hence, $g = at^l \notin C$. We conclude that $C = H = \langle a_1, a_2, a_3 \rangle$. Since C is the centraliser of the characteristic subgroup $\langle a_2, a_3 \rangle = \gamma_2(G_3)$, it is characteristic itself as well. Since $C/\gamma_2(G_3)$ and G_n/C are both isomorphic to $\mathbb{Z}$ and are generated by $a_1\gamma_2(G_3)$ and tC, respectively, we find, similarly as for $\varphi(a_2)$, that $\varphi(a_1) = a_1^{\varepsilon_1}a$ and $\varphi(t) = bt^\varepsilon$ for some $a \in \gamma_2(G_3)$, $b \in C$ and $\varepsilon_1, \varepsilon \in \{-1, 1\}$.

Now, φ respects the relation $[a_1, t] = a_2$. Hence,

$$[a_1^{\varepsilon_1}a, bt^\varepsilon] = a_2^{\varepsilon_2}a_3^k.$$

Using Lemma 4.1.4, we find that the left-hand side modulo $\gamma_3(G_3) = \langle a_3 \rangle$ equals

$$\begin{aligned} [a_1^{\varepsilon_1}a, bt^\varepsilon] &\equiv [a_1^{\varepsilon_1}, b][a_1^{\varepsilon_1}, t^\varepsilon][a, b][a, t^\varepsilon] \\ &\equiv a_2^{\varepsilon_1\varepsilon} \bmod \gamma_3(G_3), \end{aligned}$$

since $[a, t^\varepsilon] \in \gamma_3(G_3)$. Thus, $a_2^{\varepsilon_1\varepsilon} \equiv a_2^{\varepsilon_2} \bmod \gamma_3(G_3)$, which implies that $\varepsilon_1\varepsilon = \varepsilon_2$. Since $\varepsilon, \varepsilon_1, \varepsilon_2 \in \{-1, 1\}$, at least one of them must equal 1. Then one of the maps φ_1, φ_2 or φ_3 has a fixed point, which implies that $R(\varphi) = \infty$ by Theorem 4.2.1. As $\varphi \in \operatorname{Aut}(G_3)$ is arbitrary, we conclude that G_3 has the R_∞-property. ‖

4.2.2 Applications

Theorem 4.2.1 does not only provide an efficient tool to compute Reidemeister numbers on concrete groups, it can also be used to prove general relations among Reidemeister numbers of related endomorphisms.

The first one generalises Lemma T.1.7(2).

Definition 4.2.9. Let G be a group and $\varphi \in \mathrm{End}(G)$. We call φ *class preserving* if $\varphi(g)$ is conjugate to g for each $g \in G$.

Proposition 4.2.10. *Let N be a finitely generated torsion-free nilpotent group. Suppose that φ and ψ are endomorphisms where ψ is class preserving endomorphism. Then $R(\varphi \circ \psi) = R(\psi \circ \varphi) = R(\varphi)$.*

Proof. By Corollary 3.1.12, we only have to argue that $R(\varphi \circ \psi) = R(\varphi)$. Consider the adapted lower central series of N. Then ψ induces the identity map on each of the factors, as these are abelian groups. By Lemma 4.1.6(2), all the factors are torsion-free. Hence, we can apply Theorem 4.2.1 to compute $R(\varphi)$ and $R(\varphi \circ \psi)$, which yields

$$R(\varphi \circ \psi) = \prod_{i=1}^{c} R((\varphi \circ \psi)_i) = \prod_{i=1}^{c} R(\varphi_i \circ \psi_i) = \prod_{i=1}^{c} R(\varphi_i) = R(\varphi). \qquad \square$$

Remark. On a nilpotent group, each class-preserving endomorphism is automatically an automorphism. Indeed, let N be a nilpotent group and $\varphi \in \mathrm{End}(N)$ a class-preserving endomorphism. If $\varphi(n) = 1$ for some $n \in N$, then $1 = gng^{-1}$ for some $g \in G$. This implies that $n = 1$. Hence, φ is injective.

To prove that φ is surjective, note that the induced homomorphism $\bar{\varphi}$ on the abelianisation $N/\gamma_2(N)$ is the identity map. Thus, $N = \gamma_2(N)\,\mathrm{Im}\,\varphi$. By [66, Theorem 16.2.5], this equality implies that $\mathrm{Im}\,\varphi = N$. Consequently, φ is surjective. We conclude that φ is an automorphism.

Lemma 4.2.11. *Let $A \in \mathbb{Z}^{n \times n}$ be a matrix and let $k, p \geq 1$ be integers with p a prime. Then the following hold:*

(1) $\det(A - I)$ *divides* $\det(A^k - I)$*;*

(2) $\det(A - I) \equiv \det(A^p - I) \bmod p$.

Proof. The first item immediately follows from the equality

$$A^k - I = (A - I)(A^{k-1} + A^{k-2} + \ldots + I).$$

For the second, note that

$$(A - I)^p = \sum_{i=0}^{p} \binom{p}{i} (-1)^i A^{p-i} \equiv A^p - I \bmod p,$$

as p is prime. Therefore,

$$\det(A^p - I) \equiv \det((A - I)^p) \equiv (\det(A - I))^p \equiv \det(A - I) \bmod p. \qquad \square$$

Proposition 4.2.12. *Let N be a finitely generated torsion-free nilpotent group. Let $\varphi \in \operatorname{End}(N)$ and $k \geq 1$. If $R(\varphi^k)$ is finite, then so is $R(\varphi)$. Moreover, in that case, $R(\varphi)$ divides $R(\varphi^k)$.*

Proof. Let c be the nilpotency class of N and let φ_i be the induced endomorphism on the ith factor of the adapted lower central series for $i \in \{1, \ldots, c\}$. Suppose that $R(\varphi^k)$ is finite. Then $R(\varphi_i^k)$ is finite as well for each $i \in \{1, \ldots, c\}$. Fix $i \in \{1, \ldots, c\}$. Recall that $R(\varphi_i^k) = |\det(\varphi_i^k - \mathrm{Id})|_\infty$ by Proposition T.1.12. Since $R(\varphi_i^k)$ is finite, $\det(\varphi_i^k - \mathrm{Id}) \neq 0$. By Lemma 4.2.11(1), $\det(\varphi_i - \mathrm{Id})$ divides $\det(\varphi_i^k - \mathrm{Id})$. Thus, $\det(\varphi_i - \mathrm{Id})$ is finite and non-zero. Consequently, $R(\varphi_i)$ is finite, non-zero, and divides $R(\varphi_i^k)$. Therefore,

$$R(\varphi) = \prod_{i=1}^{c} R(\varphi_i)$$

is finite and divides

$$R(\varphi^k) = \prod_{i=1}^{c} R(\varphi_i^k). \qquad \square$$

Proposition 4.2.13. *Let N be a finitely generated torsion-free nilpotent group. Let $\varphi \in \operatorname{End}(N)$ and p a prime number. If both $R(\varphi^p)$ and $R(\varphi)$ are finite, then $R(\varphi^p) \equiv R(\varphi) \bmod p$.*

Proof. Let c be the nilpotency class of N and let φ_i be the induced endomorphism on the ith factor of the adapted lower central series for $i \in \{1, \ldots, c\}$. Suppose that $R(\varphi^p)$ and $R(\varphi)$ are finite. By Lemma 4.2.11(2), $R(\varphi_i^p) \equiv R(\varphi_i) \bmod p$ for each $i \in \{1, \ldots, c\}$. Therefore,

$$R(\varphi^p) = \prod_{i=1}^{c} R(\varphi_i^p) \equiv \prod_{i=1}^{c} R(\varphi_i) = R(\varphi) \bmod p.$$

$\square$

Remark. By Proposition 4.2.12, we know that if $R(\varphi^p)$ is finite, then so is $R(\varphi)$. Hence, we only have to argue that $R(\varphi^p)$ is finite in order to apply the previous corollary.

For finitely generated torsion-free nilpotent groups, the converse of Proposition 1.2.11 holds as well.

Lemma 4.2.14. *Let N be a finitely generated torsion-free nilpotent group and let C be a central subgroup such that N/C is torsion-free. Let $\varphi \in \mathrm{End}(N)$ such that $\varphi(C) \leq C$. Let $\bar{\varphi}$ denote the induced endomorphism on N/C. If $\bar{\varphi}$ has a non-trivial fixed point, then so does φ.*

Proof. Suppose that $\bar{\varphi}(nC) = nC$ for some $nC \neq C$. Let $\varphi|_C$ denote the restriction of φ to C. If $R(\varphi|_C) = \infty$, then $\varphi|_C$ has a fixed point by Proposition T.1.12. Hence, φ does as well in that case.

Therefore, suppose that $R(\varphi|_C) < \infty$. Since $\bar{\varphi}(nC) = nC$, there is a $c \in C$ such that $\varphi(n) = nc$. As C is central, $\varphi(n^m) = n^m c^m$ for all $m \geq 1$. By Proposition T.1.12, $\mathcal{R}[\varphi|_C]$ has a group structure. As $R(\varphi|_C) < \infty$, it is a finite group, so there is a $k \geq 1$ such that $c^k \sim_{\varphi|_C} 1$, say, $c^k = c_k \varphi(c_k)^{-1}$ for some $c_k \in C$. We then get

$$c_k \varphi(c_k)^{-1} = c^k = n^{-k} \varphi(n^k).$$

We derive from this that $\varphi(n^k c_k) = n^k c_k$. If $n^k c_k \in C$, then

$$(nC)^k = n^k C = n^k c_k C = C.$$

Since N/C is torsion-free, $nC = C$, which contradicts the assumption on n. Therefore, $n^k c_k \notin C$, which in particular implies that $n^k c_k \neq 1$. Thus, φ has a non-trivial fixed point. □

Proposition 4.2.15. *Let N be a finitely generated torsion-free nilpotent group. Let $\varphi \in \mathrm{End}(N)$ and suppose that $R(\varphi) = \infty$. Then $\mathrm{Stab}_\varphi(n)$ is infinite for all $n \in N$.*

Proof. Fix $\varphi \in \mathrm{End}(N)$ with $R(\varphi) = \infty$. We start by proving that φ has a non-trivial fixed point. Let

$$1 = N_0 \lhd N_1 \lhd \ldots \lhd N_c = N$$

be a φ-invariant central series with torsion-free successive quotients, e.g. the adapted lower central series. Let φ_i denote the induced endomorphism on $\frac{N_i}{N_{i-1}}$ for each $i \in \{1, \ldots, c\}$. By Theorem 4.2.1, there exists an $i \in \{1, \ldots, c\}$ such that $R(\varphi_i) = \infty$. Since N_i/N_{i-1} is finitely generated torsion-free abelian, φ_i has a non-trivial fixed point nN_{i-1} by Proposition T.1.12. If we write ψ_i for the induced endomorphism on N/N_{i-1}, it follows that ψ_i has a non-trivial fixed point as well. We prove by induction on i that we can lift this fixed point to a non-trivial fixed point of φ.

We start with the case $i = 1$. Then ψ_1 is (essentially) φ, which immediately proves the result.

Next, suppose that the result holds for i, which means that we can lift a non-trivial fixed point of ψ_i to one of φ. So, assume that ψ_{i+1} has a non-trivial fixed point nN_i. Consider the short exact sequence

$$1 \to \frac{N_i}{N_{i-1}} \to \frac{N}{N_{i-1}} \to \frac{N/N_{i-1}}{N_i/N_{i-1}} \to 1$$

We have the endomorphism ψ_i on the middle term of this sequence. Let $\bar{\psi}_i$ denote the induced endomorphism on the last (non-trivial) term in the sequence. By the third isomorphism theorem and the definition of $\bar{\psi}_i$ and ψ_{i+1}, we have the following commuting square:

$$\begin{tikzcd}
\frac{N/N_{i-1}}{N_i/N_{i-1}} \arrow[r, "\bar{\psi}_i"] \arrow[d, "\cong"] & \frac{N/N_{i-1}}{N_i/N_{i-1}} \arrow[d, "\cong"] \\
\frac{N}{N_i} \arrow[r, "\psi_{i+1}"] & \frac{N}{N_i}
\end{tikzcd}$$

Consequently, $\bar{\psi}_i$ has a non-trivial fixed point. Now, remark that N_i/N_{i-1} is a central subgroup of N/N_{i-1} and that $\frac{N/N_{i-1}}{N_i/N_{i-1}}$ is torsion-free. Therefore, Lemma 4.2.14 implies that $\psi_i \in \mathrm{End}(N/N_{i-1})$ has a non-trivial fixed point as well. The induction hypothesis now yields a non-trivial fixed point for φ. As N is torsion-free, $\mathrm{Fix}(\varphi)$ is infinite.

Finally, to prove that $\mathrm{Stab}_\varphi(n)$ is infinite for all $n \in N$, recall from Proposition 1.2.11 that $\mathrm{Stab}_\varphi(n) = \mathrm{Fix}(\tau_n \circ \varphi)$ and that $R(\tau_n \circ \varphi) = R(\varphi)$ for all $n \in N$, by Lemma T.1.7. Therefore, we can apply the result to $\tau_n \circ \varphi$ to obtain that $\mathrm{Fix}(\tau_n \circ \varphi) = \mathrm{Stab}_\varphi(n)$ is infinite for all $n \in N$. $\square$

Remark. In this section, we wanted to demonstrate the power and usefulness of the product formula on finitely generated torsion-free nilpotent groups. There is, however, an alternative way to prove Proposition 4.2.15. The main idea is the following: let N be a finitely generated torsion-free nilpotent group and $\varphi \in \mathrm{End}(N)$. One can embed N in another nilpotent group $N^{\mathbb{Q}}$, called its rational Malcev completion (see Variation 5). This group $N^{\mathbb{Q}}$ corresponds to a (rational) Lie algebra $\mathfrak{n}^{\mathbb{Q}}$. The endomorphism φ extends to an endomorphism of $N^{\mathbb{Q}}$, and this map, in turn, corresponds to a Lie algebra endomorphism φ_* on $\mathfrak{n}^{\mathbb{Q}}$.

Now, this map φ_* satisfies $R(\varphi) = |\det(\varphi_* - \mathrm{Id})|_\infty$, which resembles the formula for finitely generated free abelian groups (Proposition T.1.12). Thus, if $R(\varphi) = \infty$, then φ_* (which is also a linear map) has an eigenvalue 1. Consequently, there is a $v \in \mathfrak{n}^{\mathbb{Q}}$ such that $\varphi_*(v) = v$. Using v, one can then construct a non-trivial fixed point of φ.

Variation 5

Direct products of nilpotent groups

Mozart, Wolfgang Amadeus, *Zwölf Variationen in C über das franzözische Lied 'Ah, vous dirai-je, Maman'*, (KV 265 (300e)), Variation V (bars 1–8).

In Variation 3, we have proven that the Reidemeister spectrum of a direct product of directly indecomposable centreless groups is completely determined by the Reidemeister spectra of the individual factors. We mentioned there that these are sufficient but not necessary conditions. In this Variation, we prove a second result of this type for nilpotent groups.

The idea is to give a full and concrete description of the automorphism group of a direct product of certain nilpotent groups in terms of the factors and then use this description to determine the Reidemeister spectrum. Such descriptions for both the automorphism group and the Reidemeister spectrum are known for direct product of free nilpotent groups [117, §6.5] and 2-step nilpotent quotients of groups associated to graphs [23]. The description we give here generalises both of these. Most of the results in this Variation can be found in [106].

5.1 Preliminaries

In addition to those discussed in Variation 4, we collect here the technical results needed in the derivation of the automorphism group of a direct product of nilpotent groups.

Lemma 5.1.1. *Let N be a finitely generated torsion-free nilpotent group and let H be a finite index subgroup. Then $Z(H) \leq C_N(H) = Z(N)$.*

Proof. Let $g \in C_N(H)$ and let $n \in N$. As H has finite index in N, there is a $k \in \mathbb{N}_0$ such that $n^k \in H$. For that k, it then holds that $gn^kg^{-1} = n^k$. As N is torsion-free nilpotent, it follows that $gng^{-1} = n$. Since n was arbitrary, this shows that g lies in $Z(N)$.

The inclusions $Z(N) \leq C_N(H)$ and $Z(H) \leq C_N(H)$ are immediate. □

For finitely generated nilpotent groups, there is a useful criterion to decide whether a subgroup has finite index.

Lemma 5.1.2 (See e.g. [2, Lemma 2.8])**.** *Let N be a finitely generated nilpotent group and let H be a subgroup. If for each $n \in N$, there is a $k \in \mathbb{N}_0$ such that $n^k \in H$, then H has finite index in N.*

We need two properties of the Hirsch length of a polycyclic group.

Lemma 5.1.3 (See e.g. [104, Exercise 8])**.** *Let G be a polycyclic group, H a subgroup and N a normal subgroup. Then*

(1) $h(H) \leq h(G)$ with equality if and only if $[G : H] < \infty$;

(2) $h(G) = h(N) + h(G/N)$.

5.1.1 Group homomorphisms

The following can be seen as a Fitting lemma for finitely generated torsion-free nilpotent groups.

Lemma 5.1.4. *Let N be a finitely generated torsion-free nilpotent group. Let $\varphi : N \to N$ be an endomorphism. Then there exists a $k \in \mathbb{N}$ such that $\ker \varphi^k = \ker \varphi^{k+n}$ for all $n \geq 0$. Moreover, for such k, $\ker \varphi^k \cap \operatorname{Im} \varphi^k = 1$. In particular, if $\operatorname{Im} \varphi^k$ is normal in N, then $\operatorname{Im} \varphi^k \ker \varphi^k \cong \operatorname{Im} \varphi^k \times \ker \varphi^k$.*

Proof. Put, for each $i \in \mathbb{N}_0$, $K_i := \ker \varphi^i$. Then clearly $K_i \leq K_{i+1}$ for each i. Therefore,

$$K_1 \leq K_2 \leq K_3 \leq \dots$$

is a non-decreasing chain of subgroups. Since N is finitely generated and nilpotent, this chain must stabilise at some index, say k. Thus, for each $n \geq 0$, we have $K_k = K_{k+n}$.

Next, suppose that $x \in \ker \varphi^k \cap \operatorname{Im} \varphi^k$. Write $x = \varphi^k(y)$. Then $\varphi^{2k}(y) = \varphi^k(x) = 1$, which implies that $y \in K_{2k} = K_k$. Therefore, $x = \varphi^k(y) = 1$. □

Lemma 5.1.5. *Let G and A be two groups with A torsion-free abelian and let $\varphi : G \to A$ be a homomorphism. Then $\sqrt[G]{\gamma_2(G)} \leq \ker \varphi$.*

Proof. Since A is abelian, $\gamma_2(G) \leq \ker \varphi$. Now, let $g \in G$ and $k \in \mathbb{N}_0$ be such that $g^k \in \gamma_2(G)$. Then $\varphi(g^k) = 0$, hence $k\varphi(g) = 0$ (we write A additively), so by torsion-freeness of A, $\varphi(g) = 0$. Therefore, $g \in \ker \varphi$, which proves the lemma. □

The following lemma is a special case of [125, Theorem 1].

Lemma 5.1.6. *Let N be a finitely generated torsion-free nilpotent group. Let $\varphi \in \operatorname{End}(N)$. If φ restricts to an automorphism of $Z(N)$, then φ is an automorphism of N.*

We also need the so-called (short) five lemma for groups.

Lemma 5.1.7. *Let G_1 and G_2 be groups with respective normal subgroup N_1 and N_2. Let $\varphi : G_1 \to G_2$ be a morphism such that $\varphi(N_1) \leq N_2$. Let $\varphi|_{N_1} : N_1 \to N_2$ and $\bar{\varphi} : G_1/N_1 \to G_2/N_2$ denote the induced homomorphisms. Consider the following commuting diagram with exact rows:*

$$\begin{tikzcd}
1 \arrow[r] & N_1 \arrow[r] \arrow[d, "\varphi|_{N_1}"] & G_1 \arrow[r] \arrow[d, "\varphi"] & G_1/N_1 \arrow[r] \arrow[d, "\bar{\varphi}"] & 1 \\
1 \arrow[r] & N_2 \arrow[r] & G_2 \arrow[r] & G_2/N_2 \arrow[r] & 1
\end{tikzcd}$$

If $\varphi|_{N_1}$ and $\bar{\varphi}$ are isomorphisms, then φ is an isomorphism as well.

The technical result below is proven by J. Bidwell [7, Lemma 2.4] in the context of direct products of finite groups. His proof, however, does not use the finiteness of the groups, so it works for arbitrary groups. For the sake of completeness, we include the proof here. Recall the notation from Variation 3: given groups $G_1, \dots, G_n$ and an endomorphism φ of $G_1 \times \dots \times G_n$, we can view φ as a matrix $(\varphi_{ij})_{ij}$, where for all $i, j \in \{1, \dots, n\}$, φ_{ij} is a homomorphism from G_j to G_i.

Lemma 5.1.8 (See e.g. [7, Lemma 2.4]). *Let $G_1, \ldots, G_k$ be groups and put $G = \bigtimes_{i=1}^{k} G_i$. Let $\varphi = (\varphi_{ij})_{ij} \in \mathrm{Aut}(G)$ and write $\varphi^{-1} = (\varphi'_{ij})_{ij}$. For $i, j \in \{1, \ldots, k\}$, put*

$$\sigma_{i,j} := \sum_{\substack{l=1 \\ l \neq j}}^{k} \varphi'_{i,l}\varphi_{l,i} \in \mathrm{End}(G_i)$$

and

$$\sigma'_{i,j} := \sum_{\substack{l=1 \\ l \neq j}}^{k} \varphi_{i,l}\varphi'_{l,i} \in \mathrm{End}(G_i).$$

Then for all $i, j \in \{1, \ldots, k\}$ and $t \geq 0$, the subgroups $\mathrm{Im}(\sigma_{i,j}^{t})$ and $\mathrm{Im}\big((\sigma'_{i,j})^{t}\big)$ are normal in G_i.

Proof. We prove the result for $\mathrm{Im}(\sigma_{i,j}^{t})$; the proof for $\mathrm{Im}\big((\sigma'_{i,j})^{t}\big)$ is similar. Remark that the statement for $t = 0$ is simply the fact that G_i is normal in G_i

Let $i, j \in \{1, \ldots, k\}$ and $x \in G_i$ be arbitrary. For $p \geq 0$, we put $x_p := \varphi'_{i,j}\big(\varphi_{j,i}\big(\sigma_{i,j}^{p}(x)\big)\big)$. First, we prove, for all $m \geq 1$, that

$$x = \sigma_{i,j}^{m}(x) \prod_{l=0}^{m-1} x_l, \tag{5.1.1}$$

and that $[x_m, \mathrm{Im}\,\sigma_{i,j}] = 1$. We start with the latter. Note that

$$[\mathrm{Im}\,\sigma_{i,j}, \mathrm{Im}\,\varphi'_{i,j}\varphi_{j,i}] = 1 \tag{5.1.2}$$

by the commuting properties of the $\varphi'_{i,l}$. Therefore, $[x_m, \mathrm{Im}\,\sigma_{i,j}] = 1$ for all $m \geq 1$.

For (5.1.1), we proceed by induction on m. Since $\mathrm{Id}_{G_i} = (\varphi^{-1} \circ \varphi)_{ii} = \sum_{l=1}^{k} \varphi'_{i,l}\varphi_{l,i}$,

$$x = \sigma_{i,j}(x)\varphi'_{i,j}(\varphi_{j,i}(x)) = \sigma_{i,j}(x)x_0, \tag{5.1.3}$$

where we also used (5.1.2). This proves the case $m = 1$.

Next, suppose that (5.1.1) holds for m. Replacing x with $\sigma_{i,j}(x)$ in this equality, we get

$$\sigma_{i,j}(x) = \sigma_{i,j}^{m+1}(x) \prod_{l=0}^{m-1} \varphi'_{i,j}\big(\varphi_{j,i}\big(\sigma_{i,j}^{l+1}(x)\big)\big).$$

Therefore, combining this with (5.1.3) and (5.1.2), we obtain

$$
\begin{aligned}
x &= \varphi'_{i,j}(\varphi_{j,i}(x))\sigma_{i,j}^{m+1}(x) \prod_{l=0}^{m-1} \varphi'_{i,j}\big(\varphi_{j,i}\big(\sigma_{i,j}^{l+1}(x)\big)\big) \\
&= \sigma_{i,j}^{m+1}(x) \prod_{l=0}^{m} \varphi'_{i,j}(\varphi_{j,i}(\sigma_{i,j}^{l}(x)) \\
&= \sigma_{i,j}^{m+1}(x) \prod_{l=0}^{m} x_l.
\end{aligned}
$$

This finishes the induction proof.

Finally, let $i, j \in \{1, \ldots, k\}$ and $x, y \in G_i$ be arbitrary. Then, by (5.1.1) for $m = t$ and (5.1.2),

$$
\sigma_{i,j}^{t}(y)^{x} = \sigma_{i,j}^{t}(y)^{\sigma_{i,j}^{t}(x) \prod_{l=0}^{t-1} x_l} = \sigma_{i,j}^{t}(y)^{\sigma_{i,j}^{t}(x)} = \sigma_{i,j}^{t}(y^{x}).
$$

This proves that $\sigma_{i,j}^{t} \lhd G_i$, as x and y were arbitrary. □

5.1.2 Rational completion

Definition 5.1.9. Let N be a finitely generated torsion-free nilpotent group. The *rational Malcev completion*, or rational completion for short, of N is the unique (up to isomorphism) torsion-free nilpotent group $N^{\mathbb{Q}}$ satisfying the following properties (see e.g. [3], [104, Chapter 6]):

(1) N embeds into $N^{\mathbb{Q}}$;

(2) for all $n \in N$ and $k \in \mathbb{N}_0$, there exists a unique $m \in N^{\mathbb{Q}}$ such that $m^k = n$;

(3) for all $n \in N^{\mathbb{Q}}$, there exists a $k \in \mathbb{N}_0$ such that $n^k \in N$.

Following [1], we call a finitely generated torsion-free nilpotent group N *rationally indecomposable* if $N^{\mathbb{Q}}$ is directly indecomposable. Examples of rationally indecomposable groups include finitely generated torsion-free nilpotent groups with cyclic centre (see e.g. [1, Lemma 2]).

Proposition 5.1.10 (See e.g. [3, Proposition 5])**.** *Let N be a finitely generated torsion-free nilpotent group. Suppose that $N = N_1 \times N_2$ for some groups N_1 and N_2. Then $N^{\mathbb{Q}} = N_1^{\mathbb{Q}} \times N_2^{\mathbb{Q}}$.*

Proposition 5.1.11. *Let N be a finitely generated torsion-free nilpotent group. Then N is rationally indecomposable if and only if every finite index subgroup of N is directly indecomposable.*

In particular, every finite index subgroup of a rationally indecomposable group is itself rationally indecomposable.

Proof. First, assume that N is rationally indecomposable. Let $H \leq N$ be a subgroup of finite index. We argue that $N^{\mathbb{Q}}$ is also the rational completion of H. The result then follows from Proposition 5.1.10.

Clearly, H embeds into $N^{\mathbb{Q}}$, as it is a subgroup of N. Next, let $h \in H$ and $k \in \mathbb{N}_0$. Since $H \leq N$ and $N^{\mathbb{Q}}$ is the rational completion of N, there is a unique $n \in N^{\mathbb{Q}}$ such that $h = n^k$.

Finally, let $n \in N^{\mathbb{Q}}$. Then there is a $k \in \mathbb{N}_0$ such that $n^k \in N$. Since H has finite index in N, there is an $m \in \mathbb{N}_0$ such that $(n^k)^m \in H$. Therefore, $n^{km} \in H$, which finishes the proof that $N^{\mathbb{Q}}$ is the rational completion of H.

Conversely, suppose that $N^{\mathbb{Q}} = A \times B$ is a non-trivial decomposition. Define $N_A := A \cap N$ and $N_B := B \cap N$. We prove that neither N_A nor N_B is trivial, that $H := \langle N_A N_B \rangle$ is (isomorphic to) $N_A \times N_B$ and that H has finite index in N.

Let $a \in A$ be non-trivial. As $a \in N^{\mathbb{Q}}$, there is a $k \geq 1$ such that $a^k \in N$. Consequently, $a^k \in N_A$. As $N^{\mathbb{Q}}$ is torsion-free, $a^k \neq 1$, which proves that N_A is non-trivial. A similar argument holds for N_B.

By construction, $N_A \cap N_B = N \cap A \cap B = 1$. Also, the groups N_A and N_B commute, as $[N_A, N_B] \leq [A, B] = 1$. Thus, $H = N_A \times N_B$. We are left with proving that H has finite index in N. Let $n \in N$. Then $n = ab$ for some $a \in A$ and $b \in B$. By the properties of the Malcev completion, there exists a $k \geq 1$ such that $a^k, b^k \in N$. Then $a^k \in N_A$ and $b^k \in N_B$. Since a and b commute, $(ab)^k = a^k b^k \in N_A \times N_B = H$. Thus, some power of n lies in H. This holds for arbitrary n, so by Lemma 5.1.2, H has finite index in N.

Finally, let N be rationally indecomposable and H a finite index subgroup. Let K be a finite index subgroup of H. Since $[N : K] = [N : H][H : K]$, K has finite index in N and is therefore directly indecomposable. As K was arbitrary, every finite index subgroup of H is directly indecomposable. It follows that H is rationally indecomposable. □

Remark. There exist finitely generated torsion-free directly (but not rationally) indecomposable nilpotent groups with a subgroup of finite index that does split as a non-trivial direct product. We provide an example based on [1, § 5] and [3, § 5]. Let A be the free abelian group on a, b and c and let B be the group $A \rtimes_\theta \langle t \rangle$, where

$$\theta(t) : A \to A : a \mapsto ab, b \mapsto bc, c \mapsto c.$$

Then B is finitely generated, torsion-free and 3-step nilpotent. Next, let F be the infinite cyclic group on f and put $K = B \times F$. Fix an integer $p \geq 2$. We define the group G_p to be the subgroup of $K^{\mathbb{Q}}$ generated by K and the unique pth root of bf, which we call s. It is proven in [1, Lemma 3] that G_p is directly indecomposable (in their notation, $G_p = G(1,p)$). However, K is, by construction, a direct product.

We now argue that K has finite index in G_p. We first prove that $[t, s^{-1}]^p = c$. Indeed, $(t^{-1}st)^p = bcf$ and $s^p = bf$, so $(t^{-1}st)^p$ and s^p commute. As G_p is torsion-free nilpotent, s^t and s^{-1} commute as well. Consequently,

$$[t, s^{-1}]^p = (t^{-1}sts^{-1})^p = (t^{-1}st)^p s^{-p} = bcf(bf)^{-1} = c$$

For ease of notation, we write $c^{1/p}$ for $[t, s^{-1}]$. Since c commutes with everything in K, so does $c^{1/p}$, as we work in a torsion-free nilpotent group. Analogously, s commutes with a, b, c and f. In particular, $\left\langle f, c^{1/p} \right\rangle \leq Z(G_p)$. Using these observations and the rewriting rules coming from the definition of B and $c^{1/p}$, we can rewrite any element in G_p in the form

$$a^\alpha b^\beta f^F t^T s^S c^{C/p},$$

where $\alpha, \beta, F, T, S, C \in \mathbb{Z}$. Any right coset of K in G_p then has a representative of the form $s^S c^{C/p}$. Moreover, since $s^p = bf \in K$ and $c \in K$, it is sufficient to take $S, C \in \{0, \ldots, p-1\}$ to obtain a complete set of representatives of the right cosets of K in G_p. Therefore, $[G_p : K] \leq p^2$.

Lemma 5.1.12. *Let N be a finitely generated torsion-free nilpotent group and let $K \leq N$ be a normal subgroup.*

(1) The rational completion $K^{\mathbb{Q}}$, seen as a subgroup of $N^{\mathbb{Q}}$, is normal in $N^{\mathbb{Q}}$.

(2) Suppose that N/K is torsion-free. Then $(N/K)^{\mathbb{Q}} \cong N^{\mathbb{Q}}/K^{\mathbb{Q}}$

Proof. For the first item, we refer the reader to [71, p. 254].

For the second item, since N/K is torsion-free and finitely generated, its rational completion exists. Consider the map $\varphi : N \to N^{\mathbb{Q}}/K^{\mathbb{Q}} : n \mapsto nK^{\mathbb{Q}}$, which is the composition of the inclusion $N \hookrightarrow N^{\mathbb{Q}}$ and the projection $N^{\mathbb{Q}} \to N^{\mathbb{Q}}/K^{\mathbb{Q}}$. As $K \leq \ker \varphi$, it induces a homomorphism $\Phi : N/K \to N^{\mathbb{Q}}/K^{\mathbb{Q}}$. We prove that Φ

is injective, and that $N^{\mathbb{Q}}/K^{\mathbb{Q}}$ satisfies the properties of the rational completion for $\operatorname{Im}\Phi$.

So, suppose that $\Phi(nK) = K^{\mathbb{Q}}$ for some $n \in N$. Then $n \in K^{\mathbb{Q}}$. Hence, there is an $m \in \mathbb{N}_0$ such that $n^m \in K$. This means that $n^m K = K$ in N/K. As the latter is torsion-free, we conclude that $nK = K$, which means that $n \in K$. Hence, Φ is injective.

Next, let $nK^{\mathbb{Q}} \in \operatorname{Im}\Phi$ for some $n \in N$ and let $m \in \mathbb{N}_0$. Then there is a unique $h \in N^{\mathbb{Q}}$ such that $h^m = n$. Consequently, $(hK^{\mathbb{Q}})^m = nK^{\mathbb{Q}}$, so the second property of the rational completion is satisfied.

Finally, let $nK^{\mathbb{Q}} \in N^{\mathbb{Q}}/K^{\mathbb{Q}}$ for some $n \in N^{\mathbb{Q}}$. Then there is an $m \in \mathbb{N}_0$ such that $n^m \in N$. Consequently, $(nK^{\mathbb{Q}})^m \in \operatorname{Im}\Phi$, so the third property is satisfied. □

Proposition 5.1.13. *Let N be finitely generated torsion-free nilpotent group. Suppose that $N^{\mathbb{Q}}$ does not have an abelian direct factor. Then $Z(N) \leq \sqrt[N]{\gamma_2(N)}$.*

Proof. We proceed by contraposition. Let $\pi : N \to \frac{N}{\sqrt[N]{\gamma_2(N)}}$ be the natural projection and suppose that $z \in Z(N)$ is such that $\pi(z) \neq 1$. By Lemma 5.1.12(1), we can consider the induced projection map $\pi^{\mathbb{Q}} : N^{\mathbb{Q}} \to N^{\mathbb{Q}}/\sqrt[N]{\gamma_2(N)}^{\mathbb{Q}}$. Since $\pi(z) \neq 1$, also $\pi^{\mathbb{Q}}(z) \neq 1$. Indeed, suppose $\pi^{\mathbb{Q}}(z) = 1$. Then $z \in \sqrt[N]{\gamma_2(N)}^{\mathbb{Q}}$. Hence, $z^k \in \sqrt[N]{\gamma_2(N)}$ for some $k \geq 1$. Then $\pi(z)^k = 1$, which implies that $\pi(z) = 1$, as $\frac{N}{\sqrt[N]{\gamma_2(N)}}$ is torsion-free by Lemma 4.1.6(2).

Now, since $N^{\mathbb{Q}}/\sqrt[N]{\gamma_2(N)}^{\mathbb{Q}}$ is isomorphic to $(N/\sqrt[N]{\gamma_2(N)})^{\mathbb{Q}}$ by Lemma 5.1.12(2), we see that $N^{\mathbb{Q}}/\sqrt[N]{\gamma_2(N)}^{\mathbb{Q}}$ is abelian. Hence, it is isomorphic to $\mathbb{Q}^k$ for some $k \geq 1$. So, $\pi^{\mathbb{Q}}(z)$ is part of a $\mathbb{Q}$-basis $\mathcal{B}$ of $N^{\mathbb{Q}}/\sqrt[N]{\gamma_2(N)}^{\mathbb{Q}}$.

Finally, consider the composition

$$N^{\mathbb{Q}} \to N^{\mathbb{Q}}/\sqrt[N]{\gamma_2(N)}^{\mathbb{Q}} \to \langle \pi^{\mathbb{Q}}(z) \rangle_{\mathbb{Q}},$$

where the last group is the $\mathbb{Q}$-linear span of $\pi^{\mathbb{Q}}(z)$ and where the last map sends $\mathcal{B} \setminus \{\pi^{\mathbb{Q}}(z)\}$ to 0. Let θ be this composition map. We claim that $N^{\mathbb{Q}} \cong \ker\theta \times \langle z \rangle^{\mathbb{Q}}$. Clearly, $\ker\theta$ is normal in $N^{\mathbb{Q}}$. Since $Z(N^{\mathbb{Q}}) = Z(N)^{\mathbb{Q}}$ (see e.g. [71, p. 257]), the subgroup $\langle z \rangle^{\mathbb{Q}}$ lies in $Z(N^{\mathbb{Q}})$, so it is central and hence normal. Finally, $\ker\theta \cap \langle z \rangle^{\mathbb{Q}} = 1$ by construction, so $N^{\mathbb{Q}}$ is the (internal) direct product of $\ker\theta$ and $\langle z \rangle^{\mathbb{Q}}$. The latter is abelian, which concludes the proof of the contraposition of the statement. □

5.2 Main results

5.2.1 Automorphism group

The goal of this section is to prove the following:

Theorem 5.2.1. *Let $N_1, \ldots, N_k$ be finitely generated torsion-free non-abelian nilpotent groups such that N_i is rationally indecomposable for each $i \in \{1, \ldots, k\}$. Put $N = \bigtimes_{i=1}^{k} N_i$. Let $\varphi = (\varphi_{ij})_{ij} \in \operatorname{Aut}(N)$. Then the following hold:*

(1) For each $i \in \{1, \ldots, k\}$, there is a unique $\sigma(i) \in \{1, \ldots, k\}$ such that $\varphi_{i\sigma(i)}$ is an isomorphism.

(2) For each $j \in \{1, \ldots, k\}$, there is a unique $i \in \{1, \ldots, k\}$ such that φ_{ij} is an isomorphism.

(3) For each $i \in \{1, \ldots, k\}$ and for each $j \in \{1, \ldots, k\}$ different from $\sigma(i)$, we have $\operatorname{Im} \varphi_{ij} \leq Z(N_i)$.

To prove this, we proceed by induction on the number of distinct Hirsch lengths among the N_i. First, we prove a last technical lemma, which is a major tool in the proof of the theorem above.

Lemma 5.2.2. *Let $N_1, \ldots, N_k$ be finitely generated torsion-free non-abelian nilpotent groups such that N_i is rationally indecomposable for each $i \in \{1, \ldots, k\}$. Put $N = \bigtimes_{i=1}^{k} N_i$ and let $\varphi = (\varphi_{ij})_{ij} \in \operatorname{Aut}(N)$. Then for each $i \in \{1, \ldots, k\}$, there is a $j \in \{1, \ldots, k\}$ such that $\varphi_{ij} : N_j \to N_i$ is injective.*

Moreover, if for a given $i \in \{1, \ldots, k\}$ the obtained index $j \in \{1, \ldots, k\}$ satisfies $h(N_j) = h(N_i)$, the following hold:

(1) For all $l \in \{1, \ldots, k\}$ with $l \neq j$, $\operatorname{Im} \varphi_{il} \leq Z(N_i)$;

(2) The index $l \in \{1, \ldots, k\}$ for which φ_{il} is injective is unique (namely j);

(3) The map φ_{ij} is an isomorphism.

Proof. Write $\varphi^{-1} = (\varphi'_{ij})_{ij}$ and fix $i \in \{1, \ldots, k\}$. As N_i is not abelian and N_i is generated by the commuting subgroups $\operatorname{Im} \varphi_{i1}$ up to $\operatorname{Im} \varphi_{ik}$, by Lemma 3.1.2(1), there is a $j \in \{1, \ldots, k\}$ such that $K := \operatorname{Im} \varphi_{ij}$ is non-abelian. Let $x \in K$. Then

$$\left(\sum_{l=1}^{k} \varphi_{il} \varphi'_{li} \right)(x) = x,$$

as this is just part of the identity $\varphi \circ \varphi^{-1} = \mathrm{Id}_N$. We can rewrite this as $\sigma'_{i,j}(x)(\varphi_{ij} \circ \varphi'_{ji})(x) = x$, where $\sigma'_{i,j}$ is defined as in Lemma 5.1.8. As $(\varphi_{ij} \circ \varphi'_{ji})(x) \in \operatorname{Im} \varphi_{ij} = K$, it follows that $\sigma'_{i,j}(x) \in K$. Since $\operatorname{Im} \varphi_{il}$ commutes with $\operatorname{Im} \varphi_{ij}$ for $l \neq j$, $\operatorname{Im} \sigma'_{i,j}$ also commutes with $\operatorname{Im} \varphi_{ij} = K$. As $\operatorname{Im} \sigma'_{i,j}$ is contained in K, we obtain the inclusion $\operatorname{Im} \sigma'_{i,j} \leq Z(K)$.

Now, put $\hat{K} := \operatorname{Im}(\varphi_{ij} \circ \varphi'_{ji})$. Since $x = \sigma'_{i,j}(x)(\varphi_{ij} \circ \varphi'_{ji})(x)$ for each $x \in K$, we see that $K = Z(K)\hat{K}$. Since K is non-abelian, $\hat{K}$ must be non-abelian as well.

Next, let $x \in \ker \varphi_{ij}$. As

$$\left(\sum_{l=1}^{k} \varphi'_{jl} \varphi_{lj} \right)(x) = x,$$

we get $\sigma_{j,i}(x) = x$. Applying Lemma 5.1.4 to $\sigma_{j,i} \in \operatorname{End}(N_j)$ and using Lemma 5.1.8, we get a $t \geq 1$ such that $H := \operatorname{Im} \sigma_{j,i}^t \ker \sigma_{j,i}^t \cong \operatorname{Im} \sigma_{j,i}^t \times \ker \sigma_{j,i}^t$. Moreover, H has finite index in N_j. Indeed, if we compute the Hirsch length of H using Lemma 5.1.3, we see that

$$h(H) = h(\ker \sigma_{j,i}^t) + h(\operatorname{Im} \sigma_{j,i}^t) = h(N_j).$$

So, H does have finite index in N_j by the same lemma. Since N_j is rationally indecomposable, Proposition 5.1.11 implies that either $\ker \sigma_{j,i}^t = 1$ or $\operatorname{Im} \sigma_{j,i}^t = 1$. Suppose that $\ker \sigma_{j,i}^t = 1$. Then $\sigma_{j,i}^t$ is injective, which implies that $\sigma_{j,i}$ is injective as well. Then $h(\operatorname{Im} \sigma_{j,i}) = h(N_j)$, which implies that $\operatorname{Im} \sigma_{j,i}$ has finite index in N_j. Now, remark that $[\operatorname{Im} \sigma_{j,i}, \operatorname{Im} \varphi'_{j,i}] = 1$, as $\operatorname{Im} \varphi'_{j,i}$ commutes with $\operatorname{Im} \varphi'_{j,l}$ if $l \neq i$. Therefore, $\operatorname{Im} \varphi'_{j,i} \leq C_{N_j}(\operatorname{Im} \sigma_{j,i})$, which lies in $Z(N_j)$ due to Lemma 5.1.1 since $\operatorname{Im} \sigma_{j,i}$ has finite index in N_j. On the other hand, recall that $\hat{K} = \operatorname{Im}(\varphi_{i,j} \circ \varphi'_{j,i})$ is non-abelian. Therefore, $\operatorname{Im} \varphi'_{j,i}$ is non-abelian as well. This contradicts the previously proven inclusion $\operatorname{Im} \varphi'_{j,i} \leq Z(N_j)$. Consequently, the assumption that $\ker \sigma_{j,i}^t = 1$ is false, which implies that $\operatorname{Im} \sigma_{j,i}^t = 1$.

Going back to the equality $\sigma_{j,i}(x) = x$, we derive from $\operatorname{Im} \sigma_{j,i}^t = 1$ that $x = 1$. As we took $x \in \ker \varphi_{ij}$, this finally proves that $\ker \varphi_{ij} = 1$, which means that φ_{ij} is injective. Since i was arbitrary, the result follows.

Next, fix $i \in \{1, \ldots, k\}$ and let $j \in \{1, \ldots, k\}$ be an index such that φ_{ij} is injective. Suppose that $h(N_i) = h(N_j)$. Lemma 5.1.3 then implies that $\operatorname{Im} \varphi_{ij}$ has finite index in N_i. Let $l \in \{1, \ldots, k\}$ be distinct from j. From Lemma 5.1.1 and the commuting conditions on $(\varphi_{ij})_{ij}$ we derive that $\operatorname{Im} \varphi_{il} \leq C_{N_i}(\operatorname{Im} \varphi_{ij}) \leq Z(N_i)$, from which the first item follows. Moreover, N_l is non-abelian, so φ_{il} cannot be injective, which proves the second item.

Finally, to prove that φ_{ij} is an isomorphism, note that by Lemma 5.1.5 and Proposition 5.1.13, the inclusions $Z(N_l) \leq \sqrt[N_l]{\gamma_2(N_l)} \leq \ker \varphi_{il}$ hold for each $l \in \{1, \dots, k\}$ different from j. Let $\varphi|_{Z(N)}$ denote the induced automorphism on $Z(N)$. Then the previous inclusions yield that the ith row of $\varphi|_{Z(N)}$ consists of zero maps everywhere except for the jth column; there we have $\varphi_{ij}|_{Z(N_j)}$. As $\varphi|_{Z(N)}$ is an automorphism, this restricted map must map into $Z(N_i)$ and must be surjective. The map φ_{ij} is injective, therefore, so is it restriction. We conclude that $\varphi_{ij}|_{Z(N_j)}$ is an isomorphism between $Z(N_j)$ and $Z(N_i)$. In particular, both groups have the same Hirsch length.

Now, we consider $\bar{\varphi}$, the induced automorphism on $N/Z(N)$. Since $\operatorname{Im} \varphi_{il} \leq Z(N_i)$ for $l \neq j$, the ith row of $\bar{\varphi}$ also consists of zero maps everywhere except for the jth column; there we have $\bar{\varphi}_{ij} : N_j/Z(N_j) \to N_i/Z(N_i)$. As $\bar{\varphi}$ is an automorphism, $\bar{\varphi}_{ij}$ must be surjective. Recall that we assume that $h(N_j) = h(N_i)$ and that we have already proven that $h(Z(N_i)) = h(Z(N_j))$. Therefore, by Lemma 5.1.3, we find that $h(N_j/Z(N_j)) = h(N_i/Z(N_i))$. Applying the first isomorphism theorem to $\bar{\varphi}_{ij}$ and using Lemma 5.1.3, we get

$$h(\ker \bar{\varphi}_{ij}) = h(N_j/Z(N_j)) - h(\operatorname{Im} \bar{\varphi}_{ij}) = h(N_j/Z(N_j)) - h(N_i/Z(N_i)) = 0,$$

since $\bar{\varphi}_{ij}$ is surjective. Consequently, $\ker \bar{\varphi}_{ij}$ is finite. Since $N_j/Z(N_j)$ is torsion-free, $\ker \bar{\varphi}_{ij}$ must be trivial, which implies that $\bar{\varphi}_{ij}$ is injective as well. We conclude that $\bar{\varphi}_{ij}$ is an isomorphism.

Summarised, both $\varphi_{ij}|_{Z(N_j)}$ and $\bar{\varphi}_{ij}$ are isomorphisms. Lemma 5.1.7 then implies that φ_{ij} is an isomorphism too, which proves the last item. □

Proof of Theorem 5.2.1. As mentioned earlier, we proceed by induction on the number of distinct Hirsch lengths, say m, among the N_i. We start with the case $m = 1$. By Lemma 5.2.2, we find for each $i \in \{1, \dots, k\}$ a unique $\sigma(i) \in \{1, \dots, k\}$ such that $\varphi_{i\sigma(i)}$ is an isomorphism, since all factors have the same Hirsch length, and such that for $j \in \{1, \dots, k\}$ with $j \neq \sigma(i)$, we have that $\operatorname{Im} \varphi_{ij} \leq Z(N_i)$. Consequently, both the first and third item are proven. We only have to argue that each column of $(\varphi_{ij})_{ij}$ contains an isomorphism. In other words, we have to prove that $\sigma : \{1, \dots, k\} \to \{1, \dots, k\}$ is surjective.

To that end, note that the induced automorphism $\bar{\varphi}$ on $N/Z(N)$ only has one non-zero map on each row, namely $\bar{\varphi}_{i\sigma(i)}$. As $\bar{\varphi}$ is an automorphism, there cannot be a zero column, which implies that σ must indeed be surjective. This proves the case $m = 1$.

Now, suppose the result holds for m and that there are now $m + 1$ distinct Hirsch lengths. Suppose, after reordering, that N_{l+1} up to N_k are all the groups that have the lowest Hirsch length among the N_i. For $i \in \{l + 1, \dots, k\}$, Lemma 5.2.2 implies there is a $j \in \{1, \ldots, k\}$ such that $\varphi_{ij} : N_j \to N_i$ is injective. Then $h(N_j) \leq h(N_i)$, which proves that $j \in \{l+1, \ldots, k\}$. Therefore, $h(N_j) = h(N_i)$, so the moreover-part of Lemma 5.2.2 yields that $j =: \sigma(i)$ is the only index in $\{1, \ldots, k\}$ such that φ_{ij} is injective, such that $\operatorname{Im} \varphi_{ip} \leq Z(N_i)$ for all $p \neq \sigma(i)$, and such that $\varphi_{i\sigma(i)}$ is an isomorphism. Furthermore, Lemma 5.1.5 and Proposition 5.1.13 combined imply that $\varphi_{ip}(Z(N_p)) = 1$ for $p \neq \sigma(i)$.

Note that the same conclusions hold for $\varphi^{-1} = (\varphi'_{ij})_{ij}$, i.e. for each $i \in \{l+1, \ldots, k\}$, there is a unique $\tau(i) \in \{l+1, \ldots, k\}$ such that $\varphi'_{i\tau(i)}$ is an isomorphism, such that $\operatorname{Im} \varphi'_{i\tau(i)}$ has finite index in N_i, and such that, for $j \neq \tau(i)$, both $\operatorname{Im} \varphi'_{ij} \leq Z(N_i)$ and $\varphi'_{ij}(Z(N_j)) = 1$ hold.

At this point, we thus have maps $\sigma, \tau : \{l+1, \ldots, k\} \to \{l+1, \ldots, k\}$. We argue that both are bijective. To do so, we look at the induced maps $\bar{\varphi}$, $\bar{\varphi}^{-1}$ on $N/Z(N)$. For each $i \in \{l+1, \ldots, k\}$, the ith row of $\bar{\varphi}$ and $\bar{\varphi}^{-1}$ consists of zero maps everywhere except for the $\sigma(i)$th and $\tau(i)$th column, respectively. From the equality $\mathrm{Id}_{N/Z(N)} = \bar{\varphi} \circ \bar{\varphi}^{-1}$ we derive that

$$\mathrm{Id}_{N_i} = \bar{\varphi}_{i\sigma(i)} \circ \bar{\varphi}'_{\sigma(i)i}$$

holds for all $i \in \{l+1, \ldots, k\}$. The only non-zero map on the $\sigma(i)$th row is $\bar{\varphi}'_{\sigma(i)\tau(\sigma(i))}$. Therefore, τ is surjective. By swapping the roles of φ and φ^{-1}, we derive that σ is surjective as well.

Next, write $M_1 = \bigtimes_{i=1}^{l} N_i$ and $M_2 = \bigtimes_{i=l+1}^{k} N_i$. Then $M_1 \times M_2 = N$. Write

$$\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

in the notation of an endomorphism on $M_1 \times M_2$. Now, in this notation, we have proven in particular that $\gamma(M_1) \leq Z(M_2)$. It follows from the definitions of the lower central series, a direct product of groups, and the isolator of a subgroup that

$$\sqrt[M_1]{\gamma_2(M_1)} = \bigtimes_{i=1}^{l} \sqrt[N_i]{\gamma_2(N_i)}.$$

Applying Proposition 5.1.13 to each N_i and using Lemma 5.1.5, we get that $Z(M_1) \leq \sqrt[M_1]{\gamma_2(M_1)} \leq \ker \gamma$. Hence, the restriction $\varphi|_{Z(N)}$ of φ to $Z(N)$ has matrix representation

$$\varphi|_{Z(N)} = \begin{pmatrix} \alpha|_{Z(M_1)} & \beta|_{Z(M_2)} \\ 0 & \delta|_{Z(M_2)} \end{pmatrix}.$$

Note that $\varphi|_{Z(N)}$ is an automorphism of $Z(N)$. If we define the map

$$\psi := \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix},$$

then ψ is an endomorphism of N, as $\operatorname{Im}\alpha$ and $\operatorname{Im}\beta$ commute. Its restriction $\psi|_{Z(N)}$ to $Z(N)$ coincides with $\varphi|_{Z(N)}$, so Lemma 5.1.6 implies that $\psi \in \operatorname{Aut}(N)$. Therefore, α is injective. Consequently, $\operatorname{Im}\alpha$ has finite index in M_1. Since $\operatorname{Im}\beta$ commutes with $\operatorname{Im}\alpha$, this implies that $\operatorname{Im}\beta \leq Z(M_1)$ by Lemma 5.1.1. By a similar argument as for γ, we get that $Z(M_2) \leq \ker\beta$. Putting everything together, we get

$$\psi|_{Z(N)} = \begin{pmatrix} \alpha|_{Z(M_1)} & 0 \\ 0 & \delta|_{Z(M_2)} \end{pmatrix}.$$

We deduce that $\alpha|_{Z(M_1)}$ is an automorphism of $Z(M_1)$, so α is an automorphism itself, again by Lemma 5.1.6.

Finally, we are able to use the induction hypothesis on M_1, since M_1 is a direct product of finitely generated torsion-free non-abelian nilpotent rationally indecomposable groups, where there are m different Hirsch lengths among them. More precisely, the induction hypothesis applied to α yields the following:

- For each $i \in \{1, \ldots, l\}$, there exists a unique $\sigma(i) \in \{1, \ldots, l\}$ such that $\varphi_{i\sigma(i)}$ is an isomorphism;
- For each $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, l\}$ with $j \neq \sigma(i)$, the map φ_{ij} satisfies $\operatorname{Im}\varphi_{ij} \leq Z(N_i)$. Note, however, that we can immediately extend this to $j \in \{1, \ldots, k\}$ with $j \neq \sigma(i)$, as $\operatorname{Im}\varphi_{ij}$ commutes with $\operatorname{Im}\varphi_{i\sigma(i)} = N_i$;
- The map $\sigma : \{1, \ldots, l\} \to \{1, \ldots, l\}$ is bijective.

Therefore, combining this with what we proved earlier, we get a bijective map $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$ such that for all $i, j \in \{1, \ldots, k\}$ with $j \neq \sigma(i)$, the map $\varphi_{i\sigma(i)}$ is an isomorphism and $\operatorname{Im}\varphi_{ij} \leq Z(N_i)$. This finishes the proof. □

Theorem 5.2.1 yields necessary conditions on the matrix representation of an automorphism of a direct product. We now prove the converse (under slightly weaker conditions on the N_i), namely that each matrix $(\varphi_{ij})_{ij}$ of morphisms satisfying the necessary conditions yields an automorphism of N.

Proposition 5.2.3. *Let $N_1, \ldots, N_k$ be finitely generated torsion-free non-abelian nilpotent groups such that, for each $i \in \{1, \ldots, k\}$, ${N_i}^{\mathbb{Q}}$ has no abelian direct factors. Put $N = \bigtimes_{i=1}^{k} N_i$. Let $\sigma \in S_k$ be a permutation. Suppose that we have a matrix $(\varphi_{ij})_{ij}$ of morphisms, with $\varphi_{ij} : N_j \to N_i$ for all $i, j \in \{1, \ldots, k\}$, satisfying the following two conditions:*

(1) For each $i \in \{1, \ldots, k\}$, the map $\varphi_{i\sigma(i)}$ is an isomorphism.

(2) For all $i, j \in \{1, \ldots, k\}$ with $j \neq \sigma(i)$, the map φ_{ij} satisfies $\operatorname{Im} \varphi_{ij} \leq Z(N_i)$.

Then $\varphi := (\varphi_{ij})_{ij}$ defines an automorphism of N under the identification of Lemma 3.1.1.

Proof. The commuting conditions for $(\varphi_{ij})_{ij}$ to define an endomorphism φ are clearly met. Let $i \in \{1, \ldots, k\}$ and let $\varphi_i : N \to N_i$ be the composition of φ with the projection $N \to N_i$. Then

$$\varphi_i(Z(N)) \leq \langle \varphi_{i1}(Z(N_1)), \ldots, \varphi_{ik}(Z(N_k)) \rangle.$$

The subgroup on the right lies in $Z(N_i)$, since $\varphi_{ij}(Z(N_j)) \leq \operatorname{Im} \varphi_{ij} \leq Z(N_i)$ if $j \neq \sigma(i)$ by assumption, and $\varphi_{i\sigma(i)}(Z(N_{\sigma(i)})) = Z(N_i)$ as $\varphi_{i\sigma(i)}$ is an isomorphism. As i was arbitrary, we can deduce that $\varphi(Z(N)) \leq Z(N)$.

Now, let $\varphi|_{Z(N)} = (\psi_{ij})_{ij}$ be the induced endomorphism on $Z(N)$. For all $i, j \in \{1, \ldots, k\}$, we have $\psi_{ij} = \varphi_{ij}|_{Z(N_j)}$. For $j \neq \sigma(i)$, the map ψ_{ij} is the zero map, as $Z(N_j) \leq \sqrt[N_j]{\gamma_2(N_j)} \leq \ker \varphi_{ij}$ by Proposition 5.1.13 and Lemma 5.1.5, respectively. Each map $\psi_{i\sigma(i)}$ is an isomorphism, being the restriction of an isomorphism to the centre. Since σ is a bijection, $(\psi_{ij})_{ij}$ is a matrix containing exactly one isomorphism in each row and each column, and zero maps in all other entries. Therefore, $\varphi|_{Z(N)}$ is an automorphism of $Z(N)$. Lemma 5.1.6 then implies that φ is an automorphism of N. $\square$

5.2.2 Reidemeister spectrum

With the description of the automorphism group at hand, determining the Reidemeister spectrum is relatively straightforward.

Proposition 5.2.4. *Let $N_1, \ldots, N_k$ be finitely generated torsion-free non-abelian nilpotent groups. Put $N = \bigtimes_{i=1}^{k} N_i$ and let $\varphi = (\varphi_{ij})_{ij} \in \operatorname{Aut}(N)$. Suppose that φ satisfies the following properties:*

(1) For each $i \in \{1, \ldots, k\}$, there is a unique $\sigma(i) \in \{1, \ldots, k\}$ such that $\varphi_{i\sigma(i)}$ is an isomorphism;

(2) For each $i \in \{1, \ldots, k\}$ and $j \neq \sigma(i)$, the map $\varphi_{ij} : N_j \to N_i$ satisfies $\operatorname{Im} \varphi_{ij} \leq Z(N_i)$ and $\varphi_{ij}(Z(N_j)) = 1$.

Define $\psi := (\psi_{ij})_{ij}$, *where*

$$\psi_{ij} = \begin{cases} \varphi_{i\sigma(i)} & \text{if } j = \sigma(i) \\ 0 & \text{otherwise.} \end{cases}$$

Then ψ *is an automorphism of* N *and* $R(\varphi) = R(\psi)$.

Remark. Explicitly requiring the condition $\operatorname{Im}\varphi_{ij} \leq Z(N_i)$ for $j \neq \sigma(i)$ is redundant, as it follows from the first property and the commuting conditions on $(\varphi_{ij})_{ij}$. Indeed, since $\varphi_{i\sigma(i)}$ is an isomorphism, $\operatorname{Im}\varphi_{i\sigma(i)} = N_i$. By the commuting conditions, we then have for $j \neq \sigma(i)$ that

$$1 = [\operatorname{Im}\varphi_{ij}, \operatorname{Im}\varphi_{i\sigma(i)}] = [\operatorname{Im}\varphi_{ij}, N_i].$$

This implies that $\operatorname{Im}\varphi_{ij} \leq Z(N_i)$.

However, we include this condition for the sake of consistency with the other results.

Proof. First, note that ψ is a well-defined endomorphism of N, since there is only one map per row in the matrix representation of ψ. Secondly, let $\varphi|_{Z(N)}$ and $\psi|_{Z(N)}$ be the restrictions of φ and ψ, respectively, to $Z(N)$. They have the same matrix representations, by the assumptions on φ and the construction of ψ. By Lemma 5.1.6, ψ is then an automorphism of N. Finally, let $\bar{\varphi}$ and $\bar{\psi}$ be the induced automorphisms on $N/Z(N)$. Their matrix representations coincide as well by the assumptions on φ and the construction of ψ. Hence, combining this with Corollary 1.2.15 we obtain

$$R(\varphi) = R(\varphi|_{Z(N)})R(\bar{\varphi}) = R(\psi|_Z)R(\bar{\psi}) = R(\psi). \qquad \square$$

Corollary 5.2.5. *Let* $N_1, \ldots, N_k$ *be non-isomorphic finitely generated torsion-free non-abelian nilpotent groups and let* $r_1, \ldots, r_k$ *be positive integers. Put* $N = \bigtimes_{i=1}^{k} N_i^{r_i}$ *and* $r = r_1 + \ldots + r_k$. *Suppose every* $\varphi = (\varphi_{ij})_{ij} \in \operatorname{Aut}(N)$ *satisfies the following conditions:*

(1) For each $i \in \{1, \ldots, r\}$, *there is a unique* $\sigma(i) \in \{1, \ldots, r\}$ *such that* $\varphi_{i\sigma(i)}$ *is an isomorphism (which is then automatically an automorphism);*

(2) For each $i \in \{1, \ldots, r\}$ *and* $j \neq \sigma(i)$, *the map* $\varphi_{ij} : N_j \to N_i$ *satisfies* $\operatorname{Im}\varphi_{ij} \leq Z(N_i)$ *and* $\varphi_{ij}(Z(N_j)) = 1$.

Then

$$\operatorname{Spec}_{\mathrm{R}}(N) = \prod_{i=1}^{k}\left(\bigcup_{j=1}^{r_i} \operatorname{Spec}_{\mathrm{R}}(N_i)^{(j)}\right).$$

Proof. By Proposition 5.2.4, we can obtain the complete Reidemeister spectrum of N by only looking at the automorphisms lying in

$$\bigtimes_{i=1}^{k} (\operatorname{Aut}(N_i) \wr S_{r_i}).$$

The result now follows by combining Corollaries 3.1.6 and 3.1.9. □

Combining the previous corollary with Theorem 5.2.1, we get the following theorem:

Theorem 5.2.6. *Let $N_1, \ldots, N_k$ be non-isomorphic finitely generated torsion-free non-abelian nilpotent groups and let $r_1, \ldots, r_k$ be positive integers. Put $N = \bigtimes_{i=1}^{k} N_i^{r_i}$. Suppose that, for each $i \in \{1, \ldots, k\}$, N_i is rationally indecomposable. Then*

$$\operatorname{Spec}_{\mathrm{R}}(N) = \prod_{i=1}^{k} \left(\bigcup_{j=1}^{r_i} \operatorname{Spec}_{\mathrm{R}}(N_i)^{(j)} \right).$$

In particular, N has the R_∞-property if and only if N_i has the R_∞-property for some $i \in \{1, \ldots, k\}$.

Proof. The condition that $\varphi_{ij}(Z(N_j)) = 1$ for all $i, j \in \{1, \ldots, k\}$ with $j \neq \sigma(i)$ follows from the proof of Theorem 5.2.1. The rest of the necessary conditions to apply Corollary 5.2.5 follow from Theorem 5.2.1 itself. □

5.2.3 Abelian factors

Thus far, we have only considered direct products of non-abelian nilpotent groups. We now address the situation with abelian factors.

Theorem 5.2.7. *Let N be a finitely generated torsion-free nilpotent group such that $N^{\mathbb{Q}}$ does not have an abelian direct factor. Let $r \geq 1$ be an integer. Then*

$$\operatorname{Spec}_{\mathrm{R}}(N \times \mathbb{Z}^r) = \operatorname{Spec}_{\mathrm{R}}(N) \cdot \operatorname{Spec}_{\mathrm{R}}(\mathbb{Z}^r).$$

Proof. Let $\varphi \in \operatorname{Aut}(N \times \mathbb{Z}^r)$ and write

$$\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Since γ maps into $\mathbb{Z}^r$, Lemma 5.1.5 implies that $\sqrt[N]{\gamma_2(N)} \leq \ker \gamma$. As $N^{\mathbb{Q}}$ does not have an abelian factor, $Z(N) \leq \sqrt[N]{\gamma_2(N)}$ by Proposition 5.1.13. Therefore, if we look at the restricted automorphism on $Z(N \times \mathbb{Z}^r) = Z(N) \times \mathbb{Z}^r =: Z$, we get

$$\varphi|_Z = \begin{pmatrix} \alpha' & \beta \\ 0 & \delta \end{pmatrix},$$

where α' is the restriction of α to $Z(N)$. As $Z(N)$ is finitely generated torsion-free abelian, $R(\varphi|_Z) = R(\alpha')R(\delta)$ by Proposition T.1.12. Note that both α' and δ are automorphisms, since $\varphi|_Z$ is an automorphism on a finitely generated free abelian group. By Lemma 5.1.6, α is then an automorphism as well. The induced automorphism on $\frac{N \times \mathbb{Z}^r}{Z(N \times \mathbb{Z}^r)}$ has 'matrix' representation

$$\bar{\varphi} = \begin{pmatrix} \bar{\alpha} \end{pmatrix},$$

as $\frac{N \times \mathbb{Z}^r}{Z(N \times \mathbb{Z}^r)} = \frac{N}{Z(N)} \times 1$. Consequently, $R(\bar{\varphi}) = R(\bar{\alpha})$. By Corollary 1.2.15, $R(\varphi) = R(\bar{\varphi})R(\varphi|_Z)$. Therefore,

$$R(\varphi) = R(\bar{\varphi})R(\varphi|_Z) = R(\bar{\alpha})R(\alpha')R(\delta) = R(\alpha)R(\delta).$$

Since α and δ are automorphisms, we get the inclusion $\mathrm{Spec}_{\mathrm{R}}(N \times \mathbb{Z}^r) \subseteq \mathrm{Spec}_{\mathrm{R}}(N) \cdot \mathrm{Spec}_{\mathrm{R}}(\mathbb{Z}^r)$. The other inclusion follows from Corollary 3.1.6. □

To conclude, we combine Theorem 5.2.6 with Theorem 5.2.1.

Corollary 5.2.8. *Let $N_1, \ldots, N_k$ be non-isomorphic finitely generated torsion-free non-abelian nilpotent groups such that N_i is rationally indecomposable for each $i \in \{1, \ldots, k\}$. Let $r_1, \ldots, r_k, r$ be positive integers. Put $N = \bigtimes_{i=1}^{k} N_i^{r_i}$. Then*

$$\mathrm{Spec}_{\mathrm{R}}(N \times \mathbb{Z}^r) = \mathrm{Spec}_{\mathrm{R}}(\mathbb{Z}^r) \cdot \prod_{i=1}^{k} \left(\bigcup_{j=1}^{r_i} \mathrm{Spec}_{\mathrm{R}}(N_i)^{(j)} \right).$$

In particular, $N \times \mathbb{Z}^r$ has the R_∞-property if and only if N_i has the R_∞-property for some $i \in \{1, \ldots, k\}$.

Proof. We argue that $N^{\mathbb{Q}}$ has no abelian factors. Note that

$$N^{\mathbb{Q}} = \bigtimes_{i=1}^{k} \left(N_i^{\mathbb{Q}} \right)^{r_i},$$

by Proposition 5.1.10, and that this is a decomposition into directly indecomposable groups. By [3, Proposition 10], such a decomposition is unique up to isomorphism and order of the factors. Since none of the $N_i^{\mathbb{Q}}$ is abelian, N satisfies the conditions of Theorem 5.2.6. The result then follows after also applying Theorem 5.2.1 to N. □

5.3 Examples

As mentioned earlier, finitely generated torsion-free nilpotent groups with cyclic centre are rationally indecomposable. The following yields a sufficient condition for a finitely generated torsion-free nilpotent group to have cyclic centre:

Proposition 5.3.1. *Let N be a finitely generated torsion-free c-step nilpotent group with $c \geq 2$. If $h(N) = c + 1$, then $Z(N)$ is cyclic.*

Proof. Let $Z_1(N), \ldots, Z_c(N)$ be the upper central series of N. The factors of the upper central series are torsion-free abelian groups, i.e. $\frac{Z_i(N)}{Z_{i-1}(N)} \cong \mathbb{Z}^{r_i}$ for some integers $r_i \geq 1$. Since $h(N) = c + 1 = r_1 + \ldots + r_c$ and each r_i is at least 1, there is exactly one r_i equal to 2, the others equal 1. We claim that $r_c = 2$.

By the definition of the upper central series and the third isomorphism theorem,

$$\frac{Z_c(N)}{Z_{c-1}(N)} = \frac{N}{Z_{c-1}(N)} \cong \frac{N/Z_{c-2}(N)}{Z(N/Z_{c-2}(N))}.$$

Suppose for the sake of contradiction that $r_c = 1$. Then $\frac{Z_c(N)}{Z_{c-1}(N)} \cong \mathbb{Z}$, which means in particular that it is cyclic. The group $M := N/Z_{c-2}(N)$ then has the property that $M/Z(M)$ is cyclic. This implies that M is abelian. However, from this, it follows that

$$\frac{N}{Z_{c-2}(N)} = M = Z(M) = Z\left(\frac{N}{Z_{c-2}(N)}\right) = \frac{Z_{c-1}(N)}{Z_{c-2}(N)}.$$

Consequently, $Z_{c-1}(N) = N$, which contradicts the fact that N is c-step nilpotent. Therefore, $r_c \neq 1$, which implies that r_c must equal 2.

Thus, $r_i = 1$ for $i \in \{1, \ldots, c-1\}$. In particular,

$$\mathbb{Z} = \frac{Z_1(N)}{Z_0(N)} \cong Z(N)$$

is cyclic. □

For instance, the group G_n from Example 4.2.8 is n-step nilpotent and it is readily verified that $h(G_n) = n + 1$. Thus, $Z(G_n)$ is cyclic.

As mentioned in the beginning of this Variation, it has already been proven that the conclusion from Corollary 5.2.8 holds if all N_i belong to either of two families: the free nilpotent groups of finite rank [117], or the (indecomposable) 2-step nilpotent groups associated to graphs [23]. Here, we provide arguments to show that both are special cases of Corollary 5.2.8.

5.3.1 Free nilpotent groups

Recall that for integers $r, c \geq 2$, we let $N_{r,c}$ denote the free nilpotent group of rank r and class c.

Proposition 5.3.2. *Let $r, c \geq 2$ be integers. Then $N_{r,c}$ is rationally indecomposable.*

Proof. For $i \in \{1, \ldots, c+1\}$, let $\Gamma_i = \gamma_i(N_{r,c})$. By Witt's formula [126], the $\mathbb{Z}$-rank of Γ_2/Γ_3 equals $\frac{r^2-r}{2} = \binom{r}{2}$. Since $\gamma_i(N^{\mathbb{Q}}_{r,c}) = \Gamma_i^{\mathbb{Q}}$ for each $i \in \{1, \ldots, c\}$ by [15, Corollary 5.2.2], we also have that the $\mathbb{Q}$-dimension of $\gamma_2(N^{\mathbb{Q}}_{r,c})/\gamma_3(N^{\mathbb{Q}}_{r,c})$ equals $\binom{r}{2}$.

Under the Malcev correspondence, $N^{\mathbb{Q}}_{r,c}$ corresponds to a rational nilpotent Lie algebra $\mathfrak{n}$. Moreover, this correspondence maps $\gamma_i(N^{\mathbb{Q}}_{r,c})$ onto $\gamma_i(\mathfrak{n})$. Consequently, $\gamma_2(\mathfrak{n})/\gamma_3(\mathfrak{n})$ has $\mathbb{Q}$-dimension $\binom{r}{2}$ as well.

Now, suppose that $N^{\mathbb{Q}}_{r,c}$ splits as a direct product. Then $\mathfrak{n}$ also splits as a direct sum of Lie subalgebras (see e.g. [3, Proposition 9]), say, $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$. Consequently, $\mathfrak{n}/\gamma_2(\mathfrak{n}) = \frac{\mathfrak{n}_1}{\gamma_2(\mathfrak{n}_1)} \oplus \frac{\mathfrak{n}_2}{\gamma_2(\mathfrak{n}_2)}$. The left-hand side is an r-dimensional $\mathbb{Q}$-vector space. Hence, if r_i is the $\mathbb{Q}$-dimension of $\frac{\mathfrak{n}_i}{\gamma_2(\mathfrak{n}_i)}$, then $r_1 + r_2 = r$.

Now, since $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ and thus in particular $[\mathfrak{n}_1, \mathfrak{n}_2] = 0$, $\gamma_2(\mathfrak{n})$ is generated (as a Lie algebra) by at most

$$\binom{r_1}{2} + \binom{r_2}{2}$$

elements. Consequently, the $\mathbb{Q}$-dimension of $\gamma_2(\mathfrak{n})/\gamma_3(\mathfrak{n})$ is at most $\binom{r_1}{2} + \binom{r_2}{2}$, which yields

$$\binom{r_1}{2} + \binom{r_2}{2} \geq \binom{r}{2} = \binom{r_1 + r_2}{2}.$$

Working out both sides and multiplying by 2, we obtain

$$r_1^2 - r_1 + r_2^2 - r_2 \geq (r_1 + r_2)^2 - (r_1 + r_2).$$

Cancelling equal terms on both sides yields

$$0 \geq 2r_1r_2,$$

which proves that $r_1 = 0$ or $r_2 = 0$. Consequently, either $\mathfrak{n}_1$ or $\mathfrak{n}_2$ is trivial, which implies that $\mathfrak{n}$ does not split as a non-trivial direct sum. Therefore, $N^{\mathbb{Q}}_{r,c}$ is directly indecomposable. $\square$

By this proposition, we see that Corollary 5.2.8 applies to all free nilpotent groups of finite rank, which yields the result of S. Tertooy [117, Theorem 6.5.6].

5.3.2 2-step nilpotent groups associated to a graph

Given a finite simple graph $\Gamma(V, E)$, we can associate to it the right-angled Artin group A_Γ given by the presentation

$$\langle V \mid [v, w] \text{ if } vw \in E\rangle.$$

Here, $vw \in E$ means that v and w are joined by an edge. We then define N_Γ to be the 2-step nilpotent quotient group $A_\Gamma/\gamma_3(A_\Gamma)$.

We can also associate a rational Lie algebra to Γ. Let W be the $\mathbb{Q}$-vector space with basis V. Let U be the subspace of $\bigwedge^2 W$ generated by the set $\{v \wedge w \mid vw \notin E\}$. We define $\mathfrak{n}_\Gamma^{\mathbb{Q}}$ to be the Lie algebra with underlying vector space $W \oplus U$ and Lie bracket defined on V by

$$[v, w] = \begin{cases} v \wedge w & \text{if } vw \notin E, \\ 0 & \text{otherwise,} \end{cases}$$

and $[W, U] = [U, U] = 0$. Under the Malcev correspondence, $\mathfrak{n}_\Gamma^{\mathbb{Q}}$ corresponds to a torsion-free 2-step nilpotent group $G_\Gamma^{\mathbb{Q}} = \{e^x \mid x \in W \oplus U\}$, where the group operation is given by

$$e^x e^y = e^{x+y+\frac{1}{2}[x,y]_L}. \tag{5.3.1}$$

We write $[\cdot, \cdot]_L$ to make clear it is the Lie bracket, not the commutator bracket. We argue that $G_\Gamma^{\mathbb{Q}}$ is the rational completion of N_Γ. If we let $F(V)$ denote the free group on V, it is readily verified that the map $F(V) \to G_\Gamma^{\mathbb{Q}}$ sending v to e^v induces a group homomorphism $F : N_\Gamma \to G_\Gamma^{\mathbb{Q}}$. Now, write $V = \{v_1, \dots, v_n\}$. Then each element $x \in N_\Gamma$ can be written as

$$x = \left(\prod_{i=1}^{n} v_i^{k_i}\right) c$$

for some $k_i \in \mathbb{Z}$ and $c \in \gamma_2(N_\Gamma)$. From the product formula $e^x e^y = e^{x+y+\frac{1}{2}[x,y]_L}$, it follows that $F(x)$ is of the form

$$e^{\sum\limits_{i=1}^{n} k_i v_i + u},$$

for some $u \in U$. Suppose that $F(x) = 1$. Then $\sum\limits_{i=1}^{n} k_i v_i + u = 0$. As $\sum\limits_{i=1}^{n} k_i v_i \in W$ and $u \in U$, we must have $\sum\limits_{i=1}^{n} k_i v_i = 0$ and $u = 0$. Therefore, $k_i = 0$ for all

$i \in \{1, \ldots, n\}$, as V is a basis of W. Hence, $x = c \in \gamma_2(N_\Gamma)$. As N_Γ is 2-step nilpotent, we can write

$$c = \prod_{\substack{1 \leq i < j \leq n \\ v_i v_j \notin E}} [v_i, v_j]^{l_{ij}}$$

for some $l_{ij} \in \mathbb{Z}$. Using (5.3.1), it is readily verified that $[e^y, e^z] = e^{[y,z]_L}$ for all $y, z \in L$. Therefore, $F(x) = F(c) = e^z$, where

$$z = \sum_{\substack{1 \leq i < j \leq n \\ v_i v_j \notin E}} l_{ij}[v_i, v_j]_L.$$

Since $F(x) = 1$, we must have $z = 0$. Consequently, as $\{[v_i, v_j]_L \mid 1 \leq i < j \leq n, v_i v_j \notin E\}$ is a basis of U, we must have that $l_{ij} = 0$ for all i, j, which implies that $c = 1$ as well. We conclude that F is indeed injective, so N_Γ embeds in $G_\Gamma^{\mathbb{Q}}$.

We now verify the other two properties for $G_\Gamma^{\mathbb{Q}}$ to be the rational completion of $N_\Gamma \cong \operatorname{Im} F$. Let $e^{w+u} \in \operatorname{Im} F$ and $k \in \mathbb{N}_0$. Then

$$e^{w+u} = (e^{\frac{1}{k}(w+u)})^k.$$

Hence, all elements of $\operatorname{Im} F$ have roots in $G_\Gamma^{\mathbb{Q}}$, which are unique as $G_\Gamma^{\mathbb{Q}}$ is torsion-free and nilpotent.

Next, as $[e^y, e^z] = e^{[y,z]_L}$ for all $y, z \in L$, $e^{k[v,w]_L} \in \operatorname{Im} F$ for all $k \in \mathbb{Z}$ and $v, w \in V$. Combining this with the fact that $e^{[x,y]_L} e^{[z,w]_L} = e^{[x,y]_L + [z,w]_L}$, we get that

$$\forall u = \sum_{v,w \in V} \mu_{v,w}[v, w]_L : (\forall v, w \in V : \mu_{v,w} \in \mathbb{Z}) \implies e^u \in \operatorname{Im} F. \tag{5.3.2}$$

Furthermore, if we take $\lambda_i \in \mathbb{Z}$ for all $i \in \{1, \ldots, n\}$, then a straightforward calculation yields that

$$e^{\sum_{i=1}^n 2\lambda_i v_i} = \prod_{i=1}^n (e^{v_i})^{2\lambda_i} \cdot \prod_{1 \leq i < j \leq n} e^{-2\lambda_i \lambda_j [v_i, v_j]_L} \in \operatorname{Im} F. \tag{5.3.3}$$

Finally, let $x + u \in W \oplus U$. Write

$$x + u = \sum_{i=1}^n \lambda_i v_i + \sum_{\substack{1 \leq i < j \leq n \\ v_i v_j \notin E}} \mu_{i,j}[v_i, v_j]_L$$

for some $\lambda_i, \mu_{i,j} \in \mathbb{Q}$. Then there is an $m \in \mathbb{N}_0$ such that $m\lambda_i, m\mu_{i,j} \in 2\mathbb{Z}$ for all $1 \leq i < j \leq n$. For this m,

$$(e^{x+u})^m = e^{mx+mu} \in \operatorname{Im} F,$$

by using (5.3.2) and (5.3.3). Hence, the third property is satisfied, which shows that $G_\Gamma^{\mathbb{Q}}$ is indeed the rational completion of N_Γ.

We recall the definition of the simplicial join of two graphs. Given two graphs $\Gamma_i(V_i, E_i)$, $i \in \{1,2\}$, the *simplicial join* $\Gamma_1 * \Gamma_2$ is the graph $\Gamma(V,E)$ where $V = V_1 \sqcup V_2$ and

$$E = E_1 \sqcup E_2 \sqcup \{vw \mid v \in V_1, w \in V_2\}.$$

We say that a graph Γ *splits as a simplicial join* if $\Gamma = \Gamma_1 * \Gamma_2$ for some non-trivial graphs Γ_1 and Γ_2.

Proposition 5.3.3. *Let Γ be a finite simple graph. Then N_Γ is rationally indecomposable if and only if Γ does not split as a simplicial join.*

Proof. Suppose that $\Gamma = \Gamma_1 * \Gamma_2$ is a non-trivial simplicial join. Then $A_\Gamma \cong A_{\Gamma_1} \times A_{\Gamma_2}$. Consequently, N_Γ splits as a non-trivial direct product, which by Proposition 5.1.10 yields a decomposition of $N_\Gamma^{\mathbb{Q}} = G_\Gamma^{\mathbb{Q}}$.

Conversely, suppose that N_Γ is rationally decomposable. Similarly as in Proposition 5.3.2, the Lie algebra $\mathfrak{n}_\Gamma^{\mathbb{Q}}$ then admits a non-trivial decomposition $\mathfrak{n}_\Gamma^{\mathbb{Q}} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$. For $i \in \{1,2\}$, let β_i be a $\mathbb{Q}$-vector space basis of $\mathfrak{n}_i$. Write $\beta_i = \{w_{i,1} + u_{i,1}, \dots, w_{i,k_i} + u_{i,k_i}\}$ for some $w_{i,j} \in W, u_{i,j} \in U$. Write $\{v^*\}_{v \in V}$ for the dual basis of W^* associated to V. Define, for $i = 1,2$,

$$V_i := \{v \in V \mid \exists j \in \{1, \dots, k_i\} : v^*(w_{i,j}) \neq 0\}.$$

Since $\operatorname{span}_{\mathbb{Q}}(\beta_1 \cup \beta_2) = W \oplus U$,

$$W = \operatorname{span}_{\mathbb{Q}}(\{w_{i,j} \mid i \in \{1,2\}, j \in \{1, \dots, k_i\}\}).$$

Consequently, for each $v \in V$, there exists a $w_{i,j}$ such that $v^*(w_{i,j}) \neq 0$. Hence, $V_1 \cup V_2 = V$. Next, we argue that neither V_1 nor V_2 is empty. Suppose, for the sake of contradiction, that V_1 is empty. Then $W = \operatorname{span}_{\mathbb{Q}}(\{w_{2,j} \mid j \in \{1, \dots, k_2\}\}$, which implies that, for each $v \in V$, there exist a $u \in U$ such that $v + u \in \mathfrak{n}_2$. Since $\mathfrak{n}_2$ is closed under the Lie bracket, this implies that

$$\{[v, v'] \mid v, v' \in V\} \subseteq \mathfrak{n}_2.$$

From this, it follows that $U \subseteq \mathfrak{n}_2$. As there exists, for each $v \in V$, a $u \in U$ such that $v + u \in \mathfrak{n}_2$, V lies in $\mathfrak{n}_2$ as well. Consequently, $W = \operatorname{span}_{\mathbb{Q}}(V)$ also lies in $\mathfrak{n}_2$, which implies that $\mathfrak{n}_2 = W + U = \mathfrak{n}$. This contradicts the non-triviality of $\mathfrak{n}_1$. We conclude that V_1 is non-empty, and a similar argument holds for V_2.

As $[\mathfrak{n}_1, \mathfrak{n}_2] = 0$, we have that $[w_{1,i}, w_{2,j}] = 0$ for all $i \in \{1, \dots, k_1\}$ and $j \in \{1, \dots, k_2\}$. Consequently, every $v \in V_1$ is connected to every $w \in V_2$ in Γ (if

$v \neq w$). If $V_1 = V_2 = V$, then Γ is a complete graph, which clearly splits as a simplicial join (note that $|V| \geq 2$ in that case, as $G_\Gamma^{\mathbb{Q}} \cong \mathbb{Q}^{|V|}$ is decomposable by assumption).

So, assume either V_1 or V_2 is a strict subset of V. Then either $\{V_1 \setminus V_2, V_2\}$ or $\{V_1, V_2 \setminus V_1\}$ forms a partition $\{W_1, W_2\}$ of V such that $\Gamma = \Gamma(W_1) * \Gamma(W_2)$, which proves that Γ splits as a non-trivial simplicial join. $\square$

By this proposition, we see that Corollary 5.2.8 applies to all groups N_Γ where Γ does not split as a simplicial join, which yields the result of K. Dekimpe and M. Lathouwers [23, §5].

Finally, by Proposition 5.1.11, all finite index subgroups of free nilpotent groups or directly indecomposable 2-step nilpotent quotients of RAAGs satisfy the conditions of Corollary 5.2.8 as well.

Variation 6

Finite index subgroups

Mozart, Wolfgang Amadeus, *Zwölf Variationen in C über das franzözische Lied 'Ah, vous dirai-je, Maman'*, (KV 265 (300e)), Variation VI (bars 1–8).

At the end of Variation 3, we discussed some results regarding the R_∞-property for finite index subgroups of direct products. We continue along this thread of thought for nilpotent groups. The product formula from Theorem 4.2.1 again shows its power here.

6.1 Reidemeister number of endomorphism restricted to subgroup

Proposition 6.1.1. *Let A be a free abelian group of finite rank and let H be a finite index subgroup of A. Suppose that $\varphi \in \operatorname{End}(A)$ is such that $\varphi(H) \leq H$. Let $\varphi|_H$ denote its restriction to H. Then $R(\varphi) = R(\varphi|_H)$.*

Proof. We first prove a more general statement: let ψ be an endomorphism of A such that $\psi(H) \leq H$. Then $[A : \psi(A)] = [H : \psi(H)]$.

Indeed, since $[A : H]$ is finite, we can write

$$[H : \psi(H)] = \frac{[A : \psi(H)]}{[A : H]} = \frac{[A : \psi(A)][\psi(A) : \psi(H)]}{[A : H]}. \tag{6.1.1}$$

It is well known that $[\psi(A):\psi(H)] \leq [A:H]$. Therefore, $[H:\psi(H)] = \infty$ if and only if $[A:\psi(A)] = \infty$, which proves they are equal in that case.

So, suppose that both $[A:\psi(A)]$ and $[H:\psi(H)]$ are finite. This means in particular that ψ is injective. Indeed, we have the exact sequence

$$0 \to \ker\psi \to A \to \operatorname{Im}\psi \to 0$$

of finitely generated abelian groups. Since $\psi(A)$ has finite index in A, the Hirsch length $h(\operatorname{Im}\psi)$ of $\operatorname{Im}\psi$ equals that of A by Lemma 5.1.3. It follows from the same lemma that $h(\ker\psi) = h(A) - h(\operatorname{Im}\psi) = 0$. Thus, $\ker\psi$ is a finite subgroup of a torsion-free group, which implies that $\ker\psi = 0$. For injective homomorphisms, equality holds in $[\psi(A):\psi(H)] \leq [A:H]$. Therefore, (6.1.1) reduces to $[H:\psi(H)] = [A:\psi(A)]$.

Now, to prove the proposition itself, we know by Proposition T.1.12 that $R(\varphi) = [A:\operatorname{Im}(\varphi - \operatorname{Id}_A)]$ and $R(\varphi|_H) = [H:\operatorname{Im}(\varphi|_H - \operatorname{Id}_H)]$. Therefore, applying the above to $\psi = \varphi - \operatorname{Id}$, we conclude that $[A:\operatorname{Im}(\varphi - \operatorname{Id}_A)] = [H:\operatorname{Im}(\varphi|_H - \operatorname{Id}_H)]$. □

Theorem 6.1.2. *Let N be a finitely generated torsion-free nilpotent group and let H be a finite index subgroup. Let $\varphi \in \operatorname{End}(N)$ be such that $\varphi(H) \leq H$. Let $\varphi|_H$ denote its restriction to H. Then $R(\varphi) = R(\varphi|_H)$.*

Proof. Let $1 = N_0 \lhd N_1 \lhd \ldots \lhd N_{c-1} \lhd N_c = N$ be a central series as in Theorem 4.2.1. For each $i \in \{0,\ldots,c\}$, define $H_i := H \cap N_i$. We claim that $1 = H_0 \lhd H_1 \ldots \lhd H_{c-1} \lhd H_c = H$ is a central series for H as in Theorem 4.2.1 as well.

First, let $i \in \{0,\ldots,c-1\}$ be arbitrary. Since N_i is normal in N_{i+1}, it immediately follows that $H_i = N_i \cap H$ is normal in $H_{i+1} = N_{i+1} \cap H$. Now, let $i \in \{0,\ldots,c\}$ be arbitrary. Since $\varphi(N_i) \leq N_i$ and $\varphi(H) \leq H$, we find that

$$\varphi(H_i) = \varphi(N_i \cap H) \leq \varphi(N_i) \cap \varphi(H) \leq N_i \cap H = H_i.$$

Next, we prove that the H_i form a *central* series. For that, we have to check that $[H_i, H] \leq H_{i-1}$ for all $i \in \{1,\ldots,c\}$. Thus, fix $i \in \{1,\ldots,c-1\}$. Clearly, $[H_i, H] \leq H$. The fact that the N_i form a central series implies that

$$[H_i, H] \leq [N_i, N] \leq N_{i-1}.$$

Both inclusions together yield $[H_i, H] \leq H_{i-1}$. This proves that the H_i form a central series.

Finally, fix again $i \in \{1,\ldots,c\}$. Since $N_{i-1} \leq N_i$, we know that

$$H_{i-1} = H \cap N_{i-1} = H \cap N_i \cap N_{i-1} = H_i \cap N_{i-1}.$$

Thus, the second isomorphism theorem applied to N_{i-1} and H_i as subgroups of N_i yields

$$\frac{H_i}{H_{i-1}} = \frac{H_i}{H_i \cap N_{i-1}} \cong \frac{H_i N_{i-1}}{N_{i-1}} \leq \frac{N_i}{N_{i-1}}.$$

It follows that H_i/H_{i-1} is torsion-free, because N_i/N_{i-1} is.

We conclude that $1 = H_0 \lhd H_1 \ldots \lhd H_{c-1} \lhd H_c = H$ is a central series as in Theorem 4.2.1. Therefore, if we let φ_i denote the induced endomorphism on N_i/N_{i-1} and $\varphi_{H,i}$ the one on H_i/H_{i-1} for each $i \in \{1, \ldots, c\}$, then

$$R(\varphi) = \prod_{i=1}^{c} R(\varphi_i) \quad \text{and} \quad R(\varphi|_H) = \prod_{i=1}^{c} R(\varphi_{H,i}). \tag{6.1.2}$$

Now, let $i \in \{1, \ldots, c\}$. Let f be the canonical isomorphism between $\frac{H_i}{H_{i-1}}$ and $\frac{H_i N_{i-1}}{N_{i-1}} \leq \frac{N_i}{N_{i-1}}$. There are two ways to define an endomorphism on $\frac{H_i N_{i-1}}{N_{i-1}}$ using φ: we can either restrict φ_i to $\frac{H_i N_{i-1}}{N_{i-1}}$, since $\varphi(H_i) \leq H_i$, which yields a map $\tilde{\varphi}_i$; or we can use f and define $\psi_i := f \circ \varphi_{H,i} \circ f^{-1}$. This yields the following diagram:

$$\begin{tikzcd}
\frac{H_i}{H_{i-1}} \arrow[r, "f"] \arrow[d, "\varphi_{H,i}"] & \frac{H_i N_{i-1}}{N_{i-1}} \arrow[r, hook] \arrow[d, shift right, "\psi_i"'] \arrow[d, shift left, "\tilde{\varphi}_i"] & \frac{N_i}{N_{i-1}} \arrow[d, "\varphi_i"] \\
\frac{H_i}{H_{i-1}} \arrow[r, "f"] & \frac{H_i N_{i-1}}{N_{i-1}} \arrow[r, hook] & \frac{N_i}{N_{i-1}}
\end{tikzcd}$$

We claim that $\psi_i = \tilde{\varphi}_i$. Indeed, let $h \in H_i$. Then

$$\begin{aligned}
\tilde{\varphi}_i(hN_{i-1}) &= \varphi_i(hN_{i-1}) \\
&= \varphi(h)N_{i-1} \\
&= f(\varphi(h)H_{i-1}) \\
&= f(\varphi_{H,i}(hH_{i-1})) \\
&= f(\varphi_{H,i}(f^{-1}(hN_{i-1}))) \\
&= \psi_i(hN_{i-1}),
\end{aligned}$$

which shows that $\psi_i = \tilde{\varphi}_i$. Therefore, their Reidemeister numbers are equal and since f is an isomorphism, we get by Lemma T.1.5 that

$$R(\varphi_{H,i}) = R(\psi_i) = R(\tilde{\varphi}_i).$$

Since N_i/N_{i-1} is a torsion-free abelian group, we want to invoke Proposition 6.1.1 to prove that $R(\varphi_i) = R(\tilde{\varphi}_i)$ as well. To be able to do so, we need to verify that $\frac{H_i N_{i-1}}{N_{i-1}}$ has finite index in $\frac{N_i}{N_{i-1}}$. We know that

$$h(N) = \sum_{i=1}^{c} h\left(\frac{N_i}{N_{i-1}}\right) \quad \text{and} \quad h(H) = \sum_{i=1}^{c} h\left(\frac{H_i}{H_{i-1}}\right). \tag{6.1.3}$$

We also know that

$$h\left(\frac{H_i}{H_{i-1}}\right) = h\left(\frac{H_i N_{i-1}}{N_{i-1}}\right) \leq h\left(\frac{N_i}{N_{i-1}}\right)$$

with equality if and only if $\frac{H_i N_{i-1}}{N_{i-1}}$ has finite index in $\frac{N_i}{N_{i-1}}$. If equality does not hold for some $i \in \{1, \ldots, c\}$, then (6.1.3) would imply that $h(N) > h(H)$, which contradicts the fact that H has finite index in N. Therefore, each $\frac{H_i N_{i-1}}{N_{i-1}}$ has finite index in $\frac{N_i}{N_{i-1}}$. Consequently, Proposition 6.1.1 implies that $R(\varphi_i) = R(\tilde{\varphi}_i)$. Thus, combining this with $R(\varphi_{H,i}) = R(\tilde{\varphi}_i)$ and with (6.1.2), we find that

$$\begin{aligned} R(\varphi) &= \prod_{i=1}^{c} R(\varphi_i) \\ &= \prod_{i=1}^{c} R(\varphi_{H,i}) \\ &= R(\varphi|_H), \end{aligned}$$

which proves the theorem. □

Remark. Similarly as for Proposition 4.2.15, one can prove Theorem 6.1.2 using Lie algebras. Again, we provide the main idea: let N, H and φ be as in the statement of Theorem 6.1.2. Let $N^{\mathbb{Q}}$ be the rational Malcev completion of N and $\mathfrak{n}^{\mathbb{Q}}$ be its corresponding rational Lie algebra. Since H has finite index in N, the proof of Proposition 5.1.11 implies that $N^{\mathbb{Q}}$ is also the rational Malcev completion of H. The endomorphisms φ_* and $\varphi|_{H,*}$ induced on $\mathfrak{n}^{\mathbb{Q}}$ by φ and $\varphi|_H$, respectively, will be the same. Since $R(\varphi) = |\det(\varphi_* - \mathrm{Id})|_\infty$ and $R(\varphi|_H) = |\det(\varphi|_{H,*} - \mathrm{Id})|_\infty$, these numbers are the same as well.

Corollary 6.1.3. *Let N be a finitely generated torsion-free nilpotent group and H a characteristic subgroup of finite index. Then* $\mathrm{Spec}_{\mathrm{R}}(N) \subseteq \mathrm{Spec}_{\mathrm{R}}(H)$.

In particular, if N has full Reidemeister spectrum, then so does H.

Proof. Since H is characteristic in N, each automorphism φ of N restricts to one of H, say, $\varphi|_H$. As $R(\varphi) = R(\varphi|_H)$ by Theorem 6.1.2, each number in $\mathrm{Spec}_{\mathrm{R}}(N)$ is contained in $\mathrm{Spec}_{\mathrm{R}}(H)$. □

Example 6.1.4. Recall Theorem 4.2.6, which states that the free nilpotent group $N_{r,2}$ has full Reidemeister spectrum if and only if $r \geq 4$. Thus, if we take $r \geq 4$, then any characteristic finite index subgroup of $N_{r,2}$ has full Reidemeister spectrum. For instance, let $m \geq 2$ be an integer. Then $N_{r,2}(m) := \langle x^m \mid x \in N_{r,2} \rangle$ has finite index by Lemma 5.1.2 and is (fully) characteristic by construction. Consequently, $N_{r,2}(m)$ has full Reidemeister spectrum. ‖

6.2 Twisted conjugacy action of subgroup

We can extend the notion of twisted conjugacy to a subgroup acting on the whole group.

Definition 6.2.1. Let G be a group and H a subgroup. For $\varphi \in \mathrm{End}(H)$, we define the *$\varphi$-conjugacy action of H on G* by

$$H \times G \to G : (h, g) \mapsto h \cdot g := hg\varphi(h)^{-1}.$$

Given $g \in G$, we write $[g]_{H,G,\varphi}$ for the orbit of g under this action. We let $\mathcal{R}[H, G, \varphi]$ denote the set of orbits and $R(H, G, \varphi)$ the number of orbits.

If $G = H$, then these notions coincide with the usual ones of twisted conjugacy and Reidemeister number. Even more general, given two groups G and H and two homomorphisms $\varphi, \psi : H \to G$, one can define an action of H on G by putting

$$h \cdot g := \psi(h)g\varphi(h)^{-1}.$$

For the case above where H a subgroup, ψ is the canonical inclusion map. For more information on this generalisation, see e.g. [25, 43, 44].

Definition 6.2.2. Let G be a group and H a subgroup. For $\varphi \in \mathrm{End}(H)$ and $g \in G$, we define

$$\mathrm{Stab}(H, G, \varphi, g) := \{h \in H \mid hg\varphi(h)^{-1} = g\}.$$

Since $\mathrm{Stab}(H, G, \varphi, g)$ is the stabiliser of a group action, it is a subgroup of H.

The φ-conjugacy action of a subgroup on the whole groups satisfies similar properties as the usual twisted conjugacy action.

Lemma 6.2.3 (Cf. Corollary 1.1.2(1))**.** *Let G be a group, H a subgroup and $\varphi \in \mathrm{End}(H)$. Suppose that K is a normal subgroup of G contained in H, and that $\varphi(K) \leq K$. Let $\bar{\varphi}$ denote the induced endomorphism on H/K. Then*

$$R(H, G, \varphi) \geq R(H/K, G/K, \bar{\varphi}).$$

Proof. Let $\pi : G \to G/K$ be the natural projection. We claim that the map

$$\hat{\pi} : \mathcal{R}[H, G, \varphi] \to \mathcal{R}[H/K, G/K, \bar{\varphi}] : [g]_{H,G,\varphi} \mapsto [\pi(g)]_{H/K,G/K,\bar{\varphi}}$$

is well defined and surjective. Let $g \in G$ and $h \in H$ be arbitrary. Since $\bar{\varphi}(\pi(x)) = \pi(\varphi(x))$ for all $x \in G$, we find that

$$\begin{aligned}
\pi(h) \cdot \pi(g) &= \pi(h)\pi(g)\bar{\varphi}(h)^{-1} \\
&= \pi(h)\pi(g)\pi\Big(\varphi(h)^{-1}\Big) \\
&= \pi(hg\varphi(h)^{-1}) \\
&= \pi(h \cdot g).
\end{aligned}$$

This implies that $[\pi(g)]_{H/K,G/K,\bar{\varphi}} = [\pi(h \cdot g)]_{H/K,G/K,\bar{\varphi}}$, which shows that $\hat{\pi}$ is well defined. Furthermore, as π is surjective, so is $\hat{\pi}$.

Thus, as $\hat{\pi}$ is surjective, the inequality $R(H, G, \varphi) \geq R(H/K, G/K, \bar{\varphi})$ holds. □

Lemma 6.2.4 (Cf. Proposition 1.2.11)**.** *Let G be a finitely generated residually finite group. Let H be a normal subgroup of finite index and let $\varphi \in \mathrm{End}(H)$. If $\mathrm{Stab}(H, G, \varphi, g)$ is infinite for some $g \in G$, then $R(H, G, \varphi) = \infty$.*

In particular, if H is torsion-free and $\mathrm{Stab}(H, G, \varphi, g)$ is non-trivial for some $g \in G$, then $R(H, G, \varphi) = \infty$.

Proof. Suppose that $\mathrm{Stab}(H, G, \varphi, g)$ is infinite. Let $n \in \mathbb{N}$ and suppose that $x_1, \ldots, x_n \in \mathrm{Stab}(H, G, \varphi, g)$ are all distinct. Since G is finitely generated residually finite, we can find a normal subgroup N of G of finite index in G such that the natural projection $\pi : G \to G/N$ is injective on $\{x_1, \ldots, x_n\}$. The subgroup $N \cap H$ has finite index in H, so using Lemma 1.2.6, we can construct a subgroup K_0 such that K_0 is fully characteristic in H, has finite index in H, and is contained in $N \cap H$, and thus in N. As K_0 is characteristic in H and H is normal in G, K_0 is normal in G.

Moreover, the natural projection $\pi_0 : G \to G/K_0$ is injective on $\{x_1, \ldots, x_n\}$ as well. For if $x_iK_0 = x_jK_0$, then also $x_iN = x_jN$, since $K_0 \leq N$. Therefore, $x_i = x_j$, since π was injective on $\{x_1, \ldots, x_n\}$.

Thus, in the finite quotient group G/K_0, $\mathrm{Stab}(H/K_0, G/K_0, \bar{\varphi}, \pi_0(g))$ contains at least n distinct elements, where $\bar{\varphi}$ is the induced automorphism on H/K_0. Let r denote the number $R(H/K_0, G/K_0, \bar{\varphi})$. By a similar argument as in Proposition 1.2.11, one can prove that $|\mathrm{Stab}(H/K_0, G/K_0, \bar{\varphi}, \pi_0(g))| \leq 2^{2^r}$. Therefore,

$$R(H/K_0, G/K_0, \bar{\varphi}) \geq \log_2(\log_2(n)).$$

We also know that $R(H, G, \varphi) \geq R(H/K_0, G/K_0, \bar{\varphi})$ by Lemma 6.2.3. Therefore, $R(H, G, \varphi) \geq \log_2(\log_2(n))$. As n can be chosen arbitrarily large, this implies that $R(H, G, \varphi) = \infty$. □

Lemma 6.2.5 (Cf. Lemma T.1.7(2))**.** *Let G be a group and H a normal subgroup. Let $\varphi \in \mathrm{End}(H)$ and $g \in G$. Then $R(H, G, \varphi) = R(H, G, \tau_g \circ \varphi)$.*

Proof. Consider the map

$$F : G \to G : x \mapsto xg^{-1}.$$

We prove that F is an H-invariant map, where we equip the domain of F with the φ-conjugacy action of H, and the codomain with the $(\tau_g \circ \varphi)$-conjugacy action. Thus, let $x \in G$ and $h \in H$ be arbitrary. Then

$$\begin{aligned} F(h \cdot x) &= F(hx\varphi(h)^{-1}) \\ &= hx\varphi(h)^{-1}g^{-1} \\ &= hxg^{-1}g\varphi(h)^{-1}g^{-1} \\ &= hxg^{-1}(\tau_g \circ \varphi)(h)^{-1} \\ &= h \cdot F(x). \end{aligned}$$

As x and h are arbitrary, F is an H-invariant map. Clearly, F is a bijection. Therefore, F induces a one-to-one correspondence between the φ-conjugacy orbits and the $(\tau_g \circ \varphi)$-conjugacy orbits, which implies that $R(H, G, \varphi) = R(H, G, \tau_g \circ \varphi)$. □

For nilpotent groups, we can link $R(H, G, \varphi)$ with $R(\varphi)$ for finite index normal subgroups.

Lemma 6.2.6. *Let G be a group. Let $N \lhd G$ be a normal subgroup of G and let H be a subgroup of G such that $H \cap N = 1$. Then $[G : H] = [G : HN] \cdot |N|$.*

Proof. Since N is normal in G, the set HN is a subgroup of G and $HN = NH$. Therefore, the product formula for indices yields $[G : H] = [G : HN][HN : H]$. Now, in general, even if H is not normal, the map

$$\frac{N}{H \cap N} \to \frac{NH}{H} : n(H \cap N) \mapsto nH$$

is a well-defined bijection. Thus, $[HN : H] = [N : H \cap N] = [N : 1] = |N|$. $\square$

Lemma 6.2.7. *Let N be a finitely generated nilpotent group of class c and let H be a finite index subgroup. Then $\gamma_c(H)$ has finite index in $\gamma_c(N)$.*

Proof. Consider the c-fold tensor product $\bigotimes^c \frac{N}{\gamma_2(N)}$ of the abelian group $N/\gamma_2(N)$. Then iterated c-fold commutators induce a surjective group homomorphism $\varphi : \bigotimes^c \frac{N}{\gamma_2(N)} \to \gamma_c(N)$ (see e.g. [121, Theorem 3.1]). Since H has finite index in N, so does the image of H under the projection map $N \to N/\gamma_2(N)$. The composition map $H \hookrightarrow N \to N/\gamma_2(N)$ factors through $H/\gamma_2(H)$, so the image of $H/\gamma_2(H)$ under this composition has finite index in $N/\gamma_2(N)$. From this, it follows that the image of $\bigotimes^c \frac{H}{\gamma_2(H)}$ under the canonical inclusion has finite index in $\bigotimes^c \frac{N}{\gamma_2(N)}$.

Therefore, the image of the composition

$$\bigotimes^c \frac{H}{\gamma_2(H)} \longrightarrow \bigotimes^c \frac{N}{\gamma_2(N)} \xrightarrow{\varphi} \gamma_c(N)$$

has finite index as well. As this image is precisely $\gamma_c(H)$, the lemma follows. $\square$

Theorem 6.2.8. *Let N be a finitely generated nilpotent group and let $H \lhd N$ be a torsion-free normal subgroup of finite index. Let $\varphi \in \mathrm{End}(H)$. Then*

$$R(H, N, \varphi) = [N : H]R(\varphi).$$

Proof. We proceed by induction on the nilpotency class c of H. For $c = 1$, H is an abelian group. Consider the torsion subgroup $\tau(N)$ of N. Let $\pi : N \to N/\tau(N)$ be the canonical projection. Since H is torsion-free, $H \cap \tau(N) = 1$. Therefore, H is isomorphic to $\pi(H)$. This gives rise to an induced endomorphism $\bar{\varphi}$ of $\pi(H)$. Also note that $\pi(H)$ is normal in $N/\tau(N)$.

We claim that $R(H, N, \varphi) = |\tau(N)|R(\pi(H), N/\tau(N), \bar{\varphi})$. If $R(\pi(H), N/\tau(N), \bar{\varphi})$ is infinite, then $R(N, H, \varphi)$ is infinite as well by Lemma 6.2.3, which shows that the product formula holds in that case.

So, suppose that $R(\pi(H), N/\tau(N), \bar{\varphi}) < \infty$. Let $\{m_1\tau(N), \ldots, m_r\tau(N)\}$ be a complete set of representatives of $\mathcal{R}[\pi(H), N/\tau(N), \bar{\varphi}]$. Let $m \in N$. Then there is an $h \in H$ and an $i \in \{1, \ldots, r\}$ such that $m\tau(N) = hm_i\varphi(h)^{-1}\tau(N)$. Consequently, $m = hm_it\varphi(h)^{-1}$ for some $t \in \tau(N)$. Thus, we see that m_it lies in the orbit of m, which proves that $\{m_it \mid i \in \{1, \ldots, r\}, t \in \tau(N)\}$ is a set of representatives of $\mathcal{R}[H, N, \varphi]$.

Now, suppose that $m_it = hm_js\varphi(h)^{-1}$ for some $h \in H$, $s, t \in \tau(N)$ and $i, j \in \{1, \ldots, r\}$. Projecting onto $N/\tau(N)$ yields $m_i\tau(N) = hm_j\varphi(h)^{-1}\tau(N)$, which shows that $i = j$. Thus, we get $m_i\tau(N) = hm_i\varphi(h)^{-1}\tau(N)$, which means that $h\tau(N) \in \operatorname{Stab}(\pi(H), N/\tau(N), \bar{\varphi}, m_i)$. As $R(\pi(H), N/\tau(N), \bar{\varphi}) < \infty$ and $\pi(H)$ is torsion-free, Lemma 6.2.4 then implies that $h\tau(N) = \tau(N)$, i.e. $h \in \tau(N)$. Thus, since h lies in both H and $\tau(N)$, this means that $h = 1$. Consequently, $m_it = m_is$, which shows that $s = t$ as well. We conclude that $\{m_it \mid i \in \{1, \ldots, r\}, t \in \tau(N)\}$ is a complete set of representatives of $\mathcal{R}[H, N, \varphi]$. Consequently,

$$R(H, N, \varphi) = |\tau(N)| \cdot R(\pi(H), N/\tau(N), \bar{\varphi}). \tag{6.2.1}$$

Now, before computing $R(\pi(H), N/\tau(N), \bar{\varphi})$, we first argue that $N/\tau(N)$ is actually abelian. Since H has finite index in N, $\pi(H)$ has finite index in $N/\tau(N)$. By construction, we know that $N/\tau(N)$ is also torsion-free. Thus, $N/\tau(N)$ is a torsion-free nilpotent group that is virtually abelian. Suppose that $N/\tau(N)$ has class $k \geq 2$. Lemma 6.2.7 then implies that $1 = \gamma_k(\pi(H))$ has finite index in $\gamma_k(N/\tau(N))$, where the latter is a non-trivial torsion-free group. This is impossible, thus, $k = 1$. Therefore, $N/\tau(N)$ is abelian.

To compute $R(B, A, \psi)$ for torsion-free abelian groups $B \leq A$ where B has finite index in A and $\psi \in \operatorname{End}(B)$, we remark that $x, y \in A$ lie in the same orbit under the ψ-conjugacy action of B if and only if $x = b + y - \psi(b)$ for some $b \in B$. Therefore, if and only if $x - y \in \operatorname{Im}(\operatorname{Id}_B - \psi)$. Thus

$$R(B, A, \psi) = [A : \operatorname{Im}(\operatorname{Id}_B - \psi)] = [A : B][B : \operatorname{Im}(\operatorname{Id}_B - \psi)] = [A : B]R(\psi).$$

Applied to $A = N/\tau(N), B = \pi(H)$ and $\psi = \bar{\varphi}$, we get

$$R(\pi(H), N/\tau(N), \bar{\varphi}) = [N/\tau(N) : \pi(H)]R(\bar{\varphi}) = [N : H\tau(N)]R(\bar{\varphi}).$$

Recall that π is injective on H. Let $\pi|_H$ denote its restriction to H. Then $\bar{\varphi} = \pi|_H \circ \varphi \circ (\pi|_H)^{-1}$. Therefore, $R(\bar{\varphi}) = R(\varphi)$ by Lemma T.1.5.

For $[N : H\tau(N)]$, we can use Lemma 6.2.6 to note that

$$[N : H] = [N : H\tau(N)]|\tau(N)|.$$

Since N is finitely generated, $\tau(N)$ is a finite subgroup. Hence,

$$R(\pi(H), N/\tau(N), \bar{\varphi}) = [N : H\tau(N)]R(\bar{\varphi}) = \frac{[N : H]R(\varphi)}{|\tau(N)|}. \tag{6.2.2}$$

Finally, combining (6.2.1) and (6.2.2), we obtain

$$R(H,N,\varphi) = |\tau(N)|\frac{[N:H]R(\varphi)}{|\tau(N)|} = [N:H]R(\varphi).$$

This finishes the case where $c = 1$.

Now, suppose the result holds for all finitely generated nilpotent groups N and subgroups H of N that are torsion-free and nilpotent of class at most $c-1$. Let N be a finitely generated nilpotent group with torsion-free subgroup H of class c.

Put $C := \sqrt[H]{\gamma_c(H)}$. Since C is (fully) characteristic in H and H is normal in N, C is normal in N. Thus, we can consider the following situation:

$$\begin{tikzcd}
1 \arrow[r] & C \arrow[r] & N \arrow[r] & \frac{N}{C} \arrow[r] & 1 \\
1 \arrow[r] & C \arrow[r] \arrow[u, hook] & H \arrow[r] \arrow[u, hook] & \frac{H}{C} \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

Again since C is fully characteristic in H, we get induced endomorphisms $\varphi|_C$ on C and $\bar{\varphi}$ on H/C, which yield the numbers $R(C,C,\varphi|_C) = R(\varphi|_C)$ and $R(H/C, N/C, \bar{\varphi})$, respectively. By Lemma 4.1.6, H/C is torsion-free and has nilpotency class at most $c-1$. Therefore, the induction hypothesis yields

$$R(H/C, N/C, \bar{\varphi}) = [N/C : H/C]R(\bar{\varphi}) = [N:H]R(\bar{\varphi}).$$

Next, we prove that $R(H,N,\varphi) = R(C,C,\varphi|_C)R(H/C,N/C,\bar{\varphi})$. This equality holds if $R(H/C,N/C,\bar{\varphi}) = \infty$, for then Lemma 6.2.3 implies that $R(H,N,\varphi) = \infty$ as well. So, assume that $R(H/C,N/C,\bar{\varphi})$ is finite and let $\{n_1C,\ldots,n_rC\}$ be a complete set of representatives of $\mathcal{R}[H/C,N/C,\bar{\varphi}]$. Let $\{c_j \mid j \in \mathcal{J}\}$ be a complete set of representatives of $\mathcal{R}[\varphi|_C] = \mathcal{R}[C,C,\varphi|_C]$. Let $n \in N$. Then there is an $i \in \{1,\ldots,r\}$ and an $h \in H$ such that $nC = hn_i\varphi(h)^{-1}C$. Thus, $n = hn_i\varphi(h)^{-1}c$ for some $c \in C$. Similarly, there is a $j \in \mathcal{J}$ and an $x \in C$ such that $c = xc_j\varphi|_C(x)^{-1}$. Combining both and using the fact that C lies in $Z(H)$, we obtain

$$n = hn_i\varphi(h)^{-1}xc_j\varphi|_C(x)^{-1} = hxn_ic_j\varphi(hx)^{-1}.$$

Conversely, suppose that $n_ic_j = hn_kc_l\varphi(h)^{-1}$ for some $i,k \in \{1,\ldots,r\}$, $j,l \in \mathcal{J}$ and $h \in H$. Projecting onto N/C yields $n_iC = hn_k\varphi(h)^{-1}C$, which implies that $i = k$. Thus, $hC \in \mathrm{Stab}(H/C, N/C, \bar{\varphi}, n_i C)$. Since $R(H/C, N/C, \bar{\varphi})$ is finite and H/C is normal in N/C, Lemma 6.2.4 implies this stabiliser is trivial. Consequently, $hC = C$, which means that $h \in C$. Thus, we find that

$$n_i c_j = n_i h c_l \varphi(h)^{-1},$$

from which we conclude that $c_j = h c_l \varphi|_C(h)^{-1}$ and thus that $j = l$. This proves that $R(H, N, \varphi) = R(C, C, \varphi|_C) R(H/C, N/C, \bar{\varphi})$.

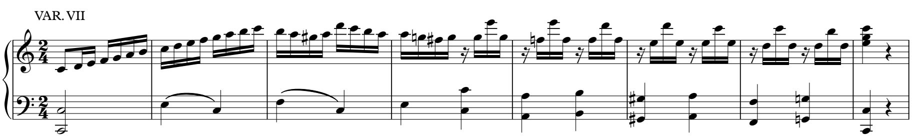
To conclude, we have proven that $R(H, N, \varphi) = R(C, C, \varphi|_C) R(H/C, N/C, \bar{\varphi})$ and that $R(H/C, N/C, \bar{\varphi}) = [N : H] R(\bar{\varphi})$. Therefore,

$$R(H, N, \varphi) = R(\varphi|_C) R(\bar{\varphi}) [N : H].$$

By Proposition 1.2.5, $R(\varphi) = R(\varphi|_C) R(\bar{\varphi})$, which shows that $R(H, N, \varphi) = R(\varphi)[N : H]$. $\square$

Variation 7

Finitely generated nilpotent groups with torsion



Mozart, Wolfgang Amadeus, *Zwölf Variationen in C über das franzözische Lied 'Ah, vous dirai-je, Maman'*, (KV 265 (300e)), Variation VII (bars 1–8).

If N is a nilpotent group with torsion, then the product formula from Theorem 4.2.1 cannot be applied (directly) anymore. There is, nonetheless, a formula to express the Reidemeister number in terms of Reidemeister numbers of automorphisms on the torsion subgroup of N. Using this formula, we find new examples of groups with full Reidemeister spectrum that cannot be constructed using direct products of groups.

7.1 Addition formula

Theorem 7.1.1. *Let N be a finitely generated nilpotent group and let $\varphi \in \operatorname{End}(N)$. Let φ_τ and $\bar{\varphi}$ denote the induced endomorphisms on $\tau(N)$ and $N/\tau(N)$, respectively. Then*

$$R(\varphi) = \sum_{[g\tau(N)]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} R(\tau_g \circ \varphi_\tau). \tag{7.1.1}$$

Proof. First, if $R(\bar{\varphi}) = \infty$, then $R(\varphi) = \infty$ as well by Corollary 1.1.2(1). Then the sum formula holds, as we defined an infinite sum of positive integers to be ∞.

So, assume that $R(\bar{\varphi}) < \infty$. Then $\mathrm{Stab}_{\bar{\varphi}}(g\tau(N))$ is trivial for all $g \in N$, by Corollary 1.2.13, as $N/\tau(N)$ is torsion-free by Proposition 4.1.13(2). Since $\mathrm{Stab}_{\bar{\varphi}}(g\tau(N)) = \mathrm{Fix}(\tau_{gN} \circ \bar{\varphi})$ for all $g \in G$ by Lemma T.1.10, we can apply Proposition 1.1.8 to derive that

$$R(\varphi) = \sum_{[g\tau(N)]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} R(\tau_g \circ \varphi_\tau). \qquad \square$$

Corollary 7.1.2. *Let N be a finitely generated nilpotent group and let $\varphi \in \mathrm{End}(N)$. Let φ_τ and $\bar{\varphi}$ denote the induced endomorphisms on $\tau(N)$ and $N/\tau(N)$, respectively. If $\tau(N) \leq Z(N)$, then $R(\varphi) = R(\varphi_\tau)R(\bar{\varphi})$.*

Proof. If $\tau(N) \leq Z(N)$, then each inner automorphism of N induces the identity on $\tau(N)$. Therefore, (7.1.1) becomes

$$R(\varphi) = \sum_{[g\tau(N)]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} R(\varphi_\tau) = R(\varphi_\tau)R(\bar{\varphi}).$$

Alternatively, we can apply Proposition 1.2.5 to N and $\tau(N)$. Indeed, $N/\tau(N)$ is nilpotent, hence residually finite, and it is also finitely generated, and by Proposition 4.1.13(2), $N/\tau(N)$ is torsion-free. $\square$

As for finitely generated torsion-free nilpotent groups, the converse of Proposition 1.2.11 holds as well for finitely generated nilpotent groups that are not necessarily torsion-free.

Proposition 7.1.3. *Let N be a finitely generated nilpotent group and $\varphi \in \mathrm{End}(N)$. If $R(\varphi)$ is infinite, then $\mathrm{Stab}_\varphi(n)$ is infinite for all $n \in N$.*

Proof. Let φ_τ and $\bar{\varphi}$ denote the induced endomorphisms on $\tau(N)$ and $N/\tau(N)$, respectively. We first prove that $\mathrm{Fix}(\varphi)$ is infinite. Since $R(\varphi)$ is infinite, $R(\bar{\varphi}) = \infty$ as well by Theorem 7.1.1. Hence, $\bar{\varphi}$ has a non-trivial fixed point $n\tau(N)$ by Proposition 4.2.15. In particular, n has infinite order.

Since $\bar{\varphi}(n\tau(N)) = n\tau(N)$, there is, for each $i \geq 1$, a $t_i \in \tau(N)$ such that $\varphi(n^i) = n^i t_i$. As $\tau(N)$ is finite, there are $1 \leq i < j$ such that $t_i = t_j$. Then

$$\varphi(n^{j-i}) = \varphi(n^j)\varphi(n^i)^{-1} = n^j t_j \left(n^i t_i\right)^{-1} = n_j t_j t_i^{-1} n^{-i} = n^{j-i}.$$

As $j-i>0$ and n has infinite order, n^{j-i} is a non-trivial fixed point of φ. Hence, $\mathrm{Fix}(\varphi)=\mathrm{Stab}_\varphi(1)$ is infinite.

Finally, the proof that $\mathrm{Stab}_\varphi(n)$ is infinite for all $n \in N$ is analogous to the one in Proposition 4.2.15. □

7.2 Examples

We compute the Reidemeister spectrum of three different families of finitely generated nilpotent groups with torsion. We restrict ourselves to groups where the torsion group is abelian, since there is a relatively easy way to compute Reidemeister numbers on finite abelian groups, and even an explicit formula for Reidemeister numbers on finite cyclic groups.

Lemma 7.2.1 (See also [62, Chapter II, Theorem 5.5])**.** *Let A be a finite abelian group and $\varphi \in \mathrm{End}(A)$. Then $R(\varphi)=|\mathrm{Fix}(\varphi)|$.*

Proof. By Lemma T.1.11, $R(\varphi)=[A : \mathrm{Im}(\mathrm{Id}-\varphi)]$. Since A is finite,

$$[A : \mathrm{Im}(\mathrm{Id}-\varphi)] = \frac{|A|}{|\mathrm{Im}(\mathrm{Id}-\varphi)|} = |\ker(\mathrm{Id}-\varphi)| = |\mathrm{Fix}(\varphi)|,$$

by the first isomorphism theorem for groups. □

We prove a generalisation of this lemma in Variation 8.

Lemma 7.2.2. *Let $n \geq 2$ and let $\varphi \in \mathrm{End}(\mathbb{Z}/n\mathbb{Z})$ be given by $\varphi(1)=k$. Then $R(\varphi)=\gcd(k-1,n)$.*

Proof. Since $R(\varphi)=|\mathrm{Fix}(\varphi)|$, we determine the fixed points of φ. We have that $\varphi(i)=i$ if and only if $i\cdot(k-1)\equiv 0 \bmod n$. Writing $d=\gcd(k-1,n)$, we see that $\frac{k-1}{d}$ is invertible modulo n, hence $i\cdot(k-1)\equiv 0 \bmod n$ if and only if $i\cdot d\equiv 0 \bmod n$. Thus, for $i\cdot d\equiv 0 \bmod n$ to hold, i must be a multiple of $\frac{n}{d}$. Since i has to lie between 0 and $n-1$ and there are d multiples of $\frac{n}{d}$ lying between 0 and $n-1$, φ has d fixed points. □

Throughout the examples, whenever φ is an automorphism on a nilpotent group N, we let φ_τ and $\bar{\varphi}$ denote the induced automorphism on $\tau(N)$ and $N/\tau(N)$, respectively. We also make extensive use of the following lemma, to prove that given maps are automorphisms:

Lemma 7.2.3. *Let N be a finitely generated nilpotent group and let $\varphi \in \operatorname{End}(N)$. If the induced map $\bar{\varphi} : N/\gamma_2(N) \to N/\gamma_2(N)$ is surjective, then φ is an automorphism.*

Proof. As $\bar{\varphi}$ is surjective, $\gamma_2(N) \operatorname{Im} \varphi = N$. By [66, Theorem 16.2.5], this equality implies that $\operatorname{Im} \varphi = N$. Therefore, φ is surjective as well. Since N is residually finite by Theorem 4.1.10, N is also Hopfian (see e.g. [12, Theorem 2.3.4]). Consequently, φ is an automorphism. □

We start with a finitely generated nilpotent group where the torsion is central. Recall that $N_{2,2}$ is the free nilpotent group of rank 2 and class 2. It has the following presentation:

$$N_{2,2} = \langle x, y, z \mid [x, y] = z, [x, z] = [y, z] = 1 \rangle.$$

Proposition 7.2.4. *Let $k \in \mathbb{N}_0$ with $k \neq 1$. Define $G_k := \frac{N_{2,2}}{\langle z^k \rangle}$. Then*

$$\operatorname{Spec}_{\mathrm{R}}(G_k) = \begin{cases} 2\mathbb{N}_0 \cup \{\infty\} & \text{if } k \text{ is even,} \\ \mathbb{N}_0 \cup \{\infty\} & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Since $Z(N_{2,2}) = \langle z \rangle$, the subgroup $\langle z^k \rangle$ is normal in $N_{2,2}$, so G_k is well defined. Put differently, $G_k = \langle x, y, z \mid [x, y] = z, [x, z] = [y, z] = 1, z^k = 1 \rangle$. We first prove that $\tau(G_k) = \langle z \rangle$. Clearly, z has finite order in G_k. Conversely, $\langle z \rangle$ is central in G_k, so also normal. The quotient group $\frac{G_k}{\langle z \rangle}$ is isomorphic to $\mathbb{Z}^2$, which is torsion-free. Hence, $\langle z \rangle$ contains $\tau(G_k)$ by Proposition 4.1.13(3), which implies that $\langle z \rangle = \tau(G_k)$.

Next, let $\varphi \in \operatorname{Aut}(G_k)$. Since $\tau(G_k) \leq Z(G_k)$, $R(\varphi) = R(\varphi_\tau)R(\bar{\varphi})$ by Corollary 7.1.2. The set $\{x\tau(G_k), y\tau(G_k)\}$ is a $\mathbb{Z}$-basis of the quotient group, so with respect to that basis, $\bar{\varphi}$ has matrix representation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{Z}$. Since $\bar{\varphi}$ is an automorphism, $\Delta := ad - bc = \pm 1$. We now compute $\varphi(z)$ using the equality $z = [x, y]$. Write $\varphi(x) = x^a y^c z^e$ and

$\varphi(y) = x^b y^d z^f$ for some $e, f \in \mathbb{Z}$. Then using Lemma 4.1.4, we obtain

$$\begin{aligned}\varphi(z) = \varphi([x,y]) &= [\varphi(x), \varphi(y)] \\ &= [x^a y^c z^e, x^b y^d z^f] \\ &= [x^a y^c, x^b y^d] \\ &= [x^a, y^d][y^c, x^b] \\ &= z^{ad-bc} = z^{\Delta}.\end{aligned}$$

Thus, by Lemma 7.2.2,

$$R(\varphi_\tau) = \gcd(\Delta - 1, k) = \begin{cases} \gcd(0,k) & \text{if } \Delta = 1 \\ \gcd(2,k) & \text{if } \Delta = -1. \end{cases}$$

Note that $\gcd(2,k) = 1$ if k is odd, and $\gcd(2,k) = 2$ if k is even. Therefore, if k is even, $R(\varphi)$ is either ∞ or even, as $R(\varphi) = R(\varphi_\tau)R(\bar{\varphi})$. This proves that $\mathrm{Spec}_{\mathrm{R}}(G_k) \subseteq 2\mathbb{N}_0 \cup \{\infty\}$ if k is even.

Conversely, we have to construct automorphisms realising all the candidate-Reidemeister numbers. For $m \in \mathbb{Z}$, define the map $\varphi_m : G_k \to G_k$ by the following images of the generators:

$$\begin{aligned}\varphi_m(x) &= y, \\ \varphi_m(y) &= xy^m, \\ \varphi_m(z) &= z^{-1}.\end{aligned}$$

It is readily verified that the map φ_m respects the relations of G_k. Moreover, the induced map $\bar{\varphi}_m$ on the abelianisation has matrix representation (w.r.t. the $\mathbb{Z}$-basis $\{x\tau(G_k), y\tau(G_k)\}$)

$$\begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix},$$

which shows that $\bar{\varphi}_m$ is surjective. By Lemma 7.2.3, φ_m is an automorphism.

Now,

$$R(\varphi_{m,\tau}) = \begin{cases} 2 & \text{if } k \text{ is even,} \\ 1 & \text{if } k \text{ is odd.} \end{cases}$$

Also, by Proposition T.1.12,

$$R(\bar{\varphi}) = \left\| \begin{matrix} 1 & -1 \\ -1 & 1-m \end{matrix} \right\|_\infty = |1 - m - 1|_\infty = |m|_\infty.$$

Thus,

$$R(\varphi) = R(\varphi_\tau)R(\bar{\varphi}) = \begin{cases} 2|m|_\infty & \text{if } k \text{ is even,} \\ |m|_\infty & \text{if } k \text{ is odd.} \end{cases}$$

This proves that

$$\mathrm{Spec}_\mathrm{R}(G_k) \supseteq \begin{cases} 2\mathbb{N}_0 \cup \{\infty\} & \text{if } k \text{ is even,} \\ \mathbb{N}_0 \cup \{\infty\} & \text{if } k \text{ is odd,} \end{cases}$$

which finishes the proof. □

The next two families of nilpotent groups still have an abelian torsion subgroup, but this subgroup is no longer central. However, to make the calculations not too tedious, the torsion subgroup is in both cases an abelian group of exponent p. These groups carry a natural $\mathbb{Z}/p\mathbb{Z}$-vector space structure, so we can represent endomorphisms by matrices. In that case, Lemma 7.2.1 takes on an easier form. If φ is an endomorphism of a finite dimensional $\mathbb{Z}/p\mathbb{Z}$-vector space, then $\mathrm{Fix}(\varphi)$ is the eigenspace associated to the eigenvalue 1. Therefore, $R(\varphi) = p^d$, where d is the dimension of that eigenspace. Consequently, to determine $R(\varphi)$, we have to determine whether φ has eigenvalue 1 and if so, what its geometric multiplicity is.

Given a prime p, we use the following notation when working modulo p: for an integer $n \in \mathbb{Z}$ and a matrix $M \in \mathbb{Z}^{r\times r}$, $r \geq 1$, we use $\bar{n}$ and $\bar{M}$, respectively, to represent their reduction modulo p. However, to avoid making the notation too heavy, we write 0 and 1 for both the integers and their reductions modulo p.

The first family consists of groups having a finite set as Reidemeister spectrum.

Proposition 7.2.5. *Let p be a prime number. Define the group H_p by means of the following presentation:*

$$H_p := \langle x, y, z \mid [x, y] = z, [x, z] = [y, z] = 1, y^p = 1\rangle.$$

Then

$$\mathrm{Spec}_\mathrm{R}(H_p) = \begin{cases} \{6, \infty\} & \text{if } p \in \{2, 3\}, \\ \{2, 2p, \infty\} & \text{if } p \geq 5. \end{cases}$$

Proof. Note that H_p is isomorphic to the quotient $\frac{N_{2,2}}{\langle\langle y^p\rangle\rangle}$, so H_p is nilpotent.

To compute $\mathrm{Spec}_\mathrm{R}(H_p)$, we first determine the torsion subgroup of H_p. Clearly, $y \in \tau(H_p)$. Also, using Lemma 4.1.4, we find

$$z^p = [x, y]^p = [x, y^p] = 1,$$

which implies that $z \in \tau(H_p)$ as well. Since z is central and $y^x = y[y,x] = yz^{-1}$, the group $\langle y, z\rangle$ is normal in H_p. The quotient group is isomorphic to $\mathbb{Z}$, as

$$H_p \to \mathbb{Z} : x \mapsto 1, y \mapsto 0, z \mapsto 0$$

is a well-defined surjective group morphism with $\langle y, z\rangle$ as kernel. Therefore, $\tau(H_p) = \langle y, z\rangle$. As y and z commute and both have order p, $\tau(H_p) \cong (\mathbb{Z}/p\mathbb{Z})^2$. Furthermore, since $\mathbb{Z}$ is free, H_p decomposes as $\tau(H_p) \rtimes \mathbb{Z}$. Consequently, Corollary 4.1.11 implies that $\infty \in \mathrm{Spec}_{\mathrm{R}}(H_p)$.

Let $\varphi \in \mathrm{Aut}(H_p)$. If $R(\bar{\varphi}) = \infty$, then $R(\varphi) = \infty$ as well by Corollary 1.1.2(1). So, assume that $R(\bar{\varphi}) < \infty$. Then $\bar{\varphi}$ is the inversion map, so we can take $\tau(H_p)$ and $x\tau(H_p)$ as representatives of $\mathcal{R}[\bar{\varphi}]$. Also, $\varphi(x) = x^{-1}y^A z^B$ for some $A, B \in \mathbb{Z}$.

Write $\varphi(y) = y^a z^c$ and $\varphi(z) = y^b z^d$ for some $a, b, c, d \in \mathbb{Z}$. Then $b = 0$, as $[H_p, H_p] = \langle z\rangle$ is characteristic. Now, since φ preserves the relation $[x, y] = z$, we find that

$$z^d = \varphi(z) = [\varphi(x), \varphi(y)] = [x^{-1}y^A z^B, y^a z^c] = [x^{-1}, y^a] = z^{-a}.$$

Thus, it follows that $a \equiv -d \bmod p$. Since $\tau(H_p)$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$ and $\{y, z\}$ is a $\mathbb{Z}/p\mathbb{Z}$-basis of this group, we can summarise all the obtained relations in a matrix representation of φ_τ with respect to this basis:

$$\varphi_\tau = \begin{pmatrix} \bar{a} & 0 \\ \bar{c} & -\bar{a} \end{pmatrix}.$$

By Theorem 7.1.1, $R(\varphi) = R(\varphi_\tau) + R(\tau_x \circ \varphi_\tau)$. To determine the matrix representation of $\tau_x \circ \varphi_\tau$, we compute the images of y and z:

$$\tau_x(\varphi(y)) = (y^a z^c)^x = (y^x)^a z^c = (yz^{-1})^a z^c = y^a z^{c-a}$$

and

$$\tau_x(\varphi(z)) = (z^d)^x = z^d.$$

Thus,

$$\tau_x \circ \varphi_\tau = \begin{pmatrix} \bar{a} & 0 \\ \bar{c} - \bar{a} & -\bar{a} \end{pmatrix}$$

First, assume that $p = 2$. For φ_τ to be an automorphism, $\det \varphi_\tau = -\bar{a}^2 \neq 0$. This implies that $a \equiv 1 \bmod 2$. As c is either 0 or 1 modulo 2, the matrix representations of φ_τ and $\tau_x \circ \varphi_\tau$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

in some order. The first matrix represents the identity map, so the corresponding automorphism has Reidemeister number equal to 4. The second matrix has an eigenvalue 1 with geometric multiplicity 1, so the corresponding automorphism has only 2 fixed points. Therefore,

$$R(\varphi) = R(\varphi_\mathcal{T}) + R(\tau_x \circ \varphi_\mathcal{T}) = 6.$$

To argue that there does indeed exist an automorphism on H_2 realising this Reidemeister number, one can verify using Lemma 7.2.3 that

$$\psi : H_2 \to H_2 : x \mapsto x^{-1}, y \mapsto y, z \mapsto z$$

is an automorphism preserving the relations of H_2 and having Reidemeister number equal to 6.

Next, assume that $p = 3$. Again, since $\varphi_\mathcal{T}$ is an automorphism, $a \not\equiv 0 \bmod 3$. Then either $a \equiv 1 \bmod 3$ or $a \equiv 2 \bmod 3$, so both $\varphi_\mathcal{T}$ and $\tau_x \circ \varphi_\mathcal{T}$ have eigenvalues 1 and -1. Therefore, $R(\varphi_\mathcal{T}) = R(\tau_x \circ \varphi_\mathcal{T}) = 3$, which implies that $R(\varphi) = 6$. Again, one can verify that

$$\psi : H_3 \to H_3 : x \mapsto x^{-1}, y \mapsto y, z \mapsto z^2$$

is an automorphism preserving the relations of H_3 and having Reidemeister number equal to 6.

Finally, assume that $p \geq 5$. If $a \not\equiv \pm 1 \bmod p$, then neither $\varphi_\mathcal{T}$ nor $\tau_x \circ \varphi_\mathcal{T}$ has an eigenvalue 1. In that case, $R(\varphi_\mathcal{T}) = R(\tau_x \circ \varphi_\mathcal{T}) = 1$, so $R(\varphi) = 2$.

If $a \equiv \pm 1 \bmod p$, then both $\varphi_\mathcal{T}$ and $\tau_x \circ \varphi_\mathcal{T}$ have eigenvalues 1 and -1. Therefore, $R(\varphi_\mathcal{T}) = R(\tau_x \circ \varphi_\mathcal{T}) = p$, which implies that $R(\varphi) = 2p$.

To end the proof, one can verify that

$$\psi_1 : H_p \to H_p : x \mapsto x^{-1}, y \mapsto y^2, z \mapsto z^{-2}$$

and

$$\psi_2 : H_p \to H_p : x \mapsto x^{-1}, y \mapsto y, z \mapsto z^{-1}$$

are well-defined automorphisms with Reidemeister number equal to 2 and $2p$, respectively. □

For the last example, we give another family of groups most of which have full Reidemeister spectrum.

Proposition 7.2.6. *Let p be a prime number. Let N_p be the group with the following presentation:*

$$N_p := \left\langle a, b, c, x, y \;\middle|\; \begin{array}{l} [a,b] = 1, [a,c] = x, [b,c] = y, \\ [x,\cdot] = [y,\cdot] = c^p = x^p = y^p = 1 \end{array} \right\rangle,$$

where $[x, \cdot] = 1$ *indicates that* x *commutes with everything. Then*

$$\mathrm{Spec}_{\mathrm{R}}(N_p) = \begin{cases} 2(2\mathbb{N}_0 - 1) \cup 6\mathbb{N}_0 \cup 20\mathbb{N}_0 \cup \{\infty\} & \text{if } p = 2, \\ \mathbb{N}_0 \cup \{\infty\} & \text{otherwise.} \end{cases}$$

Proof. We start by determining the torsion subgroup of N_p. Clearly, $T := \langle c, x, y\rangle \leq \tau(N_p)$. Furthermore, since x and y are central and both $[a, c]$ and $[b, c]$ lie in T, T is normal. The quotient group N_p/T is isomorphic to $\mathbb{Z}^2$, as

$$N_p \to \mathbb{Z}^2 : a \mapsto (1, 0), b \mapsto (0, 1), c \mapsto (0, 0), x \mapsto (0, 0), y \mapsto (0, 0)$$

is a well-defined surjective group morphism. Proposition 4.1.13(3) then implies that $T = \tau(N_p)$.

Since c, x and y all commute and have order p, the group $\tau(N_p)$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$. Let $\varphi \in \mathrm{Aut}(N_p)$. Let $M = (m_{ij})_{ij} \in \mathrm{GL}(2, \mathbb{Z})$ be the matrix representation of $\bar{\varphi}$ w.r.t. the $\mathbb{Z}$-basis $\{a\tau(N_p), b\tau(N_p)\}$. Then $\varphi(a) = a^{m_{11}} b^{m_{21}} t_a$ and $\varphi(b) = a^{m_{12}} b^{m_{22}} t_b$ for some $t_a, t_b \in \tau(N_p)$. Suppose that $\varphi(c) = c^k z$ for some $z \in \langle x, y\rangle$ and $k \in \mathbb{Z}$. We compute $\varphi(x)$ and $\varphi(y)$:

$$\varphi(x) = [\varphi(a), \varphi(c)] = [a^{m_{11}} b^{m_{21}} t_a, c^k z] = [a^{m_{11}}, c^k][b^{m_{21}}, c^k] = x^{km_{11}} y^{km_{21}}$$

and

$$\varphi(y) = [\varphi(b), \varphi(c)] = [a^{m_{12}} b^{m_{22}} t_b, c^k z] = [a^{m_{12}}, c^k][b^{m_{22}}, c^k] = x^{km_{12}} y^{km_{22}},$$

by Lemma 4.1.4. Therefore, the matrix representation of φ_τ with respect to the $\mathbb{Z}/p\mathbb{Z}$-basis $\{c, x, y\}$ of $\tau(N_p)$ has the form

$$\begin{pmatrix} \bar{k} & 0 \\ \bar{v}^{\mathsf{T}} & \bar{k}\bar{M} \end{pmatrix}$$

for some $v = (v_1, v_2) \in \mathbb{Z}^2$. The map φ_τ is an automorphism, therefore, $k \not\equiv 0 \bmod p$. In particular, if $p = 2$, then $k \equiv 1 \bmod 2$. Now, note that τ_a and τ_b restricted to $\tau(N_p)$ have matrix representations (w.r.t. the basis $\{c, x, y\}$),

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

respectively. Consequently, for $r, s \in \mathbb{Z}$, the matrix representation of $\tau_{a^r b^s}$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ \bar{r} & 1 & 0 \\ \bar{s} & 0 & 1 \end{pmatrix}.$$

If we then compute the matrix representation of $\tau_{a^r b^s} \circ \varphi_\mathcal{T}$, we obtain

$$\begin{pmatrix} \bar{k} & 0 \\ \bar{w}^\intercal & \bar{k}\bar{M} \end{pmatrix}, \tag{7.2.1}$$

where $w = v + k(r, s)$. This implies that $\tau_{a^r b^s} \circ \varphi_\mathcal{T}$ has the same eigenvalues as $\varphi_\mathcal{T}$.

Next, we prove the inclusion $\operatorname{Spec}_\mathrm{R}(N_2) \subseteq 2(2\mathbb{N}_0 - 1) \cup 6\mathbb{N}_0 \cup 20\mathbb{N}_0 \cup \{\infty\}$. So, suppose that $p = 2$. Then $k \equiv 1 \bmod 2$, so (7.2.1) becomes

$$\begin{pmatrix} 1 & 0 \\ \bar{w}^\intercal & \bar{M} \end{pmatrix}.$$

Fix $r, s \in \mathbb{Z}$. The characteristic polynomial of $\tau_{a^r b^s} \circ \varphi_\mathcal{T}$ is given by

$$(X + 1)\chi_{\bar{M}}(X) = (X + 1)(X^2 - \operatorname{Tr}(\bar{M})X + \det(\bar{M})).$$

By Proposition T.1.12, $R(\bar{\varphi}) = |\chi_M(1)|_\infty = |1 - \operatorname{Tr}(M) + \det(M)|_\infty$. We may assume that $R(\bar{\varphi}) < \infty$, since $R(\varphi) = \infty$ if $R(\bar{\varphi}) = \infty$. Since $M \in \mathrm{GL}(2, \mathbb{Z})$, $\det(M) = \pm 1$. Therefore,

$$R(\bar{\varphi}) \equiv 2 - \operatorname{Tr}(M) \equiv \operatorname{Tr}(M) \bmod 2.$$

Consequently, if $R(\bar{\varphi})$ is odd, the characteristic polynomial of $\tau_{a^r b^s} \circ \varphi_\mathcal{T}$ equals $(X + 1)(X^2 - X + 1)$, which has (in some field extension) three distinct roots, one of which is 1. Therefore, $R(\tau_{a^r b^s} \circ \varphi_\mathcal{T}) = 2$. This implies that

$$R(\varphi) = \sum_{[g\tau(N_p)]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} 2 = 2R(\bar{\varphi}) \in 2(2\mathbb{N}_0 - 1).$$

So, from now on, suppose that $R(\bar{\varphi})$ is even. Since $R(\bar{\varphi}) \equiv \operatorname{Tr}(M) \bmod 2$, the characteristic polynomial of $\tau_{a^r b^s} \circ \varphi_\mathcal{T}$ is equal to $(X + 1)(X^2 - 1) = (X + 1)^3$ (recall that we work here over a field of characteristic 2). Therefore, $\tau_{a^r b^s} \circ \varphi_\mathcal{T}$ has only one eigenvalue, namely 1.

We have to compute the dimension of the eigenspace associated to the eigenvalue 1 of the map $\tau_{a^r b^s} \circ \varphi_\mathcal{T}$. Let $E_1(r, s)$ denote this eigenspace. As $0 \equiv R(\bar{\varphi}) \equiv \operatorname{Tr}(M) \bmod 2$, the possibilities for M mod 2 are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, $\tau_{a^r b^s} \circ \varphi_\mathcal{T}$ has matrix representation equal to one of the following:

$$\begin{pmatrix} 1 & 0 & 0 \\ \bar{v}_1 + \bar{r} & 1 & 0 \\ \bar{v}_2 + \bar{s} & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ \bar{v}_1 + \bar{r} & 1 & 1 \\ \bar{v}_2 + \bar{s} & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ \bar{v}_1 + \bar{r} & 1 & 0 \\ \bar{v}_2 + \bar{s} & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ \bar{v}_1 + \bar{r} & 0 & 1 \\ \bar{v}_2 + \bar{s} & 1 & 0 \end{pmatrix}.$$

Recall that v_1 and v_2 only depend on φ. The dimension of $E_1(r,s)$ therefore depends on the parity of r and s. Thus, we get the following case distinction:

- $M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2$: $\dim E_1(r,s) = \begin{cases} 3 & \text{if } \bar{v}_1 + \bar{r} = \bar{v}_2 + \bar{s} = 0, \\ 2 & \text{otherwise.} \end{cases}$
- $M \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \bmod 2$: $\dim E_1(r,s) = \begin{cases} 2 & \text{if } \bar{v}_2 + \bar{s} = 0, \\ 1 & \text{otherwise.} \end{cases}$
- $M \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \bmod 2$: $\dim E_1(r,s) = \begin{cases} 2 & \text{if } \bar{v}_1 + \bar{r} = 0, \\ 1 & \text{otherwise.} \end{cases}$
- $M \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bmod 2$: $\dim E_1(r,s) = \begin{cases} 2 & \text{if } \bar{v}_1 + \bar{r} = \bar{v}_2 + \bar{s}, \\ 1 & \text{otherwise.} \end{cases}$

The next step in computing $R(\varphi)$ is combining the expression for $\dim E_1(r,s)$ for arbitrary $r, s \in \mathbb{Z}$ with Theorem 7.1.1. Together with what we have discussed so far, it states that

$$R(\varphi) = \sum_{[a^r b^s \tau(N_p)]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}]} 2^{\dim E_1(r,s)}. \tag{7.2.2}$$

So, to compute $R(\varphi)$, we have to determine which dimensions occur for how many representatives of $\mathcal{R}[\bar{\varphi}]$. Put $H = \operatorname{Im}(\bar{\varphi} - \mathrm{Id})$. By Proposition T.1.12, $\mathcal{R}[\bar{\varphi}] = \mathbb{Z}^2/H$. Consider the following maps, depending on the value of $M \bmod 2$:

- $M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2$: $\pi : \mathbb{Z}^2 \to (\mathbb{Z}/2\mathbb{Z})^2 : (r,s) \mapsto (r \bmod 2, s \bmod 2)$
- $M \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \bmod 2$: $\pi : \mathbb{Z}^2 \to \mathbb{Z}/2\mathbb{Z} : (r,s) \mapsto s \bmod 2$
- $M \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \bmod 2$: $\pi : \mathbb{Z}^2 \to \mathbb{Z}/2\mathbb{Z} : (r,s) \mapsto r \bmod 2$
- $M \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bmod 2$: $\pi : \mathbb{Z}^2 \to \mathbb{Z}/2\mathbb{Z} : (r,s) \mapsto r + s \bmod 2$

One can verify that $H \leq \ker \pi$ in each case, using the fact that H is spanned by the columns of $M - I$. Hence, we get a surjective group homomorphism $\hat{\pi}$ from

$\mathcal{R}[\bar{\varphi}]$ to the respective finite group, say, F. Thus, given $f \in F$, the number of cosets in $\mathcal{R}[\bar{\varphi}]$ mapped to f by $\hat{\pi}$ equals $\frac{|\mathcal{R}[\bar{\varphi}]|}{|F|} = \frac{R(\bar{\varphi})}{|F|}$.

With this map $\hat{\pi}$, each if-statement in the case distinction of $\dim E_1(r,s)$ can also be written as 'if $\hat{\pi}(r,s) = \hat{\pi}(v_1, v_2)$'. Thus, if we combine everything, (7.2.2) becomes

$$\begin{aligned} R(\varphi) &= |\{[a^r b^s \tau(N_p)]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}] \mid \hat{\pi}(r,s) = \hat{\pi}(v_1,v_2)\}| \cdot 8 \\ &\quad + |\{[a^r b^s \tau(N_p)]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}] \mid \hat{\pi}(r,s) \neq \hat{\pi}(v_1,v_2)\}| \cdot 4 \end{aligned}$$

for $M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2$ and

$$\begin{aligned} R(\varphi) &= |\{[a^r b^s \tau(N_p)]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}] \mid \hat{\pi}(r,s) = \hat{\pi}(v_1,v_2)\}| \cdot 4 \\ &\quad + |\{[a^r b^s \tau(N_p)]_{\bar{\varphi}} \in \mathcal{R}[\bar{\varphi}] \mid \hat{\pi}(r,s) \neq \hat{\pi}(v_1,v_2)\}| \cdot 2. \end{aligned}$$

in the other cases. As mentioned earlier, there are exactly $R(\bar{\varphi})/|F|$ elements in $\mathcal{R}[\bar{\varphi}]$ that are mapped to $\hat{\pi}(v_1, v_2)$. Therefore, we get the following expressions for $R(\varphi)$:

- $M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2$:

$$R(\varphi) = \frac{R(\bar{\varphi})}{4} \cdot 8 + 3 \cdot \frac{R(\bar{\varphi})}{4} \cdot 4 = 5R(\bar{\varphi}). \tag{7.2.3}$$

- Other cases:

$$R(\varphi) = \frac{R(\bar{\varphi})}{2} \cdot 4 + \frac{R(\bar{\varphi})}{2} \cdot 2 = 3R(\bar{\varphi}). \tag{7.2.4}$$

From (7.2.4), we derive that $R(\varphi) \in 6\mathbb{N}_0$ in that case, as $R(\bar{\varphi})$ is even. If $M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2$, then $R(\bar{\varphi}) \equiv 0 \bmod 4$. Indeed, in that case, $M - I$ contains even numbers in each entry of the matrix. Therefore, we can factor out two times a factor 2 in $\det(M - I)$, which implies that $R(\bar{\varphi}) \equiv 0 \bmod 4$. We conclude that $R(\varphi) = 5R(\bar{\varphi}) \in 20\mathbb{N}_0$ in that case. This finishes the proof of the inclusion $\mathrm{Spec}_{\mathrm{R}}(N_2) \subseteq 2(2\mathbb{N}_0 - 1) \cup 6\mathbb{N}_0 \cup 20\mathbb{N}_0 \cup \{\infty\}$.

We now proceed to the converse inclusion. We start with the inclusion $6\mathbb{N}_0 \cup 20\mathbb{N}_0 \subseteq \mathrm{Spec}_{\mathrm{R}}(N_2)$, by giving automorphisms that satisfy the conditions of (7.2.3) and (7.2.4). Let $m \in \mathbb{Z}$ be arbitrary. First, define the map $\varphi_m : N_2 \to N_2$

by

$$\varphi_m(a) = b,$$

$$\varphi_m(b) = ab^{2m},$$

$$\varphi_m(c) = c,$$

$$\varphi_m(x) = y,$$

$$\varphi_m(y) = x,$$

on the generators. It is readily verified that φ_m respects the relations of N_2 and that its induced map on the abelianisation is surjective. By Lemma 7.2.3, φ_m is an automorphism. Now, the matrix representation M of $\bar{\varphi}_m$ is given by

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 2m \end{pmatrix}.$$

Therefore, $M \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bmod 2$. Hence, by (7.2.4),

$$R(\varphi_m) = 3R(\bar{\varphi}_m) = 3|1 - \mathrm{Tr}(M) + \det(M)|_\infty = 3|1 - 2m - 1|_\infty = 6|m|_\infty.$$

Next, define the map $\psi_m : N_2 \to N_2$ by

$$\psi_m(a) = ab^2,$$

$$\psi_m(b) = a^{2m}b^{4m+1},$$

$$\psi_m(c) = c,$$

$$\psi_m(x) = x,$$

$$\psi_m(y) = y.$$

Again, one can verify that ψ_m defines an automorphism of N_2. The matrix representation M of $\bar{\psi}_m$ is given by

$$M = \begin{pmatrix} 1 & 2m \\ 2 & 4m+1 \end{pmatrix}.$$

Therefore, $M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2$. Hence, by (7.2.3),

$$\begin{aligned} R(\psi_m) &= 5R(\bar{\psi}_m) \\ &= 5|1 - \mathrm{Tr}(M) + \det(M)|_\infty \\ &= 5|1 - (4m+2) + 1|_\infty \\ &= 20|m|_\infty. \end{aligned}$$

As m is arbitrary, we have proven that $6\mathbb{N}_0 \cup 20\mathbb{N}_0 \subseteq \mathrm{Spec}_\mathrm{R}(N_2)$.

To end the proof, we show that $\mathrm{Spec}_\mathrm{R}(N_p)$ is full for p odd and that $\mathrm{Spec}_\mathrm{R}(N_2)$ contains $2(2\mathbb{N}_0 - 1)$. Let $m \in \mathbb{Z}$ be arbitrary. First, suppose that m is not divisible by p. Define the map $\eta_m : N_p \to N_p$ by

$$\begin{aligned} \eta_m(a) &= b, \\ \eta_m(b) &= ab^m, \\ \eta_m(c) &= c^{-1}, \\ \eta_m(x) &= y^{-1}, \\ \eta_m(y) &= x^{-1}y^{-m}. \end{aligned}$$

Again, it is readily verified, using Lemma 7.2.3, that η_m defines an automorphism of N_p.

By the general calculations done in the beginning of the proof, the matrix representations of $\bar{\eta}_m$ and $\eta_{m,\tau}$ are

$$\begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & -\bar{m} \end{pmatrix},$$

respectively. Using Proposition T.1.12, we find that

$$R(\bar{\eta}_m) = \left| \begin{vmatrix} 1 & -1 \\ -1 & 1-m \end{vmatrix} \right|_\infty = |1 - m - 1|_\infty = |m|_\infty.$$

Next, let $r, s \in \mathbb{Z}$ be arbitrary. Then the matrix representation of $\tau_{a^r b^s} \circ \eta_{m,\tau}$ equals

$$\begin{pmatrix} 1 & 0 & 0 \\ \bar{r} & 1 & 0 \\ \bar{s} & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & -\bar{m} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ -\bar{r} & 0 & -1 \\ -\bar{s} & -1 & -\bar{m} \end{pmatrix}.$$

Its characteristic polynomial is thus given by

$$(X+1)\begin{vmatrix} X & 1 \\ 1 & X+\bar{m} \end{vmatrix} = (X+1)(X^2+\bar{m}X-1).$$

As m is not divisible by p, $\bar{m} \neq 0$. Therefore, $\tau_{a^r b^s} \circ \eta_{m,\tau}$ has no eigenvalue 1 if $p \geq 3$ and three distinct eigenvalues (in some field extension), one of which is 1, if $p = 2$. Thus, Lemma 7.2.1 then implies that $R(\tau_{a^r b^s} \circ \eta_{m,\tau})$ is equal to 1 if $p \geq 3$ and equal to 2 if $p = 2$. As r and s were arbitrary, we conclude that

$$R(\eta_m) = \sum_{[g\tau(N_p)]_{\bar{\eta}_m} \in \mathcal{R}[\bar{\eta}_m]} R(\tau_g \circ \eta_{m,\tau}) = \begin{cases} 2|m|_\infty & \text{if } p = 2, \\ |m|_\infty & \text{otherwise.} \end{cases} \tag{7.2.5}$$

Thus, $2(2\mathbb{N}_0 - 1) \subseteq \mathrm{Spec}_{\mathrm{R}}(N_2)$, as m is odd if $p = 2$.

Now assume that $p \geq 3$ and that m is divisible by p. We define the map $\omega_m : N_p \to N_p$ on the generators a, b, c, x and y as follows:

$$\omega_m(a) = b^{-1},$$

$$\omega_m(b) = ab^{m+2},$$

$$\omega_m(c) = c^{-1},$$

$$\omega_m(x) = y,$$

$$\omega_m(y) = x^{-1}y^{-2}.$$

As before, a few calculations combined with Lemma 7.2.3 prove that ω_m is an automorphism of N_p. Again, by the general calculations done in the beginning of the proof, the matrix representations of $\bar{\omega}_m$ and $\omega_{m,\tau}$ are

$$\begin{pmatrix} 0 & 1 \\ -1 & m+2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix},$$

respectively.

Let $r, s \in \mathbb{Z}$ be arbitrary. The matrix representation of $\tau_{a^r b^s} \circ \omega_{m,\tau}$ equals

$$\begin{pmatrix} -1 & 0 & 0 \\ -\bar{r} & 0 & -1 \\ -\bar{s} & 1 & -2 \end{pmatrix}.$$

Consequently, its characteristic polynomial is equal to $(X+1)(X^2+2X+1)$, which does not have 1 as a root, as $p \geq 3$. Therefore, $\tau_{a^r b^s} \circ \omega_\tau$ has no non-trivial fixed points, so Lemma 7.2.1 yields $R(\tau_{a^r b^s} \circ \omega_{m,\tau}) = 1$. As r and s were arbitrary,

$$R(\omega_m) = \sum_{[g\tau(N_p)]_{\bar{\omega}_m} \in \mathcal{R}[\bar{\omega}_m]} R(\tau_g \circ \omega_{m,\tau}) = R(\bar{\omega}_m)$$

and the latter is equal to

$$\left|\begin{vmatrix} 1 & 1 \\ -1 & 1-m-2 \end{vmatrix}\right|_\infty = |-m-1+1|_\infty = |m|_\infty.$$

Combining this with (7.2.5), we find that $\mathrm{Spec}_{\mathrm{R}}(N_p) = \mathbb{N}_0 \cup \{\infty\}$ for $p \geq 3$, as m was arbitrary. $\square$

Collection III

Finite groups

Variation 8

Finite groups

Mozart, Wolfgang Amadeus, *Zwölf Variationen in C über das franzözische Lied 'Ah, vous dirai-je, Maman'*, (KV 265 (300e)), Variation VIII (bars 1–8).

Finite groups, for obvious reasons, never have the R_∞-property nor full Reidemeister spectrum. Thus, at first sight, the only remaining relevant objective is to try and completely determine the Reidemeister spectrum of a finite group. However, a possible alternative could be to decide when 1 lies in the Reidemeister spectrum of a finite group.

Proposition 8.0.1. *Let G be a finite group and $\varphi \in \operatorname{End}(G)$. Then the following are equivalent:*

(1) $R(\varphi) = 1$.

(2) The map $f : G \to G : g \mapsto g\varphi(g)^{-1}$ is surjective.

(3) The map $f : G \to G : g \mapsto g\varphi(g)^{-1}$ is injective.

(4) $\operatorname{Fix}(\varphi) = 1$.

Proof. The second and third condition are equivalent since G is finite. Note that $[1]_\varphi = \{g\varphi^{-1}(g) \mid g \in G\}$ and that $R(\varphi) = 1$ if and only if $[1]_\varphi = G$, which implies that the first and second condition are equivalent. Finally, if f is injective and $\varphi(g) = g$, then $1 = f(g) = f(1)$, hence $g = 1$. Conversely, if $\operatorname{Fix}(\varphi) = 1$, then $f(g) = f(h)$ implies that $h^{-1}g = \varphi(h^{-1}g)$ and thus $h = g$. $\square$

The previous result shows that determining what finite groups admit an automorphism with Reidemeister number equal to 1 is equivalent with determining what finite groups admit a fixed-point free automorphism. One of the major results on this topics is due to P. Rowley.

Theorem 8.0.2 (See e.g. [102])**.** *Let G be a finite group. If G admits a fixed-point free automorphism, then G is solvable.*

For more information on groups admitting a fixed-point free automorphism, we refer the interested reader to [99, §10.5],[118, 54, 112].

Up until now, little research has been done into twisted conjugacy in finite groups. Most of the few results are due to A. Fel'shtyn and R. Hill, who studied Reidemeister zeta functions on finite groups and provided an alternative way of determining Reidemeister numbers on finite groups (see e.g. [32, 28], see also Theorem 8.1.2). In the context of Nielsen fixed-point theory, there are also several results when the fundamental group is finite (see e.g. [62, Chapter II, § 5]).

Suppose, for now, that we keep our ambitious objective to try to completely determine the Reidemeister spectrum of an arbitrary finite group, say, G. By the Krull-Schmidt theorem, G can be written in an essentially unique way as a direct product $G_1 \times \ldots \times G_n$ of directly indecomposable groups. If G is centreless, then so is each G_i. In that case, if we know $\mathrm{Spec}_\mathrm{R}(G_i)$ for each $i \in \{1, \ldots, n\}$, we know $\mathrm{Spec}_\mathrm{R}(G)$ by Corollary 3.3.5. In other words, to study the Reidemeister spectrum of finite centreless groups, it is sufficient to study those that are directly indecomposable.

If G has non-trivial centre, then we cannot use Theorem 3.3.2 to express the automorphism group of G in terms of the automorphism groups of the G_i. However, there are two results due to J. Bidwell, M. Curran and D. McCaughan [7, 8] that we can use. We use the matrix notation for endomorphisms as we did in Variation 3.

Theorem 8.0.3 (See e.g. [7, Theorem 2.2])**.** *Let $G_1, \ldots, G_n$ be finite groups such that no pair of the G_i have a common direct factor. Put $G = G_1 \times \ldots \times G_n$. Then*

$$\mathrm{Aut}(G) = \left\{ (\varphi_{ij})_{ij} \;\middle|\; \begin{array}{c} \forall i \in \{1, \ldots, n\} : \varphi_{ii} \in \mathrm{Aut}(G_i) \\ \forall i \neq j \in \{1, \ldots, n\} : \varphi_{ij} \in \mathrm{Hom}(G_j, Z(G_i)) \end{array} \right\}$$

Theorem 8.0.4. *Let G be a non-abelian directly indecomposable group and $n \geq 1$ an integer. Then*

$$\mathrm{Aut}(G^n) = \left\{ (\varphi_{ij})_{ij} \;\middle|\; \begin{array}{c} \forall i \in \{1, \ldots, n\} : \varphi_{ii} \in \mathrm{Aut}(G) \\ \forall i \neq j \in \{1, \ldots, n\} : \varphi_{ij} \in \mathrm{Hom}(G, Z(G)) \end{array} \right\} \rtimes S_n,$$

where S_n is the group of automorphisms given by the permutation matrices.

Note that if all G_i are centreless, the combination of both results coincides with the content of Theorem 3.3.2.

Now, suppose G is a finite group with non-trivial centre and that $G = G_1 \times \ldots \times G_n$ is its unique decomposition into directly indecomposable groups. Let $\varphi \in \operatorname{Aut}(G)$. Theoretically speaking, one could use Theorem 1.2.3 to determine $R(\varphi)$, by modding out the centre of G. The restriction $\varphi|_{Z(G)}$ of φ to the centre of G is an automorphism on a finite abelian group. The $\varphi|_{Z(G)}$-conjugacy classes are simply the cosets of $\operatorname{Im}(\operatorname{Id}_{Z(G)} - \varphi|_{Z(G)})$ by Lemma 7.2.1. The induced automorphism $\bar{\varphi}$ on $G/Z(G)$ only contains one non-trivial map on each row and each column, so we can use Proposition 3.1.8 and its proof to determine the Reidemeister number and representatives of all Reidemeister classes of $\bar{\varphi}$. However, determining the twisted stabiliser of each representative of $\mathcal{R}[\bar{\varphi}]$ and the number of orbits of the action in the sum formula of Theorem 1.2.3 usually requires a substantial amount of time and work.

A more feasible way to determine Reidemeister spectra of finite groups would be to study specific classes, starting with (relatively speaking) easier ones, such as finite abelian groups and metacyclic groups. One can then gradually increase the complexity of the groups and perhaps use previous results. We start this approach in Variation 9 and Variation 10.

8.1 Alternative counting methods

Besides the definition, there are two additional ways to determine the Reidemeister number of an endomorphism on a finite group. The first one uses twisted conjugacy classes of other endomorphisms.

Lemma 8.1.1. *Let G be a group acting on a set X. Suppose that $x, y \in X$ are such that $y \in G \cdot x$. Then*

$$|\operatorname{Stab}(x)| = |\{h \in G \mid h \cdot x = y\}| = |\{h \in G \mid h \cdot y = x\}|.$$

Proof. We start with the first equality. Suppose that $g_0 \in H$ is such that $g_0 \cdot x = y$. Then for each $g \in \operatorname{Stab}(x)$,

$$(g_0 g) \cdot x = g_0 \cdot (g \cdot x) = g_0 \cdot x = y.$$

Thus, the map

$$f : \operatorname{Stab}(x) \to \{h \in G \mid h \cdot x = y\} : g \mapsto g_0 g$$

is well defined. It is clearly injective. To prove that f surjective as well, suppose that $h \in G$ is such that $h \cdot x = y$. Then

$$(g_0^{-1}h) \cdot x = g_0^{-1} \cdot (h \cdot x) = g_0^{-1} \cdot y = x,$$

so $g_0^{-1}h \in \mathrm{Stab}(x)$. As $h = g_0\big(g_0^{-1}h\big)$, this proves that f is surjective.

For the second equality, note that, for all $g \in G$, $g \cdot x = y$ if and only if $g^{-1} \cdot y = x$. Therefore, the map

$$F : \{h \in G \mid h \cdot x = y\} \to \{h \in G \mid h \cdot y = x\} : g \mapsto g^{-1}$$

is a well-defined bijection. The second equality now follows. □

The following is a generalisation of [32, Theorem 5]; credit goes to S. Tertooy for his help in the proof of this generalisation.

Theorem 8.1.2. *Let G be a finite group and $\varphi, \psi \in \mathrm{End}(G)$. Suppose that $\varphi \circ \psi = \psi \circ \varphi$ and that $g \sim_\varphi \psi(g)$ for all $g \in G$. Then the map*

$$\Xi : \mathcal{R}[\psi] \to \mathcal{R}[\psi] : [g]_\psi \mapsto [\varphi(g)]_\psi$$

is well defined and $R(\varphi) = |\mathrm{Fix}(\Xi)|$.

In particular, the result holds for $\psi = \mathrm{Id}_G$ and $\psi = \varphi^k$ for any $k \geq 1$.

Proof. First, we have to prove that $\Xi([g]_\psi)$ is independent of the chosen representative. Suppose that $g \sim_\psi h$ for some $g, h \in G$. Write $g = xh\psi(x)^{-1}$ for some $x \in G$. Then

$$\varphi(g) = \varphi(x)\varphi(h)\varphi\Big(\psi(x)^{-1}\Big) = \varphi(x)\varphi(h)\psi(\varphi(x))^{-1},$$

since φ and ψ commute. Therefore, $\varphi(g) \sim_\psi \varphi(h)$.

Next, we prove that $R(\varphi) = |\mathrm{Fix}(\Xi)|$. We start by rewriting $|\mathrm{Fix}(\Xi)|$:

$$\begin{aligned}
|\mathrm{Fix}(\Xi)| = \sum_{[g]_\psi \in \mathrm{Fix}(\Xi)} 1 &= \sum_{\substack{g \in G \\ [g]_\psi \in \mathrm{Fix}(\Xi)}} \frac{1}{|[g]_\psi|} \\
&= \frac{1}{|G|} \sum_{\substack{g \in G \\ [g]_\psi \in \mathrm{Fix}(\Xi)}} |\mathrm{Stab}_\psi(g)| \\
&= \frac{1}{|G|} \sum_{\substack{g \in G \\ [g]_\psi \in \mathrm{Fix}(\Xi)}} \Big|\{x \in G \mid xg\psi(x)^{-1} = g\}\Big|. \qquad (8.1.1)
\end{aligned}$$

For $g \in G$ with $[g]_\psi \in \mathrm{Fix}(\Xi)$, we can replace $\left|\{x \in G \mid xg\psi(x)^{-1} = g\}\right| = |\mathrm{Stab}_\psi(g)|$ with $\left|\{x \in G \mid x\varphi(g)\psi(x)^{-1} = g\}\right|$ by Lemma 8.1.1, as $[g]_\psi \in \mathrm{Fix}(\Xi)$ means that $g \sim_\psi \varphi(g)$. Hence, (8.1.1) becomes

$$\begin{aligned}
&\frac{1}{|G|} \sum_{\substack{g \in G \\ [g]_\psi \in \mathrm{Fix}(\Xi)}} \left|\{x \in G \mid xg\psi(x)^{-1} = g\}\right| \\
&= \frac{1}{|G|} \sum_{\substack{g \in G \\ [g]_\psi \in \mathrm{Fix}(\Xi)}} \left|\{x \in G \mid x\varphi(g)\psi(x)^{-1} = g\}\right| \\
&= \frac{1}{|G|} \sum_{\substack{g \in G \\ [g]_\psi \in \mathrm{Fix}(\Xi)}} \left|\{x \in G \mid x = g\psi(x)\varphi(g)^{-1}\}\right| \\
&= \frac{1}{|G|}\left|\{(g, x) \in G \times G \mid [g]_\psi \in \mathrm{Fix}(\Xi), x = g\psi(x)\varphi(g)^{-1}\}\right| \\
&= \frac{1}{|G|} \sum_{x \in G} \left|\{g \in G \mid [g]_\psi \in \mathrm{Fix}(\Xi), x = g\psi(x)\varphi(g)^{-1}\}\right|.
\end{aligned}$$

If $g, x \in G$ satisfy $x = g\psi(x)\varphi(g)^{-1}$, then also $x\varphi(g)\psi(x)^{-1} = g$. Consequently, $\varphi(g) \sim_\psi g$, hence $[g]_\psi \in \mathrm{Fix}(\Xi)$. So, the last summation becomes

$$\frac{1}{|G|} \sum_{x \in G} \left|\{g \in G \mid x = g\psi(x)\varphi(g)^{-1}\}\right|.$$

Again, we can invoke Lemma 8.1.1 together with the assumption that $x \sim_\varphi \psi(x)$ for all $x \in G$ to get

$$\begin{aligned}
\frac{1}{|G|} \sum_{x \in G} \left|\{g \in G \mid x = g\psi(x)\varphi(g)^{-1}\}\right| &= \frac{1}{|G|} \sum_{x \in G} \left|\{g \in G \mid x = gx\varphi(g)^{-1}\}\right| \\
&= \frac{1}{|G|} \sum_{x \in G} |\mathrm{Stab}_\varphi(x)| \\
&= \sum_{x \in G} \frac{1}{|[x]_\varphi|} \\
&= R(\varphi),
\end{aligned}$$

which finishes the proof that $R(\varphi) = |\mathrm{Fix}(\Xi)|$.

To argue that φ^k satisfies the conditions of the theorem for all $k \geq 1$, first note that clearly $\varphi \circ \varphi^k = \varphi^k \circ \varphi$. Next, recall that Lemma T.1.3 states that $g \sim_\varphi \varphi(g)$ for all $g \in G$. By induction and transitivity of φ-twisted conjugacy, we obtain that $g \sim_\varphi \varphi^k(g)$ for all $g \in G$. $\square$

The case $\psi = \mathrm{Id}_G$ is the content of [32, Theorem 5]. We state it as a separate result for reference purposes. This is also the result we referred to after Lemma 7.2.1.

Corollary 8.1.3. *Let G be a finite group and $\varphi \in \mathrm{End}(G)$. Then*

$$R(\varphi) = |\{[g] \mid [\varphi(g)] = [g]\}| = |\{[g] \mid \varphi([g]) \subseteq [g]\}|$$

In particular, if G is abelian, then $R(\varphi) = |\mathrm{Fix}(\varphi)|$.

Proof. The first equality is the content of Theorem 8.1.2 for $\psi = \mathrm{Id}$.

For the second, note that for all $g \in G$, $\varphi([g]) \subseteq [\varphi(g)]$. Therefore, $\varphi([g]) \subseteq [g]$ if and only if $[\varphi(g)] = [g]$. $\square$

If G is a group, $g \in G$ and $\varphi \in \mathrm{End}(G)$, we call the conjugacy class $[g]$ *fixed by* φ if $\varphi([g]) \subseteq [g]$.

The second alternative technique is a counting method that uses the irreducible characters of G.

Definition 8.1.4. Let G be a finite group. We define $\mathrm{Map}(G, \mathbb{C})$ to be the set of all (set-theoretical) maps from G to $\mathbb{C}$.

Given two finite groups G and H and a homomorphism $\varphi : G \to H$, we define $\widehat{\varphi} : \mathrm{Map}(H, \mathbb{C}) \to \mathrm{Map}(G, \mathbb{C}) : f \mapsto f \circ \varphi$.

In the following sections and Variations, we consider several subsets of $\mathrm{Map}(G, \mathbb{C})$ and $\mathrm{Map}(H, \mathbb{C})$. By slight abuse of notation, we always write $\widehat{\varphi}$ for the induced map between those subsets.

Definition 8.1.5. Let G be a finite group. We let $\mathrm{Irr}(G)$ denote the set of all complex-valued irreducible characters of G and $\mathcal{C}(G)$ the set of all complex-valued class functions on G, i.e. the set of maps $f : G \to \mathbb{C}$ satisfying $f(g^x) = f(g)$ for all $g, x \in G$.

We recall the inner product on $\mathcal{C}(G)$. Given $f_1, f_2 \in \mathcal{C}(G)$, we put

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

With respect to this inner product, $\mathrm{Irr}(G)$ forms an orthonormal basis of $\mathcal{C}(G)$. In particular,

$$f = \sum_{\chi \in \mathrm{Irr}(G)} \langle f, \chi \rangle \chi$$

for all $f \in \mathcal{C}(G)$.

Lemma 8.1.6. *Let G be a group, $g \in G$ and $\varphi \in \mathrm{End}(G)$. Let $[g]$ denote the (ordinary) conjugacy class of g. Then $\varphi^{-1}([g])$ is a union of conjugacy classes.*

Proof. Put $A := \varphi^{-1}([g])$ and let $x \in A$. It is sufficient to prove that $[x] \subseteq A$. Write $\varphi(x) = g^h$ for some $h \in G$. Then, for each $y \in G$,

$$\varphi(x^y) = \varphi(x)^{\varphi(y)} = g^{h\varphi(y)} \in [g].$$

This implies that $[x] \subseteq A$. $\square$

Theorem 8.1.7. *Let G be a finite group and let $\varphi \in \mathrm{End}(G)$. Then*

$$R(\varphi) = \sum_{\chi \in \mathrm{Irr}(G)} \langle \chi \circ \varphi, \chi \rangle$$

Proof. Consider the map

$$\widehat{\varphi} : \mathcal{C}(G) \to \mathcal{C}(G) : f \mapsto f \circ \varphi.$$

It is readily verified that $\widehat{\varphi}$ maps class functions to class functions and that it is linear. We can consider two bases of $\mathcal{C}(G)$: the set X and the set $\Delta := \{\Delta_{[x]} \mid [x] \in \mathcal{C}\}$ of maps

$$\Delta_{[x]} : G \to \mathbb{C} : g \mapsto \begin{cases} 1 & \text{if } g \in [x] \\ 0 & \text{if } g \notin [x], \end{cases}$$

where $\mathcal{C}$ is the set of conjugacy classes of G. We argue that the matrix representation M of $\widehat{\varphi}$ with respect to Δ is a 0-1-matrix. Let $[x] \in \mathcal{C}$ and $g \in G$ be arbitrary. Then

$$\Delta_{[x]}(\varphi(g)) = \begin{cases} 1 & \text{if } \varphi(g) \in [x] \\ 0 & \text{if } \varphi(g) \notin [x]. \end{cases}$$

In other words, $\Delta_{[x]}$ equals 1 on $\varphi^{-1}([x])$ and 0 elsewhere. By Lemma 8.1.6, $\varphi^{-1}([x])$ is a union of conjugacy classes, say, $[x_1], \ldots, [x_k]$. Then

$$\Delta_{[x]} \circ \varphi = \sum_{i=1}^{k} \Delta_{[x_i]}.$$

Consequently, M is a 0-1-matrix.

Now, there is a 1 on the diagonal of M if and only if the corresponding conjugacy class $[g]$ satisfies $\varphi([g]) \subseteq [g]$, i.e. if and only if $[g]$ is fixed by φ. As $R(\varphi)$ is the number of fixed conjugacy classes by φ by Theorem 8.1.2, $R(\varphi) = \operatorname{Tr}(\widehat{\varphi})$.

Next, we determine the matrix representation of $\widehat{\varphi}$ with respect to $\operatorname{Irr}(G)$. Suppose that $\operatorname{Irr}(G) = \{\chi_1, \ldots, \chi_c\}$. The χ_i-coordinate of $\widehat{\varphi}(\chi_j)$ equals $\langle \chi_j \circ \varphi, \chi_i \rangle$. Therefore, the matrix representation of $\widehat{\varphi}$ with respect to $\operatorname{Irr}(G)$ equals $(\langle \chi_j \circ \varphi, \chi_i \rangle)_{i,j}$. Consequently,

$$R(\varphi) = \operatorname{Tr}(\widehat{\varphi}) = \sum_{\chi \in \operatorname{Irr}(G)} \langle \chi \circ \varphi, \chi \rangle,$$

which proves the theorem. □

Corollary 8.1.8. *Let G be a finite group and $\varphi \in \operatorname{Aut}(G)$. Then the map*

$$\widehat{\varphi} : \operatorname{Irr}(G) \to \operatorname{Irr}(G) : \chi \mapsto \chi \circ \varphi$$

is a well-defined bijection and $R(\varphi) = |\operatorname{Fix}(\widehat{\varphi})|$.

Proof. Suppose that $\chi \in \operatorname{Irr}(G)$. As argued earlier, $\chi \circ \varphi$ is again a class function. To prove that $\chi \circ \varphi$ is an irreducible character as well, we compute $\langle \chi \circ \varphi, \chi \circ \varphi \rangle$:

$$\begin{aligned}
\langle \chi \circ \varphi, \chi \circ \varphi \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi(\varphi(g)) \overline{\chi(\varphi(g))} \\
&= \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi(g)} \\
&= \langle \chi, \chi \rangle \\
&= 1,
\end{aligned}$$

where we used the fact that φ is bijective in the second equality. Thus, for $\chi \in \operatorname{Irr}(G)$, we have

$$\langle \chi \circ \varphi, \chi \rangle = \begin{cases} 1 & \text{if } \chi = \chi \circ \varphi \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the expression

$$R(\varphi) = \sum_{\chi \in \operatorname{Irr}(G)} \langle \chi \circ \varphi, \chi \rangle$$

from Theorem 8.1.7 counts the number of $\chi \in \mathrm{Irr}(G)$ that are fixed by φ, i.e. $R(\varphi) = |\mathrm{Fix}(\widehat{\varphi})|$. $\square$

For an endomorphism φ on a finite abelian group, these results yield a strong connection between $\mathrm{Fix}(\varphi)$ and $\mathrm{Fix}(\widehat{\varphi})$.

Corollary 8.1.9. *Let A be a finite abelian group and let $\varphi \in \mathrm{End}(A)$. Then $|\mathrm{Fix}(\varphi)| = |\mathrm{Fix}(\widehat{\varphi})|$.*

Proof. Let $\varphi \in \mathrm{End}(A)$. By Corollary 8.1.3, $R(\varphi) = |\mathrm{Fix}(\varphi)|$, as A is abelian. By Theorem 8.1.7,

$$R(\varphi) = \sum_{\chi \in \mathrm{Irr}(A)} \langle \chi \circ \varphi, \chi \rangle.$$

Since A is abelian, all irreducible characters of A are 1-dimensional. Hence, $\chi \circ \varphi$ is also 1-dimensional for all $\chi \in \mathrm{Irr}(A)$. This implies that, for all $\chi \in \mathrm{Irr}(A)$,

$$\langle \chi \circ \varphi, \chi \rangle = \begin{cases} 1 & \text{if } \chi = \chi \circ \varphi \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, $R(\varphi) = |\{\chi \in \mathrm{Irr}(A) \mid \chi = \chi \circ \varphi\}| = |\mathrm{Fix}(\widehat{\varphi})|$. We conclude that $|\mathrm{Fix}(\varphi)| = |\mathrm{Fix}(\widehat{\varphi})|$. $\square$

We use the technique of counting fixed irreducible characters in Variation 10 to determine the Reidemeister spectrum of split metacyclic groups.

8.2 Congruences and (in)equalities

Using the alternative counting methods, we can prove various congruences, equalities and inequalities of Reidemeister numbers of endomorphisms on finite groups. We have proven several of those for finitely generated torsion-free nilpotent groups as well, using the product formula.

Proposition 8.2.1. *Let G be a finite group and $\varphi \in \mathrm{End}(G)$. Let $k \geq 0$ be an integer. Then $R(\varphi^k) \geq R(\varphi)$. In particular, if $\varphi \in \mathrm{Aut}(G)$ and k is coprime with the order of φ, then $R(\varphi^k) = R(\varphi)$.*

Proof. If we put $\psi = \varphi^k$ in Theorem 8.1.2, we immediately get

$$R(\varphi) = |\mathrm{Fix}(\Xi)| \leq \left|\mathcal{R}[\varphi^k]\right| = R(\varphi^k).$$

Now, let $\varphi \in \mathrm{Aut}(G)$ and let n be its order. Let $k \geq 0$ be coprime with n. We already have $R(\varphi^k) \geq R(\varphi)$. Since k and n are coprime, there exist $a, b \in \mathbb{Z}$ such that $ak + bn = 1$. We may assume that $a \geq 1$. Then $\varphi = \varphi^{ak+bn} = \varphi^{ak}$. Since $a \geq 1$, we have $R(\varphi^{ak}) \geq R(\varphi^k)$. Consequently, $R(\varphi) = R(\varphi^k)$. $\square$

Lemma 8.2.2. *Let G be a finite group of odd order and let $g \in G$. If $g^x = g^{-1}$ for some $x \in G$, then $g = 1$.*

Proof. Suppose that $g^x = g^{-1}$. Inductively we get that

$$g^{x^m} = g^{(-1)^m}$$

for all $m \geq 1$. Therefore, letting $2k+1$ be the order of x, we find

$$g = g^{x^{2k+1}} = g^{(-1)^{2k+1}} = g^{-1}.$$

We derive from this that $g^2 = 1$. As G has odd order, we deduce that $g = 1$. $\square$

Proposition 8.2.3. *Let G be a finite group of odd order and let $\varphi \in \mathrm{End}(G)$. Then $R(\varphi) \equiv 1 \bmod 2$.*

Proof. We use Corollary 8.1.3. Clearly, $[1] = [\varphi(1)]$, which yields one fixed point. Next, suppose that $[g] = [\varphi(g)]$. Write $g = \varphi(g)^x$ for some $x \in G$. Then

$$g^{-1} = (\varphi(g)^x)^{-1} = \varphi(g^{-1})^x,$$

which proves that $[g^{-1}] = [\varphi(g^{-1})]$ as well. By Lemma 8.2.2, if $g \neq 1$, then $[g] \neq [g^{-1}]$. Hence, if $[g] = [\varphi(g)]$, we automatically have two distinct conjugacy classes that are fixed by φ, namely $[g]$ and $[g^{-1}]$. In other words, the non-trivial conjugacy classes fixed by φ come in pairs, which implies that $R(\varphi)$ is odd. $\square$

Proposition 8.2.4. *Let G be a finite group and $\varphi \in \mathrm{End}(G)$. Let $\psi \in \mathrm{End}(G)$ be a class preserving endomorphism. Then $R(\psi \circ \varphi) = R(\varphi)$.*

Proof. Again, we use Corollary 8.1.3. Let $\Xi_\varphi : \mathcal{R}[\mathrm{Id}_G] \to \mathcal{R}[\mathrm{Id}_G] : [g] \mapsto [\varphi(g)]$ and $\Xi_\psi : \mathcal{R}[\mathrm{Id}_G] \to \mathcal{R}[\mathrm{Id}_G] : [g] \mapsto [\psi(g)]$. Then Ξ_ψ is the identity map and $(\Xi_\psi \circ \Xi_\varphi)([g]) = [(\psi \circ \varphi)(g)]$. So,

$$R(\psi \circ \varphi) = |\mathrm{Fix}(\Xi_\psi \circ \Xi_\varphi)| = |\mathrm{Fix}(\Xi_\varphi)| = R(\varphi). \qquad \square$$

For finite groups, we have an additional lower bound for central extensions.

Proposition 8.2.5. *Let G be a finite group and $C \leq Z(G)$ a central subgroup. Let $\varphi \in \mathrm{End}(G)$ such that $\varphi(C) \leq C$. Let $\varphi|_C$ denote the restricted endomorphism on C. Then $R(\varphi|_C) \leq R(\varphi)$.*

Proof. Since C is central, the C-conjugacy class of $c \in C$ is a singleton and coincides with the G-conjugacy class. Therefore, the conjugacy classes fixed by $\varphi|_C$ are also fixed by φ, which proves that $R(\varphi|_C) \leq R(\varphi)$ by Corollary 8.1.3. $\square$

The next congruences resembles Fermat's Little Theorem.

Proposition 8.2.6. *Let G be a finite group and $\varphi \in \mathrm{End}(G)$. Let p be a prime number. Then $R(\varphi^p) \equiv R(\varphi) \bmod p$.*

Proof. Yet again, we use Theorem 8.1.2, but this time with $\psi = \varphi^p$. Let $\Xi : \mathcal{R}[\varphi^p] \to \mathcal{R}[\varphi^p]$ be the map sending $[g]_{\varphi^p}$ to $[\varphi(g)]_{\varphi^p}$. Then Ξ^p is the identity map, as $g \sim_{\varphi^p} \varphi^p(g)$ for every $g \in G$ by Lemma T.1.3. Therefore, Ξ is a bijection of order either 1 or p.

If Ξ has order 1, it is the identity map, so $R(\varphi) = R(\varphi^p)$, from which the result follows. Thus, suppose that Ξ has order p. If we view Ξ as a permutation, its disjoint cycle decomposition consists of 1- and p-cycles. Let n_{p^i} be the number of p^i-cycles for $i \in \{0, 1\}$. Then $R(\varphi) = |\mathrm{Fix}(\Xi)| = n_1$ and $R(\varphi^p) = |\mathcal{R}[\varphi^p]| = n_1 + pn_p$. Hence, $R(\varphi) \equiv R(\varphi^p) \bmod p$. $\square$

Finally, we recall a well-known congruence for actions of p-groups, which also applies to Reidemeister numbers. We add a proof for the reader's convenience.

Proposition 8.2.7. *Let p be a prime number and let G be a finite p-group acting on a finite set X. Let r be the number of orbits of this action. Then $r \equiv |X| \bmod (p-1)$.*

Proof. Let p^n be the order of G. By the orbit-stabiliser theorem, each orbit has size p^i for some $i \in \{0, \ldots, n\}$ (depending on the orbit). For $i \in \{0, \ldots, n\}$, let n_i denote the number of orbits of size p^i. Then

$$|X| = \sum_{i=0}^{n} n_i p^i \equiv \sum_{i=0}^{n} n_i = r \bmod (p-1). \qquad \square$$

Corollary 8.2.8. *Let p be a prime number and let G be a finite p-group. Let $\varphi \in \mathrm{End}(G)$. Then $R(\varphi) \equiv 1 \bmod (p-1)$.*

Proof. As $R(\varphi)$ equals the number of orbits of the φ-conjugacy action of G on itself, Proposition 8.2.7 implies that $R(\varphi) \equiv |G| \bmod (p-1)$. As $|G| = p^n$ for some non-negative integer n, $R(\varphi) \equiv p^n \equiv 1 \bmod (p-1)$. $\square$

Variation 9

Finite abelian groups

Mozart, Wolfgang Amadeus, *Zwölf Variationen in C über das franzözische Lied 'Ah, vous dirai-je, Maman'*, (KV 265 (300e)), Variation IX (bars 1–8).

As mentioned earlier, we take the first steps in the odyssey of determining the Reidemeister spectra of all finite groups. We begin with the easiest finite groups, namely, the finite abelian ones. However, the odd behaviour of the prime number 2 as seen in Variation 7 is present here as well.

Recall from Lemma T.1.11 that the Reidemeister number of an endomorphism φ on an abelian group is given by $R(\varphi) = |\operatorname{coker}(\varphi - \mathrm{Id})|$. A similar and slightly more general result exists for the Reidemeister number of a self map on a topological space in terms of the induced map on the homology. This formula has in particular been studied on spaces with finite fundamental group (see e.g. [62, Chapter I, §2, Chapter II, §5]).

In addition, for finite abelian groups, the Reidemeister number of φ equals the number of fixed points of φ by Lemma T.1.11. This shows that things have come full circle here, as Reidemeister numbers in Nielsen fixed-point theory arise from trying to find estimates for the number of fixed points of a self-map on a topological space.

Most of the results in this Variation can be found in [107].

9.1 Preliminaries

Lemma 9.1.1. *Let A be a finite abelian group. Then*

$$\mathrm{Spec}_{\mathrm{R}}(A) \subseteq \mathrm{ESpec}_{\mathrm{R}}(A) \subseteq \{d \in \mathbb{N} \mid d \text{ divides } |A|\}.$$

Proof. Let $\varphi \in \mathrm{End}(A)$. By Lemma 7.2.1, $R(\varphi) = |\mathrm{Fix}(\varphi)|$. As $\mathrm{Fix}(\varphi)$ is a subgroup, its order divides the order of A. □

The following is well known; we include a proof for the reader's convenience.

Proposition 9.1.2. *Let $G = G_1 \times \ldots \times G_n$ be a direct product of finite groups such that $\gcd(|G_i|, |G_j|) = 1$ for $i \neq j$. Then for each $i \in \{1, \ldots, n\}$, G_i is fully characteristic in G. In particular,*

$$\mathrm{Aut}(G) = \bigtimes_{i=1}^{n} \mathrm{Aut}(G_i).$$

Proof. Let $\varphi \in \mathrm{End}(G)$, $i \in \{1, \ldots, n\}$ and $1 \neq g_i \in G_i$ be arbitrary. Write $\iota_i : G_i \to G$ for the inclusion, and let m be the order of g_i. Then the order of $(h_1, \ldots, h_n) =: \varphi(\iota_i(g_i))$ divides m as well, where $h_j \in G_j$ for all j. Consequently, $h_j^m = 1$ for all j. As m divides $|G_i|$, it is coprime with $|G_j|$ for $j \neq i$. Hence, $h_j = 1$ for $j \neq i$, implying that $\varphi(\iota_i(g_i)) \in G_i$. Thus, G_i is fully characteristic in G. □

Corollary 9.1.3. *Let $G = G_1 \times \ldots \times G_n$ be a direct product of finite groups such that $\gcd(|G_i|, |G_j|) = 1$ for $i \neq j$. Then*

$$\mathrm{Spec}_{\mathrm{R}}(G) = \prod_{i=1}^{n} \mathrm{Spec}_{\mathrm{R}}(G_i)$$

and

$$\mathrm{ESpec}_{\mathrm{R}}(G) = \prod_{i=1}^{n} \mathrm{ESpec}_{\mathrm{R}}(G_i).$$

Proof. This follows by combining Corollary 3.1.6 and Proposition 9.1.2. □

Since each finite abelian group A admits a unique decomposition of the form

$$A = \bigoplus_{p \in \mathcal{P}} A(p),$$

where $\mathcal{P}$ is the set of all primes and $A(p)$ is the Sylow p-subgroup of A, it is sufficient to determine the (extended) Reidemeister spectrum of finite abelian p-groups to completely determine the (extended) Reidemeister spectrum of finite abelian groups. Determining the extended Reidemeister spectrum is straightforward, as is determining the Reidemeister spectrum for odd prime numbers. For $p = 2$, on the other hand, the situation is much more complicated, both the Reidemeister spectrum itself and the proof.

In this Variation, we write the cyclic group of order n as $\mathbb{Z}/n\mathbb{Z}$ and write abelian groups additively.

Definition 9.1.4. Let n be a positive integer. We define $E(n)$ to be

$$E(n) := \{(e_1, \ldots, e_n) \in \mathbb{Z}^n \mid \forall i \in \{1, \ldots, n-1\} : 1 \leq e_i \leq e_{i+1}\}.$$

Given a prime p and $e \in E(n)$, we define the *abelian p-group of type e* to be the group

$$P_{p,e} := \bigoplus_{i=1}^{n} \mathbb{Z}/p^{e_i}\mathbb{Z}.$$

By the fundamental theorem of finite abelian groups, we know that for each finite abelian p-group P there exists an $n \geq 1$ and $e \in E(n)$ such that $P \cong P_{p,e}$. We say that P is *of type e*. We write $\pi_P : \mathbb{Z}^n \to P$ for the natural projection. If P is clear from the context, we omit the subscript and simply write π. In this Variation, we write elements in $\mathbb{Z}^n$ as column vectors x^T.

A key tool to determine the Reidemeister spectrum is a matrix description of automorphisms of finite abelian p-groups, which was proven by C. Hillar and D. Rhea [58].

Theorem 9.1.5 ([58, Theorems 3.3 & 3.6])**.** *Let P be a finite abelian p-group of type e. Put $A(P) := \{M \in \mathbb{Z}^{n \times n} \mid \forall j \leq i \in \{1, \ldots, n\} : p^{e_i - e_j} \mid M_{ij}\}$ and let $\pi : \mathbb{Z}^n \to P$ be the natural projection. Define $\Psi : A(P) \to \mathrm{End}(P) : M \mapsto \Psi(M)$ where*

$$\Psi(M) : P \to P : \pi(x^\mathsf{T}) \mapsto \Psi(M)(\pi(x^\mathsf{T})) := \pi(Mx^\mathsf{T}).$$

Then $A(P)$ is a ring under the usual matrix operations, Ψ is a well-defined ring morphism and $\mathrm{Aut}(P)$ *is precisely the image of* $\{M \in A(P) \mid M \bmod p \in \mathrm{GL}(n, \mathbb{Z}/p\mathbb{Z})\}$ *under Ψ.*

If $\varphi \in \mathrm{Aut}(P)$ is the image of M under Ψ, we say that φ is *represented by M*.

For any abelian p-group P of type $e \in E(n)$, the quotient group P/pP is an abelian group of exponent p, hence it carries a $\mathbb{Z}/p\mathbb{Z}$-vector space structure. Note that the type of P/pP is then given by the all ones vector of length n.

Lemma 9.1.6. *Let P be a finite abelian p-group of type e. Let $\varphi \in \mathrm{Aut}(P)$ be represented by M. Let $z_i^\intercal \in \mathbb{Z}^n$ be the vector with a 1 on the ith place and zeroes elsewhere and let $\rho : \mathbb{Z}^n \to P/pP$ be the projection. If we view P/pP as a vector space over $\mathbb{Z}/p\mathbb{Z}$, the matrix representation of the induced automorphism on P/pP with respect to the basis $\{\rho(z_1^\intercal), \ldots, \rho(z_n^\intercal)\}$, is the matrix $M \bmod p \in \mathrm{GL}(n, \mathbb{Z}/p\mathbb{Z})$.*

Proof. Let $\bar{\pi} : P \to P/pP$ be the projection and let $\bar{\varphi}$ be the induced automorphism on P/pP. Then $\bar{\pi} \circ \pi : \mathbb{Z}^n \to P/pP$ is the natural projection from $\mathbb{Z}^n$ onto P/pP, therefore, $\rho = \bar{\pi} \circ \pi$. We also have that

$$\bar{\varphi}(\bar{\pi}(\pi(x^\intercal))) = \bar{\pi}(\varphi(\pi(x^\intercal))) = \bar{\pi}(\pi(Mx^\intercal)).$$

Now, this implies that

$$\bar{\varphi}(\rho(z_i^\intercal)) = \rho(Mz_i^\intercal),$$

which shows that $M \bmod p$ is the matrix representation of $\bar{\varphi}$. $\square$

9.2 Extended Reidemeister spectrum and odd primes

For p an odd prime, the computation of the Reidemeister spectrum of a finite abelian p-group of type e is a straightforward application of Lemma 7.2.2 and Corollary 3.1.6. Simultaneously, we determine the extended Reidemeister spectrum of an arbitrary finite abelian group.

Lemma 9.2.1. *Let p be a prime and $n \geq 1$ a natural number. Then*

$$\mathrm{ESpec}_{\mathrm{R}}(\mathbb{Z}/p^n\mathbb{Z}) = \{p^i \mid i \in \{0, \ldots, n\}\}.$$

If p is odd, then also

$$\mathrm{Spec}_{\mathrm{R}}(\mathbb{Z}/p^n\mathbb{Z}) = \{p^i \mid i \in \{0, \ldots, n\}\}.$$

Proof. The $\subseteq$-inclusion follows from Lemma 9.1.1. For the other inclusion, we use Lemma 7.2.2. For $i \in \{0, \ldots, n\}$, the map $\varphi_i : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} : 1 \mapsto p^i + 1$ is an endomorphism of $\mathbb{Z}/p^n\mathbb{Z}$. If p is odd, then $\gcd(p^i + 1, p) = 1$, so φ_i is even an automorphism in that case. By Lemma 7.2.2, $R(\varphi_i) = p^i$. $\square$

For $e \in E(n)$ we put $\Sigma(e) = \sum_{i=1}^n e_i$.

Proposition 9.2.2. *Let p be prime and P a finite abelian p-group of type $e \in E(n)$. Then*

$$\mathrm{ESpec_R}(P) = \{p^i \mid i \in \{0, \dots, \Sigma(e)\}\}.$$

If p is odd, then also

$$\mathrm{Spec_R}(P) = \{p^i \mid i \in \{0, \dots, \Sigma(e)\}\}.$$

Proof. By Lemma 9.1.1, we only have to prove the $\supseteq$-inclusion, and this is essentially Corollary 3.1.6. Let $m \in \{0, \dots, \Sigma(e)\}$ and let $j \in \{1, \dots, n+1\}$ be the (unique) index such that

$$\sum_{l=1}^{j-1} e_l \leq m < \sum_{l=1}^{j} e_l,$$

where we put $e_{n+1} := \infty$ for convenience. By Lemma 9.2.1, there are endomorphisms (automorphisms if p is odd) φ_i of $\mathbb{Z}/p^{e_i}\mathbb{Z}$ such that

$$R(\varphi_i) = \begin{cases} p^{e_i} & \text{if } i \leq j-1, \\ p^{m - \sum_{l=1}^{j-1} e_l} & \text{if } i = j, \\ 1 & \text{if } i > j. \end{cases}$$

Then $\varphi := (\varphi_1, \dots, \varphi_n)$ is an endomorphism (automorphism if p is odd) of P and

$$R(\varphi) = \prod_{i=1}^{n} R(\varphi_i) = p^{\sum_{l=1}^{j-1} e_l + m - \sum_{l=1}^{j-1} e_l} = p^m. \qquad \square$$

Theorem 9.2.3. *Let A be a finite abelian group. Then*

$$\mathrm{ESpec_R}(A) = \{d \in \mathbb{N} \mid d \textit{ divides } |A|\}.$$

Proof. This follows from the combination of Corollary 9.1.3 and Proposition 9.2.2. $\square$

9.3 Fixed points on finite abelian p-groups

In contrast to abelian p-groups for odd p, determining the Reidemeister spectrum of an abelian 2-group is much more involved. However, the behaviour of the Reidemeister numbers fits in a more general phenomenon concerning fixed points of automorphisms, valid for all prime numbers. Let p be a prime number,

$n \geq 1$ and $e \in E(n)$. Let $P := P_{p,e}$ be the finite abelian p-group of type e. For $i \in \mathbb{Z}$ coprime with p, let μ_i denote the automorphism of P given by $\mu_i(x) = ix$. For $\varphi \in \mathrm{Aut}(P)$, we then define

$$\Pi(\varphi) := \prod_{i=1}^{p-1} |\mathrm{Fix}(\mu_i \circ \varphi)|.$$

Finally, we put $\mathrm{Spec}_{\Pi}(P) = \{\Pi(\psi) \mid \psi \in \mathrm{Aut}(P)\}$. The goal is to fully determine $\mathrm{Spec}_{\Pi}(P)$. Note that $\Pi(\varphi)$ is always a power of p, hence $\mathrm{Spec}_{\Pi}(P) \subseteq \{p^i \mid i \in \mathbb{N}\}$. If $p = 2$, then

$$\Pi(\varphi) = |\mathrm{Fix}(\mu_1 \circ \varphi)| = |\mathrm{Fix}(\varphi)| = R(\varphi)$$

by Lemma 7.2.1. Hence, $\mathrm{Spec}_{\Pi}(P) = \mathrm{Spec}_{\mathrm{R}}(P)$ in that case.

Throughout this section, we keep the notation as introduced above, unless stated otherwise.

9.3.1 Lower bound

We start by determining and proving a lower bound for $\mathrm{Spec}_{\Pi}(P)$. To formulate the lower bound, we construct a decomposition of e.

Definition 9.3.1. Given $e \in E(n)$, we construct the *abc-decomposition of* e into three types of blocks in the following way:

Step 1: Each maximal constant subsequence of $e_1, \ldots, e_n$ of length at least 2 forms one block, which we call an *a-block*.

Step 2: Among the remaining numbers, we look for successive numbers e_i and e_{i+1} such that $e_{i+1} = e_i + 1$, starting from the left. Each such pair forms one block, which we call a *b-block*.

Step 3: By Step 1 and Step 2, the remaining e_i are all distinct and differ at least 2 from each other. Each of these numbers forms one block, which we call a *c-block*.

We define $a(e), b(e)$ and $c(e)$ to be the number of a-, b- and c-blocks, respectively, in this decomposition.

For instance, consider $e = (1, 1, 2, 3, 4, 4, 6, 7, 8, 10, 12, 13)$. We go through the steps one by one and mark the blocks in e. There are two a-blocks, namely $(1, 1)$ and $(4, 4)$, hence we get

$$((1, 1), 2, 3, (4, 4), 6, 7, 8, 10, 12, 13).$$

Next, there are three b-blocks, namely $(2,3),(6,7)$ and $(12,13)$, so we get

$$((1,1),(2,3),(4,4),(6,7),8,10,(12,13)).$$

The remaining elements, 8 and 10, each form a single c-block, which yields

$$((1,1),(2,3),(4,4),(6,7),(8),(10),(12,13)).$$

Remark. This construction implies that, if a b-block of the form (e_i, e_i+1) succeeds a c-block (e_{i-1}), then $e_i \geq e_{i-1}+2$, since we form the b-blocks by starting from the left.

We now use this decomposition to formulate the lower bound of $\mathrm{Spec}_\Pi(P)$.

Theorem 9.3.2. *Let $\varphi \in \mathrm{Aut}(P)$. Then $\Pi(\varphi) \geq p^{b(e)+c(e)}$.*

The remainder of this section is devoted to proving this theorem. To do so, we construct a suitable characteristic subgroup of P. This subgroup is of the following form:

Definition 9.3.3. For non-negative integers $d_1,\ldots,d_n$ with $d_i \leq e_i$ for all i, we define $P(d_1,\ldots,d_n)$ to be the subgroup

$$\bigoplus_{i=1}^{n} p^{d_i}\mathbb{Z}/p^{e_i}\mathbb{Z}$$

of P.

Equivalently, if we let $\pi : \mathbb{Z}^n \to P$ be the natural projection, then $P(d_1,\ldots,d_n) = \pi(p^{d_1}\mathbb{Z} \oplus \ldots \oplus p^{d_n}\mathbb{Z})$.

Theorem 9.3.4. *Let $d_1,\ldots,d_n$ be non-negative integers with $d_i \leq e_i$ for all i. Then $Q := P(d_1,\ldots,d_n)$ is characteristic in P if and only if the following two conditions hold:*

(1) for all $i \in \{1,\ldots,n-1\}$ we have $d_i \leq d_{i+1}$;

(2) for all $i \in \{1,\ldots,n-1\}$ we have $e_i - d_i \leq e_{i+1} - d_{i+1}$.

Moreover, if Q is characteristic, $d_i < e_i$ for all $i \in \{1,\ldots,n\}$ and $\varphi \in \mathrm{Aut}(P)$ is represented by the matrix M as in Theorem 9.1.5, then the induced automorphism on Q is represented by the matrix $D^{-1}MD$, where $D := \mathrm{Diag}(p^{d_1},\ldots,p^{d_n})$.

Proof. For the first part, we use [69, Theorem 2.2]. There it is proven that the conditions on $d_1,\ldots,d_n$ are equivalent with the subgroup $P(e_1-d_1,\ldots,e_n-d_n)$

being characteristic. However, if the n-tuple $d := (d_1, \ldots, d_n)$ satisfies the two conditions, then so does the n-tuple $d' := (e_1 - d_1, \ldots, e_n - d_n)$, and vice versa. Indeed, the second condition for d implies the first one for d', and by symmetry, the first for d implies the second for d'. Moreover, since $0 \leq d_i \leq e_i$ for all i, also $0 \leq e_i - d_i \leq e_i$ for all i. This proves the first part.

Suppose now that Q is characteristic in P and that $d_i < e_i$ for all $i \in \{1, \ldots, n\}$. Fix $\varphi \in \mathrm{Aut}(P)$ and suppose that it is represented by M. In order to use Theorem 9.1.5 to talk about the matrix representation of automorphisms of Q, we have to write Q as a direct sum of cyclic groups of prime-power order. It is readily verified that

$$\Phi : \bigoplus_{i=1}^{n} \mathbb{Z}/p^{e_i - d_i}\mathbb{Z} \to \bigoplus_{i=1}^{n} p^{d_i}\mathbb{Z}/p^{e_i}\mathbb{Z} : (x_1, \ldots, x_n) \mapsto (p^{d_1}x_1, \ldots, p^{d_n}x_n)$$

is an isomorphism, which implies that Q is an abelian p-group of type $(e_1 - d_1, \ldots, e_n - d_n)$. Let $\tilde{Q}$ denote the group on the left-hand side. Write $\pi_P : \mathbb{Z}^n \to P$ and $\pi_{\tilde{Q}} : \mathbb{Z}^n \to \tilde{Q}$ for the natural projections onto P and $\tilde{Q}$, respectively. Then $\varphi(\pi_P(x^\mathsf{T})) = \pi_P(Mx^\mathsf{T})$ for all $x^\mathsf{T} \in \mathbb{Z}^n$. Let $\varphi|_Q$ denote the induced automorphism on Q, put $\psi := \Phi^{-1} \circ \varphi|_Q \circ \Phi$ and suppose that $x^\mathsf{T} \in \mathbb{Z}^n$ is such that $\pi_P(x^\mathsf{T}) \in Q$. Then $x = (p^{d_1}y_1, \ldots, p^{d_n}y_n)$ for some $y_1, \ldots, y_n \in \mathbb{Z}$. Put $y = (y_1, \ldots, y_n)$. Then $x^\mathsf{T} = Dy^\mathsf{T}$ and therefore, $\pi_{\tilde{Q}}(y^\mathsf{T}) = \Phi^{-1}(\pi_P(x^\mathsf{T}))$. Thus,

$$\begin{aligned}
\psi(\pi_{\tilde{Q}}(y^\mathsf{T})) &= (\Phi^{-1} \circ \varphi|_Q \circ \Phi \circ \Phi^{-1})(\pi_P(x^\mathsf{T})) \\
&= \Phi^{-1}(\varphi|_Q(\pi_P(x^\mathsf{T}))) \\
&= \Phi^{-1}(\varphi(\pi_P(x^\mathsf{T}))) \\
&= \Phi^{-1}(\pi_P(Mx^\mathsf{T})) \\
&= \Phi^{-1}(\pi_P(MDy^\mathsf{T})).
\end{aligned}$$

Note that we can rewrite the equality $\pi_{\tilde{Q}}(y^\mathsf{T}) = \Phi^{-1}(\pi_P(x^\mathsf{T}))$ as

$$\pi_{\tilde{Q}}(y^\mathsf{T}) = \Phi^{-1}(\pi_P(Dy^\mathsf{T})),$$

which holds for arbitrary $y^\mathsf{T} \in \mathbb{Z}^n$. Since we know that $\pi_P(MDy^\mathsf{T}) \in Q$, we know that $D^{-1}MDy^\mathsf{T}$ is a well-defined element of $\mathbb{Z}^n$. Thus, using the equalities above, we get

$$\begin{aligned}\psi(\pi_{\tilde{Q}}(y^{\mathsf{T}})) &= \Phi^{-1}(\pi_P(MDy^{\mathsf{T}}))\\ &= \Phi^{-1}(\pi_P(DD^{-1}MDy^{\mathsf{T}}))\\ &= \pi_{\tilde{Q}}(D^{-1}MDy^{\mathsf{T}}).\end{aligned}$$

We conclude that the matrix representation of ψ is given by $D^{-1}MD$, which finishes the proof. $\square$

We now construct the aforementioned suitable characteristic subgroup by specifying the non-negative integers d_i based on e.

Definition 9.3.5. Given $e \in E(n)$ and its abc-decomposition as in Definition 9.3.1, we define a new n-tuple $d := (d_1, \dots, d_n)$ recursively. Put $d_1 = 0$. Given d_i, we define

$$d_{i+1} = \begin{cases} d_i & \text{if } e_i \text{ and } e_{i+1} \text{ lie in the same block or } e_{i+1} \text{ lies in an } a\text{-block} \\ d_i + 1 & \text{if } e_i \text{ and } e_{i+1} \text{ do not lie in the same block and } e_{i+1} \text{ lies in a } b\text{- or } c\text{-block.} \end{cases}$$

We let $d(e)$ denote this sequence.

For example, given $e = ((1,1),(2,3),(4,4),(6,7),(8),(10),(12,13))$ as before with its abc-decomposition marked, we find that

$$d(e) = (0,0,1,1,1,1,2,2,3,4,5,5).$$

Lemma 9.3.6. *Given $e \in E(n)$, its associated n-tuple $d(e)$ has the following properties:*

(1) for all $i, j \in \{1, \dots, n\}$ with $i < j$, we have $d_i \leq d_j$ with strict inequality if e_j is the first element of a b- or c-block.

(2) for all $i, j \in \{1, \dots, n\}$ with $i < j$, we have $d_j - d_i \leq e_j - e_i$, with strict inequality if e_i is the first element of a b- or c-block.

(3) for all $i \in \{1, \dots, n\}$, we have $d_i < e_i$.

Proof. The sequence $d(e)$ is non-decreasing by construction, which proves the inequality in the first item. For the strictness part, note that it follows by construction if $i = j - 1$, and the general case follows from the chain $d_i \leq d_{j-1} < d_j$.

For the second item, we first prove it for $j = i + 1$. By definition, we have

$$d_{i+1} - d_i = \begin{cases} 0 & \text{if } e_i \text{ and } e_{i+1} \text{ lie in the same block or } e_{i+1} \text{ lies in an } a\text{-block} \\ 1 & \text{if } e_i \text{ and } e_{i+1} \text{ do not lie in the same block and } e_{i+1} \text{ lies in a} \\ & b\text{- or } c\text{-block.} \end{cases}$$

We now consider $e_{i+1} - e_i$. We distinguish several cases, based on the type of blocks in which e_{i+1} and e_i lie.

- e_i and e_{i+1} lie in the same a-block: then $e_{i+1} - e_i = 0$, by definition of an a-block.

  Since $d_{i+1} - d_i = 0$, we have $d_{i+1} - d_i \leq e_{i+1} - e_i$.

- e_i and e_{i+1} lie in the same b-block: then $e_{i+1} - e_i = 1$, by definition of a b-block.

  Since $d_{i+1} - d_i = 0$, we have $d_{i+1} - d_i < e_{i+1} - e_i$.

- e_i lies in an a- or b-block, e_{i+1} does not lie in the same block: then $e_{i+1} - e_i \geq 1$, for otherwise e_{i+1} and e_i would be part of an a-block.

  Since $d_{i+1} - d_i \leq 1$, we have $d_{i+1} - d_i \leq e_{i+1} - e_i$.

- e_i lies in a c-block, e_{i+1} lies in an a-block: then $e_{i+1} - e_i \geq 1$ for the same reason as in the previous case.

  Since $d_{i+1} - d_i = 0$, we have $d_{i+1} - d_i < e_{i+1} - e_i$.

- e_i lies in a c-block, e_{i+1} lies in a b-block: then $e_{i+1} - e_i \geq 2$ by the remark following Definition 9.3.1.

  Since $d_{i+1} - d_i = 1$, we have $d_{i+1} - d_i < e_{i+1} - e_i$.

- e_i lies in a c-block, e_{i+1} lies in a c-block: then $e_{i+1} - e_i \geq 2$, for otherwise e_{i+1} and e_i would be part of an a-block or one or more b-blocks.

  Since $d_{i+1} - d_i = 1$, we have $d_{i+1} - d_i < e_{i+1} - e_i$.

We see that in all cases the inequality $d_{i+1} - d_i \leq e_{i+1} - e_i$ holds. Moreover, in the cases where e_i is the first element of a b- or c-block, we have proven that in fact the strict inequality holds. This finishes the proof for $j = i + 1$.

We prove the general case by induction on $j - i$, with base case $j - i = 1$. Suppose it holds for all $i < j$ with $j - i < k$. Suppose that $j - i = k$. Note that

$$e_j - e_i - d_j + d_i = (e_j - e_{j-1} - d_j + d_{j-1}) + (e_{j-1} - e_i + d_i - d_{j-1}).$$

Both terms on the right-hand side are non-negative by the induction hypothesis, hence the left-hand side is non-negative as well. Moreover, if e_i is the first element of a b- or c-block, then $e_{j-1} - e_i + d_i - d_{j-1} > 0$, which implies that also $e_j - e_i - d_j + d_i > 0$.

Finally, for the third item, we again proceed by induction. For $i = 1$, we have $d_1 = 0 < 1 \leq e_1$. So, suppose $d_i < e_i$. Then by the second item, we know that $d_{i+1} - d_i \leq e_{i+1} - e_i$. Adding the inequality $d_i < e_i$ side by side yields $d_{i+1} < e_{i+1}$. □

Corollary 9.3.7. *The subgroup $P(d_1, \ldots, d_n)$ is a characteristic subgroup of P.*

Proof. By the previous lemma, $d(e)$ satisfies all the conditions from Theorem 9.3.4. □

We use the subgroup $Q := P(d_1, \ldots, d_n)$ to prove the lower bound on the number of fixed points.

Lemma 9.3.8. *Let $\varphi \in \mathrm{Aut}(P)$ be represented by a matrix $M \in \mathbb{Z}^{n\times n}$. Put $D = Diag(p^{d_1}, \ldots, p^{d_n})$ and let $i \neq j \in \{1, \ldots, n\}$. Then the following hold:*

(1) $(D^{-1}MD)_{ij} \equiv 0 \bmod p$ if e_j is the first element of a b- or c-block.

(2) $(D^{-1}MD)_{jj} \not\equiv 0 \bmod p$ if e_j is the first element of a b- or c-block.

Proof. For $a \in \mathbb{Z}$, let $\nu_p(a)$ denote the p-adic valuation of a. First, remark that $(D^{-1}MD)_{ij} = D_{ii}^{-1}M_{ij}D_{jj}$, as D is diagonal. Next, by the properties of M and the definition of D, we have that

$$\nu_p((D^{-1}MD)_{ij}) = \nu_p(D_{ii}^{-1}M_{ij}D_{jj}) \geq \begin{cases} e_i - e_j + d_j - d_i & \text{if } i > j \\ d_j - d_i & \text{if } i < j. \end{cases}$$

Suppose that e_j is the first element of a b- or c-block. Then by Lemma 9.3.6, each of the expressions above is at least 1. Therefore, $(D^{-1}MD)_{ij} \equiv 0 \bmod p$.

For $(D^{-1}MD)_{jj}$, note that $D^{-1}MD$ is the matrix representation of $\varphi|_Q$, the restriction of φ to Q, by Theorem 9.3.4. Moreover, it has to be invertible modulo p in order to define an automorphism on Q/pQ. Since the jth column of $D^{-1}MD$ is zero modulo p everywhere above and below the diagonal entry, the entry on the diagonal must be non-zero modulo p. □

Finally, we prove Theorem 9.3.2. For a matrix $A \in \mathbb{Z}^{n\times n}$, we write $\bar{A}$ for the matrix $A \bmod p \in (\mathbb{Z}/p\mathbb{Z})^{n\times n}$.

Proof of Theorem 9.3.2. Let $\varphi \in \mathrm{Aut}(P)$ be represented by $M \in \mathbb{Z}^{n\times n}$ and let $d_1, \dots, d_n, Q, \varphi|_Q$ and D be as before. Since $d_i < e_i$, the group Q has type $\tilde{e} \in E(n)$. The matrix representation of $\varphi|_Q$ is given by $N := D^{-1}MD$, by Theorem 9.3.4. Let $\bar{\varphi}$ denote the induced automorphism on the exponent-p factor group Q/pQ. By Lemma 9.1.6, the matrix representation of $\bar{\varphi}$ with respect to the basis $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ is given by $\bar{N}$. By Lemma 9.3.8, each column corresponding to a c-block and to a first element of a b-block in N is zero modulo p, except for the element on the diagonal.

Next, remark that for $i \in \{1, \dots, p-1\}$, the automorphism μ_i is represented by the matrix $X_i := \mathrm{Diag}(i, \dots, i)$. The automorphism $\mu_i \circ \varphi$ is then represented by X_iM, and the one of the induced automorphism $\bar{\mu}_i \circ \bar{\varphi}$ on Q/pQ by $\bar{X}_i\bar{N}$. Fix $j \in \{1, \dots, n\}$ such that e_j is the first element of a b- or c-block. Then $N_{jj} \not\equiv 0 \bmod p$ by Lemma 9.3.8, hence there is a unique $i \in \{1, \dots, p-1\}$ such that $iN_{jj} \equiv 1 \bmod p$. For that i, we have that the jth column of $\bar{X}_i\bar{N} - \bar{I}_n$ is zero.

Now, let $\mathcal{J}$ be the set of indices j such that e_j is the first element of a b- or c-block. For $i \in \{1, \dots, p-1\}$, let $\mathcal{J}_i = \{j \in \mathcal{J} \mid iN_{jj} \equiv 1 \bmod p\}$. Note that $\mathcal{J}$ is the disjoint union of $\mathcal{J}_1$ up to $\mathcal{J}_{p-1}$, and that $|\mathcal{J}| = b(e) + c(e)$. Then by the arguments above, $\bar{\mu}_i \circ \bar{\varphi}$ has at least $p^{|\mathcal{J}_i|}$ fixed points. Indeed, for each $j \in \mathcal{J}_i$, the jth column of $\bar{X}_i\bar{N} - \bar{I}_n$ is zero, hence $\ker(\bar{X}_i\bar{N} - \bar{I}_n)$ has at least dimension $|\mathcal{J}_i|$. By Lemma 7.2.1, we know that $R(\bar{\mu}_i \circ \bar{\varphi}) = |\mathrm{Fix}(\bar{\mu}_i \circ \bar{\varphi})|$ and $R(\mu_i \circ \varphi) = |\mathrm{Fix}(\mu_i \circ \varphi)|$. By Corollary 1.1.2(1) and Proposition 8.2.5, respectively, we know that

$$R(\bar{\mu}_i \circ \bar{\varphi}) \leq R((\mu_i \circ \varphi)|_Q) \leq R(\mu_i \circ \varphi).$$

Combining these inequalities, we conclude that

$$\prod_{i=1}^{p-1} |\mathrm{Fix}(\mu_i \circ \varphi)| \geq \prod_{i=1}^{p-1} p^{|\mathcal{J}_i|} = p^{\sum\limits_{i=1}^{p-1} |\mathcal{J}_i|} = p^{|\mathcal{J}|} = p^{b(e)+c(e)}. \qquad \square$$

With essentially the same proof, we can prove the following slightly more general version of Theorem 9.3.2:

Theorem 9.3.9. *Suppose that $S \subseteq \mathbb{Z}$ satisfies the following properties:*

(1) For each $i \in \{1, \dots, p-1\}$, there exists a unique $s \in S$ such that $i \equiv s \bmod p$.

(2) For each $s \in S$, $s \not\equiv 0 \bmod p$.

Then, for each $\varphi \in \mathrm{Aut}(P)$,

$$\prod_{s\in S} |\mathrm{Fix}(\mu_s \circ \varphi)| \geq p^{b(e)+c(e)}.$$

For example, if $p = 3$, then this generalisation applied to $S = \{1, -1\}$ states that $R(\varphi)R(-\varphi) \geq 3^{b(e)+c(e)}$ for all $\varphi \in \mathrm{Aut}(P)$.

9.3.2 Upper bound

The next result provides an upper bound for $\mathrm{Spec}_{\Pi}(P)$.

Proposition 9.3.10. *Let* $\varphi \in \mathrm{Aut}(P)$. *Then* $\Pi(\varphi) \leq p^{\Sigma(e)}$.

Proof. Fix $\varphi \in \mathrm{Aut}(P)$. We first prove that

$$\mathrm{Fix}(\mu_i \circ \varphi) \cap \langle \mathrm{Fix}(\mu_j \circ \varphi) \mid j \neq i\rangle$$

is trivial for all $i \in \{1, \dots, p-1\}$. We proceed by induction, namely by proving that, for all $k \in \{1, \dots, p-2\}$ and all $\mathcal{J} \subseteq (\{1, \dots, p-1\} \setminus \{i\})$ with $|\mathcal{J}| = k$, the intersection

$$\mathrm{Fix}(\mu_i \circ \varphi) \cap \langle \mathrm{Fix}(\mu_j \circ \varphi) \mid j \in \mathcal{J}\rangle$$

is trivial.

We start with $k = 1$, that is, with $\mathcal{J} = \{j\}$ with $j \neq i$. An element x in the intersection then satisfies $x = i\varphi(x) = j\varphi(x)$, or equivalently, $(i-j)\varphi(x) = 0$. As $i \neq j$ and both lie in $\{1, \dots, p-1\}$, we know that $i - j$ is invertible modulo p, hence $\varphi(x) = 0$. Since $x = i\varphi(x)$, we conclude that $x = 0$. This proves the claim for $k = 1$.

Now, suppose that it holds for all $\mathcal{J}$ of size k or less. Let $\mathcal{J}$ be a set of size $k+1$ not containing i and let x be an element in the intersection $\mathrm{Fix}(\mu_i \circ \varphi) \cap \langle \mathrm{Fix}(\mu_j \circ \varphi) \mid j \in \mathcal{J}\rangle$. Write $x = \sum_{j\in\mathcal{J}} x_j$, with $x_j \in \mathrm{Fix}(\mu_j \circ \varphi)$. On the one hand, we have

$$x = i\varphi(x) = i\sum_{j\in\mathcal{J}} \varphi(x_j) = \sum_{j\in\mathcal{J}} i\varphi(x_j),$$

while on the other hand, we have

$$x = \sum_{j\in\mathcal{J}} x_j = \sum_{j\in\mathcal{J}} j\varphi(x_j).$$

Therefore,

$$0 = \sum_{j\in\mathcal{J}} (j - i)\varphi(x_j).$$

Now, fix $j_0 \in \mathcal{J}$ and put $\mathcal{J}' := \mathcal{J} \setminus \{j_0\}$. We can rewrite the equality above to

$$(j_0 - i)\varphi(x_{j_0}) = \sum_{j \in \mathcal{J}'} -(j-i)\varphi(x_j).$$

Since φ is an automorphism, we can apply φ^{-1} to get

$$(j_0 - i)x_{j_0} = \sum_{j \in \mathcal{J}'} -(j-i)x_j.$$

The left-hand side lies in $\mathrm{Fix}(\mu_{j_0} \circ \varphi)$, the right-hand side is an element of $\langle \mathrm{Fix}(\mu_j \circ \varphi) \mid j \in \mathcal{J}' \rangle$. Applying the induction hypothesis to j_0 and $\mathcal{J}'$, we find that both sides are trivial, that is, $(j_0 - i)x_{j_0} = 0 = \sum_{j \in \mathcal{J}'} -(j-i)x_j$. As $i \neq j_0$, this implies $x_{j_0} = 0$. Continuing in this fashion yields $x_j = 0$ for all $j \in \mathcal{J}$, which finishes the induction. The original claim then follows from the case where $\mathcal{J} = \{1, \ldots, p-1\} \setminus \{i\}$.

From the above and Lemma 3.3.3, it follows that

$$\langle \mathrm{Fix}(\mu_i \circ \varphi) \mid i \in \{1, \ldots, p-1\} \rangle$$

is isomorphic to the direct product $\mathrm{Fix}(\mu_1 \circ \varphi) \times \ldots \times \mathrm{Fix}(\mu_{p-1} \circ \varphi)$. Consequently,

$$p^{\Sigma(e)} = |P| \geq |\langle \mathrm{Fix}(\mu_i \circ \varphi) \mid i \in \{1, \ldots, p-1\} \rangle| = \prod_{i=1}^{p-1} |\mathrm{Fix}(\mu_i \circ \varphi)| = \Pi(\varphi),$$

which proves the upper bound. □

9.3.3 Filling in the gaps

We now completely determine $\mathrm{Spec}_{\Pi}(P)$.

Theorem 9.3.11. *Let P be a finite abelian p-group of type e. Then*

$$\mathrm{Spec}_{\Pi}(P) = \{p^m \mid m \in \{b(e) + c(e), \ldots, \Sigma(e)\}\}.$$

In order to prove this theorem, we first prove it for several special cases.

Proposition 9.3.12. *Let $n \geq 1$ be a natural number. Then* $\mathrm{Spec}_{\Pi}(\mathbb{Z}/p^n\mathbb{Z}) = \{p^i \mid i \in \{1, \ldots, n\}\}$.

Proof. The $\subseteq$-inclusion follows from Theorem 9.3.2 and Proposition 9.3.10. Conversely, let $m \in \{1, \ldots, n\}$ be arbitrary. Define $\varphi_m : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$:

$1 \mapsto p^m + 1$. Since $m \geq 1$, we know that $\gcd(p^m + 1, p) = 1$. Therefore, φ_m defines an automorphism of $\mathbb{Z}/p^n\mathbb{Z}$. Moreover, for $i \in \{1, \ldots, p-1\}$, we know by Lemmata 7.2.1 and 7.2.2 that

$$|\mathrm{Fix}(\mu_i \circ \varphi_m)| = R(\mu_i \circ \varphi_m) = \gcd(i(p^m + 1) - 1, p^n) = \begin{cases} p^m & \text{if } i = 1 \\ 1 & \text{otherwise,} \end{cases}$$

as $i(p^m + 1) - 1 \equiv i - 1 \not\equiv 0 \bmod p$ when $i \not\equiv 1 \bmod p$. Therefore, $\Pi(\varphi_m) = p^m$. □

Lemma 9.3.13. *Let $P_1, \ldots, P_n$ be abelian p-groups and put $P := P_1 \oplus \ldots \oplus P_n$. For $i \in \{1, \ldots, n\}$, let $\varphi_i \in \mathrm{Aut}(P_i)$. Put $\varphi := (\varphi_1, \ldots, \varphi_n) \in \mathrm{Aut}(P)$. Then $\Pi(\varphi) = \prod_{i=1}^{n} \Pi(\varphi_i)$.*

Consequently, $\prod_{i=1}^{n} \mathrm{Spec}_\Pi(P_i) \subseteq \mathrm{Spec}_\Pi(P)$.

Proof. For $i \in \{1, \ldots, p-1\}$ and $j \in \{1, \ldots, n\}$, let μ_i be multiplication by i on P and let $\mu_{i,j}$ denote its restriction to P_j. We then have that

$$\begin{aligned} \Pi(\varphi) &= \prod_{i=1}^{p-1} |\mathrm{Fix}(\mu_i \circ \varphi)| \\ &= \prod_{i=1}^{p-1} \left|\mathrm{Fix}\left((\mu_{i,1} \circ \varphi_1, \ldots, \mu_{i,n} \circ \varphi_n)\right)\right| \\ &= \prod_{i=1}^{p-1} \prod_{j=1}^{n} |\mathrm{Fix}(\mu_{i,j} \circ \varphi_j)| \\ &= \prod_{j=1}^{n} \prod_{i=1}^{p-1} |\mathrm{Fix}(\mu_{i,j} \circ \varphi_j)| \\ &= \prod_{j=1}^{n} \Pi(\varphi_j). \end{aligned}$$

□

Proposition 9.3.14. *Let $n \geq 1$ be a natural number and put $P = \mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^{n+1}\mathbb{Z}$. Then $\mathrm{Spec}_\Pi(P) = \{p^i \mid i \in \{1, \ldots, 2n+1\}\}$.*

Proof. Again, the $\subseteq$-inclusion follows from Theorem 9.3.2 and Proposition 9.3.10. Conversely, let $m \in \{1, \ldots, 2n+1\}$. For $m \geq 2$, we can find an automorphism

φ with $\Pi(\varphi) = p^m$ using Proposition 9.3.12 and Lemma 9.3.13. Thus, suppose that $m = 1$. Consider the matrix

$$M = \begin{pmatrix} 1 & 1 \\ p & 1 \end{pmatrix}.$$

By Theorem 9.1.5, M defines an automorphism φ on P. First, we determine the fixed points of φ. If $\varphi(\pi((x,y)^\mathsf{T})) = \pi((x,y)^\mathsf{T})$, then

$$\begin{cases} x + y \equiv x \bmod p^n \\ px + y \equiv y \bmod p^{n+1}. \end{cases}$$

This implies that $y \equiv 0 \bmod p^n$ as well as $x \equiv 0 \bmod p^n$. Therefore, the fixed points of φ lie in the subgroup $\left\langle \pi((0,p^n)^\mathsf{T}) \right\rangle$ and it is easily verified that $\varphi(\pi((0,p^n)^\mathsf{T})) = \pi((0,p^n)^\mathsf{T})$. Consequently, $|\mathrm{Fix}(\varphi)| = p$.

Now, let $i \in \{2,\dots,p-1\}$ and consider $\mu_i \circ \varphi$. If $(\mu_i \circ \varphi)(\pi((x,y)^\mathsf{T})) = \pi((x,y)^\mathsf{T})$, then

$$\begin{cases} ix + iy \equiv x \bmod p^n \\ ipx + iy \equiv y \bmod p^{n+1}. \end{cases}$$

The second congruence yields $(i-1)y \equiv -ipx \bmod p^{n+1}$. Since $i \in \{2,\dots,p-1\}$, the number $i-1$ has an inverse modulo p^{n+1}, say, j. Substituting $-jipx$ in the first congruence then yields

$$x(i - i^2jp - 1) \equiv 0 \bmod p^n.$$

Since $i - i^2jp - 1 \equiv i - 1 \not\equiv 0 \bmod p$, it is invertible modulo p^n. Therefore, $x \equiv 0 \bmod p^n$. Combined with $(i-1)y \equiv -ipx \bmod p^{n+1}$, this yields $y \equiv 0 \bmod p^{n+1}$. Consequently, $|\mathrm{Fix}(\mu_i \circ \varphi)| = 1$. We conclude that $\Pi(\varphi) = p$. □

Lemma 9.3.15. *Let n, k be integers with $n \geq 2$, $k \geq 1$. Put $P = \bigoplus_{i=1}^n \mathbb{Z}/p^k\mathbb{Z}$. Let $\varphi \in \mathrm{Aut}(P)$ and let $\bar{\varphi}$ denote the induced automorphism on P/pP. If $\bar{\varphi}$ has no non-trivial fixed points, then neither does φ.*

Proof. We proceed by contraposition. Let φ be represented by M and let $\pi : \mathbb{Z}^n \to P$ be the natural projection. Suppose that $Mx^\mathsf{T} \equiv x^\mathsf{T} \bmod p^k$ for some $x^\mathsf{T} \in \mathbb{Z}^n$ with $\pi(x^\mathsf{T}) \neq 0$. Here, $Mx^\mathsf{T} \equiv x^\mathsf{T} \bmod p^k$ means that $(Mx^\mathsf{T})_i \equiv x_i^\mathsf{T} \bmod p^k$ for each $i \in \{1,\dots,n\}$. Write $x = p^l y$ with $y \in \mathbb{Z}^n$ and l maximal. Then $l < k$, otherwise $\pi(x^\mathsf{T}) = 0$. In particular, $\pi(y^\mathsf{T}) \neq 0$.

Since $Mx^\mathsf{T} \equiv x^\mathsf{T} \bmod p^k$, we find $p^l M y^\mathsf{T} \equiv p^l y^\mathsf{T} \bmod p^k$. Dividing by p^l yields $My^\mathsf{T} \equiv y^\mathsf{T} \bmod p^{k-l}$. As $l < k$, we have that $k - l \geq 1$. In particular, $My^\mathsf{T} \equiv y^\mathsf{T} \bmod p$. Thus, if $\rho : P \to P/pP$ is the canonical projection, it follows that $\rho(\pi(y^\mathsf{T}))$ is a non-trivial fixed point of $\bar{\varphi}$. □

Proposition 9.3.16. *Let n, k be integers with $n \geq 2$, $k \geq 1$. Put $P = \bigoplus_{i=1}^{n} \mathbb{Z}/p^k\mathbb{Z}$. Then* $\mathrm{Spec}_\Pi(P) = \{p^i \mid i \in \{0, \ldots, nk\}\}$.

Proof. Yet again, the $\subseteq$-inclusion follows from Theorem 9.3.2 and Proposition 9.3.10. Conversely, fix $m \in \{0, \ldots, nk\}$. For $m \geq n$, we can find an automorphism φ with $\Pi(\varphi) = p^m$ using Proposition 9.3.12 and Lemma 9.3.13. Thus, suppose that $m \leq n - 1$.

We start with $m = 0$. Using a primitive element of the finite field of p^n elements, we can find a polynomial f_n of degree n that is irreducible over $\mathbb{Z}/p\mathbb{Z}$. Its companion matrix C_{f_n} (seen as matrix over $\mathbb{Z}$) is invertible modulo p. Consequently, it induces, by Theorem 9.1.5, an automorphism φ_{f_n} of P. Since f_n has no roots in $\mathbb{Z}/p\mathbb{Z}$ (recall that $n \geq 2$), the matrix C_{f_n} has no eigenvalues in $\mathbb{Z}/p\mathbb{Z}$. Therefore, $iC_{f_n} \bmod p$ does not have eigenvalue 1 for $i \in \{1, \ldots, p-1\}$. Thus, Lemma 9.3.15 implies that $\mu_i \circ \varphi_{f_n}$ has no non-trivial fixed points for each $i \in \{1, \ldots, p-1\}$. Consequently, $\Pi(\varphi_{f_n}) = 1$.

Now, we proceed for general n. First, let $n = 2$. We already know that $\{1, p^2, p^3, \ldots, p^{2k}\} \subseteq \mathrm{Spec}_\Pi(P)$. Thus, we have to find an automorphism ψ such that $\Pi(\psi) = p$. An argument similar to the one for Proposition 9.3.14 shows that the automorphism ψ induced by the matrix

$$M = \begin{pmatrix} 1 & 1 \\ p & 1 \end{pmatrix}$$

does the job. Consequently, $\mathrm{Spec}_\Pi(P) = \{p^i \mid i \in \{0, \ldots, 2k\}\}$ for $n = 2$.

So, let $n \geq 3$ be arbitrary. If n is even, write

$$P = \bigoplus_{i=1}^{\frac{n}{2}} \left(\mathbb{Z}/p^k\mathbb{Z}\right)^2.$$

Then the result for $n = 2$ combined with Lemma 9.3.13 implies that

$$\begin{aligned} \mathrm{Spec}_\Pi\left(\left(\mathbb{Z}/p^k\mathbb{Z}\right)^2\right)^{\left(\frac{n}{2}\right)} &= \{p^i \mid i \in \{0, \ldots, 2k\}\}^{\left(\frac{n}{2}\right)} \\ &= \{p^i \mid i \in \{0, \ldots nk\}\} \\ &\subseteq \mathrm{Spec}_\Pi(P), \end{aligned}$$

which proves the result for n even. Next, suppose that n is odd. We know that $1 \in \mathrm{Spec}_\Pi(P)$ by the case $m = 0$ above. Write

$$P = \mathbb{Z}/p^k\mathbb{Z} \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} \left(\mathbb{Z}/p^k\mathbb{Z}\right)^2.$$

Then the result for $n = 2$ combined with Proposition 9.3.12 and Lemma 9.3.13 yields

$$\begin{aligned}\mathrm{Spec}_\Pi(\mathbb{Z}/p^k\mathbb{Z}) \cdot \mathrm{Spec}_\Pi\Big(\big(\mathbb{Z}/p^k\mathbb{Z}\big)^2\Big)^{\left(\frac{n-1}{2}\right)} &= \{p^i \mid i \in \{1, \dots, k\}\} \\ &\quad \cdot \{p^i \mid i \in \{0, \dots, 2k\}\}^{\left(\frac{n-1}{2}\right)} \\ &= \{p^i \mid i \in \{1, \dots, nk\}\} \\ &\subseteq \mathrm{Spec}_\Pi(P),\end{aligned}$$

which proves the result for n odd. This finishes the proof. □

Finally, we can completely determine $\mathrm{Spec}_\Pi(P)$ for arbitrary finite abelian p-groups.

Proof of Theorem 9.3.11. We factorise P using the abc-decomposition of e, i.e. we write

$$P = \left(\bigoplus_{i=1}^{a(e)} (\mathbb{Z}/p^{a_i}\mathbb{Z})^{n_i}\right) \oplus \left(\bigoplus_{i=1}^{b(e)} \big(\mathbb{Z}/p^{b_i}\mathbb{Z} \oplus \mathbb{Z}/p^{b_i+1}\mathbb{Z}\big)\right) \oplus \left(\bigoplus_{i=1}^{c(e)} \mathbb{Z}/p^{c_i}\mathbb{Z}\right),$$

where $n_i \geq 2$ for all $i \in \{1, \dots, a(e)\}$ and $c_i \geq c_{i-1} + 2$ for all $i \in \{1, \dots, c(e)\}$. By Theorem 9.3.2 and Proposition 9.3.10. we know that

$$\mathrm{Spec}_\Pi(P) \subseteq \{p^i \mid i \in \{b(e) + c(e), \dots, \Sigma(e)\}\}.$$

Conversely, by Lemma 9.3.13 and Propositions 9.3.12, 9.3.14 and 9.3.16, $\mathrm{Spec}_\Pi(P)$ contains

$$\begin{aligned}&\prod_{i=1}^{a(e)} \mathrm{Spec}_\Pi((\mathbb{Z}/p^{a_i}\mathbb{Z})^{n_i}) \cdot \prod_{i=1}^{b(e)} \mathrm{Spec}_\Pi\big(\mathbb{Z}/p^{b_i}\mathbb{Z} \oplus \mathbb{Z}/p^{b_i+1}\mathbb{Z}\big) \cdot \prod_{i=1}^{c(e)} \mathrm{Spec}_\Pi(\mathbb{Z}/p^{c_i}\mathbb{Z}) \\ &= \prod_{i=1}^{a(e)} \{p^j \mid j \in \{0, \dots, a_i n_i\}\} \cdot \prod_{i=1}^{b(e)} \{p^j \mid j \in \{1, \dots, 2b_i + 1\}\} \\ &\qquad \cdot \prod_{i=1}^{c(e)} \{p^j \mid j \in \{1, \dots, c_i\}\} \\ &= \{p^i \mid i \in \{b(e) + c(e), \dots, \Sigma(e)\}\},\end{aligned}$$

which proves the theorem. □

In particular, since $\mathrm{Spec}_\Pi(P) = \mathrm{Spec}_\mathrm{R}(P)$ for finite abelian 2-groups, we have the following:

Corollary 9.3.17. *Let P be a finite abelian 2-group of type e. Then*

$$\mathrm{Spec}_\mathrm{R}(P) = \{2^i \mid i \in \{b(e) + c(e), \dots, \Sigma(e)\}\}.$$

Proof. This immediately follows from Theorem 9.3.11, since $\mathrm{Spec}_\mathrm{R}(P) = \mathrm{Spec}_\Pi(P)$ if P is an abelian 2-group. □

By combining Corollaries 9.1.3 and 9.3.17 and Proposition 9.2.2, we can determine the Reidemeister spectrum of an arbitrary finite abelian group.

Theorem 9.3.18. *Let A be a finite abelian group. Suppose its Sylow 2-subgroup is of type e. Then*

$$\mathrm{Spec}_\mathrm{R}(A) = \{d \in \mathbb{N} \mid d \textit{ divides } |A| \textit{ and } \nu_2(d) \geq b(e) + c(e)\}.$$

Remark. If A is the cyclic group of order n, the theorem above states that

$$\mathrm{Spec}_\mathrm{R}(\mathbb{Z}/n\mathbb{Z}) = \{d \in \mathbb{N} \mid d \text{ divides } n\}$$

if n is odd and

$$\mathrm{Spec}_\mathrm{R}(\mathbb{Z}/n\mathbb{Z}) = \{d \in \mathbb{N} \mid d \text{ divides } n \text{ and } d \text{ is even}\}$$

if n is even. Since $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$, we derive from Lemma 7.2.2 that

$$\mathrm{Spec}_\mathrm{R}(\mathbb{Z}/n\mathbb{Z}) = \{\gcd(k-1, n) \mid k \in \{1, \dots, n\}, \gcd(k, n) = 1\}.$$

The explicit expressions for this set in terms of n are also determined by I. Richards in [98, Lemma 1].

This result combined with Theorem T.1.13 and Corollary 3.2.3 completely describes the (extended) Reidemeister spectrum of an arbitrary finitely generated abelian group.

Theorem 9.3.19. *Let A be a finitely generated abelian group and let $\tau(A)$ be its torsion subgroup. Suppose the Sylow 2-subgroup of $\tau(A)$ is of type e, and that $A/\tau(A)$ is free abelian of rank $r \geq 0$.*

(1) If $r = 0$, then

$$\mathrm{Spec}_\mathrm{R}(A) = \{d \in \mathbb{N} \mid d \textit{ divides } |\tau(A)| \textit{ and } \nu_2(d) \geq b(e) + c(e)\}$$

and

$$\mathrm{ESpec}_\mathrm{R}(A) = \{d \in \mathbb{N} \mid d \textit{ divides } |\tau(A)|\}.$$

(2) *If* $r = 1$, *then*

$$\mathrm{Spec}_{\mathrm{R}}(A) = \{2d \mid d \in \mathbb{N}, d \text{ divides } |\tau(A)| \text{ and } \nu_2(d) \geq b(e) + c(e)\} \cup \{\infty\}.$$

and

$$\mathrm{ESpec}_{\mathrm{R}}(A) = \mathbb{N}_0 \cup \{\infty\}.$$

(3) *If* $r \geq 2$, *then*

$$\mathrm{Spec}_{\mathrm{R}}(A) = 2^{b(e)+c(e)}\mathbb{N}_0 \cup \{\infty\}$$

and

$$\mathrm{ESpec}_{\mathrm{R}}(A) = \mathbb{N}_0 \cup \{\infty\}.$$

Proof. Since A is abelian, A is isomorphic to $\tau(A) \oplus \frac{A}{\tau(A)}$. Therefore, Corollary 3.2.3 yields $\mathrm{Spec}_{\mathrm{R}}(A) = \mathrm{Spec}_{\mathrm{R}}(\tau(A)) \cdot \mathrm{Spec}_{\mathrm{R}}(A/\tau(A))$ and $\mathrm{ESpec}_{\mathrm{R}}(A) = \mathrm{ESpec}_{\mathrm{R}}(\tau(A)) \cdot \mathrm{ESpec}_{\mathrm{R}}(A/\tau(A))$. The result then follows by combining Theorems 9.2.3, T.1.13 and 9.3.18.

Alternatively, we can also use Corollary 4.1.14 for the extended Reidemeister spectrum if $r \geq 1$. $\square$

Variation 10

Split metacyclic groups

Mozart, Wolfgang Amadeus, *Zwölf Variationen in C über das franzözische Lied 'Ah, vous dirai-je, Maman'*, (KV 265 (300e)), Variation X (bars 1–8).

The next class of finite groups for which we want tot determine the complete Reidemeister spectrum are the split metacyclic groups. As mentioned in Variation 8, we use the technique of counting fixed irreducible characters to do so.

Most of the results in this Variation can be found in [108].

10.1 Preliminaries

10.1.1 Character theory

Let G be a finite group. The *dual group* is the group $\widehat{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$ of group homomorphisms from G to $\mathbb{C}^\times$. Note that $\widehat{G}$ is abelian. By viewing $\mathbb{C}^\times$ as a subset of $\mathbb{C}$, we can view $\widehat{G}$ as a subset of $\mathrm{Map}(G, \mathbb{C})$. Thus, for two groups G, H and a homomorphism $\varphi : G \to H$, we get the map

$$\widehat{\varphi} : \widehat{H} \to \widehat{G} : \chi \mapsto \chi \circ \varphi.$$

In this case, $\widehat{\varphi}$ is a group homomorphism.

Definition 10.1.1. Let G be a finite group. For each divisor d of $|G|$, we define $\mathrm{Irr}_d(G)$ to be the set of all irreducible d-dimensional characters. We let $\mathrm{ch}_d(G)$ denote its cardinality. Given an automorphism φ of G, we write $\mathrm{ch}_{d,\varphi}(G)$ for the number of characters in $\mathrm{Irr}_d(G)$ fixed by φ.

If G is clear from the context, we simply write ch_d and $\mathrm{ch}_{d,\varphi}$.

Lemma 10.1.2 ([14, Theorem 3.11])**.** *Let A be a finite abelian group. Then* $\mathrm{Hom}(A, \mathbb{C}^\times)$ *is the group of irreducible characters of A and it is isomorphic to A.*

Proposition 10.1.3. *Let G be a finite group. Then the number of (irreducible) 1-dimensional representations of G equals $[G : \gamma_2(G)]$.*

Proof. Let $\rho : G \to \mathbb{C}^\times$ be a 1-dimensional representation of G. As $\mathbb{C}^\times$ is abelian, ρ induces a representation $\tilde{\rho} : \frac{G}{\gamma_2(G)} \to \mathbb{C}^\times$. Conversely, if $\tilde{\sigma} : \frac{G}{\gamma_2(G)} \to \mathbb{C}^\times$ is a 1-dimensional representation, then $\sigma := \tilde{\sigma} \circ \pi$ is a 1-dimensional representation of G, where $\pi : G \to \frac{G}{\gamma_2(G)}$ is the natural projection.

It is clear that if $\tilde{\sigma} \neq \tilde{\rho}$ are distinct 1-dimensional representations of $\frac{G}{\gamma_2(G)}$, then $\tilde{\sigma} \circ \pi \neq \tilde{\rho} \circ \pi$ are distinct 1-dimensional representations of G. Conversely, if $\rho \neq \sigma$ are distinct 1-dimensional representations of G, then their induced representations on $\frac{G}{\gamma_2(G)}$ are distinct as well. $\square$

Corollary 10.1.4. *Let G be a finite group. Then $\widehat{G}$ is the group of all 1-dimensional irreducible characters of G and is isomorphic to $G/\gamma_2(G)$.*

Proof. The map $\pi : G \to \frac{G}{\gamma_2(G)}$ induces a group homomorphism $\widehat{\pi} : \widehat{\frac{G}{\gamma_2(G)}} \to \widehat{G}$. By the proof of Proposition 10.1.3, this map is injective. Since both groups have the same size, $\widehat{\pi}$ is an isomorphism. Finally, since $\frac{G}{\gamma_2(G)}$ is abelian, Lemma 10.1.2 implies that $\frac{G}{\gamma_2(G)} \cong \widehat{\frac{G}{\gamma_2(G)}} \cong \widehat{G}$. $\square$

Lemma 10.1.5. *Let G be a finite group and $\varphi \in \mathrm{Aut}(G)$. Then $\mathrm{ch}_{1,\varphi} = R(\varphi^{\mathrm{ab}})$, where φ^{ab} is the induced automorphism on $G/\gamma_2(G)$.*

Proof. As mentioned earlier, we know that $\mathrm{Irr}_1(G) = \widehat{G}$ is isomorphic to $\widehat{\frac{G}{\gamma_2(G)}}$ via $\widehat{\pi}$, the dual of the canonical projection $\pi : G \to G/\gamma_2(G)$. The map $\widehat{\varphi}$ on $\mathrm{Irr}_1(G)$ is thus linked to $\widehat{\varphi^{\mathrm{ab}}}$ on $\widehat{\frac{G}{\gamma_2(G)}}$ via

$$\widehat{\varphi} = \widehat{\pi} \circ \widehat{\varphi^{\mathrm{ab}}} \circ \widehat{\pi}^{-1}.$$

Hence, their number of fixed points is be the same. Since $R(\varphi^{\mathrm{ab}}) = \left|\mathrm{Fix}(\widehat{\varphi^{\mathrm{ab}}})\right|$ by Corollary 8.1.8 and $\mathrm{ch}_{1,\varphi^{\mathrm{ab}}} = \left|\mathrm{Fix}(\widehat{\varphi^{\mathrm{ab}}})\right|$ by definition, $\mathrm{ch}_{1,\varphi} = \mathrm{ch}_{1,\varphi^{\mathrm{ab}}} = R(\varphi^{\mathrm{ab}})$. □

10.1.2 Characters of $A \rtimes C_p$

In order to apply the technique of counting fixed characters, we need an expression for the characters on semi-direct products of the form $A \rtimes C_p$, with A abelian and p a prime number. We recall a more general construction, details of which can be found in [110, §8.2].

Suppose $G = A \rtimes H$ is a semi-direct product of a finite abelian group A and another finite group H. Then H acts on $\widehat{A}$ via conjugation, i.e.

$$(h \cdot \chi)(a) := \chi(a^h)$$

for $h \in H, a \in A$ and $\chi \in \widehat{A}$. Let $\{\chi_1, \dots, \chi_m\}$ be a system of representatives of the orbits in $\widehat{A}$ of this action. For $i \in \{1, \dots, m\}$, let H_i be the stabiliser of χ_i and put $G_i := A \rtimes H_i \leq G$. We can extend χ_i to G_i by putting $\chi_i(ah) := \chi_i(a)$ for $a \in A, h \in H_i$. Since $h \cdot \chi_i = \chi_i$ for all $h \in H_i$, we see that χ_i is a 1-dimensional character of G_i. Next, let ρ be an irreducible representation of H_i. Then composing ρ with the canonical projection from G_i to H_i yields an irreducible representation $\tilde{\rho}$ of G_i. Finally, put

$$\theta_{i,\rho} := \mathrm{Ind}_{G_i}^{G}(\chi_i \otimes \tilde{\rho}),$$

the induced representation of $\chi_i \otimes \tilde{\rho}$. Note that $\dim(\theta_{i,\rho}) = [G : G_i] \dim \rho = [H : H_i] \dim \rho$.

Theorem 10.1.6 (See e.g. [110, Proposition 25 & Theorem 12]). *Let $G = A \rtimes H$ be finite with A abelian. Then the following hold:*

(1) Each irreducible representation of G is isomorphic to a representation $\theta_{i,\rho}$ of the form constructed above.

(2) Two representations $\theta_{i,\rho}$ and $\theta_{j,\sigma}$ are isomorphic if and only if $i = j$ and ρ and σ are isomorphic.

(3) The character $\overline{\chi}_{i,\rho}$ of $\theta_{i,\rho}$ is given by

$$\overline{\chi}_{i,\rho}(g) = \sum_{\substack{h \in R \\ g^h \in G_i}} (\chi_i \otimes \chi_{\tilde{\rho}})(g^h) = \frac{1}{|G_i|} \sum_{\substack{h \in G \\ g^h \in G_i}} (\chi_i \otimes \chi_{\tilde{\rho}})(g^h)$$

where R is a complete set of representatives of G/G_i.

We now consider the case where H is a cyclic group of prime order and A is any (non-trivial) finite abelian group. In this Variation, we mostly write the cyclic group of order n as C_n and write abelian groups multiplicatively, as opposed to additively as we did in Variation 9. When we have to determine solutions modulo some integer(s), we write $\mathbb{Z}/n\mathbb{Z}$. Throughout the remainder of the section, we put $G = A \rtimes_\alpha C_p$, where $\alpha : C_p \to \mathrm{Aut}(A)$, and we fix a generator y of C_p. In this notation, $a^y = \alpha(y)^{-1}(a)$.

Lemma 10.1.7. *Each irreducible character of G has dimension 1 or p.*

Proof. Let $\chi \in \widehat{A}$ be a character. Its stabiliser for the action of C_p is either 1 or C_p. If its stabiliser is C_p, then the representation ρ in the construction above is 1-dimensional. Therefore, the induced character has dimension $[G : G] \dim \rho = 1$.

If the stabiliser equals 1, then ρ and $\tilde{\rho}$ are both trivial representations. Thus, the induced character has dimension $[G : A] \dim \rho = [C_p : 1] = p$. □

If $\chi \in \widehat{A}$ is a character inducing a p-dimensional character on G, we let $\overline{\chi}$ denote this induced character. We can simplify the explicit expression for the induced characters in $\mathrm{Irr}_p(G)$.

Lemma 10.1.8. *Let χ be a character on A inducing a p-dimensional character $\overline{\chi}$ on G. Then*

$$\overline{\chi}(g) = \begin{cases} 0 & \text{if } g \notin A \\ \sum_{i=0}^{p-1} \chi(\alpha(y)^i(g)) & \text{if } g \in A \end{cases}$$

Proof. Since $\overline{\chi}$ is p-dimensional, the stabiliser of χ under the action of C_p is trivial. Therefore, the G_i from the construction equals A and the group C_p can be used as a complete set of representatives of G/A. Recall that this also implies that the representation ρ in the construction is the trivial representation. Consequently, if $g \in A$, then $(\chi \otimes \tilde{\rho})(g^{y^i}) = \chi(\alpha(y)^{-i}(g))$. Since A is normal in G, either all conjugates of g lie in A or none of them do. Hence, if $g \in A$,

$$\overline{\chi}(g) = \sum_{i=0}^{p-1} \chi(\alpha(y)^{-i}(g))$$

and if $g \notin A$, then $\overline{\chi}(g) = 0$. Switching from $-i$ to i does not change the summation, since $\alpha(y)$ has order p, so

$$\overline{\chi}(g) = \sum_{i=0}^{p-1} \chi(\alpha(y)^i(g)$$

if $g \in A$. □

Lemma 10.1.5 allows us to determine the number of characters in $\mathrm{Irr}_1(G)$ fixed by a given automorphism. For those in $\mathrm{Irr}_p(G)$, we can use the following criterion:

Lemma 10.1.9. *Let $\overline{\chi}$ be a character in $\mathrm{Irr}_p(G)$ and $\varphi \in \mathrm{Aut}(G)$ such that $\varphi(A) = A$. Let $\varphi|_A$ denote the induced automorphism on A. Then $\overline{\chi}$ is fixed by φ if and only if there is an $i \in \{0, \dots, p-1\}$ such that $\chi \circ \varphi|_A = \chi \circ \alpha(y)^i$.*

Proof. As $\varphi(A) = A$ and φ is bijective, also $\varphi(G \setminus A) = G \setminus A$, which implies that $\overline{\chi}(\varphi(g)) = 0$ if $g \notin A$. Thus, $\overline{\chi} \circ \varphi = \overline{\chi}$ if and only if

$$\sum_{i=0}^{p-1} \chi(\alpha(y)^i(g)) = \sum_{i=0}^{p-1} \chi(\alpha(y)^i(\varphi|_A(g)))$$

for all $g \in A$. We can rewrite this as

$$\sum_{i=0}^{p-1} (\chi \circ \alpha(y)^i)(g) = \sum_{i=0}^{p-1} (\chi \circ \alpha(y)^i \circ \varphi|_A)(g),$$

and as it has to hold for all $g \in A$, we get the following equality of class functions:

$$\sum_{i=0}^{p-1} \chi \circ \alpha(y)^i = \sum_{i=0}^{p-1} \chi \circ \alpha(y)^i \circ \varphi|_A. \tag{10.1.1}$$

As $\alpha(y)$ and $\varphi|_A$ are automorphisms, each term in either sum is an irreducible character of A. Moreover, no two terms in the same sum are equal. Indeed, if $\chi \circ \alpha(y)^i = \chi \circ \alpha(y)^j$ for $0 \le i, j \le p-1$, then $\chi \circ \alpha(y)^{i-j} = \chi$. This implies that y^{i-j} lies in the stabiliser of χ under the action of C_p. As this stabiliser is trivial, this means that $i \equiv j \bmod p$, and thus $i = j$. Since $\varphi|_A$ is an automorphism, we can apply the same argument to the terms $\chi \circ \alpha(y)^i \circ \varphi|_A$.

Thus, each side in (10.1.1) is a sum of distinct irreducible characters of A. As the irreducible characters on A form a basis of the vector space of class functions on A, equality in (10.1.1) holds if and only if

$$\{\chi \circ \alpha(y)^i \mid 0 \le i \le p-1\} = \{\chi \circ \alpha(y)^i \circ \varphi|_A \mid 0 \le i \le p-1\}. \tag{10.1.2}$$

This equality of sets immediately implies that $\chi \circ \varphi|_A = \chi \circ \alpha(y)^i$ for some $i \in \{0, \dots, p-1\}$.

Conversely, suppose that $\chi \circ \varphi|_A = \chi \circ \alpha(y)^i$ for some $i \in \{0, \dots, p-1\}$. Let $g \in A$. As φ is a group homomorphism, it preserves the relation $g^y = \alpha(y)^{-1}(g)$. Write $\varphi(y) = ay^k$ for some $a \in A, k \in \mathbb{Z}$. Then

$$\alpha(y)^{-k}(\varphi(g)) = \varphi(g)^{y^k} = \varphi(g)^{ay^k} = \varphi(g)^{\varphi(y)} = \varphi(g^y) = \varphi(\alpha(y)^{-1}(g)).$$

As this holds for all $g \in A$, we get the equality $\alpha(y)^{-k} \circ \varphi|_A = \varphi|_A \circ \alpha(y)^{-1}$ of automorphisms on A. By multiplying on the left by $\alpha(y)^k$, on the right by $\alpha(y)$ and swapping sides, we get the equivalent equality $\alpha(y)^k \circ \varphi|_A = \varphi|_A \circ \alpha(y)$ Note that $k \not\equiv 0 \bmod p$, as φ is an automorphism. From this equality we find that

$$\alpha(y)^{jk} \circ \varphi|_A = \varphi|_A \circ \alpha(y)^j$$

for all $j \in \mathbb{Z}$.

Using this, we proceed in proving equality in (10.1.2). Let l be the multiplicative inverse of k modulo p. For $j \in \{0, \dots, p-1\}$, we then have

$$\chi \circ \alpha(y)^j \circ \varphi|_A = \chi \circ \varphi|_A \circ \alpha(y)^{jl} = \chi \circ \alpha(y)^{i+jl}.$$

As l is invertible modulo p, the set $\{i + jl \mid j \in \{0, \dots, p-1\}\}$ forms a complete set of representatives modulo p. Consequently, we find that equality in (10.1.2) holds. □

With this characterisation we can determine an explicit formula for $\mathrm{ch}_{p,\varphi}(G)$ given an automorphism $\varphi \in \mathrm{Aut}(G)$.

Proposition 10.1.10. *Let $\varphi \in \mathrm{Aut}(G)$ be such that $\varphi(A) = A$ and let $\varphi|_A$ denote the induced automorphism on A. Then*

$$\mathrm{ch}_{p,\varphi}(G) = \frac{1}{p}\left(\sum_{i=0}^{p-1} \left|\mathrm{Fix}(\varphi|_A \circ \alpha(y)^i)\right|\right) - \left|\mathrm{Fix}(\widehat{\varphi|_A}) \cap \mathrm{Fix}(\widehat{\alpha(y)})\right|$$

Proof. Let $\varphi \in \mathrm{Aut}(G)$ with $\varphi(A) = A$ be fixed. We use Lemma 10.1.9 to determine $\mathrm{ch}_{p,\varphi}$. Define for $i \in \{0, \dots, p-1\}$ the set

$$F_{\varphi,i} := \{\chi \in \widehat{A} \mid \chi \circ \varphi|_A = \chi \circ \alpha(y)^i\}.$$

By the aforementioned lemma, if $\chi \in \widehat{A}$ induces a p-dimensional character of G, then this induced character is fixed by φ if and only if χ lies in one of the $F_{\varphi,i}$. However, not all characters in $F_{\varphi,i}$ induce a p-dimensional one on G. Therefore, we have to determine which do.

We start by computing the size of the union of all $F_{\varphi,i}$ using the inclusion-exclusion principle. For $i \neq j \in \{0, \dots, p-1\}$, consider $F_{\varphi,i} \cap F_{\varphi,j}$. A character $\chi \in \widehat{A}$ lies in this intersection if and only if

$$\chi \circ \varphi|_A = \chi \circ \alpha(y)^i = \chi \circ \alpha(y)^j.$$

The second equality is equivalent with $\chi \in \mathrm{Fix}(\widehat{\alpha(y)^{j-i}})$. As $j - i \not\equiv 0 \bmod p$ and $\widehat{\alpha(y)}$ has order p, it holds that $\mathrm{Fix}(\widehat{\alpha(y)^{j-i}}) = \mathrm{Fix}(\widehat{\alpha(y)})$. From this it also follows that $\chi \circ \varphi|_A = \chi$, which means that $\chi \in F_{\varphi,0}$. Therefore, $\chi \in \operatorname{Fix}(\widehat{\alpha(y)}) \cap F_{\varphi,0}$. Clearly, the converse holds as well, i.e. $\chi \in \operatorname{Fix}(\widehat{\alpha(y)}) \cap F_{\varphi,0}$ implies $\chi \circ \varphi|_A = \chi \circ \alpha(y)^i = \chi \circ \alpha(y)^j$. We conclude that

$$F_{\varphi,i} \cap F_{\varphi,j} = F_{\varphi,0} \cap \operatorname{Fix}(\widehat{\alpha(y)}). \tag{10.1.3}$$

Consequently, any intersection of at least two $F_{\varphi,i}$ with distinct indices equals $F_{\varphi,0} \cap \operatorname{Fix}(\widehat{\alpha(y)})$. Thus, the inclusion-exclusion principle yields

$$\begin{aligned}
\left|\bigcup_{i=0}^{p-1} F_{\varphi,i}\right| &= \sum_{i=0}^{p-1} |F_{\varphi,i}| + \sum_{i=2}^{p} (-1)^{i+1} \binom{p}{i} \left|F_{\varphi,0} \cap \operatorname{Fix}(\widehat{\alpha(y)})\right| \\
&= \sum_{i=0}^{p-1} |F_{\varphi,i}| - \left|F_{\varphi,0} \cap \operatorname{Fix}(\widehat{\alpha(y)})\right| \sum_{i=2}^{p} (-1)^{i} \binom{p}{i} \\
&= \sum_{i=0}^{p-1} |F_{\varphi,i}| - \left|F_{\varphi,0} \cap \operatorname{Fix}(\widehat{\alpha(y)})\right| \left(-\binom{p}{0} + \binom{p}{1} \right) \\
&= \sum_{i=0}^{p-1} |F_{\varphi,i}| - (p-1) \left|F_{\varphi,0} \cap \operatorname{Fix}(\widehat{\alpha(y)})\right|.
\end{aligned}$$

We now determine which characters in the union induce a p-dimensional one on G. Recall that $\chi \in \widehat{A}$ induces $\overline{\chi} \in \operatorname{Irr}_p(G)$ if and only if its stabiliser under the action of C_p on $\widehat{A}$ is trivial. As $(y^i \cdot \chi)(a) = \chi(a^{y^i}) = \chi(\alpha(y)^{-i}(a))$ for all $a \in A$, the stabiliser of χ is trivial if and only if $\chi \circ \alpha(y)^{-i} \neq \chi$ for at least one $i \in \{1, \ldots, p-1\}$. As $\alpha(y)$ has prime order, this is equivalent with $\chi \circ \alpha(y) \neq \chi$, i.e. $\chi \notin \operatorname{Fix}(\widehat{\alpha(y)})$.

Therefore, we must subtract $\left|\operatorname{Fix}(\widehat{\alpha(y)}) \cap \bigcup_{i=0}^{p-1} F_{\varphi,i}\right|$. By a similar argument as for (10.1.3) we find that

$$F_{\varphi,i} \cap \operatorname{Fix}(\widehat{\alpha(y)}) = F_{\varphi,0} \cap \operatorname{Fix}(\widehat{\alpha(y)})$$

for all $i \in \{0, \ldots, p-1\}$. Hence, we have to subtract $\left|F_{\varphi,0} \cap \operatorname{Fix}(\widehat{\alpha(y)})\right|$ and obtain

$$\sum_{i=0}^{p-1} |F_{\varphi,i}| - p\left|F_{\varphi,0} \cap \operatorname{Fix}(\widehat{\alpha(y)})\right|.$$

We have to take one last thing into account. If $\chi \in \widehat{A}$ induces a p-dimensional character on G fixed by φ, then so does $\chi \circ \alpha(y)^i$ for each $i \in \{0, \ldots, p-1\}$.

However, all these induced characters are equal. Conversely, by the second item of Theorem 10.1.6, this is the only way in which two characters of A can induce the same p-dimensional character of G. In other words, each p-dimensional character fixed by φ is counted p times in the above expression. We therefore have to divide by p to finally obtain

$$\mathrm{ch}_{p,\varphi} = \frac{1}{p}\left(\sum_{i=0}^{p-1} |F_{\varphi,i}|\right) - \left|F_{\varphi,0} \cap \mathrm{Fix}(\widehat{\alpha(y)})\right|.$$

To end the proof, note that

$$\begin{aligned} |F_{\varphi,i}| &= \left|\{\chi \in \widehat{A} \mid \chi \circ \varphi|_A = \chi \circ \alpha(y)^i\}\right| \\ &= \left|\mathrm{Fix}(\widehat{\varphi|_A \circ \alpha(y)^{-i}})\right| \\ &= \left|\mathrm{Fix}(\varphi|_A \circ \alpha(y)^{-i})\right| \end{aligned}$$

by Corollary 8.1.9, and also that $F_{\varphi,0} = \mathrm{Fix}(\widehat{\varphi|_A})$. Consequently,

$$\mathrm{ch}_{p,\varphi}(G) = \frac{1}{p}\left(\sum_{i=0}^{p-1} \left|\mathrm{Fix}(\varphi|_A \circ \alpha(y)^{-i})\right|\right) - \left|\mathrm{Fix}(\widehat{\varphi|_A}) \cap \mathrm{Fix}(\widehat{\alpha(y)})\right|.$$

Switching from $-i$ to i does not change the first summation, hence the result follows. □

The preceding results yield necessary conditions on Reidemeister numbers of automorphisms of split metacyclic groups. The main tool to prove that these conditions are also sufficient is a number theoretic result concerning the existence of an integer satisfying certain divisor properties.

Theorem 10.1.11. *Let l, n, m be non-negative integers with $n, m \geq 1$ and $l \leq m$ and let p be a prime number not dividing n. Let a be an integer such that $\gcd(a, n) = \gcd(a-1, n) = 1$ and such that $\bar{a} \in (\mathbb{Z}/n\mathbb{Z})^\times$ has order p. Suppose that $d_0, \ldots, d_{p-1}$ are pairwise coprime divisors of n, with, if $p = 2$ and $n \equiv 0 \bmod 3$, the extra condition that $d_0 d_1 \equiv 0 \bmod 3$.*

Then there exists a $\gamma \in \mathbb{Z}$ such that $\gcd(\gamma, n) = 1$, such that $\gcd(\gamma - a^i, n) = d_i$ for all $i \in \{0, \ldots, p-1\}$ and such that $\gamma - 1 \equiv p^l \bmod p^m$.

Proof. Note that n must be odd, since otherwise $\gcd(a(a-1), n) \geq 2$. First, we consider the case where p is odd. Let $n = \prod_{i=1}^{r} p_i^{e_i}$ be the prime factorisation of n, where all p_i are distinct prime numbers. For $0 \leq i \leq p-1$, let $d_i = \prod_{j=1}^{r} p_j^{e_{i,j}}$

be the prime factorisation of d_i. Consider, for all $1 \leq j \leq r$ and all $0 \leq i \leq p-1$ such that $e_{i,j} \geq 1$, the congruence

$$x - a^i \equiv p_j^{e_{i,j}} \bmod p_j^{e_j}. \tag{10.1.4}$$

Since all the d_i are coprime, we have at most one congruence modulo $p_j^{e_j}$ for each $1 \leq j \leq r$. Let q be the product of all $p_j^{e_j}$ not occurring in one of the congruences above, and consider (if $q \neq 1$), the congruence

$$x \equiv -1 \bmod q. \tag{10.1.5}$$

The system we then consider consists of all equations from (10.1.4) and (10.1.5), together with $x - 1 \equiv p^l \bmod p^m$. Since each $p_j^{e_j}$ occurs at most once and q is the product of all $p_j^{e_j}$ that did not yet occur, the conditions for the Chinese Remainder Theorem are met and we find a $\gamma \in \mathbb{Z}$ that satisfies all these congruences. Note that the product of all moduli is equal to np^m.

We now prove that γ satisfies the desired conditions.

First, let $0 \leq i \leq p-1$. We prove that $\gcd(\gamma - a^i, n) = d_i$. Let $j \in \{1, \ldots, r\}$. If $e_{i,j} \geq 1$, it immediately follows from (10.1.4) that $\gcd(\gamma - a^i, p_j^{e_j})$ equals $p_j^{e_{i,j}}$. So suppose that $e_{i,j} = 0$ and that p_j divides $\gamma - a^i$. If $e_{k,j} \geq 1$ for some $0 \leq k \leq p-1$ (which is then necessarily distinct from i), then

$$\gamma - a^i \equiv \gamma - a^k \bmod p_j.$$

This implies that $a^{i-k} \equiv 1 \bmod p_j$. Since $i \not\equiv k \bmod p$, also $a \equiv 1 \bmod p_j$, which contradicts $\gcd(a-1, n) = 1$. On the other hand, if $e_{k,j} = 0$ for all $0 \leq k \leq p-1$, then p_j divides q and we find using (10.1.5) that

$$0 \equiv \gamma - a^i \equiv -1 - a^i \bmod p_j.$$

As p and p_j are both odd, it is impossible for a^i to satisfy $a^i \equiv -1 \bmod p_j$. Therefore, we have arrived yet again at a contradiction. We therefore conclude that $\gcd(\gamma - a^i, n) = d_i$.

Next, we prove that $\gcd(\gamma, n) = 1$. Suppose that there is a $j \in \{1, \ldots, r\}$ such that p_j divides γ. If p_j divides q, then we find using (10.1.5) that

$$0 \equiv \gamma \equiv -1 \bmod p_j,$$

a contradiction. Thus, there is some $0 \leq i \leq p-1$ such that $e_{i,j} \geq 1$. Then using (10.1.4) we find that

$$0 \equiv \gamma - a^i \equiv -a^i \bmod p_j,$$

which is again a contradiction, as $\gcd(a^i, n) = 1$. We conclude that $\gcd(\gamma, n) = 1$.

Finally, we have that $\gamma - 1 \equiv p^l \bmod p^m$ by construction. This finishes the case where p is odd.

For the case where $p = 2$, we consider almost the exact same system of congruences, except we replace (10.1.5) with

$$x \equiv 3 \bmod q. \tag{10.1.6}$$

The proofs for the conditions on γ are almost identical as for the case p odd, but there are some differences.

First, let $0 \le i \le 1$. We prove that $\gcd(\gamma - a^i, n) = d_i$. Let $j \in \{1, \dots, r\}$. If $e_{i,j} \ge 1$, it again immediately follows from (10.1.4) that $\gcd(\gamma - a^i, p_j^{e_j})$ equals $p_j^{e_{i,j}}$. So suppose that $e_{i,j} = 0$ and that p_j divides $\gamma - a^i$. If $e_{k,j} \ge 1$ for some $0 \le k \le 1$ (necessarily distinct from i), then

$$\gamma - a^i \equiv \gamma - a^k \bmod p_j.$$

This implies that $a^{i-k} \equiv 1 \bmod p_j$. Since $i \not\equiv k \bmod 2$, also $a \equiv 1 \bmod p_j$, which contradicts $\gcd(a - 1, n) = 1$. On the other hand, if $e_{k,j} = 0$ for all $0 \le k \le 1$, then p_j divides q and we find

$$0 \equiv \gamma - a^i \equiv 3 - a^i \bmod p_j.$$

As $a^i \equiv \pm 1 \bmod p_j$ and p_j is odd, this congruence yields a contradiction. We therefore can conclude that $\gcd(\gamma - a^i, n) = d_i$.

Next, we prove that $\gcd(\gamma, n) = 1$. Suppose that there is a $j \in \{1, \dots, r\}$ such that p_j divides γ. If p_j divides q, then we find using (10.1.6) that

$$0 \equiv \gamma \equiv 3 \bmod p_j.$$

If we can prove that $p_j \ne 3$, we have our desired contradiction. Recall that, if $3 \mid n$, either d_0 or d_1 is a multiple of 3. Therefore, q is not divisible by 3, which implies that $p_j \ne 3$. Therefore, we have a contradiction. Thus, there is some $0 \le i \le 1$ such that $e_{i,j} \ge 1$. Then using (10.1.4) we find that

$$0 \equiv \gamma - a^i \equiv -a^i \bmod p_j,$$

which is again a contradiction, as $\gcd(a^i, n) = 1$. We conclude that $\gcd(\gamma, n) = 1$. $\square$

We end this section with a last technical lemma.

Lemma 10.1.12. *Let C be a finite cyclic group and let H_1, H_2 be two subgroups of order h_1, h_2, respectively. Then*

$$|H_1 \cap H_2| = \gcd(h_1, h_2).$$

Proof. Suppose that C has order n and let x be a generator. As H_i has order h_i, we see that $H_i = \left\langle x^{\frac{n}{h_i}} \right\rangle$. Any element in the intersection of H_1 and H_2 can thus be written in the form x^{am} for some $a \in \mathbb{Z}$ and where $m = \operatorname{lcm}(\frac{n}{h_1}, \frac{n}{h_2})$. Conversely, every element of this form lies in both H_1 and H_2. Therefore, $H_1 \cap H_2 = \langle x^m \rangle$. Since

$$\frac{n}{\operatorname{lcm}(\frac{n}{h_1}, \frac{n}{h_2})} = \gcd(h_1, h_2)$$

and $H_1 \cap H_2$ has order $\frac{n}{m}$, the result follows. □

10.2 Split metacyclic groups of the form $C_n \rtimes C_p$

We now discuss some structural properties of the groups we investigate.

Definition 10.2.1. Let G be a group. We call G *split metacyclic* if there are cyclic groups C_n and C_m such that $G \cong C_n \rtimes C_m$.

We consider split metacyclic groups of the form $(C_n \times C_{p^m}) \rtimes C_p$ where p is a prime number, m and n are non-negative integers and n is coprime with p. Due to this last condition, $C_n \times C_{p^m} \cong C_{np^m}$ is indeed cyclic.

The following is essentially a special case of [42, Lemma 3.2], but we include a full proof here due to the fact that our notation differs quite from theirs:

Proposition 10.2.2. *Let n, m and p be natural numbers with $n \geq 2$, $m \geq 0$, p a prime number and $\gcd(n, p) = 1$. Let $G := (C_n \times C_{p^m}) \rtimes_\alpha C_p$ be a semi-direct product with morphism $\alpha : C_p \to \operatorname{Aut}(C_n \times C_{p^m}) \cong (\mathbb{Z}/n\mathbb{Z})^\times \times (\mathbb{Z}/p^m\mathbb{Z})^\times$. Put*

$$H := \{g \in C_n \mid \forall x \in C_p : \alpha(x)(g) = g\}.$$

and let $n = \prod_{i=1}^r p_i^{e_i}$ be the prime factorisation of n, with $p_i \neq p_j$ if $i \neq j$. Then

(1) $H \lhd G$;

(2) there is a subset $I \subseteq \{1, \ldots, r\}$ *such that*

$$H = \prod_{i \in I} C_{p_i^{e_i}};$$

(3) $G \cong H \times ((N \times C_{p^m}) \rtimes_{\tilde{\alpha}} C_p)$ *where* $\tilde{\alpha} : C_p \to \mathrm{Aut}(N)$ *is the restriction of* α *to* $\mathrm{Aut}(N)$ *and*

$$N := \prod_{i \notin I} C_{p_i^{e_i}}.$$

Proof. If α is the trivial map, then the result is clear. Therefore, suppose that α is not the trivial map. It is clear that $H \leq C_n$. Let x, y and z be generators of C_n, C_p and C_{p^m}, respectively. Let $h \in H$. Clearly, $h^x = h$ and $h^z = h$, and as $h^y = \alpha(y)^{-1}(h) = h$, we conclude that $H \leq Z(G)$. Hence, $H \lhd G$.

Since α is non-trivial, $\alpha(y)$ has order p. Also, by Proposition 9.1.2, the map α splits as a map $\alpha = (\alpha_1, \ldots, \alpha_k, \alpha')$ where $\alpha_i : C_p \to \mathrm{Aut}(C_{p_i^{e_i}})$ and $\alpha' : C_p \to \mathrm{Aut}(C_{p^m})$. Fix $i \in \{1, \ldots, r\}$. If x_i is a generator of $C_{p_i^{e_i}}$, then $\alpha_i(y)(x_i) = x_i^{f_i}$ for some $f_i \in \mathbb{Z}$ satisfying $f_i^p \equiv 1 \bmod p_i^{e_i}$. Suppose that $f_i = \beta p_i^l + 1$ for some $\beta, l \in \mathbb{Z}$ with $\beta = 0$ or $\gcd(p_i, \beta) = 1$, and $0 \leq l < e_i$. Then

$$1 \equiv (\beta p_i^l + 1)^p \bmod p_i^{e_i}.$$

Viewing this congruence modulo p_i^{l+1}, we find that $1 \equiv p\beta p_i^l + 1 \bmod p_i^{l+1}$. If $\beta \neq 0$, then we reach a contradiction, since then $\gcd(\beta p, p_i) = 1$.

From this, it follows that either $f_i \equiv 1 \bmod p_i^{e_i}$ or $f_i \not\equiv 1 \bmod p_i$, which means that either $\alpha_i(y)$ is the identity map or that $\alpha_i(y)$ has no fixed points, by Lemma 7.2.2. Consequently, putting $I = \{i \in \{1, \ldots, r\} \mid f_i \equiv 1 \bmod p_i^{e_i}\}$, we find that

$$H = \prod_{i \in I} C_{p_i^{e_i}}.$$

Finally, put $N = \prod_{i \notin I} C_{p_i^{e_i}}$. By Proposition 9.1.2, α restricts to a map $\tilde{\alpha} : C_p \to \mathrm{Aut}(N)$. By construction, $H \cap (N \rtimes_{\tilde{\alpha}} C_p) = 1$. Thus, as G is generated by H and $(N \times C_{p^m}) \rtimes_{\tilde{\alpha}} C_p$ and as H is central, G is the internal direct product of H and $(N \times C_{p^m}) \rtimes_{\tilde{\alpha}} C_p$. □

Corollary 10.2.3. *With the notations as before, we have*

$$\mathrm{Spec}_{\mathrm{R}}(G) = \mathrm{Spec}_{\mathrm{R}}(H) \cdot \mathrm{Spec}_{\mathrm{R}}((N \times C_{p^m}) \rtimes_{\tilde{\alpha}} C_p).$$

Proof. This follows immediately from Corollary 9.1.3. □

We can use Theorem 9.3.18 to determine $\mathrm{Spec}_{\mathrm{R}}(H)$ with H as defined above. As a consequence of this proposition, we can restrict our attention to split metacyclic groups $(C_n \times C_{p^m}) \rtimes_\alpha C_p$ for which the subgroup H is trivial. This means that C_p acts freely (by automorphisms) on C_n. Moreover, this also implies that each prime factor of n is strictly greater than p. Indeed, in the notation above, $\alpha_i(y)$ is an element of order p in $\mathrm{Aut}(C_{p_i^{e_i}})$. As $\mathrm{Aut}(C_{p_i^{e_i}}) \cong (\mathbb{Z}/p_i^{e_i}\mathbb{Z})^\times$, it must hold that p divides $p_i^{e_i-1}(p_i - 1)$. Since p does not divide n, p must divide $p_i - 1$, which shows that $p < p_i$.

Next, we discuss some of the possible actions of C_p.

Lemma 10.2.4. *If C_2 acts freely by automorphisms on a cyclic group C_n with n odd, then C_2 acts by inversion.*

Proof. Let y be the non-identity element in C_2. Fix a generator x of C_n and write $y \cdot x = x^\gamma$ for some $\gamma \in \mathbb{Z}$. We prove that $\gamma \equiv -1 \bmod n$. Let p be an (odd) prime factor of n and e its exponent in the prime factorisation of n. Since $(\mathbb{Z}/p^e\mathbb{Z})^\times$ is cyclic, it has only one element of order 2, namely -1. Therefore, $\gamma \equiv \pm 1 \bmod p^e$. If $\gamma \equiv 1 \bmod p^e$, then

$$\left(x^{\frac{n}{p^e}}\right)^\gamma = x^{\frac{n\gamma}{p^e}} = x^{\frac{n}{p^e}},$$

which shows that $y \cdot x^{n/p^e} = x^{n/p^e}$. As C_2 acts without fixed points, this is a contradiction, hence $\gamma \equiv -1 \bmod p^e$. As this holds for all prime factors of n, we conclude that $\gamma \equiv -1 \bmod n$. □

The situation of the action of C_p on C_{p^m} is slightly different. Since we have already determined the Reidemeister spectrum of finite abelian groups, we may assume that $m \geq 2$, as all groups of order p^2 are abelian.

Lemma 10.2.5 (See e.g. [53, Theorem 4.4])**.** *Let p be a prime, $m \geq 2$ an integer and let $\alpha : C_p \to \mathrm{Aut}(C_{p^m})$ be an action of C_p on C_{p^m} by automorphisms. For (fixed) generators x and y of C_{p^m} and C_p, respectively, put $\alpha(y)(x) = x^\gamma$, where $\gamma \in \mathbb{Z}$.*

(1) If $p \geq 3$, then $\gamma \equiv 1 \bmod p^{m-1}$.

(2) If $p = 2$, then $\gamma \equiv \pm 1 \bmod 2^{m-1}$

Proof. Write $\gamma = ap + \beta$ for some $a \in \mathbb{Z}$ and $\beta \in \{0, \dots, p-1\}$. Since $\alpha(y)$ has order p, $\gamma^p \equiv 1 \bmod p^m$. In particular, $\beta^p \equiv 1 \bmod p$, which implies $\beta \equiv 1 \bmod p$. Hence, $\gamma \equiv 1 \bmod p$.

If $\gamma = 1$, the result follows immediately. So, suppose that $\gamma \neq 1$ and write $\gamma = a'p^k + 1$ for some $k \geq 1$ and $a' \in \mathbb{Z}$ with $\gcd(a', p) = 1$. First we treat the case $p \geq 3$. Suppose that $k \leq m - 2$. From $\gamma^p \equiv 1 \bmod p^m$ it then follows that

$$1 \equiv (a'p^k + 1)^p \equiv a'p^{k+1} + 1 \bmod p^{k+2},$$

which contradicts $\gcd(a', p) = 1$. Thus, $k \geq m - 1$, from which we conclude that $\gamma \equiv 1 \bmod p^{m-1}$.

Now, let $p = 2$. For $m = 2$, the result states $\gamma \equiv \pm 1 \bmod 2$, which we already proved. Thus, assume that $m \geq 3$. We see that

$$1 \equiv \gamma^2 \equiv 2^{2k}(a')^2 + a'2^{k+1} + 1 \equiv 2^{k+1}a'(2^{k-1}a' + 1) + 1 \bmod 2^m.$$

If $2 \leq k \leq m - 2$, this yields a contradiction modulo 2^{k+2}, as $a'(2^{k-1}a' + 1)$ is odd in that case. Thus, either $k \geq m - 1$, which implies $\gamma \equiv 1 \bmod 2^{m-1}$, or $k = 1$. For $k = 1$, note that $a' + 1$ is even. Writing $a' + 1 = 2b$ for some $b \in \mathbb{Z}$, we get

$$1 \equiv 8(2b - 1)b + 1 \bmod 2^m.$$

This forces $b \equiv 0 \bmod 2^{m-3}$, thus $a' \equiv -1 \bmod 2^{m-2}$. Consequently, $a' = A2^{m-2} - 1$ for some $A \in \mathbb{Z}$ and thus

$$\gamma = 2a' + 1 = 2^{m-1}A - 2 + 1 = 2^{m-1}A - 1,$$

which implies $\gamma \equiv -1 \bmod 2^{m-1}$. □

We now introduce and fix some notation for the rest of the Variation. The letters n, m, p represent integers with $n \geq 1$, $m \geq 0$, p prime and $\gcd(n, p) = 1$. We use $SMC(n, m, p)$ to refer to any semi-direct product of the form $(C_n \times C_{p^m}) \rtimes_\alpha C_p$ where C_p acts freely on C_n (cf. observation after Proposition 10.2.2). Here, $\alpha : C_p \to \operatorname{Aut}(C_n \times C_{p^m})$. We put $N = np^m$ and write C_N for the cyclic group $C_n \times C_{p^m}$. The symbols x and y denote fixed generators of $C_n \times C_{p^m}$ and C_p, respectively. Since

$$\operatorname{Aut}(C_n \times C_{p^m}) \cong (\mathbb{Z}/n\mathbb{Z})^\times \times (\mathbb{Z}/p^m\mathbb{Z})^\times,$$

we often regard $\alpha(y)$ as a number modulo N. In particular, $\alpha(y)$ is invertible modulo N. As $\alpha(y)$ restricted to C_n has no fixed points, Lemma 7.2.2 implies that $\gcd(\alpha(y) - 1, n) = 1$ as well. As $\alpha(y)$ has order p, this moreover implies that $\gcd(\alpha(y)^i - 1, n) = 1$ for all $i \in \mathbb{Z}$ coprime with p. In this notation, $x^y = x^{\alpha(y)^{-1}}$, where the inverse is taken modulo N.

Remark. In the notation of [42], the group $SMC(n, m, p)$ corresponds to $D(np^m, p; \alpha(y))$.

Lemma 10.2.6. *Let G be an $SMC(n,m,p)$. If $n \geq 2$, then the subgroups C_N and C_n are characteristic in G.*

Proof. Since C_N is cyclic, its subgroup C_n is characteristic in C_N. Since C_N is normal in G, we infer that C_n is normal in G as well. Let $g \in G$ be an element such that $g^n = 1$. Let $\pi : G \to G/C_n$ be the natural projection. The quotient group G/C_n has order coprime with n, so $\pi(g)^n = 1$ implies $\pi(g) = 1$. Therefore, $g \in C_n$. This implies that $C_n = \{g \in G \mid g^n = 1\}$, which in turn implies that C_n is characteristic in G.

It is clear that C_N is contained in the centraliser of C_n. Conversely, let $x^i y^j$ be an element with $j \not\equiv 0 \bmod p$. Note that C_n is generated by x^{p^m}, and $x^{p^m} \neq 1$ as $n \geq 2$. Then

$$\left(x^{p^m}\right)^{x^i y^j} = \left(x^{p^m}\right)^{y^j} = \left(x^{p^m}\right)^{\alpha(y)^{-j}} \neq x^{p^m},$$

since $\alpha(y)$ restricted to C_n has no fixed points and $j \not\equiv 0 \bmod p$. Therefore, the centraliser of C_n in G equals C_N. This implies that C_N is characteristic as well, since C_n is characteristic. □

Lemma 10.2.7. *Let G be an $SMC(n,m,p)$ with $n \geq 2$ and let $\varphi \in \mathrm{Aut}(G)$. Then there is an $a \in \mathbb{Z}$ such that $\varphi(y) = x^a y$.*

Proof. Since C_N is characteristic by Lemma 10.2.6, $\varphi(x) = x^\gamma$ for some $\gamma \in \mathbb{Z}$ coprime with N, as φ is an automorphism. Write $\varphi(y) = x^a y^b$. Since φ preserves the identity $x^y = x^{\alpha(y)^{-1}}$, we find that

$$x^{\gamma\alpha(y)^{-1}} = (x^\gamma)^{x^a y^b} = y^{-b} x^\gamma y^b = (y^{-b} x y^b)^\gamma = x^{\alpha(y)^{-b}\gamma}.$$

Thus, $\alpha(y)^{-b}\gamma \equiv \gamma\alpha(y)^{-1} \bmod N$. Since γ and $\alpha(y)$ are both invertible modulo N, we see that $\alpha(y)^{-b+1} \equiv 1 \bmod N$. The order of $\alpha(y)$ is p, hence $b \equiv 1 \bmod p$. Since y has order p as well, we may assume that $b = 1$, proving the lemma. □

Finally, as we need the number of 1-dimensional characters on G, we have to determine its commutator group.

Proposition 10.2.8 (See e.g. [42, Corollary 3.1])**.** *Let G be an $SMC(n,m,p)$. Then $[G,G] = \left\langle x^{\alpha(y)-1} \right\rangle$.*

Proof. As G is generated by x and y, $[G,G]$ equals $\langle\langle [x,y] \rangle\rangle$. Now,

$$[x,y] = x^{-1} x^y = x^{-1} x^{\alpha(y)^{-1}} = x^{\alpha(y)^{-1}-1}.$$

Since $\alpha(y)$ is invertible modulo N, the elements $x^{\alpha(y)^{-1}-1}$ and

$$x^{\alpha(y)-1} = \left(\left(x^{\alpha(y)^{-1}-1}\right)^{\alpha(y)}\right)^{-1}$$

have the same order. This proves the lemma. □

10.3 Determining $\mathrm{Spec}_{\mathrm{R}}(SMC(n,m,p))$

In this section, we determine the Reidemeister spectrum of an arbitrary $SMC(n,m,p)$. We first investigate how we can further simplify Proposition 10.1.10 for $SMC(n,m,p)$. So, let G be an arbitrary $SMC(n,m,p)$. Then, in the notation of Section 10.1.2, $A = C_N$. Suppose that C_N is characteristic (for instance, when $n \geq 2$). Let $\varphi \in \mathrm{Aut}(G)$ and put $\varphi(x) = x^{\gamma}$ with $\gcd(\gamma, N) = 1$. We have to determine $\left|\mathrm{Fix}(\varphi|_A \circ \alpha(y)^i)\right|$ for $i \in \{0, \dots, p-1\}$. By Lemma 7.2.2, we know that

$$\left|\mathrm{Fix}(\varphi|_A \circ \alpha(y)^i)\right| = \gcd(\gamma\alpha(y)^i - 1, N).$$

Strictly speaking, $\gamma\alpha(y)^i$ is an integer modulo N and no true integer, but we interpret $\gamma\alpha(y)^i$ as a representative of the congruence class. Since we are interested in the greatest common divisor with N, this yields no problems. We can get rid of the product by multiplying by $\alpha(y)^{-i}$, the ith power of (a representative of) the multiplicative inverse modulo N of $\alpha(y)$. Then the greatest common divisor does not change:

$$\left|\mathrm{Fix}(\varphi|_A \circ \alpha(y)^i)\right| = \gcd(\gamma - \alpha(y)^{-i}, N)$$

For $\left|\mathrm{Fix}(\widehat{\varphi|_A}) \cap \mathrm{Fix}(\widehat{\alpha(y)})\right|$, note that since $\widehat{C_N} \cong C_N$, we can fix a generator χ_1 of $\widehat{C_N}$. Every character of C_N is then of the form $\chi_a := \chi_1^a$ for some $a \in \mathbb{Z}$. Then we find for $a, k \in \{0, \dots, N-1\}$ that

$$(\chi_a \circ \varphi|_A)(x^k) = \chi_a(x^{\gamma k}) = \chi_a(x^k)^{\gamma} = \chi_{a\gamma}(x^k).$$

Hence, $\widehat{\varphi|_A}(\chi_a) = \chi_a^{\gamma}$ and similarly $\widehat{\alpha(y)}(\chi_a) = \chi_a^{\alpha(y)}$. Therefore, Lemma 7.2.2 yields

$$\left|\mathrm{Fix}(\widehat{\varphi|_A})\right| = \gcd(\gamma - 1, N) \quad \text{and} \quad \left|\mathrm{Fix}(\widehat{\alpha(y)})\right| = \gcd(\alpha(y) - 1, N).$$

Lemma 10.1.12 then implies that

$$\left|\mathrm{Fix}(\widehat{\varphi|_A}) \cap \mathrm{Fix}(\widehat{\alpha(y)})\right| = \gcd(\gcd(\gamma - 1, N), \gcd(\alpha(y) - 1, N))$$

which further simplifies to $\gcd(\gamma-1,\alpha(y)-1,N)$. Finally, by the definition of an $SMC(n,m,p)$, we know that $\gcd(\alpha(y)-1,n)=1$, which implies that

$$\gcd(\gamma-1,\alpha(y)-1,N)=\gcd(\gamma-1,\alpha(y)-1,p^m).$$

We therefore conclude that

$$\mathrm{ch}_{p,\varphi}=\frac{1}{p}\left(\sum_{i=0}^{p-1}\gcd(\gamma-\alpha(y)^{-i},N)\right)-\gcd(\gamma-1,\alpha(y)-1,p^m).$$

Again, switching from $-i$ to i does not change the summation, so we use

$$\mathrm{ch}_{p,\varphi}=\frac{1}{p}\left(\sum_{i=0}^{p-1}\gcd(\gamma-\alpha(y)^{i},N)\right)-\gcd(\gamma-1,\alpha(y)-1,p^m). \tag{10.3.1}$$

We distinguish several cases to effectively determine the Reidemeister spectrum of all $SMC(n,m,p)$. Each case is treated in a similar way: we start by determining the possible values for $\mathrm{ch}_{1,\varphi}$. Next, we simplify (10.3.1) based on the specific action (except for the final case, there we use a different approach). Finally, we combine both results to find candidate-Reidemeister numbers and finish by deciding which ones actually occur.

10.3.1 $\mathrm{Spec}_{\mathrm{R}}(SMC(n,m,p))$ where C_p acts trivially on C_{p^m}

Let G be an $SMC(n,m,p)$ with $n\geq 2$ where C_p acts trivially on C_{p^m}. Recall that each prime factor of n is strictly greater than p. In particular, n is odd: p is at least 2, hence each prime factor of n is at least 3. Also, $C_N=\langle x\rangle$ is characteristic in G, by Lemma 10.2.6. Moreover, Proposition 10.2.8 states that $[G,G]=\left\langle x^{\alpha(y)-1}\right\rangle$. We already assume that $\gcd(\alpha(y)-1,n)=1$ and as C_p acts trivially on C_{p^m}, $\gcd(\alpha(y)-1,p^m)=p^m$. Therefore, $[G,G]=\left\langle x^{p^m}\right\rangle=C_n$.

Proposition 10.3.1. *Let $\varphi\in\mathrm{Aut}(G)$ be an automorphism. Write $\varphi(x)=x^\gamma$ for some $\gamma\in\mathbb{Z}$ coprime with N and put $e=\nu_p(\gcd(\gamma-1,p^m))$. Then $\mathrm{ch}_{1,\varphi}\in\{p^e,p^{e+1}\}$.*

Proof. As $n\geq 2$, $\varphi(y)=x^a y$ for some $a\in\mathbb{Z}$, by Lemma 10.2.7. Also, $\varphi^{\mathrm{ab}}(\bar{x})=\bar{x}^\gamma$, thus $\varphi^{\mathrm{ab}}(\bar{x}\bar{y})=(\bar{x}^{\gamma+a}\bar{y})$ on $G/\gamma_2(G)\cong C_{p^m}\times C_p$. If $m=0$, then φ^{ab} is the identity map, hence $\mathrm{ch}_{1,\varphi}=R(\varphi^{\mathrm{ab}})=p=p^{e+1}$. Suppose $m\geq 1$. Since $\bar{y}$ has order p, so does $\varphi^{\mathrm{ab}}(\bar{y})$, hence $a\equiv 0 \bmod p^{m-1}$. As we work in an abelian group, $R(\varphi^{\mathrm{ab}}) = \left|\mathrm{Fix}(\varphi^{\mathrm{ab}})\right|$ by Corollary 8.1.3. Thus, we need to determine the number of solutions of the congruence

$$i(\gamma - 1) + ja \equiv 0 \bmod p^m,$$

where $i \in \{0, \ldots, p^m - 1\}$ and $j \in \{0, \ldots, p - 1\}$. Write $\gamma - 1 = p^e\Gamma$ and $a = p^{m-1}A$ with $A, \Gamma \in \mathbb{Z}$. Then the congruence becomes

$$i\Gamma p^e + jAp^{m-1} \equiv 0 \bmod p^m.$$

First suppose that $e = m$. Then i can take on any value, which yields p^m possibilities. If $A \equiv 0 \bmod p$, then j can also take on any value, which yields p^{m+1} total solutions. If $A \not\equiv 0 \bmod p$, then $j \equiv 0 \bmod p$, which yields 1 possibility for j and hence p^m solutions in total.

Now suppose that $e \leq m-1$. Then $\gcd(\Gamma, p) = 1$. Modulo p^{m-1}, the congruence yields $i\Gamma p^e \equiv 0 \bmod p^{m-1}$, hence $i \equiv 0 \bmod p^{m-e-1}$. Write $i = p^{m-e-1}I$ for some $I \in \mathbb{Z}$. After we divide by p^{m-1}, the congruence becomes

$$I\Gamma + jA \equiv 0 \bmod p.$$

Recall that Γ is coprime with p. If $A \equiv 0 \bmod p$, then j can take on any value and $I \equiv 0 \bmod p$. Thus, $i \equiv 0 \bmod p^{m-e}$, which yields p^e possibilities for i and a total of $p^e \cdot p = p^{e+1}$ solutions. If $A \not\equiv 0 \bmod p$, then each value of j yields a unique value for I modulo p. As $0 \leq i \leq p^m - 1$, we have that $0 \leq I \leq p^{e+1} - 1$. Hence, each value of j yields p^e possibilities for i. Thus, given j, I can take on p^e possible values. Therefore, there are p^e possibilities for i for each j. Thus, we find a total of $p^e \cdot p = p^{e+1}$ solutions to the original congruence.

Summarised, we have

- p solutions if $m = 0$;
- p^{e+1} solutions if $e \leq m - 1$ and $m \geq 1$.
- p^{e+1} solutions if $e = m$ and $a \equiv 0 \bmod p^m$.
- p^e solutions if $e = m$ and $a \not\equiv 0 \bmod p^m$. □

Remark. Note that e as defined above is at least 1 if $p = 2$ and $m \geq 1$, since then γ is odd and hence $\gamma - 1$ is even.

Proposition 10.3.2. *Let $\varphi \in \mathrm{Aut}(G)$ and write $\varphi(x) = x^\gamma$ for some $\gamma \in \mathbb{Z}$ coprime with N. Put $e = \mathrm{ord}_p(\gcd(\gamma - 1, p^m))$ and put, for $i \in \{0, \ldots, p-1\}$, $d_i = \gcd(\gamma - \alpha(y)^i, N)/p^e$. Then $d_0, \ldots, d_{p-1}$ are pairwise coprime integers, all divide n and*

$$\mathrm{ch}_{p,\varphi} = p^{e-1}\left(\sum_{i=0}^{p-1} d_i\right) - p^e.$$

Moreover, if $p=2$ *and* $n \equiv 0 \bmod 3$*, then* $d_0 d_1 \equiv 0 \bmod 3$*.*

Proof. Fix $\varphi \in \mathrm{Aut}(G)$. As C_N is characteristic, we know by (10.3.1) that

$$\mathrm{ch}_{p,\varphi} = \frac{1}{p}\left(\sum_{i=0}^{p-1} \gcd(\gamma - \alpha(y)^i, N)\right) - \gcd(\gamma - 1, \alpha(y) - 1, p^m). \tag{10.3.2}$$

Put, for $i \in \{0,\dots,p-1\}$, $a_i = \gcd(\gamma - \alpha(y)^i, N)$. Then for $i \neq j \in \{0,\dots,p-1\}$,

$$\gcd(a_i, a_j) = \gcd(\gamma - \alpha(y)^i, \gamma - \alpha(y)^j, N) = \gcd(\gamma - \alpha(y)^i, \alpha(y)^i - \alpha(y)^j, N).$$

Without loss of generality, assume $i > j$. Since $\alpha(y)$ is coprime with N, $\gcd(\alpha(y)^i - \alpha(y)^j, N) = \gcd(\alpha(y)^{i-j} - 1, N)$. By the inqualities $0 < i - j < p$ and the definition of an $SMC(n,m,p)$, $\gcd(\alpha(y)^{i-j} - 1, n) = 1$. Hence,

$$\gcd(\alpha(y)^{i-j} - 1, N) = \gcd(\alpha(y)^{i-j} - 1, p^m).$$

Furthermore, we assume here that C_p acts trivially on C_{p^m}, which implies that $\alpha(y) \equiv 1 \bmod p^m$. Thus, $\gcd(\alpha(y)^{i-j} - 1, p^m) = p^m$, which implies that

$$\gcd(a_i, a_j) = \gcd(\gamma - \alpha(y)^i, p^m) = \gcd(\gamma - 1, p^m) = p^e.$$

Thus, each $d_i = a_i / p^e$ is an integer. As a_i divides N and

$$\gcd(a_i, p^m) = \gcd(\gamma - \alpha(y)^i, p^m) = \gcd(\gamma - 1, p^m) = p^e$$

for each $i \in \{0,\dots,p-1\}$, the d_i are pairwise coprime divisors of n.

Similarly, we find that

$$\gcd(\gamma - 1, \alpha(y) - 1, p^m) = \gcd(\gamma - 1, p^m) = p^e.$$

We can therefore rewrite (10.3.2) as

$$\mathrm{ch}_{p,\varphi} = \frac{1}{p}\left(\sum_{i=0}^{p-1} p^e d_i\right) - p^e = p^{e-1}\left(\sum_{i=0}^{p-1} d_i\right) - p^e,$$

which yields the desired expression.

Finally, suppose that $p = 2$ and that $n \equiv 0 \bmod 3$. Since C_2 acts without fixed points on C_n, it acts by inversion, by Lemma 10.2.4, so $\alpha(y) \equiv -1 \bmod n$. Then $a_0 = \gcd(\gamma - 1, N)$ and $a_1 = \gcd(\gamma + 1, N)$. As $\gcd(\gamma, N) = 1$, it follows that $\gamma \equiv \pm 1 \bmod 3$, hence $a_0 a_1 \equiv 0 \bmod 3$, and thus also $d_0 d_1 \equiv 0 \bmod 3$. □

Remark. The latter condition also ties in with Theorem 9.3.9. Indeed, let f be the exponent of 3 in N. Then $\gcd(\gamma \pm 1, N)$ is divisible by $\gcd(\gamma \pm 1, 3^f)$. Also, $\gcd(\gamma + 1, 3^f) = \gcd(-\gamma - 1, 3^f)$. Let Q denote the Sylow 3-subgroup of C_N. By Lemma 7.2.2,

$$\gcd(\gamma - 1, 3^f) \gcd(-\gamma - 1, 3^f) = |\mathrm{Fix}(\varphi|_Q)| \cdot |\mathrm{Fix}(-\varphi|_Q)|.$$

Theorem 9.3.9 with $S = \{-1, 1\}$ then implies that this product is at least 3, as Q has type (f). Consequently, $a_0 a_1$ is divisible by 3.

We can now fully determine the Reidemeister spectrum. For positive integers a and b we put

$$D(a,b) := \{(d_1, \ldots, d_b) \in \mathbb{Z}^b_{\geq 0} \mid \forall i \in \{1, \ldots, b\} : d_i \mid a,$$
$$\forall i \neq j \in \{1, \ldots, b\} : \gcd(d_i, d_j) = 1\},$$

and

$$D'(a,b) := \{(d_1, \ldots, d_b) \in D(a,b) \mid \exists i \in \{1, \ldots, b\} : d_i \equiv 0 \bmod 3\}.$$

For $d \in D(a,b)$ or $D'(a,b)$, we define $\Sigma(d) := \sum_{i=1}^{p} d_i$.

Theorem 10.3.3. *Let G be an $SMC(n,m,p)$ with $n \geq 2$ and where C_p acts trivially on C_{p^m}.*

(1) If p is odd and $m = 0$, then

$$\mathrm{Spec_R}(G) = \left\{ p + \frac{\Sigma(d)}{p} - 1 \;\middle|\; d \in D(n,p) \right\}.$$

(2) If p is odd and $m \geq 1$, then

$$\mathrm{Spec_R}(G) = \left\{ p^{e+1} + p^{e-1}(\Sigma(d) - p) \;\middle|\; 0 \leq e \leq m, d \in D(n,p) \right\} \cup$$
$$\left\{ p^m + p^{m-1}(\Sigma(d) - p) \;\middle|\; d \in D(n,p) \right\}.$$

(3) If $p = 2$, $m = 0$ and $n \not\equiv 0 \bmod 3$, then

$$\mathrm{Spec_R}(G) = \left\{ 2 + \frac{\Sigma(d)}{2} - 1 \;\middle|\; d \in D(n,2) \right\}.$$

(4) If $p = 2$, $m \geq 1$ and $n \not\equiv 0 \bmod 3$, then

$$\mathrm{Spec_R}(G) = \left\{ 2^{e+1} + 2^{e-1}(\Sigma(d) - 2) \;\middle|\; 1 \leq e \leq m, d \in D(n,2) \right\} \cup$$
$$\left\{ 2^m + 2^{m-1}(\Sigma(d) - 2) \;\middle|\; d \in D(n,2) \right\}.$$

(5) *If* $p = 2$ *and* $n \equiv 0 \bmod 3$*, then* $D(n, 2)$ *has to be replaced with* $D'(n, 2)$ *in both expressions.*

Proof. For the $\subseteq$-inclusion for all cases of p and m, we combine Proposition 10.3.1 (together with the remark following it) with Proposition 10.3.2.

Next, we investigate the other inclusion for both cases of p and m. To prove that the candidate-Reidemeister numbers do indeed lie in $\mathrm{Spec}_{\mathrm{R}}(G)$, we have to solve the following problem: given $d = (d_0, \ldots, d_{p-1})$ and e satisfying the necessary conditions, find an integer $\gamma \in \mathbb{Z}$ coprime with N such that

$$\gcd(\gamma - \alpha(y)^i, N) = p^e d_i \text{ for all } i \in \{0, \ldots, p-1\}.$$

Indeed, Propositions 10.3.1 and 10.3.2 then imply that the map $\varphi : G \to G$ given by $\varphi(x) = x^\gamma, \varphi(y) = y$ is an automorphism of G with Reidemeister number equal to $p^{e+1} + p^{e-1}(\Sigma(d) - 1)$. For $e = m \geq 1$, the map $\varphi : G \to G$ given by $\varphi(x) = x^\gamma, \varphi(y) = x^{np^{m-1}} y$ is then an automorphism of G with Reidemeister number equal to $p^m + p^{m-1}(\Sigma(d) - p)$, again by Propositions 10.3.1 and 10.3.2. Note that $\varphi(y)$ has indeed order p, as $x^{np^{m-1}} \in C_{p^m}$ commutes with y and has order p as well.

Using Theorem 10.1.11, we find a $\gamma \in \mathbb{Z}$ such that $\gcd(\gamma, n) = 1$ and such that $\gcd(\gamma - \alpha(y)^i, n) = d_i$ for all $i \in \{0, \ldots, p-1\}$. If $m = 0$, γ is the desired number and we are done. So, assume that $m \geq 1$. Then we also obtain from Theorem 10.1.11 that $\gamma \equiv 1 + p^e \bmod p^m$. If p were to divide γ, then $\gamma \equiv p^e + 1 \bmod p^m$ leads to $0 \equiv 1 \bmod p$ or $0 \equiv 2 \bmod p$ when p is odd, and to $0 \equiv 1 \bmod p$ when $p = 2$, since then $e \geq 1$. In either case, we have a contradiction, which shows that $\gcd(\gamma, p) = 1$. It follows that $\gcd(\gamma, N) = 1$.

Next, let $i \in \{0, \ldots, p-1\}$. Note that

$$\gamma - \alpha(y)^i \equiv \gamma - 1 \equiv p^e \bmod p^m$$

since $\alpha(y) \equiv 1 \bmod p^m$ as C_p acts trivially on C_{p^m}. Therefore, $\gcd(\gamma - \alpha(y)^i, p^m) = p^e$. Consequently, $\gcd(\gamma - \alpha(y)^i, N) = p^e d_i$, which proves that γ is the desired number. $\square$

Example 10.3.4. Let $p < q$ be distinct primes. Then the number of non-isomorphic groups of order pq is either 1, if p does not divide $q - 1$, or 2 in the other case (see e.g. [27, Example, p. 181]). In either case, there is the abelian group $C_p \times C_q$, whose Reidemeister spectrum is given by

$$\mathrm{Spec}_{\mathrm{R}}(C_p \times C_q) = \begin{cases} \{1, p, q, pq\} & \text{if } p \text{ and } q \text{ are odd} \\ \{2, 2q\} & \text{if } p = 2 \end{cases}$$

by Theorem 9.3.18.

If p divides $q-1$, then the other group of order pq is given by $SMC(q,0,p) = C_q \rtimes C_p$, where the homomorphism $C_p \to \mathrm{Aut}(C_q)$ is non-trivial. Then

$$\mathrm{Spec}_{\mathrm{R}}(SMC(q,0,p)) = \begin{cases} \left\{p, p + \frac{q-1}{p}\right\} & \text{if } q \neq 3 \text{ or } p \neq 2 \\ \{3\} & \text{if } p = 2, q = 3 \end{cases}$$

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Example 10.3.5. We provide here a table containing the spectra of $SMC(n,m,p)$ where C_p acts trivially on C_{p^m} for some concrete values of n, m and p.

n	m	p	$\mathrm{Spec}_{\mathrm{R}}(SMC(n,m,p))$
$175 = 5^2 \cdot 7$	0	2	$\{2,4,5,7,14,17,19,89\}$
175	1	2	$\{2,4,6,8,10,12,14,26,28,32,34,36,38,176,178\}$
175	2	2	$\{4,8,10,12,14,16,20,24,28,34,38,52,56,64,68,$ $72,76,178,352,356\}$
$105 = 3 \cdot 5 \cdot 7$	0	2	$\{3,5,6,9,12,14,20,54\}$
105	1	2	$\{4,6,8,10,12,16,18,22,24,26,28,38,40,106,108\}$
105	2	2	$\{6,8,10,12,16,18,20,24,28,32,36,40,44,48,52,$ $56,76,80,108,212,216\}$
$91 = 7 \cdot 13$	0	3	$\{3,5,7,9,33\}$
91	1	3	$\{3,5,7,9,15,21,27,33,93,99\}$
91	2	3	$\{3,5,7,9,15,21,27,33,45,63,81,99,279,297\}$
$341 = 11 \cdot 31$	0	5	$\{5,7,11,13,73\}$
341	1	5	$\{5,7,11,13,15,25,35,45,55,65,73,345,365\}$
341	2	5	$\{5,7,11,13,25,35,55,65,73,75,125,175,225,275,$ $325,365,1725,1825\}$

Table 10.1: Reidemeister spectrum of $SMC(n,m,p)$ where C_p acts trivially on C_{p^m} for some concrete values of n, m and p.

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10.3.2 $\mathrm{Spec}_{\mathrm{R}}(SMC(n,m,p))$ where C_p acts non-trivially on C_{p^m}

Let G be an $SMC(n,m,p)$ where C_p acts freely on C_n and non-trivially on C_{p^m}. This implies that $m \geq 2$. If p is odd, then $\alpha(y) \equiv \beta p^{m-1} + 1 \bmod p^m$ for some $\beta \in \{1,\ldots,p-1\}$, by Lemma 10.2.5. If $p = 2$, there are three possibilities:

- $\alpha(y) \equiv -1 \bmod 2^m$
- $\alpha(y) \equiv 2^{m-1} - 1 \bmod 2^m$

- $\alpha(y) \equiv 2^{m-1} + 1 \bmod 2^m$

Remark. If $m = 2$, the second case is the trivial action and the first and third coincide. Therefore, for $m = 2$ we only consider the first case.

Either way, n is odd and each prime factor of n is strictly greater than p. A priori, it is possible that $n = 1$. However, we impose further restrictions on n, depending on the values of m and p.

10.3.2.1 $\alpha(y) \equiv 1 \bmod p^{m-1}$ and $n \geq 2$

In this section, we assume that $\alpha(y) \equiv \beta p^{m-1} + 1 \bmod p^m$ for some $\beta \in \{1, \ldots, p-1\}$ and that $n \geq 2$. Moreover, if $p = 2$, we may assume that $m \geq 3$, as remarked before. Since $n \geq 2$, Lemma 10.2.6 implies that $C_N = \langle x \rangle$ is characteristic in G. Also, Proposition 10.2.8 implies that $[G, G] = \left\langle x^{p^{m-1}} \right\rangle$.

Proposition 10.3.6. *For $\varphi \in \mathrm{Aut}(G)$, write $\varphi(x) = x^\gamma$ for some $\gamma \in \mathbb{Z}$ coprime with N. Put $e = \nu_p(\gcd(\gamma - 1, p^m))$. Then $\mathrm{ch}_{1,\varphi} = p^{\min\{m, e+1\}}$.*

Proof. By Lemma 10.2.7, $\varphi(y) = x^a y$ for some $a \in \mathbb{Z}$. By Lemma 10.2.6, φ induces an automorphism on $\frac{G}{C_n} \cong C_{p^m} \rtimes_{\bar{\alpha}} C_p$, with respective generators $\bar{x}$ and $\bar{y}$. Since $\bar{y}$ has order p, so must $\bar{x}^a \bar{y}$. Therefore,

$$1 = (\bar{x}^a \bar{y})^p = \bar{x}^{a(1+\alpha(y)+\ldots+\alpha(y)^{p-1})}. \tag{10.3.3}$$

Now,

$$\begin{aligned}
\sum_{i=0}^{p-1} \alpha(y)^i &\equiv \sum_{i=0}^{p-1} (\beta p^{m-1} + 1)^i \\
&\equiv \sum_{i=0}^{p-1} (\beta i p^{m-1} + 1) \\
&\equiv p + \beta p^{m-1} \frac{p(p-1)}{2} \bmod p^m.
\end{aligned}$$

If p is odd, $\frac{p-1}{2} \in \mathbb{Z}$. Therefore, the sum reduces to $p \bmod p^m$. From (10.3.3) it then follows that $ap \equiv 0 \bmod p^m$. Therefore, $a \equiv 0 \bmod p^{m-1}$. If $p = 2$, the sum reduces to $2 + 2^{m-1} \bmod 2^m$. As we may assume for $p = 2$ that $m \geq 3$, we see that $a(2 + 2^{m-1}) \equiv 0 \bmod 2^m$ implies that $a \equiv 0 \bmod 2^{m-1}$.

Now, $\frac{G}{\gamma_2(G)}$ is isomorphic to $C_{p^{m-1}} \times C_p$ since $\gamma_2(G) = \left\langle x^{p^{m-1}} \right\rangle$. If $\bar{x}$ and $\bar{y}$ denote the projections of x and y, respectively, on the abelianisation, the induced automorphism φ^{ab} then satisfies

$$\varphi^{\text{ab}}(\bar{x}) = \bar{x}^{\gamma}, \varphi^{\text{ab}}(\bar{y}) = \bar{y},$$

as $\bar{x}^a = \bar{1}$ in $\frac{G}{\gamma_2(G)}$. By Proposition 3.1.4 and Lemma 7.2.2, the Reidemeister number of φ^{ab} is given by $\gcd(\gamma - 1, p^{m-1}) \cdot p = p^{\min\{e, m-1\}+1}$. Note that γ is only defined modulo N, hence in particular modulo p^m, but this does not affect $\min\{e, m-1\}$. We conclude that $\text{ch}_{1,\varphi} = R(\varphi^{\text{ab}}) = p^{\min\{e+1,m\}}$, by Lemma 10.1.5. $\square$

Proposition 10.3.7. *Let $\varphi \in \text{Aut}(G)$ and write $\varphi(x) = x^{\gamma}$ for some $\gamma \in \mathbb{Z}$ coprime with N. Put $e = \nu_p(\gcd(\gamma - 1, p^m))$ and put, for $i \in \{0, \ldots, p-1\}, d_i = \gcd(\gamma - \alpha(y)^i, N)/p^{\min\{e,m-1\}}$. Then the d_i are pairwise coprime integers, all divide np, and $\text{ch}_{p,\varphi} = p^{\min\{e-1,m-2\}}(\Sigma(d) - p)$, where $d = (d_0, \ldots, d_{p-1})$.*

Moreover,

(1) $d_0 \ldots d_{p-1} \equiv 0 \bmod p$ if and only if $e \geq m - 1$;

(2) $d_0 d_1 \equiv 0 \bmod 3$ if $p = 2$ and $n \equiv 0 \bmod 3$.

Proof. Fix $\varphi \in \text{Aut}(G)$. The proof of the expression for $\text{ch}_{p,\varphi}$ is almost identical to the one in Proposition 10.3.1, but there are some subtle differences. By Lemma 10.2.6, C_N is characteristic in G. Hence, by (10.3.1), we know that

$$\text{ch}_{p,\varphi} = \frac{1}{p}\left(\sum_{i=0}^{p-1} \gcd(\gamma - \alpha(y)^i, N)\right) - \gcd(\gamma - 1, \alpha(y) - 1, p^m). \qquad (10.3.4)$$

Put, for $i \in \{0, \ldots, p-1\}$, $a_i = \gcd(\gamma - \alpha(y)^i, N)$. Then for $i \neq j \in \{0, \ldots, p-1\}$ we have

$$\gcd(a_i, a_j) = \gcd(\gamma - \alpha(y)^i, \gamma - \alpha(y)^j, N) = \gcd(\gamma - \alpha(y)^i, \alpha(y)^i - \alpha(y)^j, N).$$

Without loss of generality, assume $i > j$. Since $\alpha(y)$ is coprime with N, $\gcd(\alpha(y)^i - \alpha(y)^j, N) = \gcd(\alpha(y)^{i-j} - 1, N)$. By the inequalities $0 < i - j < p$ and by the definition of an $SMC(n, m, p)$, we have that $\gcd(\alpha(y)^{i-j} - 1, n) = 1$. Hence,

$$\gcd(\alpha(y)^{i-j} - 1, N) = \gcd(\alpha(y)^{i-j} - 1, p^m).$$

Furthermore, as we assume here that $\alpha(y) \equiv \beta p^{m-1} + 1 \bmod p^m$ for some β coprime with p, we know that $\gcd(\alpha(y)^{i-j} - 1, p^m) = p^{m-1}$, which implies that

$$\gcd(a_i, a_j) = \gcd(\gamma - \alpha(y)^i, p^{m-1}) = \gcd(\gamma - 1, p^{m-1}) = p^{\min\{e,m-1\}}.$$

Thus, since $d_i = a_i/p^{\min\{e,m-1\}}$ for each $i \in \{0,\dots,p-1\}$, we find that the d_i are integers and pairwise coprime.

Now, note that

$$\gcd(a_i, p^m) = \gcd(\gamma - \alpha(y)^i, p^m) = \gcd(\gamma - \beta i p^{m-1} - 1, p^m).$$

If $e \leq m-2$, then this equals p^e for each i. If $e = m$, then

$$\gcd(\gamma - \beta i p^{m-1} - 1, p^m) = \begin{cases} p^m & \text{if } i = 0 \\ p^{m-1} & \text{if } i \neq 0. \end{cases}$$

Finally, if $e = m-1$, then write $\gamma - 1 = p^{m-1}\Gamma$ with $\gcd(\Gamma, p) = 1$ ($\Gamma \neq 0$ as $\gamma \neq 1$ in that case). Then

$$\gcd(\gamma - \beta i p^{m-1} - 1, p^m) = p^{m-1}\gcd(\Gamma - \beta i, p) = \begin{cases} p^m & \text{if } \Gamma - \beta i \equiv 0 \bmod p \\ p^{m-1} & \text{otherwise.} \end{cases}$$

This proves that in all cases d_i divides np for all $i \in \{0,\dots,p-1\}$ and proves the first 'moreover'-claim. Similarly, we find that

$$\gcd(\gamma - 1, \alpha(y) - 1, p^m) = \gcd(\gamma - 1, p^{m-1}) = p^{\min\{e,m-1\}}.$$

We can therefore rewrite (10.3.4) as

$$\mathrm{ch}_{p,\varphi} = \frac{1}{p}\left(\sum_{i=0}^{p-1} p^{\min\{e,m-1\}} d_i\right) - p^{\min\{e,m-1\}} = p^{\min\{e-1,m-2\}}(\Sigma(d) - p).$$

which yields the desired expression.

The fact that $d_0 d_1 \equiv 0 \bmod 3$ if $p = 2$ and $n \equiv 0 \bmod 3$ is proven similarly as in Proposition 10.3.2. □

Theorem 10.3.8. *Let G be an $SMC(n,m,p)$ with $n \geq 2$ and $\alpha(y) \equiv \beta p^{m-1} + 1 \bmod p^m$ for some $\beta \in \{1,\dots,p-1\}$. In particular, $m \geq 2$.*

(1) If p is odd, then

$$\mathrm{Spec}_{\mathrm{R}}(G) = \big\{p^{e+1} + p^{e-1}(\Sigma(d) - p) \;\big|\; 0 \leq e \leq m-1, d \in D(np,p),$$
$$d_1 \dots d_p \equiv 0 \bmod p \iff e = m-1\big\}.$$

(2) If $p = 2$ and $n \not\equiv 0 \bmod 3$, then

$$\mathrm{Spec}_{\mathrm{R}}(G) = \big\{2^{e+1} + 2^{e-1}(\Sigma(d) - 2) \;\big|\; 1 \leq e \leq m-1, d \in D(2n,2),$$
$$d_1 d_2 \equiv 0 \bmod 2 \iff e = m-1\big\}.$$

(3) If $p = 2$ and $n \equiv 0 \bmod 3$, then $D(2n, 2)$ has to be replaced by $D'(2n, 2)$.

Proof. The $\subseteq$-inclusion is proven by combining Proposition 10.3.6 and Proposition 10.3.7 with the following remarks:

- It follows from the expressions for $\mathrm{ch}_{1,\varphi}$ and $\mathrm{ch}_{p,\varphi}$ that we can assume that $e \leq m - 1$;
- The condition $d_1 \dots d_p \equiv 0 \bmod p \iff e = m - 1$ is the first 'moreover'-part of Proposition 10.3.7;
- If $\varphi \in \mathrm{Aut}(G)$, then $\varphi(x) = x^\gamma$ for some $\gamma \in \mathbb{Z}$ coprime with N, by Lemma 10.2.6. If $p = 2$, this implies that γ is odd, therefore, $\gamma - 1$ is even. As the exponent e comes from $\mathrm{ord}_2(\gamma - 1)$, this shows why $1 \leq e \leq m - 1$ for $p = 2$.

For the other inclusion, let $d_0, \dots, d_{p-1}$ be divisors of np satisfying the conditions of Proposition 10.3.7 and let e be an integer satisfying the given inequalities. For $i \in \{0, \dots, p-1\}$, put $d'_i = \frac{d_i}{p^{\mathrm{ord}_p(d_i)}}$. Then Theorem 10.1.11 provides us with a $\gamma \in \mathbb{Z}$ satisfying $\gcd(\gamma, n) = 1$, $\gamma \equiv 1 + p^e \bmod p^m$ and $\gcd(\gamma - \alpha(y)^i, n) = d'_i$ for each $i \in \{0, \dots, p-1\}$. Using the same argument as in Theorem 10.3.3, it follows that $\gcd(\gamma, N) = 1$.

Now, for $\gcd(\gamma - \alpha(y)^i, p^m)$, first suppose that $e \leq m - 2$. Then none of the d_i is divisible by p, which means that $d_i = d'_i$ for all $i \in \{0, \dots, p-1\}$. Let $0 \leq i \leq p - 1$. Since $e \leq m - 2$, the congruences $\gamma - 1 \equiv p^e \bmod p^m$ and $\alpha(y) \equiv 1 \bmod p^{m-1}$ together imply that

$$\gamma - \alpha(y)^i \equiv \gamma - 1 \equiv p^e \bmod p^{m-1},$$

which shows that the highest power of p dividing $\gamma - \alpha(y)^i$ is p^e. Therefore, $\gcd(\gamma - \alpha(y)^i, N) = p^e d_i$ as desired.

For $e = m - 1$, we may assume, without loss of generality, that $d_{\beta^{-1} \bmod p} \equiv 0 \bmod p$, which means that $d'_j = d_j$ for $j \not\equiv \beta^{-1} \bmod p$ and $d_{\beta^{-1} \bmod p} = pd'_{\beta^{-1} \bmod p}$. For $i \in \{0, \dots, p-1\}$, we compute $\gamma - \alpha(y)^i \bmod p^m$:

$$\begin{aligned}
\gamma - \alpha(y)^i &\equiv \gamma - (\beta p^{m-1} + 1)^i \\
&\equiv \gamma - i\beta p^{m-1} - 1 \\
&\equiv p^{m-1} - i\beta p^{m-1} \\
&\equiv p^{m-1}(1 - i\beta) \bmod p^m,
\end{aligned}$$

where the second congruence follows from the fact that $m \geq 2$, and the third from the fact that $\gamma - 1 \equiv p^{m-1} \bmod p^m$. We conclude that $\gamma - \alpha(y)^i$ is always divisible by p^{m-1} and that it is divisible by p^m if and only if the last expression is 0 modulo p^m, hence if and only if $1 - i\beta \equiv 0 \bmod p$; in other words, if and only if $i \equiv \beta^{-1} \bmod p$. This ensures that $\gcd(\gamma - \alpha(y)^i, N) = p^e d_i$.

Propositions 10.3.6 and 10.3.7 then imply that the map $\varphi : G \to G$ given by $\varphi(x) = x^\gamma, \varphi(y) = y$ is an automorphism of G with Reidemeister number equal to $p^{e+1} + p^{e-1}(\Sigma(d) - p)$. □

Example 10.3.9. As we did after Theorem 10.3.3, we provide a table containing the spectra of $SMC(n, m, p)$ where C_p acts non-trivially on C_{p^m} for some concrete values of n, m and p.

n	m	p	$\mathrm{Spec}_{\mathrm{R}}(SMC(n, m, p))$
$175 = 5^2 \cdot 7$	2	2	$\{3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 27, 29, 37, 39, 41, 51, 53, 57, 59, 71, 73, 177, 179, 351, 353\}$
$105 = 3 \cdot 5 \cdot 7$	2	2	$\{5, 7, 9, 11, 13, 15, 17, 19, 23, 25, 29, 31, 33, 37, 39, 41, 43, 45, 47, 49, 73, 75, 107, 109, 211, 213\}$
$91 = 7 \cdot 13$	2	3	$\{3, 5, 7, 9, 11, 17, 23, 29, 33, 41, 47, 53, 101, 281\}$
$341 = 11 \cdot 31$	2	5	$\{5, 7, 11, 13, 29, 39, 59, 69, 73, 79, 109, 179, 189, 369, 1729\}$

Table 10.2: Reidemeister spectrum of $SMC(n, m, p)$ where C_p acts non-trivially on C_{p^m} for some concrete values of n, m and p.

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10.3.2.2 $p = 2$ and $\alpha(y) \equiv -1 \bmod 2^{m-1}$

By Lemma 10.2.4, C_2 acts on C_n by inversion. From the congruences of $\alpha(y)$ modulo n and 2^{m-1}, it follows that $\alpha(y) \equiv -1 \bmod 2^{m-1}n$. This yields two possibilities for $\alpha(y) \bmod N$, namely $\alpha(y) \equiv -1$ or $2^{m-1}n - 1 \bmod N$. In either case, Proposition 10.2.8 yields $[G, G] = \langle x^2 \rangle$. We do not put any restrictions on n, so $n = 1$ is possible. Also recall that $m \geq 2$ and if $\alpha(y) \equiv 2^{m-1}n - 1 \bmod N$, we may assume $m \geq 3$. This in particular implies that $\alpha(y) \equiv -1 \bmod 4$ in either case. Finally, since $p = 2$, $\alpha(y)$ has order 2, so $\alpha(y) = \alpha(y)^{-1}$.

The expression for $\mathrm{Spec}_{\mathrm{R}}(SMC(n, m, 2))$ is nearly identical in both cases. Therefore, we treat them together, pointing out the differences when they occur.

Lemma 10.3.10. *The subgroup C_N is characteristic in G.*

Proof. Let $a \in \mathbb{Z}$. Then

$$(x^a y)^2 = x^a y x^a y = x^a x^{\alpha(y)a} = x^{a(1+\alpha(y))}.$$

If $\alpha(y) \equiv -1 \bmod 2^m$, the element $x^a y$ has order 2, whereas x has order $n \cdot 2^m > 2$. If $\alpha(y) \equiv 2^{m-1} - 1 \bmod 2^m$, the element $x^a y$ has order either 2 or 4, whereas x has order $n \cdot 2^m > 4$ (recall that $m \geq 3$ in that case). Either way, every element of the same order as x lies in C_N, which shows that C_N is characteristic. □

Proposition 10.3.11. *Let φ be an automorphism of G. Then $\mathrm{ch}_{1,\varphi} \in \{2, 4\}$. Moreover, if $\alpha(y) \equiv 2^{m-1} - 1 \bmod 2^m$, then $\mathrm{ch}_{1,\varphi} = 4$.*

Proof. By Lemma 10.3.10, we know that $\varphi(x) = x^\gamma$ for some odd $\gamma \in \mathbb{Z}$. Consequently, $\varphi^{\mathrm{ab}}(\bar{x}) = \bar{x}$, as $[G, G] = \langle x^2 \rangle$, which proves that φ^{ab} has at least 2 fixed points. Since G^{ab} is abelian and has size 4, we conclude that $\mathrm{ch}_{1,\varphi} = R(\varphi^{\mathrm{ab}}) \in \{2, 4\}$.

Now assume $\alpha(y) \equiv 2^{m-1} - 1 \bmod 2^m$. Writing $\varphi(y) = x^c y^d$, we see that $d = 1$, otherwise $y \notin \mathrm{Im}(\varphi)$. The element $\varphi(y)$ must have order 2, thus

$$1 = (x^c y)^2 = x^{2^{m-1} nc}$$

which shows that $c \equiv 0 \bmod 2$. Consequently, $\varphi^{\mathrm{ab}}(\bar{y}) = \bar{y}$. We conclude that φ^{ab} is the identity map. Therefore, $\mathrm{ch}_{1,\varphi} = R(\varphi^{\mathrm{ab}}) = 4$. □

Proposition 10.3.12. *Let $\varphi \in \mathrm{Aut}(G)$. Then there exist coprime positive integers d_0, d_1 dividing n and an integer $e \geq 1$ such that*

$$\mathrm{ch}_{2,\varphi} = 2^e d_0 + d_1 - 2.$$

Moreover, if $n \equiv 0 \bmod 3$, then $d_0 d_1 \equiv 0 \bmod 3$.

Proof. By Lemma 10.3.10, we can write $\varphi(x) = x^\gamma$ for some γ coprime with N.

Again, we can use (10.3.1) to determine $\mathrm{ch}_{2,\varphi}$, since C_N is characteristic. The expression reads

$$\mathrm{ch}_{2,\varphi} = \frac{\gcd(\gamma - 1, N) + \gcd(\gamma - \alpha(y), N)}{2} - \gcd(\gamma - 1, \alpha(y) - 1, 2^m).$$

The last term equals 2, as $\alpha(y) - 1 \equiv 2 \bmod 4$ and γ is odd. For the first two terms, remark that exactly one of the numbers $\gamma - 1$ or $\gamma - \alpha(y)$ is a multiple of 4. Indeed, both are even and their difference $-\alpha(y) + 1$ is congruent to 2 modulo 4. The other is then equal to $2a$ for some odd number a. We may assume that

$\gamma - 1$ is a multiple of 4, by noting that $\tau_y \circ \varphi$ is an automorphism that maps x to $x^{\alpha(y)\gamma}$ and that has the same Reidemeister number as φ by Lemma T.1.7(2).

Thus, write $\gcd(\gamma - 1, N) = 2^f d_0$ and $\gcd(\gamma - \alpha(y), N) = 2d_1$ with $f \geq 2$ and d_0 and d_1 both dividing n. We get

$$\begin{aligned} \mathrm{ch}_{2,\varphi} &= \frac{2^f d_0 + 2d_1}{2} - 2 \\ &= 2^{f-1} d_0 + d_1 - 2. \end{aligned}$$

Recall that n is odd, hence so are d_0 and d_1. Thus, $d := \gcd(d_0, d_1)$ is an odd divisor of n dividing both $\gamma - 1$ and $\gamma - \alpha(y)$. Hence, d divides $\alpha(y) - 1$, which is coprime with n, so d must be 1. Putting $e = f - 1$, we obtain the desired expression for $\mathrm{ch}_{2,\varphi}$. The 'moreover'-part is again proven similarly as in Proposition 10.3.2. □

Before stating the Reidemeister spectrum, we first introduce some notation, which is similar to the one in Variation 3.

Definition 10.3.13. Let $A_1, \ldots, A_n$ be sets of natural numbers. We define

$$A_1 + \ldots + A_n := \mathop{+}_{i=1}^{n} A_i := \{a_1 + \ldots + a_n \mid \forall i \in \{1, \ldots, n\} : a_i \in A_i\}$$

Theorem 10.3.14. *Let G be an $SMC(n,m,2)$ where $\alpha(y) \equiv -1 \bmod 2^{m-1}n$ and $m \geq 2$.*

(1) If $\alpha(y) \equiv -1 \bmod N$ and $n \not\equiv 0 \bmod 3$, then

$$\mathrm{Spec_R}(G) = \{2,4\} + \{2^e d_0 + d_1 - 2 \mid 1 \leq e \leq m-1, (d_0, d_1) \in D(n,2)\}.$$

(2) If $\alpha(y) \equiv 2^{m-1}n - 1 \bmod N$ (and thus $m \geq 3$) and $n \not\equiv 0 \bmod 3$, then

$$\mathrm{Spec_R}(G) = \{2^e d_0 + d_1 + 2 \mid 1 \leq e \leq m-1, (d_0, d_1) \in D(n,2)\}.$$

(3) If $n \equiv 0 \bmod 3$, then $D(n,2)$ has to be replaced by $D'(n,2)$ in both cases.

Proof. Combining Proposition 10.3.11 and Proposition 10.3.12 yields the inclusion

$$\mathrm{Spec_R}(G) \subseteq \{2,4\} + \{2^e d_0 + d_1 - 2 \mid 1 \leq e \leq m-1, (d_0, d_1) \in D(n,2)\}$$

for the first case,

$$\mathrm{Spec_R}(G) \subseteq \{4\} + \{2^e d_0 + d_1 - 2 \mid 1 \leq e \leq m-1, (d_0, d_1) \in D(n,2)\}$$

for the second case, and the same with $D'(n,2)$ instead of $D(n,2)$ if $n \equiv 0 \bmod 3$.

For the other inclusion, we have to solve a similar problem as before: given 2 divisors d_0, d_1 of n satisfying the conditions of Proposition 10.3.12 and an integer e satisfying $1 \leq e \leq m-1$, find an integer $\gamma \in \mathbb{Z}$ coprime with N such that

$$\gcd(\gamma - 1, N) = 2^{e+1}d_0 \text{ and } \gcd(\gamma - \alpha(y), N) = 2d_1.$$

Indeed, by Propositions 10.3.11 and 10.3.12, the map $\varphi : G \to G$ given by $\varphi(x) = x^\gamma, \varphi(y) = y$ is then an automorphism of G with Reidemeister number equal to

$$4 + 2^e d_0 + d_1 - 2$$

and if $\alpha(y) \equiv -1 \bmod N$, $\psi : G \to G$ given by $\psi(x) = x^\gamma, \psi(y) = xy$ is one with Reidemeister number

$$2 + 2^e d_0 + d_1 - 2.$$

Theorem 10.1.11 provides us with a $\gamma \in \mathbb{Z}$ satisfying $\gcd(\gamma, n) = 1$, $\gcd(\gamma - 1, n) = d_0$, $\gcd(\gamma - \alpha(y), n) = d_1$ and $\gamma \equiv 1 + 2^{e+1} \bmod 2^m$. For $\gcd(\gamma, 2)$, observe that

$$\gamma \equiv 2^{e+1} + 1 \bmod 2^m.$$

As $e \geq 1$, this implies $\gamma \equiv 1 \bmod 4$, which proves that $\gcd(\gamma, 2) = 1$. Therefore, $\gcd(\gamma, N) = 1$.

Clearly, $\gcd(\gamma - 1, 2^m) = 2^{e+1}$. Since $e \geq 1$ and $\alpha(y) \equiv 3 \bmod 4$, we find that

$$\gamma - \alpha(y) \equiv \gamma - 3 \equiv 2 \bmod 4.$$

Therefore, $\gcd(\gamma - \alpha(y), 2^m) = 2$. As $\gcd(\gamma - \alpha(y)^i, n) = d_i$, we deduce that $\gcd(\gamma - 1, N) = 2^{e+1}d_0$ and $\gcd(\gamma - \alpha(y), N) = 2d_1$. □

Remark. As we did not put any restrictions on n, Theorem 10.3.14 also fully determines the Reidemeister spectrum of the groups $C_{2^m} \rtimes_\alpha C_2$ where $\alpha(y) \equiv -1 \bmod 2^{m-1}$.

Example 10.3.15. Again, we provide a table containing the spectra of $SMC(n, m, 2)$ where $\alpha(y) \equiv -1 \bmod 2^{m-1}n$ and $m \geq 2$ for some concrete values of n and m.

n	m	$\alpha(y) \bmod N$	$\mathrm{Spec}_{\mathrm{R}}(SMC(n,m,p))$
$175 = 5^2 \cdot 7$	3	-1	$\{3,5,7,9,11,13,15,17,19,21,23,27,29,$ $31,33,35,37,39,41,51,53,55,57,59,$ $71,73,101,103,107,109,141,143,$ $177,179,181,351,353,701,703\}$
175	3	$2^{m-1}n-1$	$\{5,7,9,11,13,17,19,21,23,29,31,35,$ $39,41,53,55,59,73,103,109,143,$ $179,181,353,703\}$
$105 = 3\cdot 5\cdot 7$	3	-1	$\{5,7,9,11,13,15,17,19,21,23,25,27,$ $29,31,33,37,39,41,43,45,47,49,61,$ $63,67,69,73,75,85,87,89,91,107,$ $109,111,143,145,211,213,421,423\}$
105	3	$2^{m-1}n-1$	$\{7,9,13,15,19,21,25,27,31,33,39,43,$ $45,49,63,69,75,87,91,109,111,145,$ $213,423\}$

Table 10.3: Reidemeister spectrum of $SMC(n,m,2)$ where $\alpha(y) \equiv -1 \bmod 2^{m-1}n$ for some concrete values of n and m.

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The last remaining case is $\alpha(y) \equiv 1 \bmod p^{m-1}$ and $n = 1$, which we treat in the next section.

10.3.3 $\mathrm{Spec}_{\mathrm{R}}(SMC(1,m,p))$ with $m \geq 2$ and non-trivial action

Here we deal with p-groups of the form $C_{p^m} \rtimes_\alpha C_p$, where we assume that $\alpha(y) \not\equiv 1 \bmod p^m$. In particular, $m \geq 2$. We distinguish two cases: $\alpha(y) \equiv 1 \bmod p^{m-1}$ and $\alpha(y) \not\equiv 1 \bmod p^{m-1}$. By Lemma 10.2.5, the latter can only occur if $p = 2$. However, by the remark following the proof of Theorem 10.3.14, the case $\alpha(y) \not\equiv 1 \bmod 2^{m-1}$ has already been dealt with. Therefore, we may assume that $\alpha(y) \equiv 1 \bmod p^{m-1}$.

We are not able to use Proposition 10.1.10 nor (10.3.1), since C_{p^m} is not necessarily characteristic in G; for instance, this is the case for $C_{p^2} \rtimes C_p$. We can, however, still use the technique of character counting, but we need another approach.

By Proposition 10.2.8, we already know the commutator subgroup of G, namely $\left\langle x^{p^{m-1}} \right\rangle$.

Lemma 10.3.16 (See e.g. [42, Corollary 3.1]). *The centre of G is given by $\langle x^p \rangle$.*

Proof. Since

$$x(x^a y^b)x^{-1} = x^{a+1}y^b x^{-1} = x^{a+1}y^b x^{-1}y^{-b}y^b = x^{a+1-\alpha(y)^b}y^b,$$

for all $a, b \in \mathbb{Z}$, we see that $x^a y^b \in Z(G)$ implies that $1 - \alpha(y)^b \equiv 0 \bmod p^m$. Since $\alpha(y) \equiv \beta p^{m-1} + 1 \bmod p^m$ for some $\beta \in \{1, \dots, p-1\}$, we find that $b \equiv 0 \bmod p$. Moreover,

$$yx^a y^{-1} = x^{\alpha(y)a}$$

and for this to be equal to x^a, we must have that $a(\alpha(y) - 1) \equiv 0 \bmod p^m$. As $\alpha(y) \equiv \beta p^{m-1} + 1 \bmod p^m$, we conclude that $a \equiv 0 \bmod p$. Therefore, $Z(G) \leq \langle x^p \rangle$. As x^p clearly commutes with both x and y, equality holds. □

Lemma 10.3.17. *For any $a, b, e \in \mathbb{Z}$ with $e \geq 0$,*

$$(x^a y^b)^e = x^{a\left(\sum_{i=0}^{e-1} \alpha(y)^{ib}\right)} y^{be},$$

Proof. We proceed by induction on e. The cases $e = 0$ and $e = 1$ are trivial. So, suppose the result holds for e. Then

$$\begin{aligned}
(x^a y^b)^{e+1} &= x^{a\left(\sum_{i=0}^{e-1} \alpha(y)^{ib}\right)} y^{be} x^a y^b \\
&= x^{a\left(\sum_{i=0}^{e-1} \alpha(y)^{-ib}\right)} y^{be} x^a y^{-be} y^{b+be} \\
&= x^{a\left(\sum_{i=0}^{e-1} \alpha(y)^{-ib}\right)} x^{a\alpha(y)^{be}} y^{b(e+1)} \\
&= x^{a\left(\sum_{i=0}^{e} \alpha(y)^{ib}\right)} y^{b(e+1)},
\end{aligned}$$

which proves that the result holds for $e + 1$. □

Lemma 10.3.18 (See e.g. [103, Proposition 1])**.** *Let $\varphi \in \mathrm{Aut}(G)$ and write $\varphi(x) = x^a y^b$, $\varphi(y) = x^c y^d$. Then*

(1) $c \equiv 0 \bmod p^{m-1}$*;*

(2) $\gcd(a, p) = 1$ *and* $d \equiv 1 \bmod p$*;*

(3) $\mathrm{ch}_{1,\varphi} \in \{p^i \mid 1 \leq i \leq m\}$ *if p is odd,* $\mathrm{ch}_{1,\varphi} \in \{2^i \mid 2 \leq i \leq m\}$ *if $p = 2$;*

(4) $\mathrm{ch}_{1,\varphi} = p^m$ *if and only if* $a \equiv 1 \bmod p^{m-1}$ *and* $b \equiv 0 \bmod p$*.*

Proof. For the first item, we compare both sides of the equality $1 = \varphi(y)^p$. By Lemma 10.3.17, we find

$$1 = x^{c\left(\sum_{i=0}^{p-1} \alpha(y)^{id}\right)}.$$

Therefore,

$$c\left(\sum_{i=0}^{p-1} \alpha(y)^{id}\right) \equiv 0 \bmod p^m.$$

If $d \equiv 0 \bmod p$, then the sum on the left-hand side equals cp. Consequently, $c \equiv 0 \bmod p^{m-1}$. So, suppose that $d \not\equiv 0 \bmod p$. Write $\alpha(y) \equiv \beta p^{m-1} + 1 \bmod p^m$ for some $\beta \in \{1, \ldots, p-1\}$. By the fact that $d \not\equiv 0 \bmod p$ and that $\alpha(y)$ has (multiplicative) order p modulo p^m, the sum equals

$$\begin{aligned} \sum_{i=0}^{p-1} \alpha(y)^{id} &\equiv \sum_{i=0}^{p-1} (\beta p^{m-1} + 1)^{id} \\ &\equiv \sum_{i=0}^{p-1} (di\beta p^{m-1} + 1) \\ &\equiv p + d\beta p^{m-1} \frac{p(p-1)}{2} \bmod p^m, \end{aligned}$$

where the third line follows from the fact that $m \geq 2$. For odd p, we find that the last expression is congruent to p modulo p^m. Therefore,

$$c\left(\sum_{i=0}^{p-1} \alpha(y)^{id}\right) \equiv cp \bmod p^m,$$

which implies that $c \equiv 0 \bmod p^{m-1}$. For $p = 2$, the expression becomes $2^{m-1} + 2 \bmod 2^m$. Since the case $m = 2$ corresponds to the inversion action of C_2 on C_4, which has been dealt with in Theorem 10.3.14, we may assume that $m \geq 3$ in this case. Then

$$c\left(\sum_{i=0}^{1} \alpha(y)^{id}\right) \equiv c(2 + 2^{m-1}) \bmod 2^m,$$

which implies that $c \equiv 0 \bmod 2^{m-1}$.

For the second item, first note that $\varphi^{\mathrm{ab}}(\bar{x}) = \bar{x}^a\bar{y}^b$ must have order p^{m-1}, which implies that $\gcd(a, p) = 1$. Subsequently, we compare both sides of the equality $\varphi(x^{\alpha(y)}) = \varphi(y)\varphi(x)\varphi(y)^{-1}$, using Lemma 10.3.17. This yields

$$x^{a\left(\sum_{i=0}^{\alpha(y)-1}\alpha(y)^{ib}\right)}y^{\alpha(y)b} = x^c y^d x^a y^b y^{-d} x^{-c}.$$

Since $c \equiv 0 \bmod p^{m-1}$, $x^c \in Z(G)$. Therefore,

$$x^c y^d x^a y^b y^{-d} x^{-c} = y^d x^a y^{-d} y^b = x^{a\alpha(y)^d} y^b.$$

Thus, we end up with the congruence

$$a\left(\sum_{i=0}^{\alpha(y)-1}\alpha(y)^{ib}\right) \equiv a\alpha(y)^d \bmod p^m,$$

where we can cancel out the a left and right, since we already proved that $\gcd(a, p) = 1$. This yields

$$\sum_{i=0}^{\alpha(y)-1}\alpha(y)^{ib} \equiv \alpha(y)^d \bmod p^m.$$

With $\alpha(y) \equiv \beta p^{m-1} + 1 \bmod p^m$ as before, the right-hand side equals

$$\alpha(y)^d \equiv d\beta p^{m-1} + 1 \bmod p^m,$$

whereas the sum in left-hand side equals

$$\sum_{i=0}^{\alpha(y)-1}\alpha(y)^{ib} \equiv \sum_{i=0}^{\alpha(y)-1}(ib\beta p^{m-1} + 1) \equiv \frac{\alpha(y)(\alpha(y)-1)}{2}b\beta p^{m-1} + \alpha(y) \bmod p^m.$$

For p odd, this reduces to $\alpha(y) \bmod p^m$, as $m \geq 2$ and $\alpha(y) \equiv 1 \bmod p$. Thus, we get the congruence

$$\beta p^{m-1} + 1 \equiv d\beta p^{m-1} + 1 \bmod p^m,$$

which implies that $d \equiv 1 \bmod p$. For $p = 2$, we may, as before, assume that $m \geq 3$. Recall that $\alpha(y) \equiv 1 \bmod 2^{m-1}$. Thus, $\frac{\alpha(y)-1}{2}b\beta 2^{m-1}$ is divisible by $2^{m-2} \cdot 2^{m-1} = 2^{2m-3}$. As $m \geq 3$, it follows that $2m - 3 \geq m$. Therefore,

$$\frac{\alpha(y)(\alpha(y)-1)}{2}b\beta 2^{m-1} + \alpha(y) \equiv \alpha(y) \bmod 2^m.$$

Since we assume that C_2 acts non-trivially on C_{2^m} and since $\alpha(y) \equiv 1 \bmod 2^{m-1}$, we know that $\alpha(y) \equiv 2^{m-1}+1 \bmod 2^m$. Consequently, we get the congruence

$$d\beta 2^{m-1}+1 \equiv \alpha(y) \equiv 2^{m-1}+1 \bmod 2^m,$$

which yields $d \equiv 1 \bmod 2$ again.

For the third item, since $\mathrm{Fix}(\varphi^{\mathrm{ab}})$ is a subgroup of G^{ab}, it follows that $\mathrm{ch}_{1,\varphi} = R(\varphi^{\mathrm{ab}})$ divides p^m. We furthermore note that $\varphi^{\mathrm{ab}}(\bar{y}^i) = \bar{x}^{ci}\bar{y}^i = \bar{y}^i$ for all $i \in \{0,\dots,p-1\}$, as $c \equiv 0 \bmod p^{m-1}$ and $d \equiv 1 \bmod p$. Therefore, φ^{ab} has at least p fixed points, which implies that $R(\varphi^{\mathrm{ab}}) = \left|\mathrm{Fix}(\varphi^{\mathrm{ab}})\right| \geq p$. If $p=2$, also

$$\varphi^{\mathrm{ab}}\left(\bar{x}^{2^{m-2}}\right) = \bar{x}^{2^{m-2}}$$

as $a \equiv 1 \bmod 2$, y has order 2 and $m \geq 3$ in that case. This implies that $\mathrm{ch}_{1,\varphi} \geq 4$ if $p=2$. The final item follows immediately from the fact that $\mathrm{ch}_{1,\varphi} = p^m$ if and only if φ^{ab} is the identity map. $\square$

If χ_a is a character on C_{p^m} inducing a 1-dimensional character on G, then

$$\chi_a(x^e) = \chi_a\left(x^{\alpha(y)^i e}\right)$$

for all $e \in \{0,\dots,p^m-1\}$ and $i \in \{0,\dots,p-1\}$. This can only hold if $\chi_a = \chi_{a\alpha(y)^i}$ for all $i \in \{0,\dots,p-1\}$, i.e. if and only if $a(\alpha(y)^i - 1) \equiv 0 \bmod p^m$. As $\alpha(y) \equiv \beta p^{m-1}+1 \bmod p^m$ for some $\beta \in \{1,\dots,p-1\}$, this is equivalent to $a \equiv 0 \bmod p$. Therefore, the characters χ_a on C_{p^m} inducing a p-dimensional character on G are those for which $\gcd(a,p)=1$.

Lemma 10.3.19. *Let $\overline{\chi}_l \in \mathrm{Irr}_p(G)$ be a character with $\gcd(l,p)=1$ and let $\zeta = \exp\left(\frac{2\pi i}{p^m}\right)$ be a primitive p^mth root of unity. Write $\alpha(y) \equiv \beta p^{m-1}+1 \bmod p^m$ for a $\beta \in \{1,\dots,p-1\}$. Then, for $e \in \{0,\dots,p^m-1\}$,*

$$\overline{\chi}_l(x^e) = \zeta^{le}\sum_{j=0}^{p-1}\zeta^{le\beta j p^{m-1}} = \begin{cases} 0 & \text{if } \gcd(e,p)=1 \\ p\zeta^{le} & \text{if } \gcd(e,p)=p. \end{cases}$$

In particular, $\overline{\chi}_l$ is zero outside $Z(G)$.

Proof. On C_{p^m}, the character χ_l satisfies $\chi_l(x^e) = \zeta^{le}$ for all $e \in \{0, \ldots, p^m - 1\}$. Now let $e \in \{0, \ldots, p^m - 1\}$. We compute $\overline{\chi}_l(x^e)$ using Lemma 10.1.8:

$$\begin{aligned}\overline{\chi}_l(x^e) &= \sum_{j=0}^{p-1} \chi_l\Big(x^{e\alpha(y)^j}\Big) \\ &= \sum_{j=0}^{p-1} \chi_l\Big(x^{e(\beta j p^{m-1}+1)}\Big) \\ &= \sum_{j=0}^{p-1} \zeta^{le\beta j p^{m-1}+le} \\ &= \zeta^{le} \sum_{j=0}^{p-1} \zeta^{le\beta j p^{m-1}}.\end{aligned}$$

Recall that β is coprime with p. Thus, if $\gcd(e,p) = 1$, the last sum equals $1 + \zeta^{p^{m-1}} + \ldots + \zeta^{(p-1)p^{m-1}}$, which is zero. If $\gcd(e,p) = p$, the last sum equals p, as $\zeta^{p^m} = 1$.

Since $Z(G) = \langle x^p \rangle \leq C_{p^m}$ and $\overline{\chi}_l$ is already zero outside C_{p^m}, we conclude that $\overline{\chi}_l$ is zero outside $Z(G)$. $\square$

Proposition 10.3.20. *Let $\varphi \in \mathrm{Aut}(G)$ and $\overline{\chi}_a$ be a character in $\mathrm{Irr}_p(G)$. If $\overline{\chi}_a$ is fixed by φ, then all characters in $\mathrm{Irr}_p(G)$ are fixed by φ.*

In particular, $\mathrm{ch}_{p,\varphi}(G) \in \{0, p^{m-2}(p-1)\}$.

Proof. Let $\overline{\chi}_a$ be a character in $\mathrm{Irr}_p(G)$. By Lemma 10.3.19, we know that $\overline{\chi}_a$ is zero outside $Z(G)$. As $Z(G)$ is characteristic in G, also $\varphi(G \setminus Z(G)) = G \setminus Z(G)$, which implies that $\overline{\chi}_a \circ \varphi$ is also zero outside $Z(G)$. Since $Z(G) = \langle x^p \rangle$, we can write $\varphi(x^p) = x^{\gamma p}$ for some $\gamma \in \mathbb{Z}$ coprime with p. Then, using Lemma 10.3.19 again, we find the following equivalence:

$$\overline{\chi}_a \circ \varphi = \overline{\chi}_a \iff \forall e \in \mathbb{Z} : p\zeta^{ape} = p\zeta^{a\gamma pe},$$

where $\zeta = \exp\Big(\frac{2\pi i}{p^m}\Big)$ is a primitive p^mth root of unity. The last condition is clearly equivalent with $\zeta^{ap(\gamma-1)} = 1$.

So, suppose that $\overline{\chi}_a \circ \varphi = \overline{\chi}_a$. Then $\zeta^{ap(\gamma-1)} = 1$. Let $b \in \{0, \ldots, p^m - 1\}$ be coprime with p. Then there exists a $c \in \mathbb{Z}$ such that $ac \equiv b \bmod p^m$. Consequently,

$$\zeta^{bp(\gamma-1)} = \zeta^{acp(\gamma-1)} = \Big(\zeta^{ap(\gamma-1)}\Big)^c = 1,$$

which proves that $\overline{\chi}_b \circ \varphi = \overline{\chi}_b$.

Thus, either all or none of the characters in $\mathrm{Irr}_p(G)$ are fixed by φ. Since

$$|G| = \mathrm{ch}_1 + p^2 \,\mathrm{ch}_p = \left|G^{\mathrm{ab}}\right| + p^2 \,\mathrm{ch}_p = p^m + p^2 \,\mathrm{ch}_p,$$

we find that $\mathrm{ch}_p = \frac{p^{m+1}-p^m}{p^2} = p^{m-1} - p^{m-2} = p^{m-2}(p-1)$. Thus, either $\mathrm{ch}_{p,\varphi} = 0$ or $\mathrm{ch}_{p,\varphi} = p^{m-2}(p-1)$. □

Theorem 10.3.21. *Let G be an $SMC(1,m,p)$ with $m \geq 2$ if p is odd, $m \geq 3$ if $p = 2$, and $\alpha(y) \equiv \beta p^{m-1} + 1 \bmod p^m$ for some $\beta \in \{1,\dots,p-1\}$.*

(1) If p is odd, then

$$\mathrm{Spec}_{\mathrm{R}}(G) = \{p^i \mid 1 \leq i \leq m-1\} \cup \{2p^{m-1} - p^{m-2}, p^m + p^{m-1} - p^{m-2}\}.$$

(2) If $p = 2$, then

$$\mathrm{Spec}_{\mathrm{R}}(G) = \{2^i \mid 2 \leq i \leq m-1\} \cup \{2^m - 2^{m-2}, 2^m + 2^{m-2}\}.$$

Proof. Combining Lemma 10.3.18 and Proposition 10.3.20 yields the inclusions

$$\mathrm{Spec}_{\mathrm{R}}(G) \subseteq \{p^i \mid 1 \leq i \leq m\} + \{0, p^{m-2}(p-1)\}$$

for odd p and

$$\mathrm{Spec}_{\mathrm{R}}(G) \subseteq \{2^i \mid 2 \leq i \leq m\} + \{0, 2^{m-2}\}$$

for $p = 2$. For the $\subseteq$-inclusion, we therefore still need to prove the following:

- If $\mathrm{ch}_{p,\varphi} = p^{m-2}(p-1)$ for $\varphi \in \mathrm{Aut}(G)$, then $\mathrm{ch}_{1,\varphi} \in \{p^{m-1}, p^m\}$.
- If $\mathrm{ch}_{1,\varphi} = p^m$ for $\varphi \in \mathrm{Aut}(G)$, then $\mathrm{ch}_{p,\varphi} = p^{m-2}(p-1)$.

Fix $\varphi \in \mathrm{Aut}(G)$. First, suppose that $\mathrm{ch}_{p,\varphi} = p^{m-2}(p-1)$. Write $\varphi(x) = x^a y^b$ with $a \in \{0,\dots,p^m-1\}$ and $b \in \{0,\dots,p-1\}$. It is easily checked that $\varphi(x^p) = x^{pa}$ for odd p and $\varphi(x^2) = x^{(1+\alpha(y)^b)a}$ for $p = 2$. As all characters in $\mathrm{Irr}_p(G)$ are fixed, it follows from Lemma 10.3.19 that for p odd,

$$p\zeta^{lpe} = p\zeta^{lape}$$

for all $e \in \{0,\dots,p^{m-1}-1\}$ and l coprime with p, where again $\zeta = \exp\!\left(\frac{2\pi i}{p^m}\right)$. This can only hold if $\zeta^{p(a-1)} = 1$, i.e. if $a \equiv 1 \bmod p^{m-1}$. For $p = 2$, we get

$$2\zeta^{2le} = 2\zeta^{(1+\alpha(y)^b)ale}$$

for all $e \in \{0, \dots, p^{m-1} - 1\}$ and l coprime with p. This can only hold if $a + a\alpha(y)^b - 2 \equiv 0 \bmod 2^m$. Viewing this modulo 2^{m-1} yields $2a - 2 \equiv 0 \bmod 2^{m-1}$, which shows that $a - 1 \equiv 0 \bmod 2^{m-2}$. Thus, for all p, we have that $a - 1 \equiv 0 \bmod p^{m-2}$.

Writing $\varphi(y) = x^c y$ with $c \equiv 0 \bmod p^{m-1}$ (due to Lemma 10.3.18), we now determine $\mathrm{ch}_{1,\varphi}$ by counting the number of fixed points of φ^{ab} on G^{ab}. In order to do so, we have to count the number of solutions (A, B) in $\mathbb{Z}/p^{m-1}\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ to the system of congruences

$$\begin{cases} aA \equiv A \bmod p^{m-1} \\ bA + B \equiv B \bmod p. \end{cases}$$

If $A \equiv 0 \bmod p$, then $A(a-1) \equiv 0 \bmod p^{m-1}$. Therefore, this systems has at least $p^{m-2} \cdot p = p^{m-1}$ solutions, and at most $\left|G^{ab}\right| = p^m$. We conclude that $\mathrm{ch}_{1,\varphi} \in \{p^{m-1}, p^m\}$.

Next, suppose that $\mathrm{ch}_{1,\varphi} = p^m$. By Lemma 10.3.18, we then know that $\varphi(x) = x^{Ap^{m-1}+1}$ for some $A \in \mathbb{Z}$. Fix $\overline{\chi}_l \in \mathrm{Irr}_p(G)$ and let ζ be as before. Note that

$$\zeta^{lp(Ap^{m-1}+1-1)} = \zeta^{lAp^m} = 1.$$

Thus, by the proof of Proposition 10.3.20, this implies that $\overline{\chi}_l \circ \varphi = \overline{\chi}_l$. As $\overline{\chi}_l$ was arbitrary, we conclude that $\mathrm{ch}_{p,\varphi} = p^{m-2}(p-1)$.

Thus, we have proven that

$$\mathrm{Spec}_{\mathrm{R}}(G) \subseteq \begin{cases} \{p^i \mid 1 \leq i \leq m-1\} & \\ \quad \cup \{2p^{m-1} - p^{m-2}, p^m + p^{m-1} - p^{m-2}\} & \text{if } p \text{ is odd,} \\ \{2^i \mid 2 \leq i \leq m-1\} & \\ \quad \cup \{2^m - 2^{m-2}, 2^m + 2^{m-2}\} & \text{if } p = 2 \end{cases}$$

We are left with providing automorphisms of G realising the candidate-Reidemeister numbers on the right. For $i \in \{0, \dots, m-1\}$, define

$$\varphi_i : G \to G : x \mapsto x^{p^i+1}, y \mapsto y.$$

It is readily verified that φ_i preserves the relations of G and that φ_i is surjective (and therefore injective) for all $i \in \{0, \dots, m-1\}$ if p is odd, and for all $i \in \{1, \dots, m-1\}$ if $p = 2$. The map φ_i^{ab} is given by $\varphi_i^{\mathrm{ab}}(\bar{x}^A \bar{y}^B) = \bar{x}^{A(p^i+1)} \bar{y}^B$, which has p^{i+1} fixed points, which shows that $\mathrm{ch}_{1,\varphi_i} = p^{i+1}$. If $i \leq m-2$, then

$$\overline{\chi}_1(\varphi(x^p)) = \overline{\chi}_1(x^{p^{i+1}+p}) = p\zeta^{p^{i+1}+p} \neq p\zeta^p = \overline{\chi}_1(x^p),$$

which shows that $\overline{\chi}_1$ is not fixed by φ_i. Therefore, $\mathrm{ch}_{p,\varphi_i} = 0$ if $i \leq m-2$. If $i = m-1$, then the equality $p\zeta^{p^{i+1}+1} = p\zeta^p$ holds and thus $\mathrm{ch}_{p,\varphi_i} = p^{m-2}(p-1)$. Consequently, we see that

$$R(\varphi_i) = \begin{cases} p^{i+1} & \text{if } i \leq m-2 \\ p^m + p^{m-2}(p-1) & \text{if } i = m-1. \end{cases}$$

Finally, define $\psi : G \to G : x \mapsto x^{p^{m-1}+1}y, y \mapsto y$. Again, ψ is a well-defined automorphism of G. For $\overline{\chi}_l \in \mathrm{Irr}_p(G)$ and $e \in \{0, \dots, p^{m-1}-1\}$, we see that

$$\overline{\chi}_l(\psi(x^{pe})) = \overline{\chi}_l(x^{p^m e + pe}) = \overline{\chi}_l(x^{pe}).$$

Thus, $\overline{\chi}_l$ and $\overline{\chi}_l \circ \psi$ match on $Z(G)$. As $\overline{\chi}_l$ is zero outside $Z(G)$, it follows that $\overline{\chi}_l$ is fixed by ψ. Therefore, $\mathrm{ch}_{p,\psi} = p^{m-2}(p-1)$, as $\overline{\chi}_l$ was arbitrary. Now, on G^{ab} we see that $\psi(\bar{x}^A\bar{y}^B) = \bar{x}^A\bar{y}^{B+A}$. Thus, $\bar{x}^A\bar{y}^B$ is a fixed point if and only if $A \equiv 0 \bmod p$. This implies that ψ^{ab} has $p^{m-2} \cdot p = p^{m-1}$ fixed points, which that

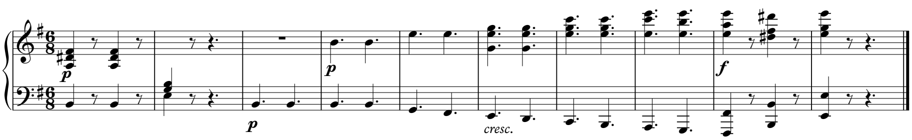
$$R(\psi) = R(\psi^{\mathrm{ab}}) + \mathrm{ch}_{p,\psi} = p^{m-1} + p^{m-2}(p-1).$$

This finishes the proof. $\square$

Codas

Coda A

Inverse Reidemeister problem



van Beethoven, Ludwig, *Prestissimo* from *Piano Sonata No. 30, Op. 109* (bars 168–177).[1]

In this first Coda, we introduce and study the inverse Reidemeister problem.

A.1 Positioning and statement of the problem

Determining the Reidemeister spectrum of a group is one of many examples of the following general construction in mathematics: given an object X in some category C, we associate to it an object $F(X)$ in a category D, which is not necessarily distinct form C. Here, F does not need to be a functor. A natural question to ask then is how many objects in D we reach via this way; more precisely, whether there exists for each $Y \in \mathsf{D}$ an $X \in \mathsf{C}$ such that $F(X) = Y$.

For example, to each topological space, we can associate its fundamental group. Conversely, using complexes, one can prove that each group is the fundamental group of some topological space (see e.g. [76, Proposition 2.3]). Another example is the Galois group of field extension. However, the so-called *inverse Galois problem* is still not fully solved, and asks whether each finite group is the Galois group of some Galois extension of $\mathbb{Q}$. For more information on the latter, we refer the reader to e.g. [61].

[1] Excerpt adopted from [5].

In the context of twisted conjugacy, the natural inverse question is thus the following:

Question 1 (Inverse Reidemeister problem)**.** *Let $S \subseteq \mathbb{N}_0 \cup \{\infty\}$ be non-empty. Does there exist a group G such that $S = \mathrm{Spec}_\mathrm{R}(G)$?*

As stated, this is an extremely ambitious question. Determining the Reidemeister spectrum of a given group is already difficult, trying to find a way of constructing a group with given Reidemeister spectrum borders on the impossible. In addition, the sets considered range from tame to wild; for instance, while there are several groups known whose Reidemeister spectrum is $2\mathbb{N}_0 \cup \{\infty\}$, does there exist one whose Reidemeister spectrum is $2\mathbb{N}_0 \setminus \{2k\} \cup \{\infty\}$, where k is a given positive integer? Although the set in Proposition 7.2.6 (in case $p = 2$) already seems quite exotic, what about the set of all numbers whose decimal representation contains exactly three times the digit 1?

A milder question would therefore be the following:

Question 2. *Let $S \subseteq \mathbb{N}_0 \cup \{\infty\}$ be non-empty. Does there exist a group G such that $S \subseteq \mathrm{Spec}_\mathrm{R}(G)$ and such that $\mathrm{Spec}_\mathrm{R}(G)$ is 'small' in comparison to S?*

For a finite set S, we could use for instance $|\mathrm{Spec}_\mathrm{R}(G) \setminus S|$ or $\frac{|\mathrm{Spec}_\mathrm{R}(G)\setminus S|}{|S|}$ as a measure. For an infinite set S, we could use the notions of upper and/or lower asymptotic density of sets of natural numbers to quantify how small $\mathrm{Spec}_\mathrm{R}(G)$ is in comparison to S.

A.2 First partial results

A first small step towards investigating the general problem would be to consider 'tame' sets: $k\mathbb{N}_0 \cup \{\infty\}$ or $\{k, \infty\}$, where k is a positive integer.

Proposition A.2.1 (See e.g. [18, Proposition 3.2])**.** *Let A be an abelian divisible group. Then $\mathrm{Spec}_\mathrm{R}(A) \subseteq \{1, \infty\}$.*

Equality holds for $A = \mathbb{Q}^r, \mathbb{R}^r, \mathbb{C}^r$ for any $r \geq 1$

Proof. Let $\varphi \in \mathrm{Aut}(A)$. By Lemma T.1.11, $R(\varphi) = [A : \mathrm{Im}(\varphi - \mathrm{Id})]$. If this number is finite, then it must be 1 by Corollary 2.2.18. Hence, $\mathrm{Spec}_\mathrm{R}(A) \subseteq \{1, \infty\}$.

Suppose now that $A = \mathbb{Q}^r, \mathbb{R}^r$ or $\mathbb{C}^r$ for an integer $r \geq 1$. The identity map has infinite Reidemeister number, so $\infty \in \mathrm{Spec}_\mathrm{R}(A)$. The map $\psi : A \to A : x \mapsto 2x$

is an automorphism with inverse $\omega : A \to A : x \mapsto x/2$. Since $\psi - \mathrm{Id} = \mathrm{Id}$, $R(\psi) = [A : \mathrm{Im}\,\mathrm{Id}] = [A : A] = 1$. Thus, $\mathrm{Spec}_{\mathrm{R}}(A) = \{1, \infty\}$. □

Recall that the *holomorph* of a group G, denoted by $\mathrm{Hol}(G)$, is defined as the semi-direct product $G \rtimes \mathrm{Aut}(G)$.

Lemma A.2.2. *Let p be an odd prime number. Put $G = \mathrm{Hol}(\mathbb{Z}/p\mathbb{Z})$. Then the following hold:*

(1) $\mathrm{Aut}(G) = \mathrm{Inn}(G)$*;*

(2) $Z(G) = 1$*;*

(3) G is directly indecomposable.

(4) G has p conjugacy classes.

Proof. For the first item, we refer the reader to [78, p.134-135].

Next, let x be a generator of $\mathbb{Z}/p\mathbb{Z}$ and α one of $\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^{\times}$. Suppose that $x^i\alpha^j \in Z(G)$ for some $i, j \in \mathbb{Z}$. Then $\alpha x^i \alpha^j \alpha^{-1} = x^i \alpha^j$, hence $\alpha(x^i) = x^i$. By Lemma 7.2.2, only the identity map on $\mathbb{Z}/p\mathbb{Z}$ has a non-trivial fixed point. Therefore, x^i is trivial. Since $x = \alpha^j x \alpha^{-j}$ must hold as well, $\alpha^j(x) = x$. Consequently, α^j is the identity map. We conclude that $Z(G) = 1$.

For the third item, we note that $\mathbb{Z}/p\mathbb{Z}$ is directly indecomposable and that it has non-inner automorphisms. It then follows from [79, Theorem 1] that $\mathrm{Hol}(\mathbb{Z}/p\mathbb{Z})$ is directly indecomposable.

Finally, we determine the number of conjugacy classes of G. It is well known that this is equal to the number of complex irreducible representations of G. [96, Theorem 6.1(3)] states that G has exactly $1 + p - 1 = p$ such representations. Hence, G has p conjugacy classes. □

Proposition A.2.3. *Let p be a prime number. Then* $\mathrm{Spec}_{\mathrm{R}}(\mathrm{Hol}(\mathbb{Z}/p\mathbb{Z})) = \{p\}$.

Proof. Since $\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z})$ is trivial, $\mathrm{Hol}(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Therefore, Theorem 9.3.18 implies that $\mathrm{Spec}_{\mathrm{R}}(\mathrm{Hol}(\mathbb{Z}/2\mathbb{Z})) = \{2\}$.

Now suppose that p is odd. By Lemma A.2.2(1), $\mathrm{Hol}(\mathbb{Z}/p\mathbb{Z})$ only has inner automorphisms. Therefore, Lemma T.1.7(2) implies that $\mathrm{Spec}_{\mathrm{R}}(\mathrm{Hol}(\mathbb{Z}/p\mathbb{Z})) = \{R(\mathrm{Id})\}$. Using Corollary 8.1.3 and Lemma A.2.2(4), we conclude that $\mathrm{Spec}_{\mathrm{R}}(\mathrm{Hol}(\mathbb{Z}/p\mathbb{Z})) = \{p\}$. □

Proposition A.2.4. *Let k be a square-free integer. Then for each of the following sets S, there exists a group G such that* $\mathrm{Spec}_{\mathrm{R}}(G) = S$*:*

(1) $S = \{k\}$*;*

(2) $S = \{k, \infty\}$*;*

(3) $S = k\mathbb{N}_0 \cup \{\infty\}$*.*

Proof. For a prime p, let H_p denote the holomorph of $\mathbb{Z}/p\mathbb{Z}$. Let $k = p_1 \dots p_n$ be the prime factorisation of k. As k is square-free, all p_i are distinct. Since each H_p is centreless and directly indecomposable by Lemma A.2.2(2)&(3), Corollary 8.1.3 implies that

$$\mathrm{Spec_R}(H_{p_1} \times \ldots \times H_{p_n}) = \prod_{i=1}^{n} \mathrm{Spec_R}(H_{p_i}) = \prod_{i=1}^{n} \{p_i\} = \{k\},$$

where we also used Proposition A.2.3. This proves the first item.

Next, put $H := H_{p_1} \times \ldots \times H_{p_n}$. Consider the group $G := H \times \mathbb{Q}$. Since $\mathbb{Q}$ is torsion-free and H is finite, H is characteristic in G. By Proposition 2.2.19, $\mathbb{Q}$ is characteristic in G as well.

Hence, by Corollary 3.1.6,

$$\mathrm{Spec_R}(G) = \mathrm{Spec_R}(H) \cdot \mathrm{Spec_R}(\mathbb{Q}) = \{k\} \cdot \{1, \infty\} = \{k, \infty\}.$$

Lastly, applying Corollary 3.2.3 to H and $\mathbb{Z}^2$ and using Theorem T.1.13, we find

$$\mathrm{Spec_R}(H \times \mathbb{Z}^2) = \mathrm{Spec_R}(H) \cdot \mathrm{Spec_R}(\mathbb{Z}^2) = \{k\} \cdot (\mathbb{N}_0 \cup \{\infty\}) = k\mathbb{N}_0 \cup \{\infty\}. \qquad \square$$

Proposition A.2.5. *Let* $S \subseteq \mathbb{N}_0$ *be a non-empty finite set. Then there exists a finite group* G *such that* $S \subseteq \mathrm{Spec_R}(G)$*.*

Proof. Let m be the least common multiple of all elements in S. Suppose that $m = 2^k n$ for some odd integer n and $k \geq 0$. Consider the finite abelian group $A := \mathbb{Z}/n\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{k+2}$ of order $2^{k+2}n = 4m$. Then the Sylow 2-subgroup of A has type $(1, 1, \dots, 1)$ ($k + 2$ times). Therefore, Theorem 9.3.18 implies that

$$\mathrm{Spec_R}(A) = \{d \in \mathbb{N} \mid d \text{ divides } 4m\}.$$

Since each element in S divides m, $S \subseteq \mathrm{Spec_R}(A)$. $\square$

Note that we can just take $A = \mathbb{Z}/m\mathbb{Z}$ if m is odd. The size of $\mathrm{Spec_R}(A)$ in the proof can be quite big in comparison to $|S|$. For instance, if $S = \{p_1, \dots, p_k\}$, where each p_i is an odd prime, then $m = p_1 \dots p_k$. If we take $A = \mathbb{Z}/m\mathbb{Z}$, then $|\mathrm{Spec_R}(A)|$ equals the number of divisors of m, which is equal to 2^k. However, S has size k.

We end with a non-existence result.

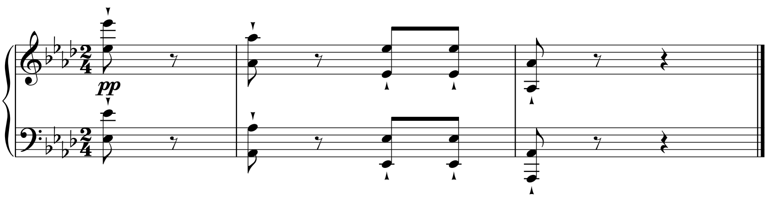
Proposition A.2.6. *Let $S \subseteq \mathbb{N}_0$ be a non-empty finite set. Let $m = \max S$ and suppose that $m-1 \in S$. Then there is no finite group G such that $\operatorname{Spec}_{\mathrm{R}}(G) = S$.*

Proof. We prove the contrapositive: if G is a finite group and $M = \max(\operatorname{Spec}_{\mathrm{R}}(G))$, then $M - 1 \notin \operatorname{Spec}_{\mathrm{R}}(G)$.

It follows from Corollary 8.1.3 that M equals the number of conjugacy classes of G. Let $\varphi \in \operatorname{Aut}(G)$. Again by Corollary 8.1.3, $R(\varphi)$ equals the number of conjugacy classes of G fixed by φ. If φ fixes $M - 1$ conjugacy classes, it must fix all of them, since φ permutes the conjugacy classes. Therefore, $R(\varphi) \neq M - 1$. We conclude that $M - 1 \notin \operatorname{Spec}_{\mathrm{R}}(G)$. □

Coda B

Reidemeister numbers as class function



van Beethoven, Ludwig, *Scherzo* from *Piano Sonata No. 18, Op. 31 No. 3* (bars 169–171).[1]

In Variation 8, we proved that the Reidemeister number of an automorphism φ on a finite group can be computed by counting the number of irreducible characters, which are class functions, that are fixed by φ. In this Coda, we discuss how one can view the map that associates to an automorphism its Reidemeister number as a class function itself. This point of view leads to an alternative proof of Menon's Identity.

Let G be a finite group and $\varphi \in \mathrm{End}(G)$. In Theorem 8.1.7, we proved that $R(\varphi)$ equals the trace of the linear map

$$\widehat{\varphi} : \mathcal{C}(G) \to \mathcal{C}(G) : f \mapsto f \circ \varphi,$$

where $\mathcal{C}(G)$ is the vector space of complex-valued class functions on G. If φ is an automorphism, $\widehat{\varphi}$ is an isomorphism as well. So, we obtain a map

$$\mathrm{Aut}(G) \to \mathrm{GL}(\mathcal{C}(G)) : \varphi \mapsto \widehat{\varphi}.$$

[1]Excerpt adopted from [4].

However, this map is not a group homomorphism, but a group anti-homomorphism. Indeed, let $\varphi, \psi \in \mathrm{Aut}(G)$ and $f \in \mathcal{C}(G)$. Then

$$\widehat{\varphi}(\widehat{\psi}(f)) = (f \circ \psi) \circ \varphi = f \circ (\psi \circ \varphi) = (\widehat{\psi \circ \varphi})(f).$$

To obtain a representation, we have to consider

$$\rho : \mathrm{Aut}(G) \to \mathrm{GL}(\mathcal{C}(G)) : \varphi \mapsto \widehat{\varphi}^{-1}.$$

By Theorem 8.1.7, the character of this representation is the map $\chi_\rho : \mathrm{Aut}(G) \to \mathbb{N}_0 : \varphi \mapsto R(\varphi^{-1})$. By Lemma T.1.8, this map equals the map $R : \mathrm{Aut}(G) \to \mathbb{N}_0 : \varphi \mapsto R(\varphi)$. For the remainder of this section, we refer to the latter map as the *Reidemeister map*.

Given a group G, then $\mathrm{Aut}(G)$ acts (on the left) in a natural way on both $\mathcal{R}[\mathrm{Id}_G]$ and $\mathrm{Irr}(G)$. The action on $\mathcal{R}[\mathrm{Id}_G]$ is given by

$$\mathrm{Aut}(G) \times \mathcal{R}[\mathrm{Id}_G] \to \mathcal{R}[\mathrm{Id}_G] : (\varphi, [g]) \mapsto [\varphi(g)],$$

the one on $\mathrm{Irr}(G)$ by

$$\mathrm{Aut}(G) \times \mathrm{Irr}(G) \to \mathrm{Irr}(G) : (\varphi, \chi) \mapsto \chi \circ \varphi^{-1}.$$

We need an inverse in order for it to be a left action.

Proposition B.0.1. *Let G be a finite group and let R be the Reidemeister map. Let 1 be the trivial character on $\mathrm{Aut}(G)$. Then $\langle R, 1\rangle$ equals both the number of orbits of the natural action of $\mathrm{Aut}(G)$ on $\mathrm{Irr}(G)$, and the number of orbits of the natural action of $\mathrm{Aut}(G)$ on $\mathcal{R}[\mathrm{Id}_G]$, the set of standard conjugacy classes.*

Proof. It is a standard result that, given a finite group F and two representations $\rho : F \to V, \sigma : F \to W$ with associated characters χ_ρ and χ_σ, the inner product $\langle \chi_\rho, \chi_\sigma\rangle$ equals the dimension of $\mathrm{Hom}_F(V, W)$, the F-invariant linear maps from V to W (see e.g. [110, §7.2 Lemma 2]). Applied to R and 1, we get

$$\langle R, 1\rangle = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Aut}(G)}(\mathcal{C}(G), \mathbb{C}).$$

Let $f \in \mathrm{Hom}_{\mathrm{Aut}(G)}(\mathcal{C}(G), \mathbb{C})$. Fix $\chi \in \mathrm{Irr}(G)$ and let $\varphi \in \mathrm{Aut}(G)$. Since f is an $\mathrm{Aut}(G)$-invariant map, it must hold that

$$f(\chi) = \varphi \cdot f(\chi) = f(\varphi \cdot \chi) = f(\chi \circ \varphi^{-1}).$$

This means that f is constant on the set $\{\chi \circ \psi^{-1} \mid \psi \in \mathrm{Aut}(G)\}$, i.e. the orbit of χ under the action of $\mathrm{Aut}(G)$. Therefore, if $\{\chi_1, \ldots, \chi_k\}$ is a complete set of representatives of the orbits, then f is completely determined by the values $f(\chi_1), \dots, f(\chi_k)$. For $i \in \{1, \dots, k\}$, let $f_i : \mathcal{C}(G) \to \mathbb{C}$ be the linear map determined by the images

$$f_i(\chi) = \begin{cases} 1 & \text{if } \chi \in \mathrm{Aut}(G) \cdot \chi_i \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_i \in \mathrm{Hom}_{\mathrm{Aut}(G)}(\mathcal{C}(G), \mathbb{C})$, by construction. Moreover, the set $\{f_1, \dots, f_k\}$ is a $\mathbb{C}$-basis of $\mathrm{Hom}_{\mathrm{Aut}(G)}(\mathcal{C}(G), \mathbb{C})$. Indeed, let $f \in \mathrm{Hom}_{\mathrm{Aut}(G)}(\mathcal{C}(G), \mathbb{C})$. Then

$$f(\chi) = \sum_{i=1}^{k} f(\chi_i) f_i(\chi)$$

for all $\chi \in \mathrm{Irr}(G)$. For, given $\chi \in \mathrm{Irr}(G)$, there exists a $\varphi \in \mathrm{Aut}(G)$ and a unique χ_i such that $\chi = \chi_i \circ \varphi$. Then

$$f(\chi) = f(\chi_i \circ \varphi) = f(\chi_i) = \sum_{j=1}^{k} f(\chi_j) f_j(\chi),$$

as $f_j(\chi) = 0$ for $j \neq i$. Since $\mathrm{Irr}(G)$ is a $\mathbb{C}$-basis of $\mathcal{C}(G)$, $f = \sum_{i=1}^{k} f(\chi_i) f_i$.

Next, suppose that $\sum_{i=1}^{k} \lambda_i f_i$ is identically 0. Let $i \in \{1, \dots, k\}$ be arbitrary. Evaluating $\sum_{i=1}^{k} \lambda_i f_i$ in χ_i yields $\lambda_i = 0$. As i was arbitrary, we deduce that $\{f_1, \dots, f_k\}$ is a linearly independent set. Therefore, it is a $\mathbb{C}$-basis of $\mathrm{Hom}_{\mathrm{Aut}(G)}(\mathcal{C}(G), \mathbb{C})$, which implies that $\langle R, 1 \rangle$ equals the number of orbits of the natural action of $\mathrm{Aut}(G)$ on $\mathrm{Irr}(G)$.

Since the number of orbits of the natural action of $\mathrm{Aut}(G)$ on $\mathcal{R}[\mathrm{Id}_G]$ equals the number of orbits of the natural action of $\mathrm{Aut}(G)$ on $\mathrm{Irr}(G)$ (see e.g. [67, Corollary 5.6]), the second part of the result follows. $\square$

As an application of this result, we provide a new proof of Menon's Identity. There already exists numerous proofs of this identity and its generalisations (see e.g. [119]). The approach below resembles the one of I. Richards in [98, Theorem], where the identity is proven by counting cyclic groups. However, he does not explicitly mention the term 'Menon's Identity'.

Proposition B.0.2. *Let $n \geq 1$ be an integer. Let $d(n)$ denote the number of divisors of n and $\varphi(n)$ the Euler totient function of n, i.e. the cardinality of*

$$S := \{s \in \mathbb{Z} \mid 1 \leq s \leq n, \gcd(s, n) = 1\}.$$

Then

$$\sum_{s \in S} \gcd(s - 1, n) = \varphi(n) d(n).$$

Proof. Consider the cyclic group $\mathbb{Z}/n\mathbb{Z}$. For $k \in \{1, \ldots, n\}$, define $\varphi_k : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} : x \mapsto kx$. Then $R(\varphi_k) = |\mathrm{Fix}(\varphi_k)| = \gcd(k-1, n)$ by Lemma 7.2.2. Thus, the sum in the left-hand side equals

$$\sum_{s \in S} R(\varphi_s).$$

Since φ_k is an automorphism if and only if $k \in S$, this sum is also equal to $|\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})|\langle R, 1\rangle$. As $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$, we have $|\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})| = \varphi(n)$. By Proposition B.0.1, $\langle R, 1\rangle$ equals the number of orbits of the action of $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ on the conjugacy classes of $\mathbb{Z}/n\mathbb{Z}$. This is simply the action of $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ on $\mathbb{Z}/n\mathbb{Z}$.

Let $x, y \in \mathbb{Z}/n\mathbb{Z}$. If $\varphi \in \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ satisfies $\varphi(x) = y$, then x and y have the same order, since φ is an automorphism. Conversely, suppose that x and y have the same order, say d. Then $x = x'\frac{n}{d}$ and $y = y'\frac{n}{d}$ for some $x', y' \in \mathbb{Z}$ coprime with n. Let z be the multiplicative inverse of x' modulo n. Then $\varphi_{y'z}(x) = y$. We conclude that $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ acts transitively on the elements in $\mathbb{Z}/n\mathbb{Z}$ of the same order.

Hence, the orbits are in one-to-one correspondence with the divisors of n, which implies that the number of orbits equals $d(n)$. Therefore, we conclude that

$$\sum_{s \in S} \gcd(s-1, n) = \sum_{s \in S} R(\varphi_s) = |\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})|\langle R, 1\rangle = \varphi(n)d(n). \qquad \square$$

Bibliography

[1] G. Baumslag. "Direct decompositions of finitely generated torsion-free nilpotent groups". In: *Mathematatische Zeitschrift* 145.1 (1975), pp. 1–10. DOI: 10.1007/bf01214492.

[2] G. Baumslag. *Lecture Notes on Nilpotent Groups*. Vol. 2. Regional Conference Series in Mathematics. Rhode Island: American Mathematical Society, 1971.

[3] G. Baumslag, C. F. Miller and G. Ostheimer. "Decomposability of finitely generated torsion-free nilpotent groups". In: *International Journal of Algebra and Computation* 26.8 (2016), pp. 1529–1546. DOI: 10.1142/S0218196716500673.

[4] L. van Beethoven. "Piano Sonata No. 18, Op. 31 No. 3". In: *Trois sonates pour le Piano-forte, Composées par Louis van Beethoven*. Bonn, Plate 345: N. Simrock, n.d. [1802].

[5] L. van Beethoven. *Piano Sonata No. 30, Op. 109*. Holograph manuscript. URL: http://vmirror.imslp.org/files/imglnks/usimg/1/1f/IMSLP621234-PMLP1487-Beethoven_Op_109_Manuscript.pdf (visited on 04/11/2022).

[6] G. Berhuy. *An Introduction to Galois Cohomology and its Applications*. Vol. 377. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 2010. DOI: 10.1017/CBO9781139107051.

[7] J. N. S. Bidwell. "Automorphisms of direct products of finite groups II". In: *Archiv der Mathematik* 91.2 (2008), pp. 111–121. DOI: 10.1007/s00013-008-2653-5.

[8] J. N. S. Bidwell, M. J. Curran and D. J. McCaughan. "Automorphisms of direct products of finite groups". In: *Archiv der Mathematik* 86.6 (2006), pp. 481–489. DOI: 10.1007/s00013-005-1547-z.

[9] C. Bleak, A. L. Fel'shtyn and D. L. Gonçalves. "Twisted conjugacy classes in R. Thompson's group F". In: *Pacific Journal of Mathematics* 238.1 (2008), pp. 1–6. DOI: 10.2140/pjm.2008.238.1.

[10] M. R. Bridson and C. F. Miller. "Recognition of subgroups of direct products of hyperbolic groups". In: *Proceedings of the American Mathematical Society* 132.1 (2004), pp. 59–65. DOI: 10.1090/S0002-9939-03-07008-4.

[11] R. F. Brown. *The Lefschetz fixed point theorem*. Chicago: Scott-Foresman, 1971.

[12] T. Ceccherini-Silberstein and M. Coornaert. *Cellular Automata and Groups*. Springer Monographs in Mathematics. Berlin, Heidelberg: Springer-Verlag, 2010. DOI: 10.1007/978-3-642-14034-1.

[13] A. E. Clement, S. Majewicz and M. Zyman. *The Theory of Nilpotent Groups*. Cham: Birkhäuser, 2017. DOI: 10.1007/978-3-319-66213-8.

[14] K. Conrad. *Characters of Finite Abelian Groups*. URL: https://kconrad.math.uconn.edu/blurbs/grouptheory/charthy.pdf (visited on 04/11/2022).

[15] L. J. Corwin and F. P. Greenleaf. *Representations of nilpotent Lie Groups and their applications. Part I: Basic theory and examples*. Vol. 18. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1990.

[16] D. Curtiss. "On Kellogg's Diophantine Problem". In: *The American Mathematical Monthly* 29.10 (1922), pp. 380–387. DOI: 10.2307/2299023.

[17] K. Dekimpe. *Almost-Bieberbach groups: affine and polynomial structures*. Vol. 1639. Lecture Notes in Mathematics. Springer Verlag, Berlin, 1996.

[18] K. Dekimpe and D. L. Gonçalves. "The R_∞ property for abelian groups". In: *Topological Methods in Nonlinear Analysis* 46.2 (2015), pp. 773–784. DOI: 10.12775/TMNA.2015.066.

[19] K. Dekimpe and D. L. Gonçalves. "The R_∞ property for free groups, free nilpotent groups and free solvable groups". In: *Bulletin of the London Mathematical Society* 46.4 (2014), pp. 737–746. DOI: 10.1112/blms/bdu029.

[20] K. Dekimpe and D. L. Gonçalves. "The R_∞ property for nilpotent quotients of surface groups". In: *Transactions of the London Mathematical Society* 3.1 (2016), pp. 28–46. DOI: 10.1112/tlms/tlw002.

[21] K. Dekimpe, D. L. Gonçalves and O. Ocampo. "The R_∞ property for pure Artin braid groups". In: *Monatshefte für Mathematik* 195.1 (2021), pp. 15–33. DOI: 10.1007/s00605-020-01484-7.

[22] K. Dekimpe, T. Kaiser and S. Tertooy. "The Reidemeister spectra of low dimensional crystallographic groups". In: *Journal of Algebra* 533 (2019), pp. 353–375. DOI: 10.1016/j.jalgebra.2019.04.038.

[23] K. Dekimpe and M. Lathouwers. *The Reidemeister spectrum of 2-step nilpotent groups determined by graphs*. 2021. DOI: 10.48550/arXiv.2202.03000. arXiv: 2202.03000 [math.GR].

[24] K. Dekimpe and P. Senden. "The R_∞-property for right-angled Artin groups". In: *Topology and its Applications* 293 (2021). DOI: 10.1016/j.topol.2020.107557.

[25] K. Dekimpe and S. Tertooy. "Algorithms for twisted conjugacy classes of polycyclic-by-finite groups". In: *Topology and its Applications* 293 (2021), p. 107565. DOI: 10.1016/j.topol.2020.107565.

[26] K. Dekimpe, S. Tertooy and A. R. Vargas. "Fixed points of diffeomorphisms on nilmanifolds with a free nilpotent fundamental group". In: *The Asian Journal of Mathematics* 24.1 (2020), pp. 147–164. DOI: 10.4310/AJM.2020.v24.n1.a6.

[27] D. S. Dummit and R. M. Foote. *Abstract Algebra*. John Wiley and Sons, Inc., 2004.

[28] A. L. Fel'shtyn. "Dynamical zeta functions, Nielsen theory and Reidemeister torsion". In: *Memoirs of the American Mathematical Society* 147.699 (2000), pp. xii+146. DOI: 10.1090/memo/0699.

[29] A. L. Fel'shtyn. "The Reidemeister number of any automorphism of a Gromov hyperbolic group is infinite". In: *Rossiiskaya Akademiya Nauk. Sankt-Peterburgskoe Otdelenie. Matematicheskii Institut im. V. A. Steklova. Zapiski Nauchnykh Seminarov (POMI)* 279.Geom. i Topol. 6 (2001), pp. 229–240, 250. DOI: 10.1023/B:JOTH.0000008749.42806.e3.

[30] A. L. Fel'shtyn and D. L. Gonçalves. "Reidemeister spectrum for metabelian groups of the form $Q^n \rtimes \mathbb{Z}$ and $\mathbb{Z}[1/p]^n \rtimes \mathbb{Z}$, p prime". In: *International Journal of Algebra and Computation* 21.3 (2011), pp. 505–520. DOI: 10.1142/S0218196711006297.

[31] A. L. Fel'shtyn and D. L. Gonçalves. "Twisted conjugacy classes of automorphisms of Baumslag-Solitar groups". In: *Algebra and Discrete Mathematics* 5.3 (2006), pp. 36–48.

[32] A. L. Fel'shtyn and R. Hill. "The Reidemeister Zeta Function with Applications to Nielsen Theory and a Connection with Reidemeister Torsion". In: *K-Theory* 8.4 (1994), pp. 367–393. DOI: 10.1007/BF00961408.

[33] A. L. Fel'shtyn and J. B. Lee. "The Nielsen and Reidemeister numbers of maps on infra-solvmanifolds of type (R)". In: *Topology and its Applications* 181 (2015), pp. 62–103. DOI: 10.1016/j.topol.2014.12.003.

[34] A. L. Fel'shtyn, Y. Leonov and E. Troitsky. "Twisted conjugacy classes in saturated weakly branch groups". In: *Geometriae Dedicata* 134.1 (2008), pp. 61–73. DOI: 10.1007/s10711-008-9245-1.

[35] A. L. Fel'shtyn and T. Nasybullov. "The R_∞ and S_∞ properties for linear algebraic groups". In: *Journal of Group Theory* 19.5 (2016), pp. 901–921. DOI: 10.1515/jgth-2016-0004.

[36] A. L. Fel'shtyn and E. Troitsky. "Aspects of the property R_∞". In: *Journal of Group Theory* 18.6 (2015), pp. 1021–1034. DOI: 10.1515/jgth-2015-0022.

[37] A. L. Fel'shtyn and E. Troitsky. "Twisted Burnside-Frobenius theory for discrete groups". In: *Journal für die Reine und Angewandte Mathematik* 2007.613 (2007), pp. 193–210. DOI: 10.1515/CRELLE.2007.097.

[38] N. J. Fullarton. "On the number of outer automorphisms of the automorphism group of a right-angled Artin group". In: *Mathematical Research Letters* 23.1 (2016), pp. 145–162. DOI: 10.4310/MRL.2016.v23.n1.a8.

[39] M. Furi, M. P. Pera and M. Spadini. "On the uniqueness of the fixed point index on differentiable manifolds". In: *Fixed Point Theory and Algorithms for Sciences and Engineering* 2004.4 (2004), p. 478686. DOI: 10.1155/S168718200440713X.

[40] G. Gandini and N. Wahl. "Homological Stability for automorphism groups of Raags". In: *Algebraic & Geometric Topology* 16.4 (2016), pp. 2421–2441. DOI: 10.2140/agt.2016.16.2421.

[41] G. Gershwin. *3 Preludes*. Plate N. W. 50-11, New York: New World Music Co., 1927.

[42] M. Golasiński and D. L. Gonçalves. "On automorphisms of split metacyclic groups". In: *manuscripta mathematica* 128.2 (2009), p. 251. DOI: 10.1007/s00229-008-0233-4.

[43] D. L. Gonçalves. "Coincidence Reidemeister classes on nilmanifolds and nilpotent fibrations". In: *Topology and its Applications* 83.3 (1998), pp. 169–186. DOI: 10.1016/S0166-8641(97)00106-5.

[44] D. L. Gonçalves. "Coincidence theory". In: *Handbook of Topological Fixed Point Theory*. Ed. by R. Brown. Dordrecht: Springer, 2005, pp. 3–42. DOI: 10.1007/1-4020-3222-6_1.

[45] D. L. Gonçalves and D. H. Kochloukova. “Sigma theory and twisted conjugacy classes”. In: *Pacific Journal of Mathematics* 247.2 (2010), pp. 335–352. DOI: 10.2140/pjm.2010.247.335.

[46] D. L. Gonçalves and P. Sankaran. “Sigma theory and twisted conjugacy, II: Houghton groups and pure symmetric automorphism groups”. In: *Pacific Journal of Mathematics* 280.2 (2016), pp. 349–369. DOI: 10.2140/pjm.2016.280.349.

[47] D. L. Gonçalves, P. Sankaran and P. N. Wong. “Twisted conjugacy in free products”. In: *Communications in Algebra* 48.9 (2020), pp. 3916–3921. DOI: 10.1080/00927872.2020.1751848.

[48] D. L. Gonçalves and P. N. Wong. “Homogeneous spaces in coincidence theory II”. In: *Forum Mathematicum* 17.2 (2005), pp. 297–313. DOI: 10.21711/231766361997/rmc134.

[49] D. L. Gonçalves and P. N. Wong. “Twisted conjugacy classes in exponential growth groups”. In: *Bulletin of the London Mathematical Society* 35.2 (2003), pp. 261–268. DOI: 10.1112/S0024609302001832.

[50] D. L. Gonçalves and P. N. Wong. “Twisted Conjugacy Classes in Nilpotent Groups”. In: *Journal für die Reine und Angewandte Mathematik* 2009.633 (2009), pp. 11–27. DOI: 10.1515/CRELLE.2009.058.

[51] D. L. Gonçalves and P. N. Wong. “Twisted conjugacy classes in wreath products”. In: *International Journal of Algebra and Computation* 16.5 (2006), pp. 875–886. DOI: 10.1142/S0218196706003219.

[52] D. L. Gonçalves and P. N. Wong. “Twisted Conjugacy for Virtually Cyclic Groups and Crystallographic Groups”. In: *Combinatorial and Geometric Group Theory.* 2010, pp. 119–147. DOI: 10.1007/978-3-7643-9911-5_5.

[53] D. Gorenstein. *Finite Groups.* 2nd ed. New York: Chelsea Publishing Company, 1980.

[54] D. Gorenstein and I. N. Herstein. “Finite Groups Admitting a Fixed-Point Free Automorphism of Order 4”. In: *American Journal of Mathematics* 83.1 (1961), pp. 71–78. DOI: 10.2307/2372721.

[55] M. Gromov. “Hyperbolic Groups”. In: *Essays in Group Theory.* Ed. by S. M. Gersten. New York: Springer New York, 1987, pp. 75–263. DOI: 10.1007/978-1-4613-9586-7_3.

[56] P. R. Heath. “Product Formulae for Nielsen Numbers”. In: *Pacific Journal of Mathematics* 117.2 (1985), pp. 267–289. DOI: 10.2140/pjm.1985.117.267.

[57] P. R. Heath and E. Keppelmann. “Fibre techniques in Nielsen periodic point theory on nil and solvmanifolds I”. In: *Topology and its Applications* 76.3 (1997), pp. 217–247. DOI: 10.1016/S0166-8641(96)00100-9.

[58] C. J. Hillar and D. L. Rhea. “Automorphisms of Finite Abelian Groups”. In: *The American Mathematical Monthly* 114.10 (2007), pp. 917–923. DOI: 10.1080/00029890.2007.11920485.

[59] T. Hsu and D. Wise. “Ascending HNN extensions of polycyclic groups are residually finite”. In: *Journal of Pure and Applied Algebra* 182.1 (2003), pp. 65–78. DOI: 10.1016/S0022-4049(02)00310-9.

[60] E. Jabara. “Automorphisms with finite Reidemeister number in residually finite groups”. In: *Journal of Algebra* 320 (2008), pp. 3671–3679. DOI: 10.1016/j.jalgebra.2008.09.001.

[61] C. U. Jensen, A. Ledet and N. Yui. *Generic Polynomials. Constructive Aspects of the Inverse Galois Problem.* Cambridge: Cambridge University Press, 2002.

[62] B. Jiang. *Lectures on Nielsen Fixed Point Theory.* Vol. 14. Contemporary Mathematics. Providence, Rhode Island: American Mathematical Society, 1983.

[63] J. H. Jo, J. B. Lee and S. R. Lee. “The R_∞-property for Houghton’s groups”. In: *Algebra and Discrete Mathematics* 23.2 (2017), pp. 249–262.

[64] F. Johnson. “Automorphisms of direct products and their geometric realisations”. In: *Mathematische Annalen* 263.3 (1983), pp. 343–364. DOI: 10.1007/BF01457136.

[65] A. Juhász. “Twisted Conjugacy in Certain Artin Groups”. In: *Ischia Group Theory 2010.* 2012, pp. 175–195. DOI: 10.1142/9789814350051_0014.

[66] M. I. Kargapolov and J. I. Merzljakov. *Fundamentals of the Theory of Groups.* Vol. 62. Graduate Texts in Mathematics. New York: Springer, New York, 1979.

[67] G. Karpilovsky. *Group Representations Volume 1 Part B: Introduction to Group Representations and Characters.* Vol. 1. Group Representations. Elsevier Science, 1992.

[68] A. Karrass, W. Magnus and D. Solitar. *Combinatorial Group Theory. Presentations of Groups in Terms of Generators and Relations.* New York: Dover Publications, Inc., 1976.

[69] B. Kerby and E. Rode. “Characteristic Subgroups of Finite Abelian Groups”. In: *Communications in Algebra* 39.4 (2011), pp. 1315–1343. DOI: 10.1080/00927871003591843.

[70] S. W. Kim, J. B. Lee and K. B. Lee. “Averaging Formula for Nielsen Numbers”. In: *Nagoya Mathematical Journal* 178 (2005), pp. 37–53. DOI: 10.1017/S0027763000009107.

[71] A. G. Kurosh. *Theory of Groups*. Vol. 2. American Mathematical Society, 1960.

[72] G. Levitt. "On the automorphism group of generalized Baumslag-Solitar groups". In: *Geometry & Topology* 11.1 (2007), pp. 473–515. DOI: 10.2140/gt.2007.11.473.

[73] G. Levitt and M. Lustig. "Most automorphisms of a hyperbolic group have very simple dynamics". In: *Annales scientifiques de l'École Normale Supérieure* Ser. 4, 33.4 (2000), pp. 507–517. DOI: 10.1016/s0012-9593(00)00120-8.

[74] C. Löh. *Geometric Group Theory: an Introduction*. Universitext. Springer International Publishing, 2017.

[75] R. C. Lyndon. "Two notes on Rankin's book on the modular group". In: *Journal of the Australian Mathematical Society* 16.4 (1973), pp. 454–457. DOI: 10.1017/S1446788700015433.

[76] R. C. Lyndon and P. Schupp. *Combinatorial Group Theory*. Springer Verlag Berlin Heidelberg, 1977.

[77] A. Mal'cev. "On the faithful representation of infinite groups by matrices". In: *American Mathematical Society Translations: Series 2* 45.1 (1965), pp. 1–18. DOI: 10.1090/trans2/045.

[78] G. Miller. "On the multiple holomorphs of a group". In: *Mathematische Annalen* 66.1 (1908), pp. 133–142. DOI: 10.1007/BF01450918.

[79] W. Mills. "Decomposition of Holomorphs". In: *Pacific Journal of Mathematics* 11.4 (1961), pp. 1443–1446. DOI: 10.2140/pjm.1961.11.1443.

[80] W. A. Mozart. "Zwölf Variationen in C über das franszösische Lied 'Ah, vous dirai-je Maman' (KV 265 (300e))". In: *Neue Mozart-Ausgabe, Serie IX, Werkgruppe 26, Variationen für Klavier [NMA IX/26]*. Ed. by K. von Fischer. Plate BA 4525: Kassel: Bärenreiter-Verlag, 1961.

[81] W. A. Mozart. *Zwölf Variationen in C über das franszösische Lied 'Ah, vous dirai-je, Maman' (KV 265 (300e))*. Holograph manuscript. URL: http://vmirror.imslp.org/files/imglnks/usimg/0/03/IMSLP472376-PMLP55775-Mozart_variations_sur_Ah_vous_dirais-je_Maman_k265_autograph_facsimile_manuscript_-Unlocked_by_www.freemypdf.com-.pdf (visited on 03/11/2022).

[82] T. Mubeena and P. Sankaran. "Twisted Conjugacy Classes in Abelian Extensions of Certain Linear Groups". In: *Canadian Mathematical Bulletin* 57.1 (2014), pp. 132–140. DOI: 10.4153/CMB-2012-013-7.

[83] J. R. Munkres. *Topology*. Pearson Education Limited, 2014.

[84] T. K. Naik, N. Nanda and M. Singh. "Some remarks on twin groups". In: *Journal of Knot Theory and Its Ramifications* 29.10 (2020), p. 2042006. DOI: 10.1142/S0218216520420067.

[85] T. Nasybullov. "Chevalley groups of types B_n, C_n, D_n over certain fields do not possess the R_∞-property". In: *Topological Methods in Nonlinear Analysis* 56.2 (2020), pp. 401–417. DOI: 10.12775/TMNA.2019.113.

[86] T. Nasybullov. "The R_∞-property for Chevalley groups of types B_l, C_l, D_l over integral domains". In: *Journal of Algebra* 446 (2016), pp. 489–498. DOI: 10.1016/j.jalgebra.2015.09.030.

[87] T. Nasybullov. "Twisted conjugacy classes in Chevalley groups". In: *Algebra and Logic* 53.6 (2015), pp. 481–501. DOI: 10.1007/s10469-015-9310-4.

[88] T. Nasybullov. "Twisted conjugacy classes in general and special linear groups". In: *Algebra and Logic* 51.3 (2012), pp. 220–231. DOI: 10.1007/s10469-012-9185-6.

[89] T. Nasybullov. "Twisted conjugacy classes in unitriangular groups". In: *Journal of Group Theory* 22.2 (2019), pp. 253–266.

[90] P. Neumann. "On the structure of standard wreath products of groups". In: *Mathematische Zeitschrift* 84.4 (1964), pp. 343–373. DOI: 10.1007/BF01109904.

[91] M. Newman. "A bound for the number of conjugacy classes in a group". In: *Journal of the London Mathematical Society* 43.1 (1968), pp. 108–110. DOI: 10.1112/jlms/s1-43.1.108.

[92] J. Nielsen. "Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen". In: *Acta Mathematica* 50 (1927), pp. 189–358. DOI: 10.1007/BF02421324.

[93] J. Nielsen. "Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen II". In: *Acta Mathematica* 53 (1929), pp. 1–76. DOI: 10.1007/BF02547566.

[94] J. Nielsen. "Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen III". In: *Acta Mathematica* 58 (1932), pp. 87–167. DOI: 10.1007/BF02547775.

[95] D. S. Passman. *The Algebraic Structure of Group Rings*. New York: John Wiley and Sons, Inc., 1977.

[96] S. S. Pradhan and B. Sury. "Rational and quasi-permutation representations of holomorphs of cyclic p-groups". In: *International Journal of Group Theory* 11.3 (2022), pp. 151–174. DOI: 10.22108/ijgt.2021.128359.1686.

[97] K. Reidemeister. “Automorphismen von Homotopiekettenringen”. In: *Mathematische Annalen* 112.1 (1936), pp. 586–593. DOI: 10.1007/BF01565432.

[98] I. M. Richards. “A Remark on the Number of Cyclic Subgroups of a Finite Group”. In: *The American Mathematical Monthly* 91.9 (1984), pp. 571–572. DOI: 10.1080/00029890.1984.11971498.

[99] D. J. S. Robinson. *A Course in the Theory of Groups*. Vol. 80. Graduate Texts in Mathematics. New York: Springer-Verlag, 1996. DOI: 10.1007/978-1-4419-8594-1.

[100] V. Roman’kov. “Twisted conjugacy classes in nilpotent groups”. In: *Journal of Pure and Applied Algebra* 215.4 (2011), pp. 664–671. DOI: 10.1016/j.jpaa.2010.06.015.

[101] J. J. Rotman. *Advanced Modern Algebra*. Upper Sadle River, New Jersey: Prentice Hall, 2003.

[102] P. Rowley. “Finite Groups Admitting a Fixed-Point-Free Automorphism Group”. In: *Journal of Algebra* 174.2 (1995), pp. 724–727. DOI: 10.1006/jabr.1995.1148.

[103] M. Schulte. “Automorphisms of Metacyclic p-groups with Cyclic Maximal Subgroups”. In: *Rose-Hulman Undergraduate Mathematics Journal* 2.2 (2001).

[104] D. Segal. *Polycyclic Groups*. Cambridge Tracts in Mathematics. Cambridge: Cambridge University Press, 1983.

[105] A. Selberg. “On discontinuous groups in higher-dimensional symmetric spaces”. In: *Matematika* 6.3 (1962), pp. 3–16.

[106] P. Senden. *The Reidemeister spectrum of direct products of nilpotent groups*. 2022. DOI: 10.48550/arXiv.2206.13853. arXiv: 2206.13853 [math.GR].

[107] P. Senden. *The Reidemeister spectrum of finite abelian groups*. 2022. DOI: 10.48550/arXiv.2205.15740. arXiv: 2205.15740 [math.GR].

[108] P. Senden. *The Reidemeister spectrum of split metacyclic groups*. 2021. DOI: 10.48550/arXiv.2109.12892. arXiv: 2109.12892 [math.GR].

[109] P. Senden. “Twisted conjugacy in direct products of groups”. In: *Communications in Algebra* 49.12 (2021), pp. 5402–5422. DOI: 10.1080/00927872.2021.1945615.

[110] J.-P. Serre. *Linear Representations of Finite Groups*. Vol. 42. Graduate Texts in Mathematics. New York: Springer, New York, 1977.

[111] S. Shokranian. *The Selberg-Arthur Trace Formula. Based on Lectures by James Arthur*. Vol. 1503. Lecture Notes in Mathematics. Berlin Heidelberg: Springer-Verlag, 1992. DOI: 10.1007/BFb0092305.

[112] P. Shumyatsky and A. Tamarozzi. "On finite groups with fixed-point free automorphisms". In: *Communications in Algebra* 30.6 (2002), pp. 2837–2842. DOI: doi:10.1081/AGB-120003992.

[113] M. Stein, J. Taback and P. N. Wong. "Automorphisms of higher rank lamplighter groups". In: *International Journal of Algebra and Computation* 25.8 (2015), pp. 1275–1299. DOI: 10.1142/S0218196715500411.

[114] J. J. Sylvester. "XXXIX. On the equation to the secular inequalities in the planetary theory". In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 16.100 (1883), pp. 267–269. DOI: 10.1080/14786448308627430.

[115] J. Taback and P. N. Wong. *A note on twisted conjugacy classes and generalized Baumslag-Solitar groups.* 2008. DOI: 10.48550/arXiv.math/0606284. arXiv: 0606284v3 [math.GR].

[116] J. Taback and P. N. Wong. "Twisted conjugacy and quasi-isometry invariance for generalized solvable Baumslag-Solitar groups". In: *Journal of the London Mathematical Society* 75.3 (2007), pp. 705–717. DOI: 10.1112/jlms/jdm024.

[117] S. Tertooy. "Reidemeister spectra for almost-crystallographic groups". PhD thesis. KU Leuven, Oct. 2019.

[118] J. Thompson. "Finite Groups with Fixed-Point Free Automorphisms of Prime Order". In: *Proceedings of the National Academy of Sciences of the United States of America* 45.4 (1959), pp. 578–581. DOI: 10.2307/90107.

[119] L. Tóth. *Proofs, generalizations and analogs of Menon's identity.* 2021. DOI: 10.48550/arXiv.2110.07271. arXiv: 2110.07271v1 [math.GR].

[120] I. Van den Bussche. "Reidemeister numbers for maps on solvmanifolds". MA thesis. KU Leuven, 2017.

[121] R. B. J. Warfield. *Nilpotent Groups.* Vol. 513. Lecture Notes in Mathematics. Springer-Verlag Berlin Heidelberg, 1976. DOI: 10.1007/BFb0080152.

[122] F. Wecken. "Fixpunktklassen I". In: *Mathematische Annalen* 117.1 (1940), pp. 659–671. DOI: 10.1007/BF01450034.

[123] F. Wecken. "Fixpunktklassen II". In: *Mathematische Annalen* 118.1 (1941), pp. 216–234. DOI: 10.1007/BF01487362.

[124] F. Wecken. "Fixpunktklassen III". In: *Mathematische Annalen* 118.1 (1941), pp. 544–577. DOI: 10.1007/BF01487386.

[125] B. A. F. Wehrfritz. "Endomorphisms of polycyclic-by-finite groups". In: *Mathematische Zeitschrift* 264.3 (2010), pp. 629–632. DOI: 10.1007/s00209-009-0482-2.

[126] E. Witt. “Treue Darstellung Liescher Ringe”. In: *Journal für die Reine und Angewandte Mathematik* 1937.177 (1937), pp. 152–160. DOI: 10.1515/crll.1937.177.152.

[127] P. N. Wong. “Reidemeister zeta function for group extensions”. In: *Journal of the Korean Mathematical Society* 38.6 (2001), pp. 1107–1116.

FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
ALGEBRAIC TOPOLOGY & GROUP THEORY
Etienne Sabbelaan 53
B-8500 Kortrijk
pieter.senden@kuleuven.be
www.kuleuven-kulak.be/algtop