Centre For Space Physics

Honours

Background-first draft

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Contents

1 Cosmology

The aim of Cosmology is to understand the underlying laws of physics, that are assumed to hold throughout the universe, by explaining the origin and evolution of the universe and all processes contained within it.

1.1 Newtonian cosmology

Newton considered the universe to be homogeneous (uniformly distributed) and isotropic (has the same properties in all directions) and thus that the distribution of matter in the universe would look the same to all observers regardless of their position [3]. Therefore no

2 Friedman equations and Equation of state

2.1 Equation of State from Newtonian Mechanics

From the first law of thermodynamics we have that

$$dE = -PdV + dQ \tag{1}$$

With dE the change in energy content, P the pressure, dV the change in volume and dQ the change in heat. Considering an Isotropic and homogeneous universe [5],the expansion of the universe will be adiabatic with the Pressure P constant and there will be no change in the heat (Q) content of the universe since there is nowhere for the heat to come from [5]. From this we can then conclude that the dQ term in equation (1) will be $dQ = 0$ and reduces to.

$$dE = -PdV \tag{2}$$

This implies that

$$\dot{E} = -P\frac{dV}{dt} \tag{3}$$

since P is a constant. We have that for c the speed of light

$$E = \frac{4\pi}{3}c^2r^3\rho \tag{4}$$

where ρ is the mass density as a function of time. If we now consider a expanding universe where the relative motion of galaxies are not due to intrinsic peculiar velocities, but to the expansion of the universe then we can write r as $r = a(t)\chi$ where χ is the actual distance between the two points (galaxies) and $a(t)$ the expansion parameter defined so that $a(t) = 1$ at the present time. Equation (4) then becomes

$$E = \frac{4\pi}{3} c^2 a^3 \chi^3 \rho \tag{5}$$

From this then follows that

$$\dot{E} = \frac{4\pi\chi^3 c^2}{3} \left(3a^2 \dot{a}\rho + \dot{\rho}a^3\right) \tag{6}$$

Substituting this back into equation (3) yields

$$\dot{\rho} + 3H\left(\rho + P\right) = 0 \tag{7}$$

Where H is the Hubble parameter. If we now let $P = \rho\omega$ (which holds true for a perfect fluid), where ω can be a function of time, in such a way that P remains a constant. then equation (7) becomes

$$\dot{\rho} + 3H\left(1 + \omega\right)\rho = 0 \tag{8}$$

This ODE has a solution of the form

$$\rho = Ca^{-3(1+\omega)} \tag{9}$$

with C constant. For matter $\omega = 0$, for radiation $\omega = \frac{1}{3}$ and for the cosmological constant $\omega = -1$ [5]. From this then follows that for matter

$$\rho = C_d a^{-3} \tag{10}$$

for radiation

$$\rho = C_{rad} a^{-4} \tag{11}$$

and for the cosmological constant

$$\rho = C_{DE} \tag{12}$$

2.2 The Friedman equation

Consider a isolated, isotropic sphere with mass M and radius $r = a(t)\chi$, where $a(t)$ is the expansion coefficient and χ the distance between any two points in the sphere [5]. From

Newtons shell theorem [4], objects (with mass m) on the surface of the sphere will experience the gravitational force due to the mass within the sphere as if all the mass is at the center of the shell. From this the equation of motion for points on the surface of the sphere can be found from Newton‘s second law:

$$m\ddot{r} = -\frac{GMm}{r^2} \tag{13}$$

Multiplying both sides of the equation with $\dot{r}$, dividing by m and integrating over time gives

$$\frac{1}{2}\dot{r}^2 = \frac{GM}{r} + U \tag{14}$$

Where U is an integration constant. Since we have that $M = \frac{4\pi}{3}r^3\rho$ and $r = a\chi$, it follows that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho + \frac{2U}{a^2\chi^2} \tag{15}$$

which is the expansion equation, with ρ the mass-density.

2.3 The Acceleration equation

From the result of the Expansion equation (15) and the Equation of State (9) the Acceleration equation can be found as:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G\left(\rho + 3P\right) \tag{16}$$

where $P = \rho\omega$ is the pressure.

2.4 Solving the Friedman equation for different epochs and curvatures

Equation (15) can be rewritten as [5]

$$\begin{aligned}\dot{a}^2 &= \frac{8\pi G}{3c^2}\epsilon a^2 - \kappa\frac{c^2}{R_0^2} \\ &= A\epsilon a^2 - \kappa B\end{aligned} \tag{17}$$

with ϵ energy density, κ curvature and R_0 the present value of curvature radius. Let curvature take the values $\kappa = -1$ for open space, $\kappa = 1$ for closed space and $\kappa = 0$ for flat space [2], then using the solutions to equation (9) the solutions to the Friedman equation (15) can be summarised as

Solving for Matter dominated Epochs

curvature κ	a	t
0	$\sqrt[3]{AC_d}\left(\frac{3}{2}\right)^{\frac{2}{3}}(t-t_0)^{\frac{2}{3}}$	-
1	$\frac{AC_d}{2B}\left(1-\cos\left(\sqrt{B}\,(\eta)\right)\right)$	$\frac{AC_d}{2B}\left(\eta-\frac{1}{\sqrt{B}}\sin\left(\sqrt{B}\,(\eta)\right)\right)$
-1	$\frac{AC_d}{2B}\left(\cosh\left(\sqrt{B}\,(\eta)\right)-1\right)$	$\frac{AC_d}{2B}\left(\frac{1}{\sqrt{B}}\sinh\left(\sqrt{B}\,(\eta)\right)-\eta\right)$

Solving for Radiation dominated Epochs

curvature κ	a	t
0	$\sqrt{2}\sqrt[4]{AC_{rad}}\,(t-t_0)^{\frac{1}{2}}$	-
1	$\sqrt{\frac{AC_{rad}}{B}}\sin\left(\sqrt{B}\,(\eta)\right)$	$-\sqrt{\frac{AC_{rad}}{B^2}}\cos\left(\sqrt{B}\,(\eta)\right)$
-1	$\sqrt{AC_{rad}}\sinh\left(\sqrt{B}\,(\eta)\right)$	$\sqrt{\frac{AC_{rad}}{B}}\cosh\left(\sqrt{B}\,(\eta)\right)$

Solving for Dark Energy dominated Epochs

curvature κ	a	t
0	$e^{\sqrt{AC_{DE}}(t-t_0)}$	-
1	$\sqrt{\frac{B}{AC_{DE}}}\cosh\left(\sqrt{AC_{DE}}\,(t-t_0)\right)$	-
-1	$-\sqrt{\frac{B}{AC_{DE}}}\operatorname{csch}\sqrt{B}\,(\eta)$	$-\sqrt{\frac{1}{AC_{DE}}}ln\left\lvert\tanh\frac{\sqrt{B}}{2}\,(\eta)\right\rvert$

Where the a new parametrization, known as the conformal time parametrization has been introduced. The new parametrization is defined by,

$$\int dt = \int a d\eta \tag{18}$$

where η is the farthest distance that information could have travelled since the beginning of the universe[1].

Apendix A

Equation of State from Newtonian Mechanics

From the first law of thermodynamics we have that

$$dE = -PdV + dQ \tag{19}$$

With dE the change in energy content, P the pressure, dV the change in volume and dQ the change in heat. Considering an Isotropic and homogeneous universe [5],the expansion of the universe will be adiabatic with the Pressure P constant and there will be no change in the heat (Q) content of the universe since there is nowhere for the heat to come from [5]. From this we can then conclude that the dQ term in equation 1 will be $dQ = 0$ and reduces to.

$$dE = -PdV \tag{20}$$

this impies that

$$\begin{aligned} E(t) &= -PV(t) \\ \Rightarrow \dot{E} &= \frac{d\,(-PV)}{dt} \\ &= -\frac{dP}{dt}V - P\frac{dV}{dt} \\ &= -P\frac{dV}{dt} \end{aligned} \tag{21}$$

since P is a constant. We have that for M the total mass and c the speed of light

$$\begin{aligned} V &= \frac{4\pi}{3}r^3 \\ E &= Mc^2 \\ &= \frac{4\pi}{3}c^2 r^3 \rho \end{aligned} \tag{22}$$

where ρ is the mass density as a function of time. If we now consider a expanding universe where the relative motion of galaxies are not due to intrinsic peculiar velocities, but to the expansion of the universe then we can write r as $r = a(t)\chi$ where χ is the actual distance between the two points (galaxies) and $a(t)$ the expansion parameter defined so that $a(t) = 1$ at the present time. Equations 4 then becomes

$$\begin{aligned} V &= \frac{4\pi}{3}a^3\chi^3 \\ E &= Mc^2 \\ &= \frac{4\pi}{3}c^2 a^3 \chi^3 \rho \end{aligned} \tag{23}$$

From this then follows that

$$\dot{V} = 4\pi\chi^3 a^2 \dot{a} \tag{24}$$

and

$$\dot{E} = \frac{4\pi\chi^3 c^2}{3}\left(3a^2\dot{a}\rho + \dot{\rho}a^3\right) \tag{25}$$

Substituting this back into equation 3 yields

$$\begin{aligned}
\frac{4\pi\chi^3 c^2}{3}\left(3a^2\dot{a}\rho + \dot{\rho}a^3\right) &= -P4\pi\chi^3 a^2\dot{a}\\
\Rightarrow \dot{\rho} + 3\frac{\dot{a}}{a}\rho + 3P\frac{\dot{a}}{a} &= 0\\
\Rightarrow \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + P\right) &= 0\\
\Rightarrow \dot{\rho} + 3H\left(\rho + P\right) &= 0
\end{aligned} \tag{26}$$

Where H is the Hubble parameter. If we now let $P = \rho\omega$, where ω can be a function of time, in such a way that P remains a constant. then equation 8 becomes

$$\begin{aligned}
\dot{\rho} + 3H\left(\rho + \rho\omega\right) &= 0\\
\dot{\rho} + 3H\left(1 + \omega\right)\rho &= 0
\end{aligned} \tag{27}$$

We can solve this ODE

$$\begin{aligned}
\dot{\rho} + 3H\left(1 + \omega\right)\rho &= 0\\
\Rightarrow \frac{\dot{\rho}}{\rho} &= -3H\left(1 + \omega\right)\\
\Rightarrow \frac{\dot{\rho}}{\rho} &= -3\left(1 + \omega\right)\frac{\dot{a}}{a}\\
\Rightarrow ln(\rho) &= -3\left(1 + \omega\right)ln(a) + k\\
\Rightarrow \rho &= Ca^{-3(1+\omega)}
\end{aligned} \tag{28}$$

with C and k constants. For matter $\omega = 0$, for radiation $\omega = \frac{1}{3}$ and for the cosmological constant $\omega = -1$ [5]. From this then follows that for matter

$$\rho = Ca^{-3} \tag{29}$$

for radiation

$$\rho = Ca^{-4} \tag{30}$$

and for the cosmological constant

$$\rho = C \tag{31}$$

The Expansion equation

Consider a isolated, isotropic sphere with mass M and radius $R = a(t)r$, where $a(t)$ is the expansion coefficient and r the distance between any two points in the sphere [5]. From Newtons shell theorem [4], objects (with mass m) on the surface of the sphere will experience the gravitational force due to the mass within the sphere as if all the mass is at the center of the shell. From this the equation of motion for points on the surface of the sphere can be found from Newton's second law:

$$m\ddot{R} = -\frac{GMm}{R^2} \tag{32}$$

Multiplying both sides of the equation with $\dot{R}$, dividing by m and integrating over time gives

$$\begin{aligned} \ddot{R}\dot{R} &= -\frac{GM}{R^2}\dot{R} \\ \Rightarrow \frac{1}{2}\dot{R}^2 &= \frac{GM}{R} + U \end{aligned} \tag{33}$$

Where U is an integration constant. Since we have that $M = \frac{4\pi}{3}R^3\rho$ and $R = ar$, it follows that

$$\begin{aligned} \frac{1}{2}\dot{R}^2 &= \frac{GM}{R} + U \\ \Rightarrow \frac{1}{2}\dot{a}^2 r^2 &= \frac{\frac{4\pi}{3}a^3 r^3 \rho G}{ar} + U \\ \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3}G\rho + \frac{2U}{a^2 r^2} \end{aligned} \tag{34}$$

which is the expansion equation, with ρ the mass-density.

The Acceleration equation

From the result of equation (3) (The Expansion equation) and the equation of state:

$$\dot{\rho} + 3H\left(1+\omega\right)\rho = 0 \tag{35}$$

we can find the Acceleration equation. From the The Expansion equation (3) we have that

$$\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3}G\rho + \frac{2U}{a^2r^2} \\
\Rightarrow \dot{a}^2 &= \frac{8\pi}{3}G\rho a^2 + \frac{2U}{r^2} \\
\Rightarrow \frac{d}{dt}\left(\dot{a}^2\right) &= \frac{d}{dt}\left(\frac{8\pi}{3}G\rho a^2 + \frac{2U}{r^2}\right) \\
\Rightarrow 2\dot{a}\ddot{a} &= \frac{8\pi}{3}G\left(2\rho a\dot{a} + \dot{\rho}a^2\right)
\end{aligned} \tag{36}$$

dividing this by $2a\dot{a}$ and substituting in the Equation of State (with $H = \frac{\dot{a}}{a}$) gives

$$\begin{aligned}
\Rightarrow \frac{\ddot{a}}{a} &= \frac{4\pi}{3}G\left(2\rho + \dot{\rho}\frac{a}{\dot{a}}\right) \\
\Rightarrow \frac{\ddot{a}}{a} &= \frac{4\pi}{3}G\left(2\rho - 3H\left(1+\omega\right)\rho\frac{a}{\dot{a}}\right) \\
\Rightarrow \frac{\ddot{a}}{a} &= \frac{4\pi}{3}G\left(2\rho - 3\left(1+\omega\right)\rho\right) \\
\Rightarrow \frac{\ddot{a}}{a} &= -\frac{4\pi}{3}G\left(\rho + 3\omega\rho\right) \\
\Rightarrow \frac{\ddot{a}}{a} &= -\frac{4\pi}{3}G\left(\rho + 3P\right)
\end{aligned} \tag{37}$$

where $P = \rho\omega$ is the pressure.The result of (6) is the Acceleration equation.

Solving the Friedman equations for different epochs and curvatures

Equation (3) can be rewritten as [5]

$$\begin{aligned} \dot{a}^2 &= \frac{8\pi G}{3c^2}\epsilon a^2 - \kappa \frac{c^2}{R_0^2} \\ &= A\epsilon a^2 - \kappa B \end{aligned} \tag{38}$$

with ϵ energy density, κ curvature and R_0 the present value of curvature radius.

Solving for flat space

For flat space $\kappa = 0$

matter dominated epoch For matter dominated epoch $\epsilon = C_0 a^{-3}$, from which follows that equation (7) becomes

$$\begin{aligned} \dot{a}^2 &= AC_0 a^{-1} - 0 \\ \Rightarrow \sqrt{a}\dot{a} &= \sqrt{AC_0} \\ \Rightarrow \int \sqrt{a} da &= \sqrt{AC_0} \int dt \\ \Rightarrow a^{\frac{3}{2}} &= \sqrt{AC_0}\frac{3}{2}(t - t_0) \\ \Rightarrow a &= \sqrt[3]{AC_0}\left(\frac{3}{2}\right)^{\frac{2}{3}}(t - t_0)^{\frac{2}{3}} \end{aligned} \tag{39}$$

Radiation dominated epoch For matter dominated epoch $\epsilon = C_0 a^{-4}$, from which follows that equation (7) becomes

$$\begin{aligned} \dot{a}^2 &= AC_0 a^{-2} - 0 \\ \Rightarrow a\dot{a} &= \sqrt{AC_0} \\ \Rightarrow \int a da &= \sqrt{AC_0} \int dt \\ \Rightarrow a^2 &= 2\sqrt{AC_0}(t - t_0) \\ \Rightarrow a &= \sqrt{2}\sqrt[4]{AC_0}(t - t_0)^{\frac{1}{2}} \end{aligned} \tag{40}$$

For dark energy dominated epoch For matter dominated epoch $\epsilon = C_0$, from which follows that equation (7) becomes

$$\begin{aligned} \dot{a}^2 &= AC_0 a^2 - 0 \\ \Rightarrow \frac{\dot{a}}{a} &= \sqrt{AC_0} \\ \Rightarrow \int \frac{da}{a} &= \sqrt{AC_0} \int dt \\ \Rightarrow ln\,(a) &= \sqrt{AC_0}\,(t - t_0) \\ \Rightarrow a &= e^{\sqrt{AC_0}(t-t_0)} \end{aligned} \tag{41}$$

Solving for closed space

For closed space $\kappa = 1$

$$\dot{a}^2 = A\epsilon a^2 - B \tag{42}$$

For matter dominated epoch For matter dominated epochs $\epsilon = C_0 a^{-3}$, from which follows that equation (11) becomes

$$\begin{aligned} \dot{a}^2 &= AC_0 a^{-1} - B \\ \Rightarrow \dot{a} &= \sqrt{AC_0 a^{-1} - B} \\ \Rightarrow \frac{\sqrt{a}\dot{a}}{\sqrt{AC_0 - Ba}} &= 1 \\ \Rightarrow \int \frac{\sqrt{a}da}{\sqrt{AC_0 - Ba}} &= \int dt \end{aligned} \tag{43}$$

Introducing a new parametrization

$$\int dt = \int a d\eta \tag{44}$$

with η the conformal time. Inserting this into equation (12) yields

$$\begin{aligned} \int \frac{da}{\sqrt{a}\sqrt{AC_0 - Ba}} &= \int d\eta \\ \Rightarrow \end{aligned} \tag{45}$$

Letting $u = \sqrt{a}$

$$\begin{aligned}
\frac{2}{\sqrt{B}} \int \frac{du}{\sqrt{\frac{AC_0}{B} - u^2}} &= \int d\eta \\
\Rightarrow \arcsin \frac{u\sqrt{B}}{\sqrt{AC_0}} &= \frac{\sqrt{B}}{2} (\eta - \eta_0) \\
\Rightarrow u &= \frac{\sqrt{AC_0}}{\sqrt{B}} \sin \left(\frac{\sqrt{B}}{2} (\eta - \eta_0) \right) \\
\Rightarrow a &= \frac{AC_0}{B} \sin^2 \left(\frac{\sqrt{B}}{2} (\eta - \eta_0) \right)
\end{aligned} \tag{46}$$

and

$$\begin{aligned}
\int dt &= \int a d\eta \\
\Rightarrow \int dt &= \int \frac{AC_0}{B} \sin^2 \left(\frac{\sqrt{B}}{2} (\eta - \eta_0) \right) d\eta \\
\Rightarrow \int dt &= \frac{AC_0}{2B} \int 1 - \cos \left(\sqrt{B} (\eta - \eta_0) \right) d\eta \\
\Rightarrow t - t_0 &= \frac{AC_0}{2B} \left(\eta - \frac{1}{\sqrt{B}} \sin \left(\sqrt{B} (\eta - \eta_0) \right) \right)
\end{aligned} \tag{47}$$

For $t = t_0 = 0$, $\eta_0 = 0$, from which follows that

$$\begin{aligned}
\Rightarrow a &= \frac{AC_0}{B} \sin^2 \left(\frac{\sqrt{B}}{2} (\eta) \right) \\
&= \frac{AC_0}{2B} \left(1 - \cos \left(\sqrt{B} (\eta) \right) \right) \\
\Rightarrow t &= \frac{AC_0}{2B} \left(\eta - \frac{1}{\sqrt{B}} \sin \left(\sqrt{B} (\eta) \right) \right)
\end{aligned} \tag{48}$$

For radiation dominated epoch For matter dominated epochs $\epsilon = C_0 a^{-4}$, from which follows that equation (11) becomes

$$\begin{aligned}
\dot{a}^2 &= AC_0 a^{-1} - B \\
\Rightarrow \dot{a} &= \sqrt{AC_0 a^{-2} - B} \\
\Rightarrow \dot{a} &= \sqrt{\frac{AC_0 - Ba^2}{a^2}} \\
\Rightarrow \int \frac{1}{\sqrt{AC_0 - Ba^2}} da &= \int d\eta \\
\Rightarrow \frac{1}{\sqrt{B}} \int \frac{1}{\sqrt{\frac{AC_0}{B} - a^2}} da &= \int d\eta \\
\Rightarrow a &= \sqrt{\frac{AC_0}{B}} \sin \left(\sqrt{B} (\eta - \eta_0) \right)
\end{aligned} \tag{49}$$

Substituting this back into equation (13)

$$\begin{aligned}
\int dt &= \int a d\eta \\
\Rightarrow \int dt &= \int \sqrt{\frac{AC_0}{B}} \sin\left(\sqrt{B}\,(\eta - \eta_0)\right) d\eta \\
\Rightarrow t - t_0 &= -\sqrt{\frac{AC_0}{B^2}} \cos\left(\sqrt{B}\,(\eta - \eta_0)\right)
\end{aligned} \tag{50}$$

From which follows

$$\begin{aligned}
\Rightarrow a &= \sqrt{\frac{AC_0}{B}} \sin\left(\sqrt{B}\,(\eta)\right) \\
\Rightarrow t &= -\sqrt{\frac{AC_0}{B^2}} \cos\left(\sqrt{B}\,(\eta)\right)
\end{aligned} \tag{51}$$

For Dark energy dominated epoch For matter dominated epochs $\epsilon = C_0$, from which follows that equation (11) becomes

$$\begin{aligned}
\dot{a}^2 &= AC_0 a^2 - B \\
\Rightarrow \dot{a} &= \sqrt{AC_0 a^2 - B} \\
\Rightarrow \dot{a} &= \sqrt{B}\sqrt{a^2 \frac{AC_0}{B} - 1} \\
\Rightarrow \int \frac{da}{\sqrt{a^2 \frac{AC_0}{B} - 1}} &= \sqrt{B} \int dt
\end{aligned} \tag{52}$$

Letting $u = a\sqrt{\frac{AC_0}{B}}$

$$\begin{aligned}
\int \frac{da}{\sqrt{a^2 \frac{AC_0}{B} - 1}} &= \sqrt{B} \int dt \\
\Rightarrow \frac{1}{\sqrt{\frac{AC_0}{B}}} \int \frac{du}{\sqrt{u^2 - 1}} &= \sqrt{B} \int dt \\
\Rightarrow \int \frac{du}{\sqrt{u^2 - 1}} &= \sqrt{AC_0} \int dt \\
\Rightarrow ln\left|u + \sqrt{u^2 - 1}\right| &= \sqrt{AC_0}\,(t - t_0) \\
\Rightarrow \cosh^{-1} u &= \sqrt{AC_0}\,(t - t_0) \\
\Rightarrow u &= \cosh\left(\sqrt{AC_0}\,(t - t_0)\right) \\
\Rightarrow a &= \sqrt{\frac{B}{AC_0}} \cosh\left(\sqrt{AC_0}\,(t - t_0)\right)
\end{aligned} \tag{53}$$

Solving for open space

For open space $\kappa = -1$

$$\dot{a}^2 = A\epsilon a^2 + B \tag{54}$$

matter dominated epoch For matter dominated epochs $\epsilon = C_0 a^{-3}$, from which follows that equation (23) becomes

$$\begin{aligned}
\dot{a}^2 &= AC_0 a^{-1} + B \\
\Rightarrow \dot{a} &= \sqrt{\frac{AC_0 + Ba}{a}} \\
\Rightarrow \int \sqrt{\frac{a}{AC_0 + Ba}} da &= \int a d\eta \\
\Rightarrow \int \frac{1}{\sqrt{a}\sqrt{AC_0 + Ba}} da &= \int d\eta
\end{aligned} \tag{55}$$

letting $u = \sqrt{a}\sqrt{\frac{B}{AC_0}}$ equation (24) becomes

$$\begin{aligned}
\int \frac{1}{\sqrt{aAC_0}\sqrt{1 + \frac{Ba}{AC_0}}} da &= \int d\eta \\
\Rightarrow \int \frac{1}{\sqrt{1+u^2}} da &= \frac{\sqrt{B}}{2}\left(\eta - \eta_0\right) \\
\Rightarrow ln\left(u + \sqrt{1+u^2}\right) &= \frac{\sqrt{B}}{2}\left(\eta - \eta_0\right) \\
\Rightarrow \sinh^{-1} u &= \frac{\sqrt{B}}{2}\left(\eta - \eta_0\right) \\
\Rightarrow u &= \sinh\left(\frac{\sqrt{B}}{2}\left(\eta - \eta_0\right)\right) \\
\Rightarrow a &= \frac{AC_0}{B}\sinh^2\left(\frac{\sqrt{B}}{2}\left(\eta - \eta_0\right)\right) \\
\Rightarrow a &= \frac{AC_0}{2B}\left(\cosh\left(\sqrt{B}\left(\eta - \eta_0\right)\right) - 1\right)
\end{aligned} \tag{56}$$

Inserting this into the conformal time parametrization

$$\begin{aligned}
\int dt &= \int a d\eta \\
\int dt &= \int \frac{AC_0}{2B}\left(\cosh\left(\sqrt{B}\left(\eta - \eta_0\right)\right) - 1\right) d\eta \\
t - t_0 &= \frac{AC_0}{2B}\left(\frac{1}{\sqrt{B}}\sinh\left(\sqrt{B}\left(\eta - \eta_0\right)\right) - \eta\right)
\end{aligned} \tag{57}$$

from which follows

$$\begin{aligned}a &= \frac{AC_0}{2B}\left(\cosh\left(\sqrt{B}\,(\eta)\right)-1\right)\\ t &= \frac{AC_0}{2B}\left(\frac{1}{\sqrt{B}}\sinh\left(\sqrt{B}\,(\eta)\right)-\eta\right)\end{aligned} \tag{58}$$

For radiation dominated epoch For radiation dominated epochs $\epsilon = C_0 a^{-4}$, from which follows that equation (23) becomes

$$\begin{aligned}\dot{a}^2 &= AC_0 a^{-2} + B\\ \Rightarrow \dot{a} &= \sqrt{\frac{AC_0 + Ba^2}{a^2}}\\ \Rightarrow \int \sqrt{\frac{a^2}{AC_0 + Ba^2}}da &= \int a d\eta\\ \Rightarrow \int \frac{1}{\sqrt{1+\frac{Ba^2}{AC_0}}}da &= \sqrt{AC_0}\int d\eta\end{aligned} \tag{59}$$

Letting $u = \sqrt{\frac{B}{AC_0}}a$

$$\begin{aligned}\int \frac{1}{\sqrt{1+\frac{Ba^2}{AC_0}}}da &= \sqrt{AC_0}\int d\eta\\ \Rightarrow \int \frac{1}{\sqrt{1+u^2}}du &= \sqrt{B}\int d\eta\\ \Rightarrow \sinh^{-1} u &= \sqrt{B}\,(\eta-\eta_0)\\ \Rightarrow u &= \sinh\left(\sqrt{B}\,(\eta-\eta_0)\right)\\ \Rightarrow a &= \sqrt{AC_0}\sinh\left(\sqrt{B}\,(\eta-\eta_0)\right)\end{aligned} \tag{60}$$

from which follows

$$\begin{aligned}\int dt &= \int a d\eta\\ \int dt &= \int \sqrt{AC_0}\sinh\left(\sqrt{B}\,(\eta-\eta_0)\right)d\eta\\ t - t_0 &= \sqrt{\frac{AC_0}{B}}\cosh\left(\sqrt{B}\,(\eta-\eta_0)\right)\end{aligned} \tag{61}$$

from which follows

$$\begin{aligned}a &= \sqrt{AC_0}\sinh\left(\sqrt{B}\,(\eta)\right)\\ t &= \sqrt{\frac{AC_0}{B}}\cosh\left(\sqrt{B}\,(\eta)\right)\end{aligned} \tag{62}$$

For dark energy dominated epoch For dark energy dominated epochs $\epsilon = C_0$, from which follows that equation (23) becomes

$$\begin{aligned}
\dot{a}^2 &= AC_0a^2 + B \\
\Rightarrow \dot{a} &= \sqrt{B}\sqrt{\frac{AC_0a^2}{B} + 1} \\
\Rightarrow \int \frac{1}{\sqrt{\frac{AC_0a^2}{B} + 1}} da &= \sqrt{B}\int a d\eta \\
\Rightarrow \int \frac{1}{a\sqrt{\frac{AC_0a^2}{B} + 1}} da &= \sqrt{B}\left(\eta - \eta_0\right)
\end{aligned} \tag{63}$$

letting $u = \sqrt{\frac{AC_0}{B}}a$

$$\begin{aligned}
\int \frac{1}{a\sqrt{\frac{AC_0a^2}{B} + 1}} da &= \sqrt{B}\left(\eta - \eta_0\right) \\
\Rightarrow \int \frac{1}{u\sqrt{u^2 + 1}} du &= \sqrt{B}\left(\eta - \eta_0\right) \\
\Rightarrow -ln\left(u^{-1} + \sqrt{u^{-2} + 1}\right) &= \sqrt{B}\left(\eta - \eta_0\right) \\
\Rightarrow \sinh^{-1} u^{-1} &= -\sqrt{B}\left(\eta - \eta_0\right) \\
\Rightarrow u^{-1} &= \sinh -\sqrt{B}\left(\eta - \eta_0\right) \\
\Rightarrow u &= -\operatorname{csch}\sqrt{B}\left(\eta - \eta_0\right) \\
\Rightarrow a &= -\sqrt{\frac{B}{AC_0}}\operatorname{csch}\sqrt{B}\left(\eta - \eta_0\right)
\end{aligned} \tag{64}$$

substituting this back into the conformal time parametrization gives

$$\begin{aligned}
\int dt &= \int a d\eta \\
\int dt &= -\int \sqrt{\frac{B}{AC_0}}\operatorname{csch}\sqrt{B}\left(\eta - \eta_0\right) d\eta \\
t - t_0 &= -\sqrt{\frac{1}{AC_0}} ln\left|\tanh\frac{\sqrt{B}}{2}\left(\eta - \eta_0\right)\right|
\end{aligned} \tag{65}$$

From which follows

$$\begin{aligned}
a &= -\sqrt{\frac{B}{AC_0}}\operatorname{csch}\sqrt{B}\left(\eta\right) \\
t &= -\sqrt{\frac{1}{AC_0}} ln\left|\tanh\frac{\sqrt{B}}{2}\left(\eta\right)\right|
\end{aligned} \tag{66}$$

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