Centre For Space Physics

Honours

Acceleration and Expansion equations

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1 The Expansion equation

Consider a isolated, isotropic sphere with mass M and radius $R = a(t)r$, where $a(t)$ is the expansion coefficient and r the distance between any two points in the sphere (*Weinberg*, 2013). From Newtons shell theorem (*Surowski*, 2011), objects (with mass m) on the surface of the sphere will experience the gravitational force due to the mass within the sphere as if all the mass is at the center of the shell. From this the equation of motion for points on the surface of the sphere can be found from Newton‘s second law:

$$m\ddot{R} = -\frac{GMm}{R^2} \tag{1}$$

Multiplying both sides of the equation with $\dot{R}$, dividing by m and integrating over time gives

$$\begin{aligned} \ddot{R}\dot{R} &= -\frac{GM}{R^2}\dot{R} \\ \Rightarrow \frac{1}{2}\dot{R}^2 &= \frac{GM}{R} + U \end{aligned} \tag{2}$$

Where U is an integration constant. Since we have that $M = \frac{4\pi}{3}R^3\rho$ and $R = ar$, it follows that

$$\begin{aligned} \frac{1}{2}\dot{R}^2 &= \frac{GM}{R} + U \\ \Rightarrow \frac{1}{2}\dot{a}^2r^2 &= \frac{\frac{4\pi}{3}a^3r^3\rho G}{ar} + U \\ \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3}G\rho + \frac{2U}{a^2r^2} \end{aligned} \tag{3}$$

which is the expansion equation, with ρ the mass-density.

2 The Acceleration equation

From the result of equation (3) (The Expansion equation) and the equation of state:

$$\dot{\rho} + 3H\left(1+\omega\right)\rho = 0 \tag{4}$$

we can find the Acceleration equation. From the The Expansion equation (3) we have that

$$\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3}G\rho + \frac{2U}{a^2r^2} \\
\Rightarrow \dot{a}^2 &= \frac{8\pi}{3}G\rho a^2 + \frac{2U}{r^2} \\
\Rightarrow \frac{d}{dt}\left(\dot{a}^2\right) &= \frac{d}{dt}\left(\frac{8\pi}{3}G\rho a^2 + \frac{2U}{r^2}\right) \\
\Rightarrow 2\dot{a}\ddot{a} &= \frac{8\pi}{3}G\left(2\rho a\dot{a} + \dot{\rho}a^2\right)
\end{aligned} \tag{5}$$

dividing this by $2a\dot{a}$ and substituting in the Equation of State (with $H = \frac{\dot{a}}{a}$) gives

$$\begin{aligned}
\Rightarrow \frac{\ddot{a}}{a} &= \frac{4\pi}{3}G\left(2\rho + \dot{\rho}\frac{a}{\dot{a}}\right) \\
\Rightarrow \frac{\ddot{a}}{a} &= \frac{4\pi}{3}G\left(2\rho - 3H\left(1+\omega\right)\rho\frac{a}{\dot{a}}\right) \\
\Rightarrow \frac{\ddot{a}}{a} &= \frac{4\pi}{3}G\left(2\rho - 3\left(1+\omega\right)\rho\right) \\
\Rightarrow \frac{\ddot{a}}{a} &= -\frac{4\pi}{3}G\left(\rho + 3\omega\rho\right) \\
\Rightarrow \frac{\ddot{a}}{a} &= -\frac{4\pi}{3}G\left(\rho + 3P\right)
\end{aligned} \tag{6}$$

where $P = \rho\omega$ is the pressure.The result of (6) is the Acceleration equation.

3 Solving the Friedman equations for different epochs and curvatures

Equation (3) can be rewritten as (*Weinberg*, 2013)

$$\begin{aligned} \dot{a}^2 &= \frac{8\pi G}{3c^2}\epsilon a^2 - \kappa \frac{c^2}{R_0^2} \\ &= A\epsilon a^2 - \kappa B \end{aligned} \tag{7}$$

with ϵ energy density, κ curvature and R_0 the present value of curvature radius.

3.1 Solving for flat space

For flat space $\kappa = 0$

For matter dominated epoch

For matter dominated epoch $\epsilon = C_0 a^{-3}$, from which follows that equation (7) becomes

$$\begin{aligned} \dot{a}^2 &= AC_0 a^{-1} - 0 \\ \Rightarrow \sqrt{a}\dot{a} &= \sqrt{AC_0} \\ \Rightarrow \int \sqrt{a} da &= \sqrt{AC_0} \int dt \\ \Rightarrow a^{\frac{3}{2}} &= \sqrt{AC_0}\frac{3}{2}(t - t_0) \\ \Rightarrow a &= \sqrt[3]{AC_0}\left(\frac{3}{2}\right)^{\frac{2}{3}} (t - t_0)^{\frac{2}{3}} \end{aligned} \tag{8}$$

For Radiation dominated epoch

For matter dominated epoch $\epsilon = C_0 a^{-4}$, from which follows that equation (7) becomes

$$\begin{aligned} \dot{a}^2 &= AC_0 a^{-2} - 0 \\ \Rightarrow a\dot{a} &= \sqrt{AC_0} \\ \Rightarrow \int a da &= \sqrt{AC_0} \int dt \\ \Rightarrow a^2 &= 2\sqrt{AC_0}(t - t_0) \\ \Rightarrow a &= \sqrt{2}\sqrt[4]{AC_0}(t - t_0)^{\frac{1}{2}} \end{aligned} \tag{9}$$

For dark energy dominated epoch

For matter dominated epoch $\epsilon = C_0$, from which follows that equation (7) becomes

$$\begin{aligned} \dot{a}^2 &= AC_0 a^2 - 0 \\ \Rightarrow \frac{\dot{a}}{a} &= \sqrt{AC_0} \\ \Rightarrow \int \frac{da}{a} &= \sqrt{AC_0} \int dt \\ \Rightarrow ln\,(a) &= \sqrt{AC_0}\,(t - t_0) \\ \Rightarrow a &= e^{\sqrt{AC_0}(t-t_0)} \end{aligned} \tag{10}$$

3.2 Solving for open space

For open space $\kappa = 1$

$$\dot{a}^2 = A\epsilon a^2 - B \tag{11}$$

For matter dominated epoch

For matter dominated epochs $\epsilon = C_0 a^{-3}$, from which follows that equation (11) becomes

$$\begin{aligned} \dot{a}^2 &= AC_0 a^{-1} - B \\ \Rightarrow \dot{a} &= \sqrt{AC_0 a^{-1} - B} \\ \Rightarrow \frac{\sqrt{a}\dot{a}}{\sqrt{AC_0 - Ba}} &= 1 \\ \Rightarrow \int \frac{\sqrt{a}da}{\sqrt{AC_0 - Ba}} &= \int dt \end{aligned} \tag{12}$$

Introducing a new parametrization

$$\int dt = \int a d\eta \tag{13}$$

with η the conformal time. Inserting this into equation (12) yields

$$\begin{aligned} \int \frac{da}{\sqrt{a}\sqrt{AC_0 - Ba}} &= \int d\eta \\ &\Rightarrow \end{aligned} \tag{14}$$

Letting $u = \sqrt{a}$

$$\begin{aligned}
\frac{2}{\sqrt{B}} \int \frac{du}{\sqrt{\frac{AC_0}{B} - u^2}} &= \int d\eta \\
\Rightarrow \arcsin \frac{u\sqrt{B}}{\sqrt{AC_0}} &= \frac{\sqrt{B}}{2} (\eta - \eta_0) \\
\Rightarrow u &= \frac{\sqrt{AC_0}}{\sqrt{B}} \sin\left(\frac{\sqrt{B}}{2} (\eta - \eta_0)\right) \\
\Rightarrow a &= \frac{AC_0}{B} \sin^2\left(\frac{\sqrt{B}}{2} (\eta - \eta_0)\right)
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
\int dt &= \int a d\eta \\
\Rightarrow \int dt &= \int \frac{AC_0}{B} \sin^2\left(\frac{\sqrt{B}}{2} (\eta - \eta_0)\right) d\eta \\
\Rightarrow \int dt &= \frac{AC_0}{2B} \int 1 - \cos\left(\sqrt{B}\,(\eta - \eta_0)\right) d\eta \\
\Rightarrow t - t_0 &= \frac{AC_0}{2B} \left(\eta - \frac{1}{\sqrt{B}} \sin\left(\sqrt{B}\,(\eta - \eta_0)\right)\right)
\end{aligned} \tag{16}$$

For $t = t_0 = 0$, $\eta_0 = 0$, from which follows that

$$\begin{aligned}
\Rightarrow a &= \frac{AC_0}{B} \sin^2\left(\frac{\sqrt{B}}{2} (\eta)\right) \\
&= \frac{AC_0}{2B} \left(1 - \cos\left(\sqrt{B}\,(\eta)\right)\right) \\
\Rightarrow t &= \frac{AC_0}{2B} \left(\eta - \frac{1}{\sqrt{B}} \sin\left(\sqrt{B}\,(\eta)\right)\right)
\end{aligned} \tag{17}$$

For radiation dominated epoch

For matter dominated epochs $\epsilon = C_0 a^{-4}$, from which follows that equation (11) becomes

$$\begin{aligned}
\dot{a}^2 &= AC_0 a^{-1} - B \\
\Rightarrow \dot{a} &= \sqrt{AC_0 a^{-2} - B} \\
\Rightarrow \dot{a} &= \sqrt{\frac{AC_0 - Ba^2}{a^2}} \\
\Rightarrow \int \frac{1}{\sqrt{AC_0 - Ba^2}} da &= \int d\eta \\
\Rightarrow \frac{1}{\sqrt{B}} \int \frac{1}{\sqrt{\frac{AC_0}{B} - a^2}} da &= \int d\eta \\
\Rightarrow a &= \sqrt{\frac{AC_0}{B}} \sin\left(\sqrt{B}\,(\eta - \eta_0)\right)
\end{aligned} \tag{18}$$

Substituting this back into equation (13)

$$\begin{aligned}
\int dt &= \int a d\eta \\
\Rightarrow \int dt &= \int \sqrt{\frac{AC_0}{B}} \sin\left(\sqrt{B}\,(\eta - \eta_0)\right) d\eta \\
\Rightarrow t - t_0 &= -\sqrt{\frac{AC_0}{B^2}} \cos\left(\sqrt{B}\,(\eta - \eta_0)\right)
\end{aligned} \tag{19}$$

From which follows

$$\begin{aligned}
\Rightarrow a &= \sqrt{\frac{AC_0}{B}} \sin\left(\sqrt{B}\,(\eta)\right) \\
\Rightarrow t &= -\sqrt{\frac{AC_0}{B^2}} \cos\left(\sqrt{B}\,(\eta)\right)
\end{aligned} \tag{20}$$

For Dark energy dominated epoch

For matter dominated epochs $\epsilon = C_0$, from which follows that equation (11) becomes

$$\begin{aligned}
\dot{a}^2 &= AC_0 a^2 - B \\
\Rightarrow \dot{a} &= \sqrt{AC_0 a^2 - B} \\
\Rightarrow \dot{a} &= \sqrt{B}\sqrt{a^2 \frac{AC_0}{B} - 1} \\
\Rightarrow \int \frac{da}{\sqrt{a^2 \frac{AC_0}{B} - 1}} &= \sqrt{B} \int dt
\end{aligned} \tag{21}$$

Letting $u = a\sqrt{\frac{AC_0}{B}}$

$$\begin{aligned}
\int \frac{da}{\sqrt{a^2\frac{AC_0}{B} - 1}} &= \sqrt{B}\int dt \\
\Rightarrow \frac{1}{\sqrt{\frac{AC_0}{B}}}\int \frac{du}{\sqrt{u^2 - 1}} &= \sqrt{B}\int dt \\
\Rightarrow \int \frac{du}{\sqrt{u^2 - 1}} &= \sqrt{AC_0}\int dt \\
\Rightarrow ln\left|u + \sqrt{u^2 - 1}\right| &= \sqrt{AC_0}\,(t - t_0) \\
\Rightarrow \cosh^{-1} u &= \sqrt{AC_0}\,(t - t_0) \\
\Rightarrow u &= \cosh\left(\sqrt{AC_0}\,(t - t_0)\right) \\
\Rightarrow a &= \sqrt{\frac{B}{AC_0}}\cosh\left(\sqrt{AC_0}\,(t - t_0)\right)
\end{aligned} \tag{22}$$

3.3 Solving for closed space

For closed space $\kappa = -1$

$$\dot{a}^2 = A\epsilon a^2 + B \tag{23}$$

For matter dominated epoch

For matter dominated epochs $\epsilon = C_0 a^{-3}$, from which follows that equation (23) becomes

$$\begin{aligned} \dot{a}^2 &= AC_0 a^{-1} + B \\ \Rightarrow \dot{a} &= \sqrt{\frac{AC_0 + Ba}{a}} \\ \Rightarrow \int \sqrt{\frac{a}{AC_0 + Ba}} da &= \int a d\eta \\ \Rightarrow \int \frac{1}{\sqrt{a}\sqrt{AC_0 + Ba}} da &= \int d\eta \end{aligned} \tag{24}$$

letting $u = \sqrt{a}\sqrt{\frac{B}{AC_0}}$ equation (24) becomes

$$\begin{aligned} \int \frac{1}{\sqrt{aAC_0}\sqrt{1 + \frac{Ba}{AC_0}}} da &= \int d\eta \\ \Rightarrow \int \frac{1}{\sqrt{1+u^2}} da &= \frac{\sqrt{B}}{2} (\eta - \eta_0) \\ \Rightarrow ln\left(u + \sqrt{1+u^2}\right) &= \frac{\sqrt{B}}{2} (\eta - \eta_0) \\ \Rightarrow \sinh^{-1} u &= \frac{\sqrt{B}}{2} (\eta - \eta_0) \\ \Rightarrow u &= \sinh\left(\frac{\sqrt{B}}{2} (\eta - \eta_0)\right) \\ \Rightarrow a &= \frac{AC_0}{B} \sinh^2\left(\frac{\sqrt{B}}{2} (\eta - \eta_0)\right) \\ \Rightarrow a &= \frac{AC_0}{2B}\left(\cosh\left(\sqrt{B} (\eta - \eta_0)\right) - 1\right) \end{aligned} \tag{25}$$

Inserting this into the conformal time parametrization

$$\begin{aligned} \int dt &= \int a d\eta \\ \int dt &= \int \frac{AC_0}{2B}\left(\cosh\left(\sqrt{B} (\eta - \eta_0)\right) - 1\right) d\eta \\ t - t_0 &= \frac{AC_0}{2B}\left(\frac{1}{\sqrt{B}} \sinh\left(\sqrt{B} (\eta - \eta_0)\right) - \eta\right) \end{aligned} \tag{26}$$

from which follows

$$\begin{aligned} a &= \frac{AC_0}{2B}\left(\cosh\left(\sqrt{B}\,(\eta)\right) - 1\right) \\ t &= \frac{AC_0}{2B}\left(\frac{1}{\sqrt{B}}\sinh\left(\sqrt{B}\,(\eta)\right) - \eta\right) \end{aligned} \tag{27}$$

For radiation dominated epoch

For radiation dominated epochs $\epsilon = C_0 a^{-4}$, from which follows that equation (23) becomes

$$\begin{aligned} \dot{a}^2 &= AC_0 a^{-2} + B \\ \Rightarrow \dot{a} &= \sqrt{\frac{AC_0 + Ba^2}{a^2}} \\ \Rightarrow \int \sqrt{\frac{a^2}{AC_0 + Ba^2}}\,da &= \int a\,d\eta \\ \Rightarrow \int \frac{1}{\sqrt{1 + \frac{Ba^2}{AC_0}}}\,da &= \sqrt{AC_0}\int d\eta \end{aligned} \tag{28}$$

Letting $u = \sqrt{\frac{B}{AC_0}}a$

$$\begin{aligned} \int \frac{1}{\sqrt{1 + \frac{Ba^2}{AC_0}}}\,da &= \sqrt{AC_0}\int d\eta \\ \Rightarrow \int \frac{1}{\sqrt{1+u^2}}\,du &= \sqrt{B}\int d\eta \\ \Rightarrow \sinh^{-1} u &= \sqrt{B}\,(\eta - \eta_0) \\ \Rightarrow u &= \sinh\left(\sqrt{B}\,(\eta - \eta_0)\right) \\ \Rightarrow a &= \sqrt{AC_0}\sinh\left(\sqrt{B}\,(\eta - \eta_0)\right) \end{aligned} \tag{29}$$

from which follows

$$\begin{aligned} \int dt &= \int a\,d\eta \\ \int dt &= \int \sqrt{AC_0}\sinh\left(\sqrt{B}\,(\eta - \eta_0)\right)d\eta \\ t - t_0 &= \sqrt{\frac{AC_0}{B}}\cosh\left(\sqrt{B}\,(\eta - \eta_0)\right) \end{aligned} \tag{30}$$

from which follows

$$\begin{aligned} a &= \sqrt{AC_0}\sinh\left(\sqrt{B}\,(\eta)\right) \\ t &= \sqrt{\frac{AC_0}{B}}\cosh\left(\sqrt{B}\,(\eta)\right) \end{aligned} \tag{31}$$

For dark energy dominated epoch

For dark energy dominated epochs $\epsilon = C_0$, from which follows that equation (23) becomes

$$\begin{aligned}
\dot{a}^2 &= AC_0a^2 + B \\
\Rightarrow \dot{a} &= \sqrt{B}\sqrt{\frac{AC_0a^2}{B}+1} \\
\Rightarrow \int \frac{1}{\sqrt{\frac{AC_0a^2}{B}+1}}da &= \sqrt{B}\int ad\eta \\
\Rightarrow \int \frac{1}{a\sqrt{\frac{AC_0a^2}{B}+1}}da &= \sqrt{B}\left(\eta-\eta_0\right)
\end{aligned} \tag{32}$$

letting $u = \sqrt{\frac{AC_0}{B}}a$

$$\begin{aligned}
\int \frac{1}{a\sqrt{\frac{AC_0a^2}{B}+1}}da &= \sqrt{B}\left(\eta-\eta_0\right) \\
\Rightarrow \int \frac{1}{u\sqrt{u^2+1}}du &= \sqrt{B}\left(\eta-\eta_0\right) \\
\Rightarrow -ln\left(u^{-1}+\sqrt{u^{-2}+1}\right) &= \sqrt{B}\left(\eta-\eta_0\right) \\
\Rightarrow \sinh^{-1}u^{-1} &= -\sqrt{B}\left(\eta-\eta_0\right) \\
\Rightarrow u^{-1} &= \sinh -\sqrt{B}\left(\eta-\eta_0\right) \\
\Rightarrow u &= -\operatorname{csch}\sqrt{B}\left(\eta-\eta_0\right) \\
\Rightarrow a &= -\sqrt{\frac{B}{AC_0}}\operatorname{csch}\sqrt{B}\left(\eta-\eta_0\right)
\end{aligned} \tag{33}$$

substituting this back into the conformal time parametrization gives

$$\begin{aligned}
\int dt &= \int ad\eta \\
\int dt &= -\int\sqrt{\frac{B}{AC_0}}\operatorname{csch}\sqrt{B}\left(\eta-\eta_0\right)d\eta \\
t-t_0 &= -\sqrt{\frac{1}{AC_0}}ln\left|\tanh\frac{\sqrt{B}}{2}\left(\eta-\eta_0\right)\right|
\end{aligned} \tag{34}$$

From which follows

$$\begin{aligned}
a &= -\sqrt{\frac{B}{AC_0}}\operatorname{csch}\sqrt{B}\left(\eta\right) \\
t &= -\sqrt{\frac{1}{AC_0}}ln\left|\tanh\frac{\sqrt{B}}{2}\left(\eta\right)\right|
\end{aligned} \tag{35}$$

Bibliography

Surowski, D., David surowski, https://www.math.ksu.edu/~dbski/writings/shell.pdf, 2011.

Weinberg, D., Webpage, http://www.astronomy.ohio-state.edu/~dhw/A5682/notes4.pdf, 2013.