Centre For Space Physics

Honours

Acceleration and Expansion equations

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March 26, 2018

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1 The Expansion equation

Consider a isolated, isotropic sphere with mass M and radius $R = a(t)r$, where $a(t)$ is the expansion coefficient and r the distance between any two points in the sphere (*Weinberg*, 2013). From Newtons shell theorem (*Surowski*, 2011), objects (with mass m) on the surface of the sphere will experience the gravitational force due to the mass within the sphere as if all the mass is at the center of the shell. From this the equation of motion for points on the surface of the sphere can be found from Newton's second law:

$$m\ddot{R} = -\frac{GMm}{R^2} \tag{1}$$

Multiplying both sides of the equation with $\dot{R}$, dividing by m and integrating over time gives

$$\begin{aligned} \ddot{R}\dot{R} &= -\frac{GM}{R^2}\dot{R} \\ \Rightarrow \frac{1}{2}\dot{R}^2 &= \frac{GM}{R} + U \end{aligned} \tag{2}$$

Where U is an integration constant. Since we have that $M = \frac{4\pi}{3}R^3\rho$ and $R = ar$, it follows that

$$\begin{aligned} \frac{1}{2}\dot{R}^2 &= \frac{GM}{R} + U \\ \Rightarrow \frac{1}{2}\dot{a}^2 r^2 &= \frac{\frac{4\pi}{3}a^3 r^3 \rho G}{ar} + U \\ \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3}G\rho + \frac{2U}{a^2 r^2} \end{aligned} \tag{3}$$

which is the expansion equation, with ρ the mass-density.

2 The Acceleration equation

From the result of equation (3) (The Expansion equation) and the equation of state:

$$\dot{\rho} + 3H\left(1+\omega\right)\rho = 0 \tag{4}$$

we can find the Acceleration equation. From the The Expansion equation (3) we have that

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3}G\rho + \frac{2U}{a^2 r^2} \\ \Rightarrow \dot{a}^2 &= \frac{8\pi}{3}G\rho a^2 + \frac{2U}{r^2} \\ \Rightarrow \frac{d}{dt}\left(\dot{a}^2\right) &= \frac{d}{dt}\left(\frac{8\pi}{3}G\rho a^2 + \frac{2U}{r^2}\right) \\ \Rightarrow 2\dot{a}\ddot{a} &= \frac{8\pi}{3}G\left(2\rho a\dot{a} + \dot{\rho}a^2\right) \end{aligned} \tag{5}$$

dividing this by $2a\dot{a}$ and substituting in the Equation of State (with $H = \frac{\dot{a}}{a}$) gives

$$\begin{aligned} \Rightarrow \frac{\ddot{a}}{a} &= \frac{4\pi}{3}G\left(2\rho + \dot{\rho}\frac{a}{\dot{a}}\right) \\ \Rightarrow \frac{\ddot{a}}{a} &= \frac{4\pi}{3}G\left(2\rho - 3H\left(1+\omega\right)\rho\frac{a}{\dot{a}}\right) \\ \Rightarrow \frac{\ddot{a}}{a} &= \frac{4\pi}{3}G\left(2\rho - 3\left(1+\omega\right)\rho\right) \\ \Rightarrow \frac{\ddot{a}}{a} &= -\frac{4\pi}{3}G\left(\rho + 3\omega\rho\right) \\ \Rightarrow \frac{\ddot{a}}{a} &= -\frac{4\pi}{3}G\left(\rho + 3P\right) \end{aligned} \tag{6}$$

where $P = \rho\omega$ is the pressure.The result of (6) is the Acceleration equation.

Bibliography

Surowski, D., David surowski, https://www.math.ksu.edu/~dbski/writings/shell.pdf, 2011.

Weinberg, D., Webpage, http://www.astronomy.ohio-state.edu/~dhw/A5682/notes4.pdf, 2013.