Centre For Space Physics

Honours

Equation of state

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1 Equation of State from Newtonian Mechanics

From the first law of thermodynamics we have that

$$dE = -PdV + dQ \tag{1}$$

With dE the change in energy content, P the pressure, dV the change in volume and dQ the change in heat. Considering an Isotropic and homogeneous universe (*Weinberg*, 2013),the expansion of the universe will be adiabatic with the Pressure P constant and there will be no change in the heat (Q) content of the universe since there is nowhere for the heat to come from (*Weinberg*, 2013). From this we can then conclude that the dQ term in equation 1 will be $dQ = 0$ and reduces to.

$$dE = -PdV \tag{2}$$

this impies that

$$\begin{aligned} E(t) &= -PV(t) \\ \Rightarrow \dot{E} &= \frac{d\,(-PV)}{dt} \\ &= -\frac{dP}{dt}V - P\frac{dV}{dt} \\ &= -P\frac{dV}{dt} \end{aligned} \tag{3}$$

since P is a constant. We have that for M the total mass and c the speed of light

$$\begin{aligned} V &= \frac{4\pi}{3}r^3 \\ E &= Mc^2 \\ &= \frac{4\pi}{3}c^2r^3\rho \end{aligned} \tag{4}$$

where ρ is the mass density as a function of time. If we now consider a expanding universe where the relative motion of galaxies are not due to intrinsic peculiar velocities, but to the expansion of the universe then we can write r as $r = a(t)\chi$ where χ is the actual distance between the two points (galaxies) and $a(t)$ the expansion parameter defined so that $a(t) = 1$ at the present time. Equations 4 then becomes

$$\begin{aligned} V &= \frac{4\pi}{3}a^3\chi^3 \\ E &= Mc^2 \\ &= \frac{4\pi}{3}c^2a^3\chi^3\rho \end{aligned} \tag{5}$$

From this then follows that

$$\dot{V} = 4\pi\chi^3 a^2 \dot{a} \tag{6}$$

and

$$\dot{E} = \frac{4\pi\chi^3 c^2}{3}\left(3a^2\dot{a}\rho + \dot{\rho}a^3\right) \tag{7}$$

Substituting this back into equation 3 yields

$$\begin{aligned} \frac{4\pi\chi^3 c^2}{3}\left(3a^2\dot{a}\rho + \dot{\rho}a^3\right) &= -P4\pi\chi^3 a^2\dot{a} \\ \Rightarrow \dot{\rho} + 3\frac{\dot{a}}{a}\rho + 3P\frac{\dot{a}}{a} &= 0 \\ \Rightarrow \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + P\right) &= 0 \\ \Rightarrow \dot{\rho} + 3H\left(\rho + P\right) &= 0 \end{aligned} \tag{8}$$

Where H is the Hubble parameter. If we now let $P = \rho\omega$, where ω can be a function of time, in such a way that P remains a constant. then equation 8 becomes

$$\begin{aligned} \dot{\rho} + 3H\left(\rho + \rho\omega\right) &= 0 \\ \dot{\rho} + 3H\left(1 + \omega\right)\rho &= 0 \end{aligned} \tag{9}$$

We can solve this ODE

$$\begin{aligned} \dot{\rho} + 3H\left(1+\omega\right)\rho = 0& \\ \Rightarrow \frac{\dot{\rho}}{\rho} &= -3H\left(1+\omega\right) \\ \Rightarrow \frac{\dot{\rho}}{\rho} &= -3\left(1+\omega\right)\frac{\dot{a}}{a} \\ \Rightarrow ln(\rho) &= -3\left(1+\omega\right)ln(a) + k \\ \Rightarrow \rho &= Ca^{-3(1+\omega)} \end{aligned} \tag{10}$$

with C and k constants. For matter $\omega = 0$, for radiation $\omega = \frac{1}{3}$ and for the cosmological constant $\omega = -1$ (*Weinberg*, 2013). From this then follows that for matter

$$\rho = Ca^{-3} \tag{11}$$

for radiation

$$\rho = Ca^{-4} \tag{12}$$

and for the cosmological constant

$$\rho = C \tag{13}$$

(*Weinberg*, 2013)

Bibliography

Weinberg, D., Webpage, http://www.astronomy.ohio-state.edu/~dhw/A5682/notes4.pdf, 2013.