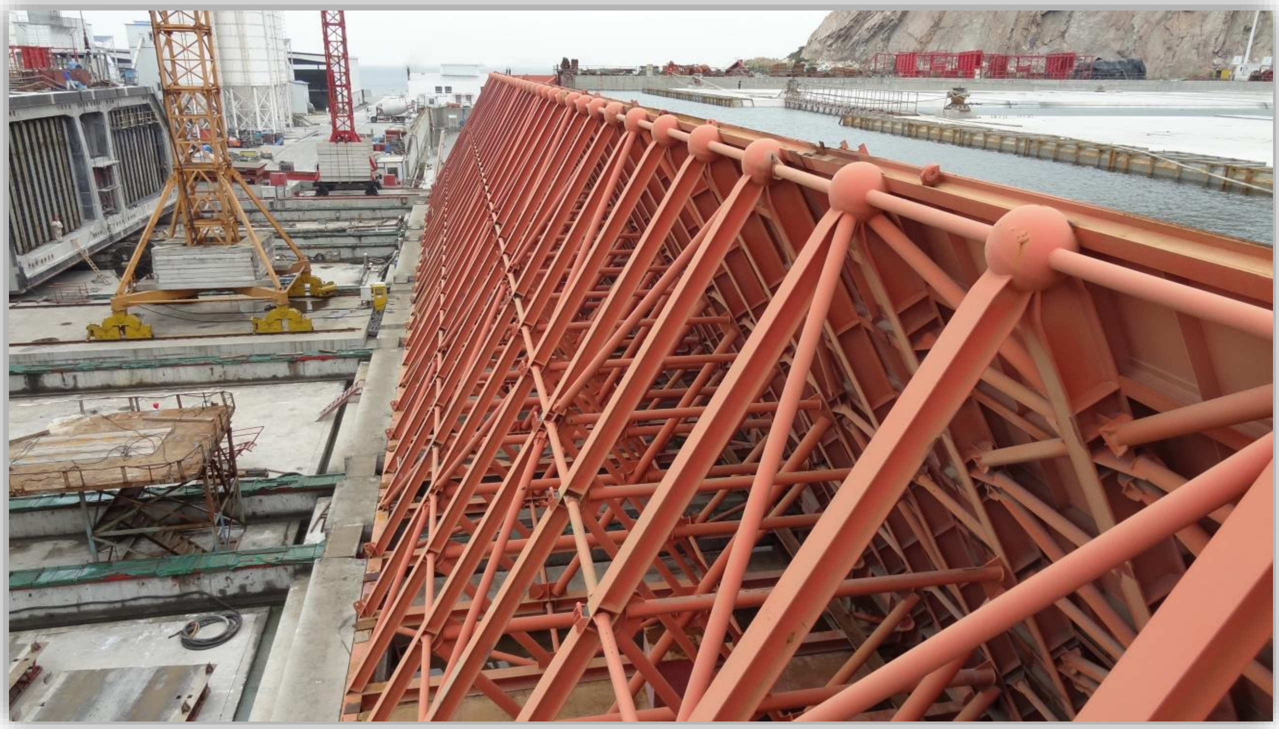
Practical deflection of a beam

5R4



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INHOUDSOPGAVE

1.0 PHASE I – THEORY

In this report the behaviour of the following differential equation is described:

$$EI * \frac{d^4y}{dx^4} = q(x), 0 \leq x \leq L \qquad (1)$$

The boundary conditions are as follows:

$$y|_{x=0} = 0 \qquad \left.\frac{d^2y}{dx^2}\right|_{x=0} = 0$$
$$y|_{x=L} = 0 \qquad \left.\frac{d^2y}{dx^2}\right|_{x=L} = 0 \qquad (1 - I)$$

The constant variables are:

L = 10 [m];
B = 0.04 [m];
d = 0.2 [m];
*E = 2*10^11 [N/m²];* ($E = 2*10^{11}\ [N/m^2]$)
ρ = 7800 [kg/m³];
g = 9.8 [m/s²];
s = 2 [m];
m = 500 [kg].

1.1

Determine the mass of the beam. Is the additional mass of the same order?

The assignment description yields:

$$q(x) = q_{eig} + q_m(x) \qquad (2)$$

In this assignment the q_m is given by:

$$q_m(x) = \begin{cases} 0, & 0 < x < \bar{x} - \frac{s}{2}, \bar{x} + \frac{s}{2} < x < L \\ \frac{m}{s} g, & \bar{x} - \frac{s}{2} < x < \bar{x} + \frac{s}{2} \end{cases}, \bar{x} = \frac{L}{2} \qquad (2 - I)$$

The mass of both the beam and the additional load are $q_m * L$.

$$q_{eig} = \rho * B * d * g \qquad (2)$$

Result:

Both masses are of the second Order.

$$M = \frac{q_{eig} * L}{g} = 7800 * 0.04 * 0.2 * 10 = 624\ kg$$

$$m = 500\ kg \qquad (2\text{– II})$$

The mass of the beam is 624 kg this is of order $c*10^2$, the additional mass is 500 kg this is of the order $k*10^2$. This means that the additional mass and the weight of the beam of the same order.

1.2, 1.3, 1.4 & 1.5

For clarification a bar is made (fig. 1) with the individual step sizes (h) for visualizing the process.

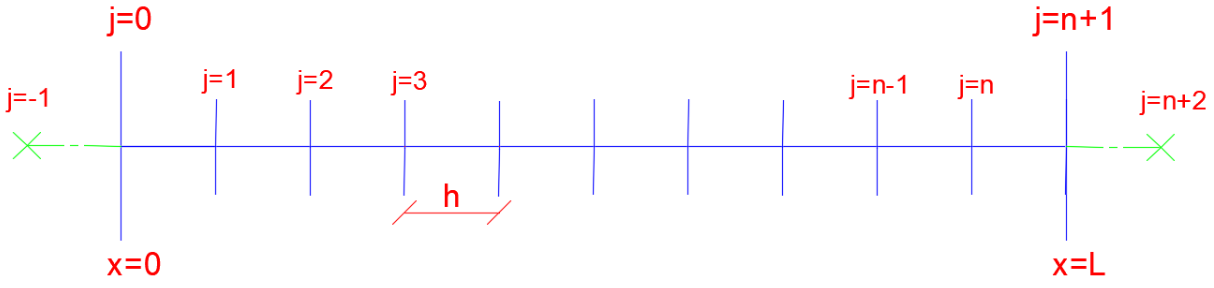

Figure 1 – Visualization of the steps.

Firstly the difference equation is given by:

$$Q(h) \approx EI * \frac{f(x+2h)-4f(x+h)+6f(x)-4f(x-h)+f(x-2h)}{h^4} \rightarrow \frac{EI}{h^4}\left(w_{j-2} - 4 * w_{j-1} + 6w_j - 4 * w_{j+1} + w_{j+2}\right) = q_{j(x)} \tag{3}$$

Secondly the central difference scheme for the second derivatives yields in combination with the boundary conditions w0 = 0 and w_{n+1}=0:

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 0 \xrightarrow{yields} f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \rightarrow w_j = \frac{w_{j-1} - 2 * w_j + w_{j+1}}{h^2} \tag{4}$$

Result:

$W_{-1} = -W_1$

$$y|_{\,x=0} = 0 \rightarrow w_0 = 0$$

$$j = 0 \rightarrow w_0 = \frac{w_{-1} - 2 * w_0 + w_1}{h^2} = 0 \tag{4– I}$$

$$w_{-1} = -w_1$$

The same process is repeated for the second boundary condition:

$$\left.\frac{d^2y}{dx^2}\right|_{x=L} = 0 \xrightarrow{yields} f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \rightarrow w_j = \frac{w_{j-1} - 2 * w_j + w_{j+1}}{h^2} \tag{5}$$

Result:

$W_{n+2} = -W_n$

$$y|_{\,x=L} = 0 \rightarrow w_{n+1} = 0$$

$$j = n + 1 \rightarrow w_{n+1} = \frac{w_n - 2 * w_{n+1} + w_{n+2}}{h^2} = 0 \tag{5– I}$$

$$w_{n+2} = -w_n$$

The only thing left to do is to fill in the difference equation for the now known W_{-1}, W_0, W_{n+1} and W_{n+2}. This will be done for j = {1,2,3 … n-2, n-1 and n}.

	$\frac{EI}{h^4}\left(w_{j-2} - 4 * w_{j-1} + 6w_j - 4 * w_{j+1} + w_{j+2}\right) = q_j(x)$	(6)
$j = 1 \rightarrow$	$\frac{EI}{h^4}\left(w_{-1} - 4 * w_0 + 6w_1 - 4 * w_2 + w_3\right) = q_1(x) \rightarrow \frac{EI}{h^4}\left(5w_1 - 4 * w_2 + w_3\right) = q_1(x)$	(6–I)
$j = 2 \rightarrow$	$\frac{EI}{h^4}\left(w_0 - 4 * w_1 + 6w_2 - 4 * w_3 + w_4\right) = q_2(x) \rightarrow \frac{EI}{h^4}\left(-4 * w_1 + 6w_2 - 4 * w_3 + w_4\right) = q_2(x)$	(6–II)
$j = 3 \rightarrow$	$\frac{EI}{h^4}\left(w_1 - 4 * w_2 + 6w_3 - 4 * w_4 + w_5\right) = q_3(x)$	(6–III)
$j = j \rightarrow$	$\frac{EI}{h^4}\left(w_{j-2} - 4 * w_{j-1} + 6w_j - 4 * w_{j+1} + w_{j+2}\right) = q_j(x)$	(6–IV)
$j = n - 2 \rightarrow$	$\frac{EI}{h^4}\left(w_{n-4} - 4 * w_{n-3} + 6w_{n-2} - 4 * w_{n-1} + w_n\right) = q_{n-2}(x)$	(6–V)
$j = n - 1 \rightarrow$	$\frac{EI}{h^4}\left(w_{n-3} - 4 * w_{n-2} + 6w_{n-1} - 4 * w_n + w_{n+1}\right) = q_{n-1}(x) \rightarrow$ $\frac{EI}{h^4}\left(w_{n-3} - 4 * w_{n-2} + 6w_{n-1} - 4 * w_n\right) = q_{n-1}(x)$	(6–VI)
$j = n \rightarrow$	$\frac{EI}{h^4}\left(w_{n-2} - 4 * w_{n-1} + 6w_n - 4 * w_{n+1} + w_{n+2}\right) = q_n(x) \rightarrow$ $\frac{EI}{h^4}\left(w_{n-2} - 4 * w_{n-1} + 5w_n\right) = q_n(x)$	(6–VII)

Now a matrix can be constructed (eq. 7):

$$
\begin{bmatrix}
5 & -4 & 1 & 0 & 0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 \\
-4 & 6 & -4 & 1 & 0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 \\
1 & -4 & 6 & -4 & 1 & 0 & \dots & \dots & \dots & \dots & 0 & 0 \\
0 & 1 & -4 & 6 & -4 & 1 & & & & & \vdots & \vdots \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \vdots & \vdots \\
\vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \vdots & \vdots \\
\vdots & \vdots & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots \\
\vdots & \vdots & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & & & & & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\
\vdots & \vdots & & & & & 1 & -4 & 6 & -4 & 1 & 0 \\
0 & 0 & & & & & 0 & 1 & -4 & 6 & -4 & 1 \\
0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 & 1 & -4 & 6 & -4 \\
0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 & 0 & 1 & -4 & 5
\end{bmatrix}
*
\begin{bmatrix}
w_1 \\ w_2 \\ w_3 \\ w_4 \\ \vdots \\ \vdots \\ w_j \\ \vdots \\ \vdots \\ w_{n-3} \\ w_{n-2} \\ w_{n-1} \\ w_n
\end{bmatrix}
=
\begin{bmatrix}
q_1(x) \\ q_2(x) \\ q_3(x) \\ q_4(x) \\ \vdots \\ \vdots \\ q_j(x) \\ \vdots \\ \vdots \\ q_{n-3}(x) \\ q_{n-2}(x) \\ q_{n-1}(x) \\ q_n(x)
\end{bmatrix}
\tag{7}
$$

One can simply see that the matrix is symmetric and has n x n dimensions.

1.6

Consider the local truncation error for the difference equations associated with interior grid points, such as presented in question 2. Of what order in h is this local truncation error?

For the numerical method of solving the boundary problem is only one approximation used. This approximation is the approximation for the fourth derivative of y. The approximation that should be used is given in equation (8) and the error of this approximation is given in equation (9) of the assignment. The error of the method is $O(h^2)$. The fact that the method is exact except for this approximation means that the global truncation error of the method $O(h^2)$.

In equation (8) of the document we derived the relation between the local and the global truncation error. This derivations yield that the local truncation error is always smaller than the global truncation error.

In the equations below ε is defined as: **ε** ≤ |**A* y - A w**|, **y** as the vector containing the exact results and **w** the vector containing the approximate result find with the numerical method.

$$\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{w} \tag{8}$$

Result:

$\boldsymbol{e} \leq O(h^2)$

$$\begin{aligned} &\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{w} = \boldsymbol{A}^{-1} * \boldsymbol{A} * (\boldsymbol{y} - \boldsymbol{w}) = \boldsymbol{A}^{-1} * (\boldsymbol{A} * \boldsymbol{y} - \boldsymbol{A} * \boldsymbol{w}) \\ &\boldsymbol{A}^{-1} * (\boldsymbol{A} * \boldsymbol{y} - \boldsymbol{A} * \boldsymbol{w}) = \boldsymbol{A}^{-1} * \boldsymbol{\varepsilon} \\ &||\boldsymbol{e}|| = \left||\boldsymbol{A}^{-1} * \boldsymbol{\varepsilon}|\right| \leq \left||\boldsymbol{A}^{-1}|\right| * ||\boldsymbol{\varepsilon}|| \\ &\left||\boldsymbol{A}^{-1}|\right| * ||\boldsymbol{\varepsilon}|| = O(h^2) \\ &\boldsymbol{e} \leq O(h^2) \end{aligned} \tag{8 – I}$$

1.7

The matrix is already given in eq. 7. (*A short summery, the matrix is symmetric and the dimensions are n x n.*)

There are multiple ways to solve a system of linear equations. This can be done numerically by using the Newton-Rapson method for linear systems.

It can also be done by calculating the inverse of the matrix and multiplying by the results vector. This method is not a numerical approximation but the exact answer. However it becomes a numerical approximation if the inverse is calculated with a numerical method.

The exercise states that the system can be solved numerically, which we explained above. However didn't we create a new function in assignment 11 – 15. Writing such a function is very labour intensive and we think this is outside the curriculum of this practicum. Given the fact that the numpy package is a reliable method, which gives a fast and easy answer.

Which of the methods (algorithms) is used by lg.spsolve is unknown to us. A sparse matrix is a matrix mainly filled with zeros. Therefor it is likely that the function lg.spsolve(sparse matrix solve) is specialized in solving matrixes with many zeros. Hence it is most likely favourable to use lg.spsolve over the numpy.solve in case of large matrixes containing lots of zeros. However we used the method from the numpy package because we made the notebook earlier than this assignment. And the numpy method gives no difficulties. So we choose not to revise the notebook to prevent making small untraceable mistakes.

1.8

Determine the analytical value of Q_a of Q.

$$Q_a = \int_{\bar{x}-\frac{s}{2}}^{\bar{x}+\frac{s}{2}} q_m dx \qquad (8)$$

Result:

$$Q_a = \frac{h*m}{2s}$$

$$Q_a = \frac{h}{2}*0 + \frac{h}{2}*\frac{m}{s}*g = \frac{h*m}{2s}$$

$$Q_a = \frac{h*m}{2s} \qquad (8-I)$$

This answer is the exact answer to equation (8).

1.9

Determine the discrete value of Q_d of Q.

$$Q_d = 2*\frac{h}{2}*\ q_m^i \qquad (9)$$

Result:

$$Q_d = \frac{h*m*g}{s}$$

$$Q_d = 2*\frac{h}{2}*\frac{m}{s}*g \qquad (9-I)$$

This is not the exact value but an approximation of the true integral.

1.10

A natural demand, resulting from a finite volume derivation of the discrete equations, is $Q_d = Q_a$. Verify that this implies that q_m^i must be chosen as the average of the values q_m on the left and right of the interface.

$$Q_a = Q_d \qquad (10)$$

Result:

$$q_m^i = \frac{m*g}{2*s}$$

$$\int_{\bar{x}-\frac{s}{2}}^{\bar{x}+\frac{s}{2}} q_m dx = 2*\frac{h}{2}*\ q_m^i \qquad (10\text{–I})$$

$$q_m^i = \frac{m*g}{2*s} \qquad (10\text{–II})$$

Solving equation 7 – I yields $q_m^i = \frac{m*g}{2*s}$. This proves the statement in the assignment