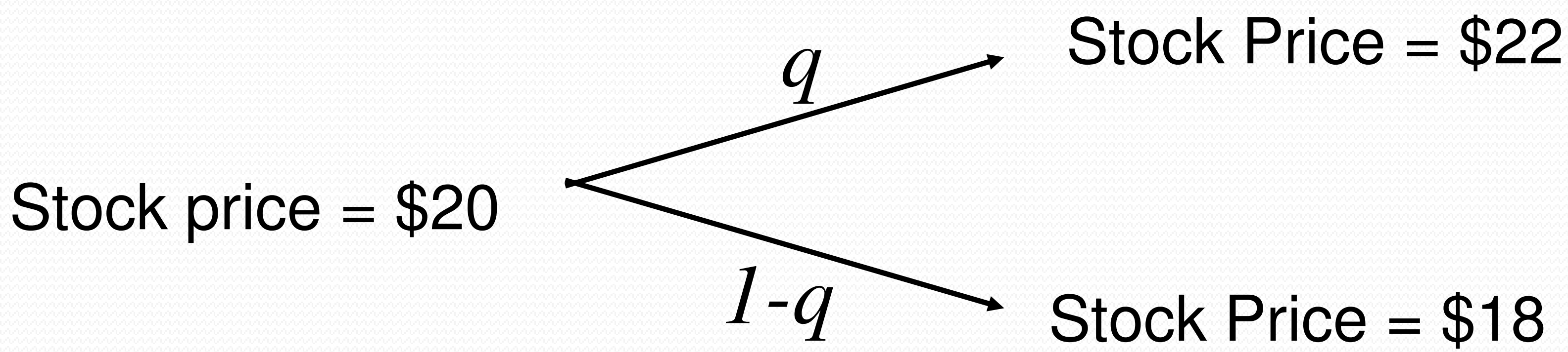
Option Pricing using Binomial Trees

What are the main assumptions?

- Simple two-state economy
- Interest rate is constant
- No dividend payments
- Arbitrage-free economy
- No transaction cost
- European option

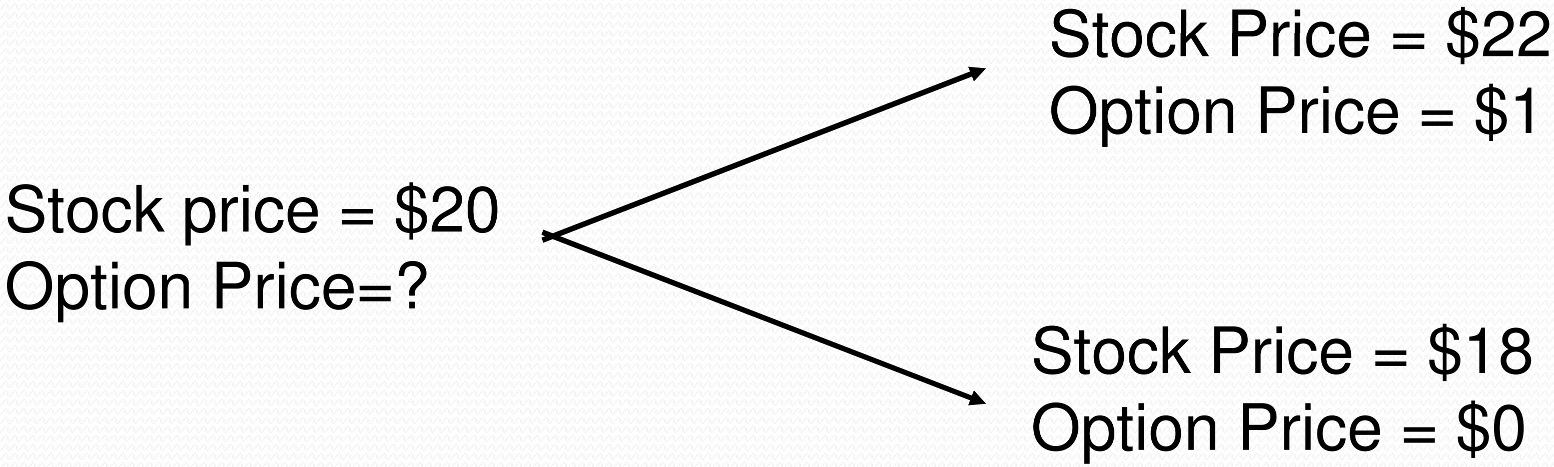
A Simple Binomial Model

- A stock price is currently $20
- In three months it will be e.g. either $22 or $18



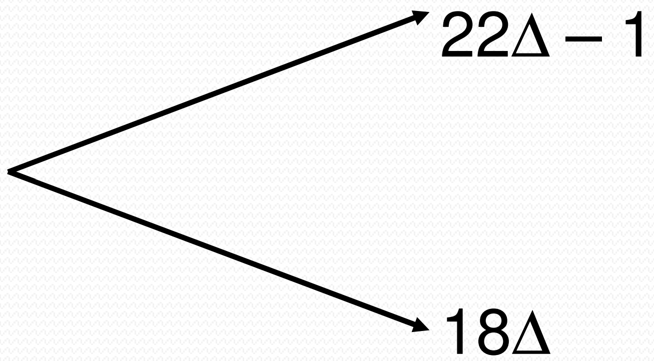
A Call Option

A 3-month call option on the stock has a strike price of 21.



Setting Up a Riskless Portfolio

- Consider the Portfolio: long Δ shares and short 1 call option



- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$

Valuing the Portfolio

- Risk-Free Rate is 12%
- The riskless portfolio is:

  long 0.25 shares
  short 1 call option
- The value of the portfolio in 3 months is

$$22\times 0.25 - 1 = 4.50$$

- (No arbitrage) The value of the portfolio today is

$$4.5e^{-0.12\times 0.25} = 4.3670$$

Valuing the Option

- The portfolio that is

  long 0.25 shares
  short 1 option

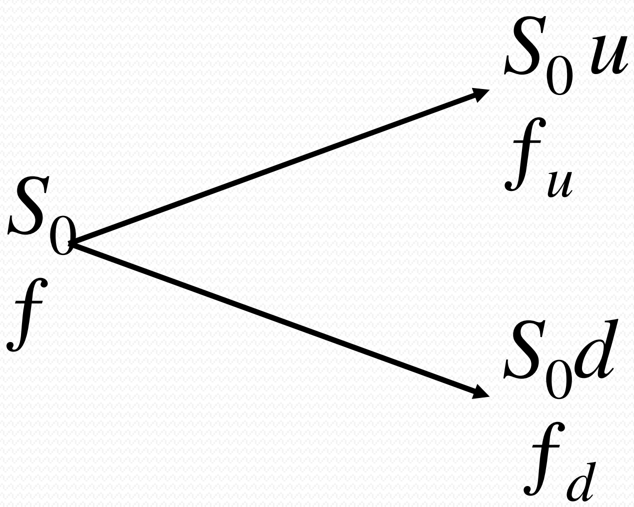
  is worth 4.367
- The value of the shares is

  5.000 (= 0.25×20)
- The value of the option is therefore

  0.633 (= 5.000 – 4.367)

Generalization

- A derivative expires at time T and is dependent on a stock



Generalization (continued)

- Consider the portfolio that is long Δ shares and short 1 derivative

$$
\begin{tikzcd}[column sep=large]
 & S_0 u\Delta - f_u \\
\Delta S_0 - f \arrow[ur] \arrow[dr] & \\
 & S_0 d\Delta - f_d
\end{tikzcd}
$$

- The portfolio is riskless when $S_0 u\Delta - f_u = S_0 d\,\Delta - f_d$ or

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

Generalization (continued)

- Value of the portfolio at time T is $S_0 u\,\Delta - f_u$
- Value of the portfolio today is $(S_0 u\,\Delta - f_u)e^{-rT}$
- Another expression for the portfolio value today is $S_0\Delta - f$
- Hence

$$f = S_0\Delta - (S_0 u\,\Delta - f_u)e^{-rT}$$

Generalization (continued)

- Substituting for Δ we obtain

$$f = \left(S_0\Delta(e^{rT} - u) + f_u\right)e^{-rT} = \left(S_0 \tfrac{f_u - f_d}{S_0(u-d)}(e^{rT} - u) + f_u\right)e^{-rT} =$$

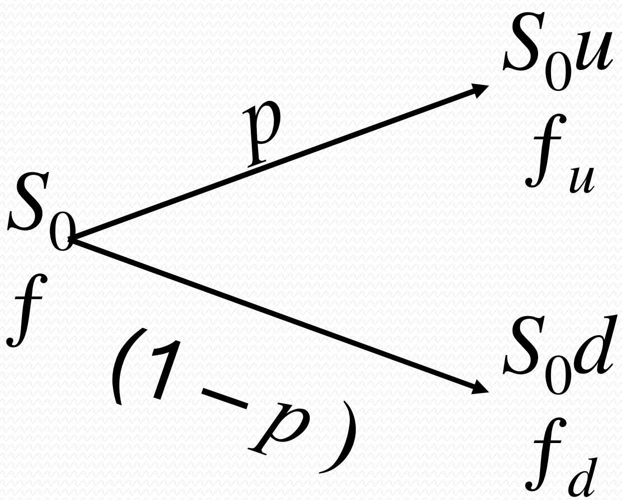
$$\left(\tfrac{(e^{rT} - u + u - d)}{u-d} f_u + \tfrac{u - e^{rT}}{u-d} f_d\right)e^{-rT} = \left(pf_u + (1-p)f_d\right)e^{-rT}$$

$$p = \frac{e^{rT} - d}{u - d}$$

$$1 - p = \frac{u - d - e^{rT} + d}{u - d} = \frac{u - e^{rT}}{u - d}$$

Risk-Neutral Valuation

- $f = [\, p\, f_u + (1 - p\,)f_d\,]e^{-rT}$
- The variables p and $(1 - p\,)$ can be interpreted as the risk-neutral probabilities of up and down movements
- The value of a derivative is its *expected* payoff in a risk-neutral world *discounted* at the risk-free rate



Risk-Neutral Expectation of Stock Price

$$E[S_T] = pS_u + (1-p)S_d$$

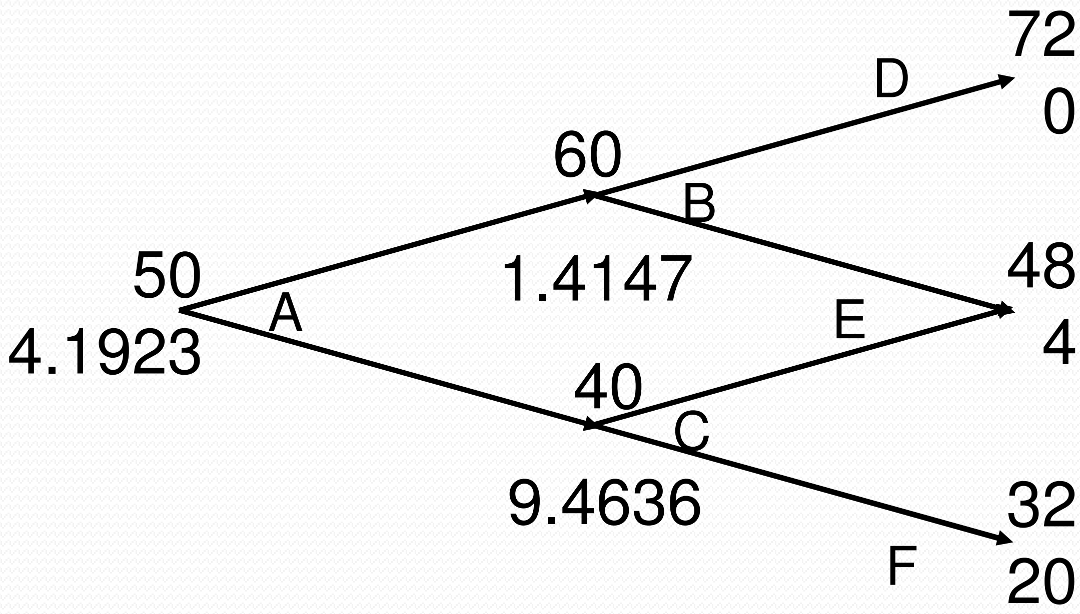
$$E[S_T] = \frac{e^{rT}-d}{u-d} uS_o + \frac{u-e^{rT}}{u-d} dS_o$$

$$E[S_T] = e^{rT} S_0$$

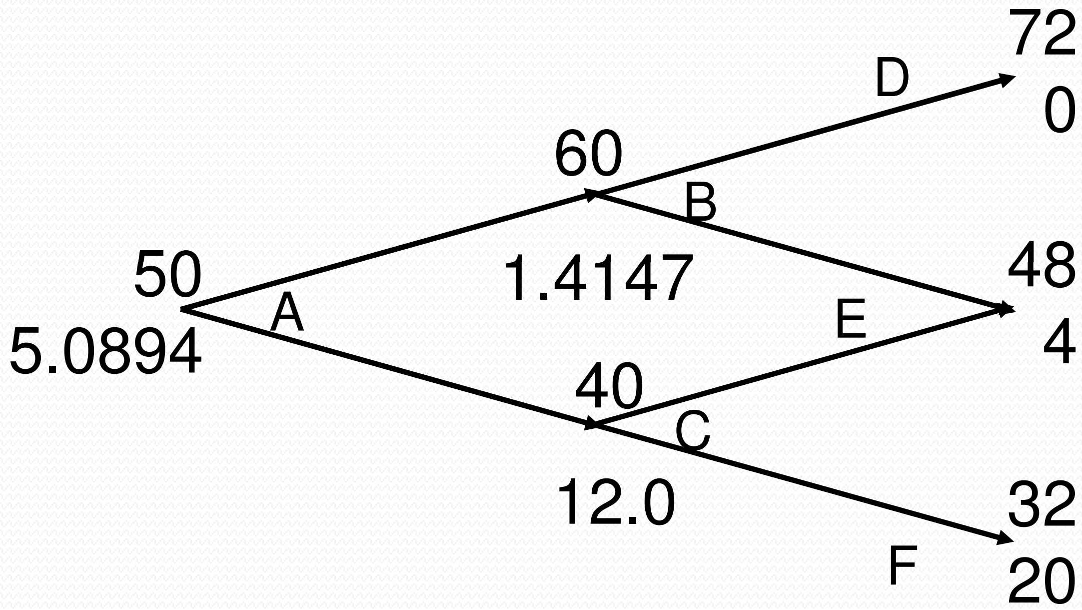
Irrelevance of Stock’s Expected Return

When we are valuing an option in terms of the underlying stock the expected return on the stock is irrelevant

A Put Option Example; K=52
A Two-Step Tree



What Happens When an Option is American



Tree Parameters

- We choose the tree parameters p, u, and d so that the tree gives correct values for the mean & standard deviation of the stock price changes in a risk-neutral world

$$e^{r\,\delta t} = pu + (1-p)d$$

$$\sigma^2\delta t = pu^2 + (1-p)d^2 - [pu + (1-p)d]^2$$

- A further condition often imposed is $u = 1/d$

Tree Parameters from Stock Price Movements

When δt is small (Cox, Ross, and Rubinstein)

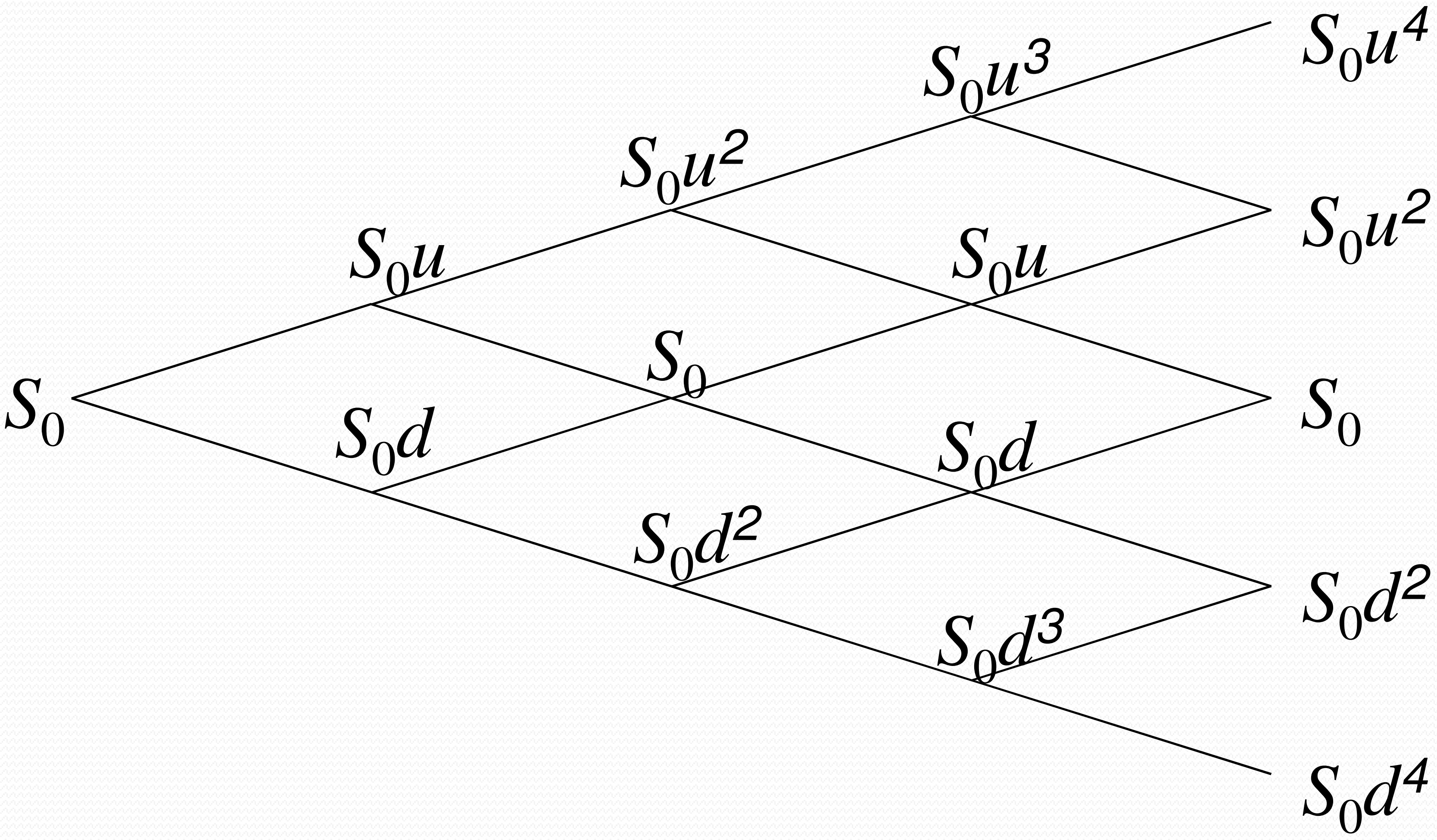
$$u = e^{\sigma\sqrt{\delta t}}$$

$$d = e^{-\sigma\sqrt{\delta t}}$$

$$p = \frac{a-d}{u-d}$$

$$a = e^{r\,\delta t}$$

Tree with Multiple Levels



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Backwards Induction

- We know the value of the option at the final nodes
- We work back through the tree using risk-neutral valuation to calculate the value of the option at each node, testing for early exercise when appropriate

Example: Put Option

$$S_0 = 50; \; X = 50; \; r = 10\%; \; \sigma = 40\%;$$

$$T = 5 \text{ months} = 0.4167;$$

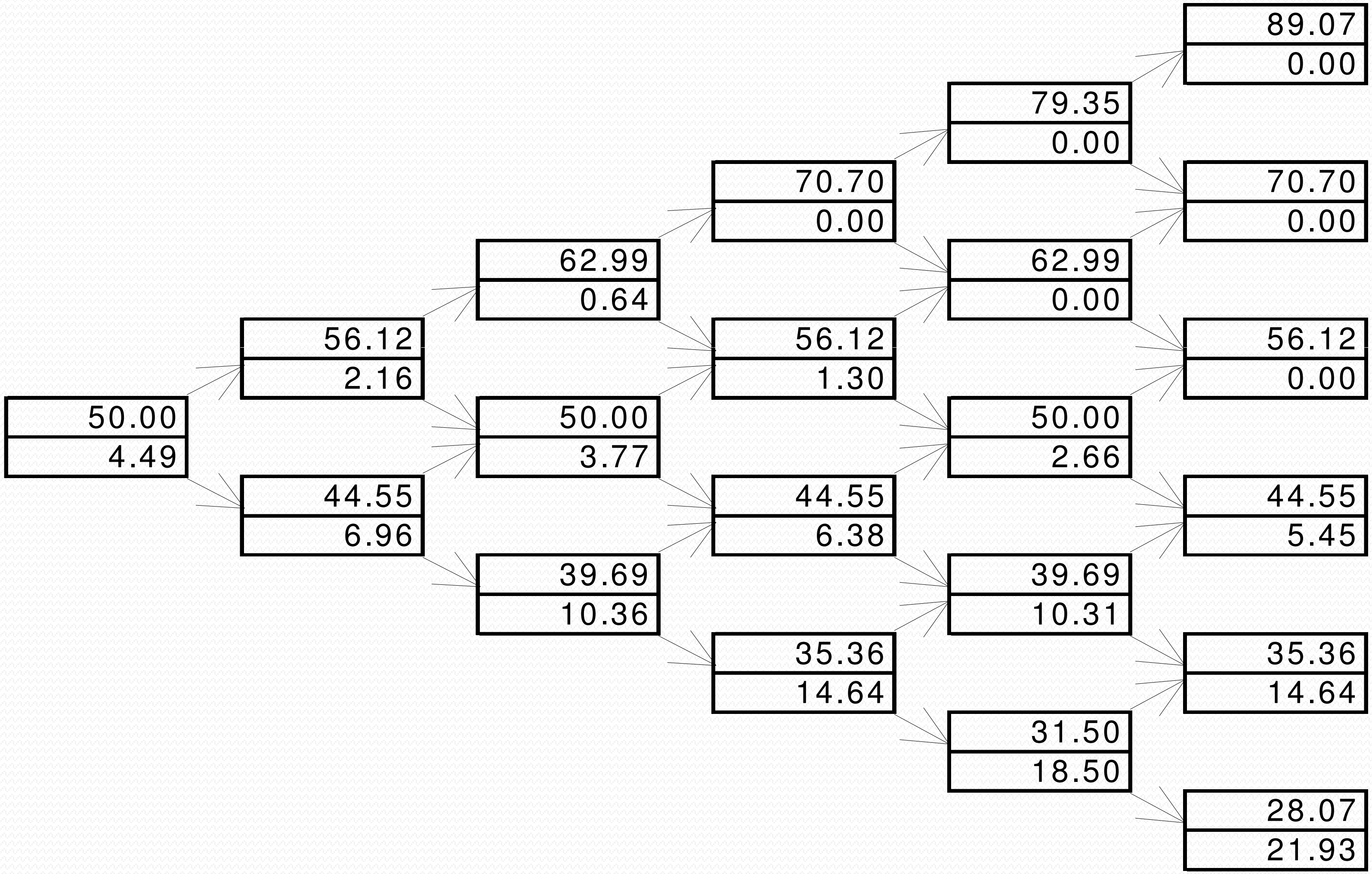
$$\delta t = 1 \text{ month} = 0.0833$$

The parameters imply

$$u = 1.1224; \; d = 0.8909;$$

$$a = 1.0084; \; p = 0.5076$$

Example (continued)



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Calculation of Delta

Delta is the hedge parameter

Delta is calculated from the nodes at time δt

$$\Delta = \frac{\Delta f}{\Delta S} = \frac{2.16 - 6.96}{56.12 - 44.55} = -0.41$$