Lab Assignments Computational Finance

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Submission guidelines

These assignments can be done in groups of three students. Reports with a *clear and concise description of the assignment, the methods, the results and discussion* should be submitted before the deadlines. You are free to choose the programming language/environment in which you would like to write your computer programs. If you have questions about the assignments do not hesitate to contact the teaching assistant or the lecturer.

Grading scheme

- Each of the three assignments carries equal weight of 20% and the exam is worth 40%;
- The score of the exam should be 5 points (on the scale of 1 to 10) and higher for passing the course;

Assignment 1	Assignment 2	Assignment 3	Exam
20%	20%	20%	40%

Assignment 1: Black-Scholes Model and Binomial Tree Methods

Part I

Option Valuation

A commonly used approach to compute the price of an option is the so-called binomial tree method. Suppose that the maturity of an option on a non-dividend-paying stock is divided into N subintervals of length δt . We will refer to the j^{th} node at time $i^{\delta t}$ as the (i, j) node. The stock price at the (i, j) node is $S_{i,j} = S_0 u^j d^{i-j}$ (with u and d the upward and down-ward stock price movements, respectively). In the binomial tree approach, option prices are computed through a back-ward induction scheme:

1. The value of a call option at its expiration date is $MAX(0, S_{N,j} - K)$;
2. Suppose that the values of the option at time $(i+1)^{\delta t}$ is known for all j. There is a probability p of moving from the (i, j) node at time $i^{\delta t}$ to the $(i+1, j+1)$ node at time $(i+1)^{\delta t}$, and a probability $1-p$ of moving from the (i, j) node at time $i^{\delta t}$ to the $(i+1, j)$ node at time $(i+1)^{\delta t}$. Risk-neutral valuation gives

$$f_{i,j} = e^{-r\delta t}(pf_{i+1,j+1} + (1-p)f_{i+1,j})$$

Consider a European call option on a non-dividend-paying stock with a maturity of one year and strike price of €99. Let the one year interest rate be 6% and the current price of the stock be €100. Furthermore, assume that the volatility is 20%.

1. Write a binomial tree program to approximate the price of the option. Take a tree with 50 steps. How does your estimate compare to the analytical value? Experiment for different values of the volatility.
2. Study the convergence of the method for increasing number of steps in the tree. What is the computational complexity of this algorithm as a function of the number of steps in the tree?
3. Compute the hedge parameter from the Binomial Tree model. Compare with the analytical values. Experiment for different values of the volatility.
4. Now suppose that the option is American. Change the code such that it can handle early exercise opportunities. What is the value of the American put and call for the corresponding parameters? Experiment for different values of the volatility.

Part II
Hedging Simulations

The fundamental idea behind the Black-Scholes model is that of dynamic replication of the claim by taking positions in the underlying. In practice this means that a trader should apply a dynamic hedging strategy in order to ensure that the claim is replicated at expiry. In this part of the assignment we will apply a delta hedging of a European Call option.
We make the following assumptions:

- The dynamics of the stock price S is given by the following equation

$$dS = rSdt + \sigma SdZ$$

- The option and the corresponding delta sensitivities is based on the Black-Scholes model.

Consider again a short position in a European call option on a non-dividend-paying stock with a maturity of one year and strike price of €99. Let the one year interest rate be 6% and the current price of the stock be €100. Furthermore, assume that the volatility is 20%. Perform hedging simulation where the volatility in the stock price process is matching the volatility used in the option valuation (set both equal to 20%). Vary the frequency of the hedge adjustment (from daily to weekly) and explain the results. Perform numerical experiments where the volatility in the stock price process is not matching the volatility used in the option valuation. Experiment for various levels and explain the results.

Appendices

In the two sections below, some additional theoretical background for the assignment is presented. The first section treats the parameter derivation of the binomial tree model. The second presents the analytical formulas for the computation of option prices and their hedge parameters. This can be used as a benchmark for the binomial tree approximation.

A Binomial tree parameters

We consider a binomial tree model for the evolution of a stock S. Our assumptions are the following:

1. The stock price S_1 after a period of time Δt can only have two possible outcomes: $S_0 \cdot u$ or $S_0 \cdot d$ with $0 < d < u$.
2. The risk-neutral probability of an up movement is denoted p.
3. In line with the no-arbitrage argument, the expected return of the asset is that of the risk-free rate: $\mathbb{E}\left(S_1\right) = S_0 \cdot e^{r\Delta t}$.
4. For small values of Δt, the variance of the stock price change is approximately $S_0^2 \sigma^2 \Delta t$

Under these assumptions, we will derive expressions for the parameters p, u and d in line with a risk-neutral valuation.

A.1 The parameter p

First we consider the expectation of the stock under the binomial model. In the binomial model, there are only two possible outcomes for S_1, by which it follows that

$$\mathbb{E}\left(S_1\right) = p \cdot S_0 \cdot u + (1-p) \cdot S_0 \cdot d$$

Then, following assumption 3, we can write

$$S_0 \cdot e^{r\Delta t} = p \cdot S_0 \cdot u + (1-p) \cdot S_0 \cdot d$$

Which solves for p as

$$\begin{aligned} p &= \frac{e^{r\Delta t} - d}{u-d} \\ 1-p &= \frac{u - e^{r\Delta t}}{u-d} \end{aligned}$$

A.2 The parameters u and d

For the derivation of u and d we consider the variance of the stock under the binomial model. Also we impose one more assumption, namely $u \cdot d = 1$. Recall that $\mathbb{E}\left(S_1\right) = S_0 \cdot e^{r\Delta t}$, then according to the definition of the variance we have

$$\begin{aligned} \mathrm{Var}\left(S_1\right) &= \mathbb{E}\left(S_1^2\right) - \mathbb{E}\left(S_1\right)^2 \\ &= p \cdot \left(S_0 \cdot u\right)^2 + (1-p) \cdot \left(S_0 \cdot d\right)^2 - \left(S_0 \cdot e^{r\Delta t}\right)^2 \end{aligned}$$

Now substitute our derived value of p to find

$$\begin{aligned} \operatorname{Var}(S_1) &= S_0^2\left(\frac{e^{r\Delta t}-d}{u-d}\cdot u^2+\frac{u-e^{r\Delta t}}{u-d}\cdot d^2-e^{2r\Delta t}\right) \\ &= S_0^2\left(e^{r\Delta t}(u+d)-u\cdot d-e^{2r\Delta t}\right) \end{aligned}$$

Now we use assumption 4, which says $\operatorname{Var}(S_1)=S_0^2\sigma^2\Delta t$. Divide both sides by S_0^2 and substitute $d=\frac{1}{u}$ so that

$$\begin{aligned} \sigma^2\Delta t &= e^{r\Delta t}\left(u+\frac{1}{u}\right)-1-e^{2r\Delta t} \\ u+\frac{1}{u} &= e^{-r\Delta t}\left(\sigma^2\Delta t+1+e^{2r\Delta t}\right) \\ &= e^{-r\Delta t}\sigma^2\Delta t+e^{-r\Delta t}+e^{r\Delta t} \end{aligned}$$

Now use Taylor expansions to approximate $e^{-r\Delta t}\approx 1-r\Delta t$ and $e^{r\Delta t}\approx 1+r\Delta t$. If we neglect all terms with $(\Delta t)^2$ and higher powers, we have

$$\begin{aligned} u+\frac{1}{u} &\approx (1-r\Delta t)\,\sigma^2\Delta t+(1-r\Delta t)+(1+r\Delta t) \\ &\approx \sigma^2\Delta t+2 \end{aligned}$$

This we can easily rewrite as

$$u^2-\left(\sigma^2\Delta t+2\right)u+1=0$$

which we can solve by using the quadratic formula and again ignoring higher powers of Δt:

$$\begin{aligned} u &\approx \frac{\sigma^2\Delta t+2\pm\sqrt{\left(\sigma^2\Delta t+2\right)^2-4}}{2} \\ &\approx \frac{1}{2}\sigma^2\Delta t+1\pm\sigma\sqrt{\Delta t} \end{aligned}$$

The solution with the minus-sign corresponds to the d parameter $\left(=\frac{1}{u}\right)$, therefore we will discard it for now. As a final step, consider the second-order Taylor expansion of $f(x)=e^{\sigma x}$ around zero. This would yield $f(x)\approx 1+\sigma x+\frac{1}{2}\sigma^2x^2$. Therefore it follows that

$$\begin{aligned} u &\approx 1+\sigma\sqrt{\Delta t}+\frac{1}{2}\sigma^2\Delta t \\ &\approx f\left(\sqrt{\Delta t}\right)=e^{\sigma\sqrt{\Delta t}} \end{aligned}$$

And hence we conclude:

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}} \\ d &= e^{-\sigma\sqrt{\Delta t}} \end{aligned}$$

B Black-Scholes formula for European call options

B.1 Option price

The Black-Scholes formula allows us to calculate analytically the price of European call and put options. At time t, prior to maturity T, the price C_t of a European call option with strike price K on a non-dividend-paying stock is

$$C_t = N(d_1)S_t - N(d_2)Ke^{-r(T-t)}$$

with d_1 and d_2 satisfying

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$$

$$\begin{aligned} d_2 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right] \\ &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

and $N(x)$ the standard normal cumulative probability distribution function

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-\frac{1}{2}z^2}\,dz$$

S_t is the price of the underlying stock at time t, r is the risk-free interest rate, and σ is the volatility of the underlying stock returns.

B.2 Hedge parameter

The hedge parameter Δ_t at time $t \le T$ can be found by partially differentiating the call price C_t with respect to S_t

$$\Delta_t = \frac{\partial C_t}{\partial S_t} = N(d_1)$$