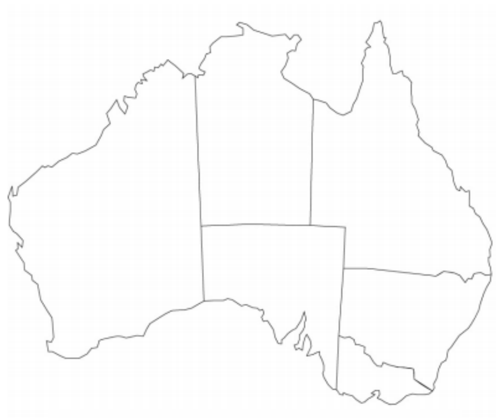
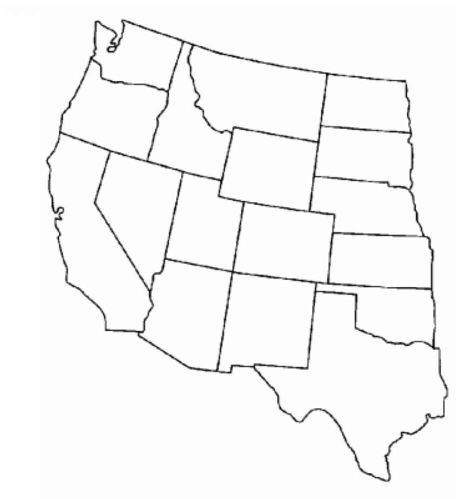
The Four Color Theorem Worksheet:





Notes on Notation:[1]

- Given a graph G and a vertex v in G , G − v is the graph removing v and all its shared edges from G . The reverse of G − v is (G − v) + v , and (G − v) + v = G .

- G(i, j) is a subgraph of G consisting of the vertices that are colored with colors i and j only, and edges connecting two of them. G (i, j,v) c is the connected component of G(i, j) containing vertex v.

- A path in G(i, j) , called a Kempe chain and denoted by Ch(i, j,u,v) , joining vertices u and v , that is a sequence of edges and vertices painted only with colors i and j .

- v is a vertex of graph G , which is denoted by v ∈G , and on the contrary v is not a vertex of graph G , which is denoted by v ∉G .

- A circle is a closed path. When vertex w∉Ch(i, j,u,v) , Ch(i, j,u,v) together with w as well as its two edges connected to u and v forms a Kempe circle, which is denoted by Ch(i, j,u,v) + w .

- Suppose n(v) , n(e) , and n(f) are the number of vertices, edges, and faces in a planar graph. Since each edge is shared by two faces and each face is bounded by three edges at least, 2n(e) ≥ 3n(f) which together with Euler's formula n(v) − n(e) + n(f) = 2 can be used to show 6n(v) − 2n(e) ≥12 . Now, if ( ) n di is the number of vertices of degree di and D is the maximum degree, with [2]Euler's Theorem 2: $2n(e)=\sum_{i=1}^{D} i\times n(d_i)$, $6n(v)-2n(e)=6\sum_{i=1}^{D} n(d_i)-\sum_{i=1}^{D} i\times n(d_i)$ $=\sum_{i=1}^{D}(6-i)n(d_i)\geq 12.$

- Lemma 1. For any planar graph Gn with n (n ≥ 6) vertices, there are vertices n v , n−1 v , …, 6 v such that d(vi) ≤ 5 and i i i G = G − v −1 are also planar graphs for i from n down to .

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[1] Extracted from Xiang's Formal Proof of the four color theorem

[2] But since 12 > 0 and 6 − i ≤ 0 for all i ≥ 6 , this demonstrates that there is at least one vertex of degree 5 or less in a planar graph [1]. Thus, the following lemma holds.