Pojet MADS partie 1

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0.1. equations describing the variation of temperature and salinity of the system

$$\begin{aligned}
\frac{dT_p}{dt} &= -\frac{|q|}{V}(T_p - T_e) + \frac{1}{\tau}(T_p^a - T_p) \\
\frac{dT_e}{dt} &= \frac{|q|}{V}(T_p - T_e) + \frac{1}{\tau}(T_e^a - T_e) \\
\frac{dS_p}{dt} &= -\frac{|q|}{V}(S_p - S_e) + F_p \\
\frac{dS_e}{dt} &= \frac{|q|}{V}(S_p - S_e) + F_e
\end{aligned} \tag{1}$$

0.2. equations describing the variation of flow and density of water

$$\begin{aligned}
&q = \gamma(\rho_p - \rho_e) \\
&\rho_i = \rho_0 - \alpha_T(T_i - T_0) + \alpha_S(S_i - S_0)
\end{aligned} \tag{2}$$

1. Rewrite the differential equations 1 in terms of the four state variables T_i, S_i. This requires finding an expression for q in terms of the state variables.

Rewriting the equations 2 we have:

$$\begin{aligned}
&q = \gamma((\rho_0 - \alpha_T(T_p - T_0) + \alpha_S(S_p - S_0)) - (\rho_0 - \alpha_T(T_e - T_0) + \alpha_S(S_e - S_0))) \\
&q = \gamma(-\alpha_T(T_p - T_0) + \alpha_S(S_p - S_0) + \alpha_T(T_e - T_0) - \alpha_S(S_e - S_0)) \\
&q = \gamma(-\alpha_T T_p + \alpha_S S_p + \alpha_T T_e - \alpha_S S_e) \\
&q = \gamma(\alpha_S(S_p - S_e) - \alpha_T(T_p - T_e))
\end{aligned}$$

we thus have:

$$q = \gamma(\alpha_S(S_p - S_e) - \alpha_T(T_p - T_e)) \tag{3}$$

It is worth noting that the dependency on T_0 and S_0 does not appear anymore, but this is still an equation only valid around the reference values.
This means that these equations are only valid if $T_p \approx T_e$ and $S_p \approx S_e$.

Using equation 3, equations 1 become:

$$\begin{aligned}
\frac{dT_p}{dt} &= -\frac{\gamma}{V}|\alpha_S(S_p - S_e) - \alpha_T(T_p - T_e)|(T_p - T_e) + \frac{1}{\tau}(T_p^a - T_p) \\
\frac{dT_e}{dt} &= \frac{\gamma}{V}|\alpha_S(S_p - S_e) - \alpha_T(T_p - T_e)|(T_p - T_e) + \frac{1}{\tau}(T_e^a - T_e) \\
\frac{dS_p}{dt} &= -\frac{\gamma}{V}|\alpha_S(S_p - S_e) - \alpha_T(T_p - T_e)|(S_p - S_e) + F_p \\
\frac{dS_e}{dt} &= \frac{\gamma}{V}|\alpha_S(S_p - S_e) - \alpha_T(T_p - T_e)|(S_p - S_e) + F_e
\end{aligned} \tag{4}$$

2. Discuss the temperature and salinity conditions necessary to have a pole-ward surface flow ($q > 0$).

Using equation 3 we can see that $sign(q) = sign(\alpha_S(S_p - S_e) + \alpha_T(T_e - T_p))$.
We thus see that the more the equator is hotter than the pole, the more positive is the flow, this is counterbalanced by the salinity : the less saline the pole is in confront of the equator, the more negative is the flow.
We add these two effects with α_T, α_S being proportionnality coefficient.

3. The presence of an absolute value (a non-differentiable function!) in the differential equations may look 'dangerous' at first sight. Justify that this does not jeopardize the existence and unicity of local solutions for the initial value problem.

The conditions needed for existence and unicity of local solutions for the initial value porblem are:

1. Existence : continuity of $f(x, t)$
2. Unicity : Lipschitz condition

Exept the absolute value, the rest of the expressions is linear and then continuous with respect to the Lipschitz condition. Since q is a linear expression, we need to check both conditions on the absolute value of a linear expression in S_p, S_e, T_p, T_e : $|\alpha_S(S_p - S_e) - \alpha_T(T_p - T_e)|$.

Proof of existence

Since the absolute value is a continuous function everywhere (even if non-differentiable), and linear functions are continuous, the absolute value of a sum (or difference) of continous functions is continuous as well.

Proof of unicity

A function $f(x)$ is said to be *Lipschitz continuous* on an interval I if there exists a constant $L > 0$ such that:

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2|, \quad \forall x_1, x_2 \in I,$$

where L is called the *Lipschitz constant*.

Proof for linear functions: Let $f(x) = ax + b$, where $a, b \in \mathbb{R}$. We want to show that $f(x)$ is Lipschitz continuous.

We have:

$$|f(x_1) - f(x_2)| = |(ax_1 + b) - (ax_2 + b)| = |ax_1 - ax_2| = |a||x_1 - x_2|.$$

Thus, we can write:

$$|f(x_1) - f(x_2)| \leq |a||x_1 - x_2|.$$

Here, the Lipschitz constant L is simply $|a|$.

2. The absolute value preserves Lipschitz continuity Let $f(x)$ be a Lipschitz continuous function with Lipschitz constant $L > 0$. That is,

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2|, \quad \forall x_1, x_2 \in I.$$

We want to show that $g(x) = |f(x)|$ is also Lipschitz continuous.

Proof: Consider two points $x_1, x_2 \in I$. The difference between $g(x_1)$ and $g(x_2)$ is:

$$|g(x_1) - g(x_2)| = ||f(x_1)| - |f(x_2)||.$$

We get:

$$||f(x_1)| - |f(x_2)|| \leq |f(x_1) - f(x_2)|.$$

Since $f(x)$ is Lipschitz continuous, we know:

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2|.$$

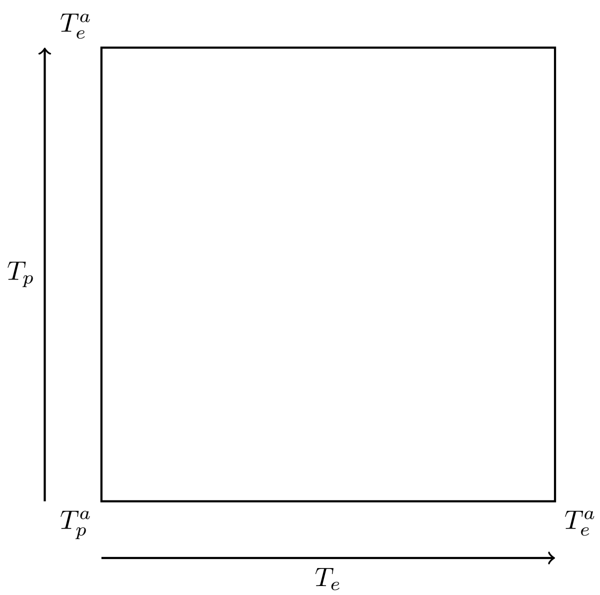
Combining these two results, we obtain:

$$|g(x_1) - g(x_2)| \leq L|x_1 - x_2|.$$

Thus, $g(x) = |f(x)|$ is Lipschitz continuous with the same Lipschitz constant L.

4. Prove that the set $\{(T_p, T_e, S_p, S_e) : T_p^a \leq T_p \leq T_e^a$ and $T_p^a \leq T_e \leq T_e^a\}$ is invariant.

We can see the set geometrically :



To prove that this is invariant we will use the Bony's theorem: Let f be locally Lipschitz continuous vector field defined on an open set $\Omega \in \mathbb{R}$, and X a closed set of Ω (i.e. the intersection of a closed set of $\mathbb{R}$ with Ω). If $<f(x), n(x)> \leq 0$ for every $x \in \delta X$ and every vector n(x) outward normal to X at x, then X is (positively) invariant for f.

We will check for every bound :

1. Bound $T_p = T_p^a$: the normal vector is $-e_p$ (vector of direction of T_p). Then we need

$$\frac{dT_p}{dt}.(-e_p) = -\frac{dT_p}{dt} \leq 0$$

. We know that $\frac{dT_p}{dt} = -\frac{\gamma}{V}|\alpha_S(S_p - S_e) - \alpha_T(T_p - T_e)|(T_p - T_e) + \frac{1}{\tau}(T_p^a - T_p)$. When we evaluate this expression at the bound (in $T_p = T_p^a$), we get:

$$\frac{dT_p}{dt} = -\frac{\gamma}{V}|\alpha_S(S_p - S_e) - \alpha_T(T_p^a - T_e)|(T_p^a - T_e)$$

Since $T_e^a \leq T_e$, the difference between both is negative. We multiply it by an absulute value and $-\frac{\gamma}{V}$. Thus, $\frac{dT_p}{dt}$ is positive and $\frac{dT_p}{dt} \leq 0$ as needed.

2. Bound $T_p = T_e^a$: the normal vector is $+e_p$ (vector in the direction of T_p). Then we need:

$$\frac{dT_p}{dt} \cdot e_p = \frac{dT_p}{dt} \leq 0.$$

We know that:

$$\frac{dT_p}{dt} = -\frac{\gamma}{V}|\alpha_S(S_p - S_e) - \alpha_T(T_p - T_e)|(T_p - T_e) + \frac{1}{\tau}(T_p^a - T_p).$$

At the bound $T_p = T_e^a$, we evaluate:

$$\frac{dT_p}{dt} = -\frac{\gamma}{V}|\alpha_S(S_p - S_e) - \alpha_T(T_e^a - T_e)|(T_e^a - T_e) + \frac{1}{\tau}(T_p^a - T_e^a).$$

The term $(T_e^a - T_e)$ is positive since $T_e^a \geq T_e$, making the product with $-\frac{\gamma}{V}|\cdot|$ negative. The second term $(T_p^a - T_e^a)$ is non-positive because $T_p^a \leq T_e^a$. Hence:

$$\frac{dT_p}{dt} \leq 0,$$

which satisfies the condition.

3. Bound $T_e = T_p^a$: the normal vector is $-e_e$ (vector in the direction of T_e). Then we need:

$$\frac{dT_e}{dt} \cdot (-e_e) = -\frac{dT_e}{dt} \leq 0.$$

We know that:

$$\frac{dT_e}{dt} = \frac{\gamma}{V}|\alpha_S(S_p - S_e) - \alpha_T(T_p - T_e)|(T_p - T_e) + \frac{1}{\tau}(T_e^a - T_e).$$

At the bound $T_e = T_p^a$, we evaluate:

$$\frac{dT_e}{dt} = \frac{\gamma}{V}|\alpha_S(S_p - S_e) - \alpha_T(T_p - T_p^a)|(T_p - T_p^a) + \frac{1}{\tau}(T_e^a - T_p^a).$$

The term $(T_p - T_p^a)$ is positive since $T_p \geq T_p^a$, making the product with $\frac{\gamma}{V}|\cdot|$ positive. The second term $(T_e^a - T_p^a)$ is non-negative because $T_e^a \geq T_p^a$. Thus:

$$-\frac{dT_e}{dt} \leq 0,$$

which satisfies the condition.

4. Bound $T_e = T_e^a$: the normal vector is $+e_e$ (vector in the direction of T_e). Then we need:

$$\frac{dT_e}{dt} \cdot e_e = \frac{dT_e}{dt} \leq 0.$$

We know that:

$$\frac{dT_e}{dt} = \frac{\gamma}{V}|\alpha_S(S_p - S_e) - \alpha_T(T_p - T_e)|(T_p - T_e) + \frac{1}{\tau}(T_e^a - T_e).$$

At the bound $T_e = T_e^a$, we evaluate:

$$\frac{dT_e}{dt} = \frac{\gamma}{V}|\alpha_S(S_p - S_e) - \alpha_T(T_p - T_e^a)|(T_p - T_e^a) + \frac{1}{\tau}(T_e^a - T_e^a).$$

The term $(T_p - T_e^a)$ is non-positive since $T_p \leq T_e^a$, making the product with $\frac{\gamma}{V}|\cdot|$ negative. The second term is zero because $(T_e^a - T_e^a) = 0$. Thus:

$$\frac{dT_e}{dt} \leq 0,$$

which satisfies the condition.

5. Prove that the set $\{(T_p, T_e, S_p, S_e) : S_p \geq 0$ and $S_e \geq 0\}$ is invariant if and only if $F_e \geq 0$ and $F_p \geq 0$. This condition is not satisfied in our case. On the other hand, a negative salinity is clearly unphysical. What part of the modelling process causes this undesired property? Is it a serious problem for our modelling and analysis?

We will answer the part of the question the same way than the privious question. Since $0 \leq S_p$, the normal vector is $-(e_p)$. We want

$$\frac{dS_p}{dt} \cdot -(e_p) \leq 0$$

Then, we want $\frac{dS_p}{dt}$ to be positive. We know that :

$$\frac{dS_p}{dt} = -\frac{\gamma}{V}|\alpha_S(S_p - S_e) - \alpha_T(T_p - T_e)|(S_p - S_e) + F_p$$

Evaluating this expression at $S_p = 0$, we get :

$$\frac{dS_p}{dt} = -\frac{\gamma}{V}|\alpha_S(-S_e) - \alpha_T(T_p - T_e)|(-S_e) + F_p$$

Given that $0 \leq S_e$, the first term cannot be negative but in the case where it is null, F_p cannot be negative to fulfill the condition. The same way for the bound $S_e = 0$ with the normal vector $+e_e$, we get that $0 \leq F_e$. This is indeed incompatible with our problem since $F_p = -F_e$.

Unphysical negative salinity and source of the issue:

A negative salinity is unphysical. This undesired property arises from the assumption that F_p and F_e are constants. In reality, these fluxes depend on various natural processes (e.g., evaporation and precipitation) and are not constant. The simplification neglects variability in salinity changes, leading to non-physical solutions.

Implications for Modeling and Analysis:

The non-invariance of the set shows potential issues with the model. While this might be acceptable for qualitative analysis, it compromises the quality of quantitative predictions. The results may therefore not be physically right, necessitating a better model taking account of variable fluxes or additional mechanisms.

6. The presence of non-linear (including quadratic) terms in the equations may look 'dangerous': remember that the ODE $\frac{dx}{dt} = x^2$ may blow up in finite time, which looks physically dubious. Justify that this cannot happen here.

A blow up cannot happens here because of the last term of the expressions. Let's take the first equation :

$$\frac{dT_p}{dt} = -\frac{\gamma}{V}|\alpha_S(S_p - S_e) - \alpha_T(T_p - T_e)|(T_p - T_e) + \frac{1}{\tau}(T_p^a - T_p)$$

the term $\frac{1}{\tau}(T_p^a - T_p)$ avoid the expression to blow up. Indeed if T_p rises, it will be higher than T_p^a and then $(T_p^a - T_p)$ will be negative making the derivative of T_p lower blocking this way the rise of T_p. It is the same for T_e.

7. Start the system with initial conditions: $[T_p, T_e, S_p, S_e] = [10°C, 26°C, 34ppt, 36ppt]$. Then, simulate the system of differential equations. Use a timestep of 86400 sec (1 day) and simulate the evolution over 500 years. Illustrate and comment the evolution of temperatures and salinity in this case.

Small technical detail: in our simulation we condidered that one year is always 365 days.

After running our simulation, we see in Figure 1 that:

- In the polar box, the hot and salty water coming from the equator tends to increase the salinity and decrease the temperature of the water, despite the fact that the melting continously decreases salinity and that the surface air is colder than even the initial sea temperature.
- In the equatorial box, the cooler and fresher water coming from the pole tends to decrease the salinity and increase the temperature of the water, despite the fact that evaporation continously increases salinity and that the surface air is colder than enven the initial sea temperature.

After a transition period, we see around 300 years after the beginning of our simulation that the exchange of salt and heat from the AMOC stabilizes itself with the effect of evaporation and melting on the salinity and the effect of heat exchange with the air on temperature. The system seems to be in equilibrium.

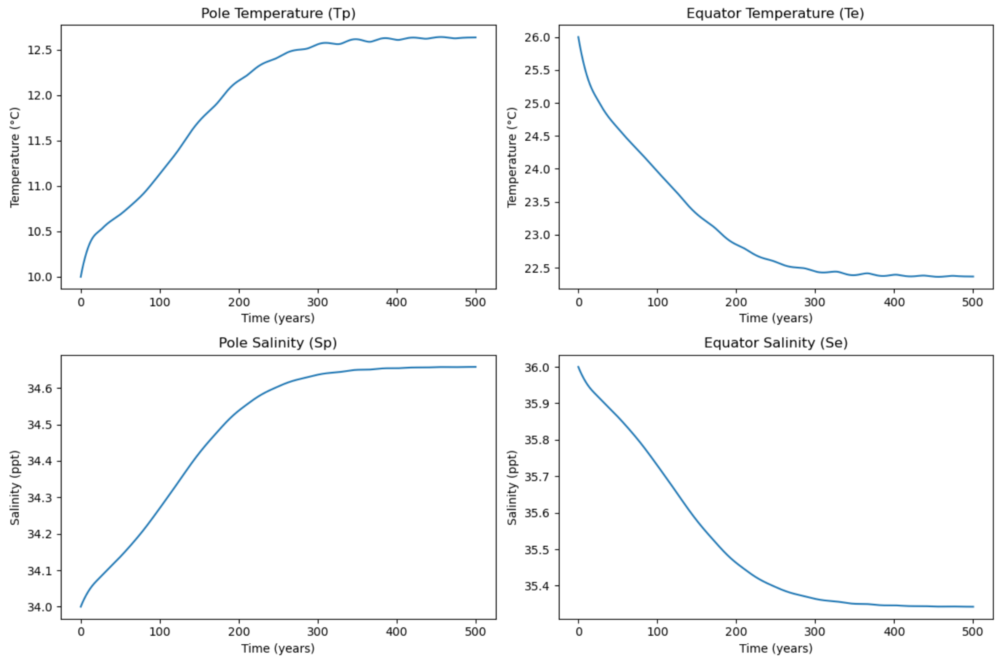


Figure 1: Simulated evolution of ocean temperature and salinity over 500 years, no climate change

8. Now we suppose that climate change increases the temperature of the atmosphere at the pole of 2°C, i.e. we have $T_p^a = 9°C$. Simulate the system with the same initial conditions as before. Compare the evolutions of salinity and temperatures in this scenario with the ones obtained in the previous question. Then, plot the evolution of water flux q in both cases. Are the two trajectories similar to each other? How is the direction of the surface flow evolving in each case? How do you relate this to the press excerpts?

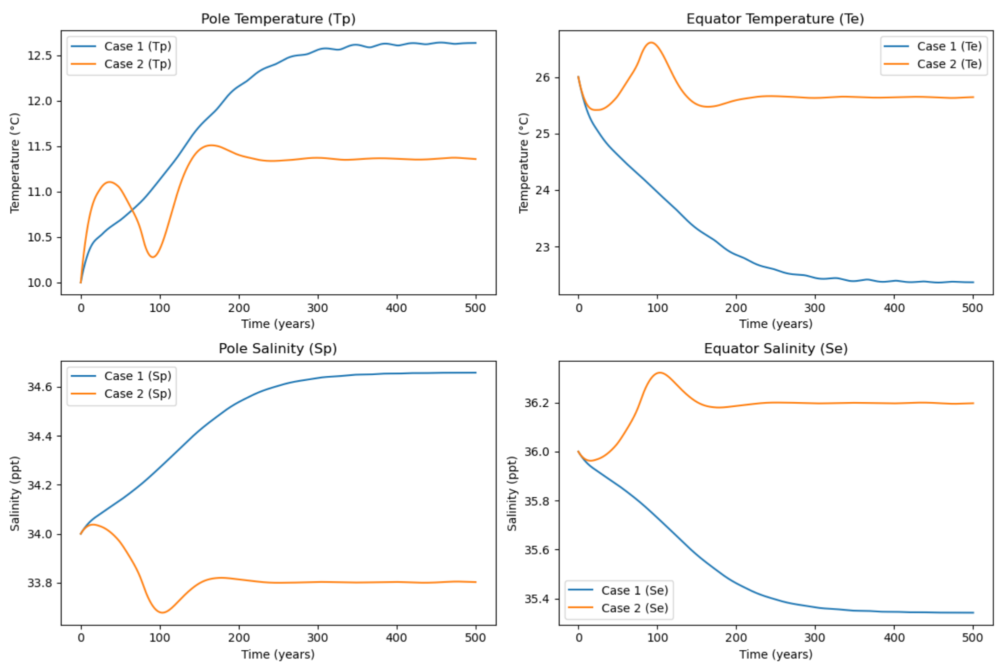


Figure 2: Simulated evolution of ocean temperature and salinity over 500 years, with and without climate change compared

8.1. Results for the boxes

We see in Figure 2 a strange behaviour: despite the increase in surface temperature, the sea temperature at the pole after convergence to the equilibrium is smaller in the case with climate change than in the case without climate change.

At the equator, mirroring the situation at the pole the temperature after convergence is higher in the case with climate change than in the case without climate change.

This is due to the decrease in the AMOC circulation as you see in Figure 3

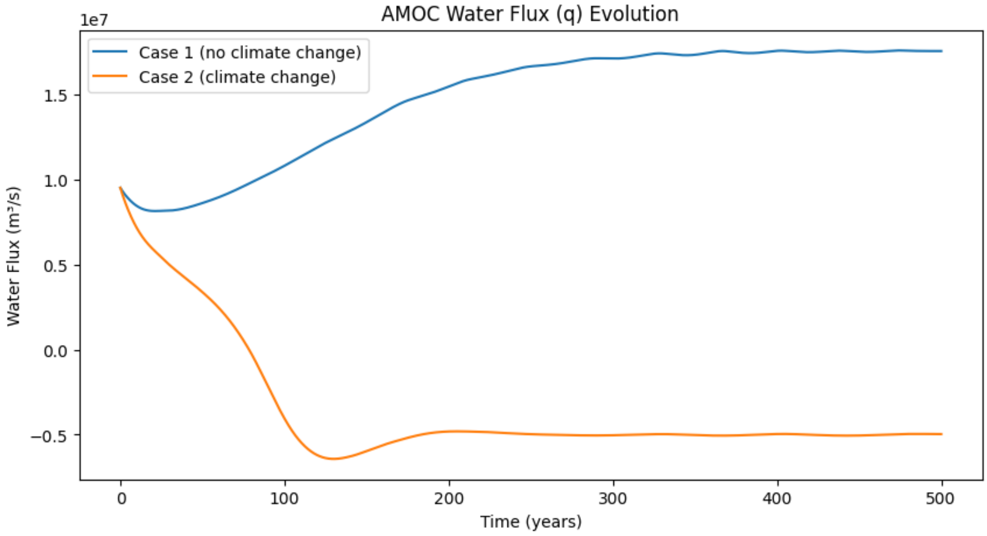


Figure 3: Simulated evolution of AMOC circulation over 500 years, with and without climate change compared

What is really interesting is that not only the absolute value of the AMOC decreases but we also see a compelitely reversal of the circulation.

8.2. Generalisation of our findings to reality

If this model fits reality, we see a diminution of sea temperature at the poles: that could explain the press excerpts:

- "Icy winds howl across the thames, ice flows blocks shipping in the Mersey docks, crops failing across the uk" can be explained by the decrease in sea temperature in the poles.
- "Rising sea in the east coast" can possibly be explained by the reversal of AMOC?
- "Wet and dry season switch around in the rain forest" no idea where it comes from

But this holds only if we can generalize our findings to reality: first we can ask ourself if two boxes really are a good enough approximation of reality.

Then we can question the choice of assuptions on the effect of climate change: why we have an inrcrease in atmosphere pole temperature only at the poles? Do we also need to model the effect of sea temperature on atmosphere temperature? Also don't we have an increase in melting with higher temperature at the poles?

If we change these assumptions for our modelisation we have very different results so one can think we modelize what happens in reality not because we have a good model but only by chance.