The Poisson Process, meaning, properties and simulations

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1 Meaning and Properties

A Poisson process is a specific type of stochastic process that models the occurrence of events in continuous time. It is named after the French mathematician Siméon Denis Poisson, who introduced it in the early 19th century. The key characteristics of a Poisson process are:

1. **Constant rate**: Events in a Poisson process happen at a constant rate λ per unit of time. Mathematically:

$$\lambda(t) = \lambda, \forall \lambda$$

2. **Memoryless property**: The probability of an event happening in the future is independent of the past. Mathematically:

$$P(X > s + t | X > s) = P(X > t); s, t \geq 0$$

3. **Discretness**: While the Poisson process is continous in time events are counted within time intervals, the number of events in a fixed time interval follows a Poisson distribution. Mathematically, if $N(t)$ represents the number of events that happened up to time t:

$$P(N(t) = k) = \frac{(\lambda t)^k e^{\lambda t}}{k!}$$

4. **Inter-Arrival Times**: The times between consecutive events, called inter-arrival times, follow an exponential distribution, meaning that if T_n is the time between the $(n-1)$th event and the nth event, then:

$$P(T_n > t) = e^{-\lambda t}$$

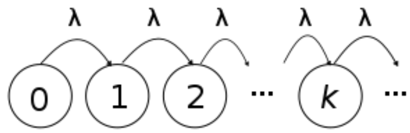

Figure 1: Visual representation of a Poisson process

The Poisson process is a fundamental tool in probability theory and stochastic processes, providing a simple yet powerful model for events occurring in a random and independent manner over time.

2 Simulation

With the following Python script I can select a certain rate λ and a certain number of samples and obtain a graph of the Poisson process Frequency and Numer of Events.

```python
import streamlit as st
import numpy as np
import matplotlib.pyplot as plt

# Simulate the Poisson process
lambda_val = 100  # Poisson rate parameter
num_samples = 100  # Number of samples to generate
samples = np.random.poisson(lambda_val, num_samples)

# Display the result in Streamlit
st.title("Poisson Process Simulation")
st.write(f"Lambda (rate parameter): {lambda_val}")
st.write(f"Number of samples: {num_samples}")
st.write("Generated samples:")
st.write(samples)

# Plot the Poisson process
plt.hist(samples, bins=range(max(samples)+1),
    align='left', rwidth=0.8)
plt.xlabel("Number of events")
plt.ylabel("Frequency")
plt.title("Poisson Process")
st.pyplot(plt)
```

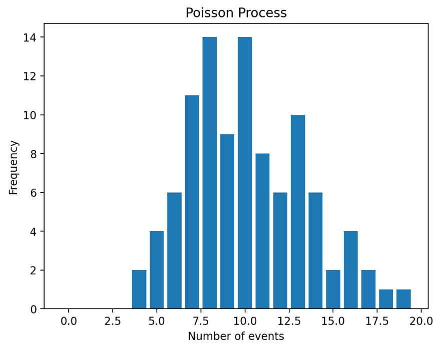

Figure 2: Poisson process with $\lambda = 10$.

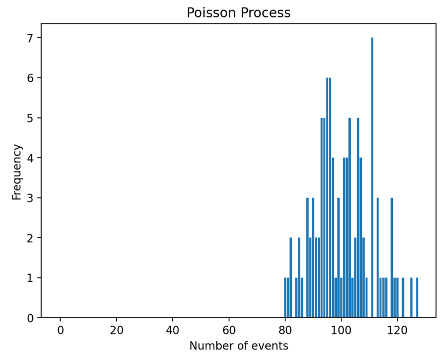

Figure 3: Poisson process with $\lambda = 100$.