Numerical Methods Comparison

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1 Euler-Murayama

The Euler-Maruyama method for the stochastic differential equation

$$dX(t) = a(X(t), t)\, dt + b(X(t), t)\, dW(t)$$

with initial condition $X(0) = X_0$ is given by the update rule

$$X_{n+1} = X_n + a(X_n, t_n)\Delta t + b(X_n, t_n)\sqrt{\Delta t} Z_n$$

where X_n is the approximation of $X(t_n)$ at time step n, $t_n = n\Delta t$, $\Delta t = \frac{T}{N}$ is the time step size, and Z_n is a standard normal random variable.

2 Milstein

The Milstein method for the stochastic differential equation

$$dX(t) = a(X(t), t)\, dt + b(X(t), t)\, dW(t)$$

with initial condition $X(0) = X_0$ is given by the update rule

$$\begin{aligned} X_{n+1} = X_n &+ a(X_n, t_n)\Delta t + b(X_n, t_n)\Delta W_n \\ &+ \frac{1}{2} b(X_n, t_n) \frac{\partial b(X_n, t_n)}{\partial x} (\Delta W_n^2 - \Delta t), \end{aligned}$$

where X_n is the approximation of $X(t_n)$ at time step n, $t_n = n\Delta t$, $\Delta t = \frac{T}{N}$ is the time step size, ΔW_n is the increment of a Wiener process at time step n, and $\frac{\partial b(X_n, t_n)}{\partial x}$ is the partial derivative of $b(X, t)$ with respect to X.

3 Runge Kutta

The Runge-Kutta method for the stochastic differential equation

$$dX(t) = a(X(t), t)\, dt + b(X(t), t)\, dW(t)$$

with initial condition $X(0) = X_0$ is given by the update rule

$$\begin{aligned}
k_1 &= a(X_n, t_n)\Delta t + b(X_n, t_n)\sqrt{\Delta t} Z_{n,1}, \\
k_2 &= a(X_n + \frac{k_1}{2}, t_n + \frac{\Delta t}{2})\Delta t + b(X_n + \frac{k_1}{2}, t_n + \frac{\Delta t}{2})\sqrt{\Delta t} Z_{n,2}, \\
k_3 &= a(X_n + \frac{k_2}{2}, t_n + \frac{\Delta t}{2})\Delta t + b(X_n + \frac{k_2}{2}, t_n + \frac{\Delta t}{2})\sqrt{\Delta t} Z_{n,3}, \\
k_4 &= a(X_n + k_3, t_n + \Delta t)\Delta t + b(X_n + k_3, t_n + \Delta t)\sqrt{\Delta t} Z_{n,4}, \\
X_{n+1} &= X_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).
\end{aligned}$$

where X_n is the approximation of $X(t_n)$ at time step n, $t_n = n\Delta t$, $\Delta t = \frac{T}{N}$ is the time step size, and $Z_{n,i}$ are independent standard normal random variables.

4 Heun

The Heun method for the stochastic differential equation

$$dX(t) = a(X(t), t)\, dt + b(X(t), t)\, dW(t)$$

with initial condition $X(0) = X_0$ is given by the update rule

$$\begin{aligned}
Y_{n+1} &= X_n + a(X_n, t_n)\Delta t + b(X_n, t_n)\sqrt{\Delta t} Z_n, \\
X_{n+1} &= X_n + \frac{1}{2}\left[a(X_n, t_n) + a(Y_{n+1}, t_{n+1})\right]\Delta t + \frac{1}{2}\left[b(X_n, t_n) + b(Y_{n+1}, t_{n+1})\right]\sqrt{\Delta t} Z_{n+1},
\end{aligned}$$

where X_n is the approximation of $X(t_n)$ at time step n, $t_n = n\Delta t$, $\Delta t = \frac{T}{N}$ is the time step size, Z_n is a standard normal random variable, and Y_{n+1} is an intermediate approximation.