Stochastic Processes and SDE's

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1 Stochastic Processes

Stochastic processes are a way of modeling phenomena that change over time in a random or unpredictable manner. For example, the weather, the stock market, the number of customers in a queue, the movement of particles, etc. are all examples of stochastic processes. A stochastic process can be described by a collection of random variables indexed by time $\{X_t\}_{t\geq 0}$ that represent the state of the system at different times. Alternatively, a stochastic process can be described as a probability distribution upon a space of paths. The distribution and dependence of these random variables determine the properties and behavior of the stochastic process. There are different types of stochastic processes, such as discrete or continuous, stationary (meaning that the parameters mean μ and variance σ^2 do not change over time) or non-stationary, Markov (also known as Markov chain, which describes a series of events where the probability of each event depends only on the state of the previous event) or non-Markov, etc.

So it is clear that a stochastic process involves some response variable, e.g. Y_t, that takes values varying randomly in some way over time t. An observed value or realization of the process is denoted as y_t and it represents the state of the stochastic process at time t. More in general the probability of the process being in some defined state at a certain instant t depends on the events, i.e. changes of the so called "space state".

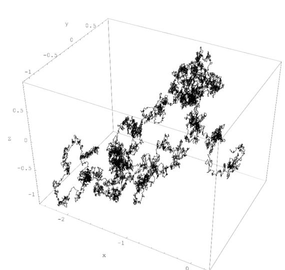

Figure 1: A single computer-simulated sample function or realization, among other terms, of a three-dimensional Wiener or Brownian motion.

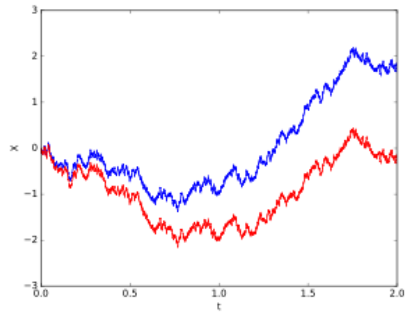

Figure 2: Wiener process, with drift in blue, without drift in red.

Let's consider some examples of stochastic processes:

1. **Random Walk**: A random walk is a stochastic process defined as a sum of i.i.d. random variables or random vectors in Euclidean space, i.e. they are processes that change in discrete time. A random walk starts at 0 and with equal probability moves of $+1$ or -1.

2. **Wiener Process**: Also known as Brownian motion, is considered the most important among stochastic processes. Its index set T and state space S are the non-negative numbers and real numbers, respectively, so it has both continuous index set and states space. If the mean of any increment is zero, then the resulting Wiener or Brownian motion process is said to have zero drift. If, instead, the mean of the increment for any two points in time is equal to the time difference multiplied by some constant μ , which is a real number, then the resulting stochastic process is said to have drift equal to μ. Below I show an example of Wiener process:

3. **Poisson Process**: A Poisson process is characterized as a counting process, a type of stochastic process depicting the random count of occurrences or events until a specific time. The count of events within the interval from zero to a designated time follows a Poisson random variable determined by that specific time and a given parameter λ. This process is defined over the natural numbers as its state space, and its index set consists of non-negative numbers.

2 Stochastic Differential Equations (SDEs)

Stochastic differential equations (SDEs) are a type of differential equations that involve random terms or noise. They are used to model various phenomena that have both deterministic and stochastic components, such as physical systems with thermal fluctuations, biological processes with random mutations, or financial markets with stock price fluctuations.

SDEs can be classified into different types depending on the form of the noise term, such as additive noise, multiplicative noise, or jump noise.
SDEs can also be written in different forms, such as the Itô form or the Stratonovich form, which have different interpretations and properties. Solving SDEs is not as straightforward as solving ordinary differential equations, because the noise term makes the solution a random process rather than a function. Therefore, one needs to use special methods, such as the Itô calculus or the Stratonovich calculus, to define and manipulate the stochastic integrals that appear in SDEs. There are also various numerical methods, such as the Euler-Maruyama method or the Milstein method, to approximate the solution of SDEs by discretizing the time and space variables.
SDEs have many applications in various fields, such as physics, biology, chemistry, engineering, economics, and finance. Some examples of SDEs are the Langevin equation, which describes the motion of a particle in a viscous fluid with random collisions, the Ornstein-Uhlenbeck process, which models the velocity of a particle undergoing Brownian motion, the Black-Scholes equation, which models the price of a financial option, and the Lotka-Volterra equation, which models the dynamics of a predator-prey system with random fluctuations.

In general, a SDE is presented in the form:

$$dX(t,\omega) = f(t, X(t,\omega))dt + g(t, X(t,\omega))dW(t,\omega)$$

Note that ω denotes that X is random variabls which posses the intial state $X(0,\omega) = X_0$, also we know that $f(t, X(t,\omega)) \in \Re$, $g(t, X(t,\omega)) \in \Re$ and $W(t,\omega) \in \Re$. The previous equation can also be written in the integral form as:

$$X(t,\omega) = X_0 + \int_0^t f(s, X(s,\omega))ds + \int_0^t g(s, X(s,\omega))dW(s,\omega)$$

The simplest SDE considerable is the one that governs the **Arithmetic Brownian Motion**:

$$dX_t = \mu dt + \sigma dB_t$$

Where B is a Wiener process.

We can also expand the previous SDE obtaining the **Geometric Brownian Motion (GBM)**, used to model stock prices in the Black-Scholes pricing model.

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

Another example of SDE is the one that governs the **Ornstein-Uhlenback** process:

$$dX_t = \theta(\mu - X_t)dt + \sigma dB_t$$

Where θ is the mean-reversion rate. This SDE is used in finance to compute the returns of a stock with Log-normal distribution.