The implicit function theorem

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The first part of the following theorem is the standard version of the implicit function theorem. The second part offers a partial converse. For the sake of notational ease, we denote the Jacobian of the function f with respect to the variable p at a point (p,s) by $D_p f(p,s)$ at times.

Theorem 1 (Implicit Function Theorem)**.** *Let $f\colon \mathbb{R}^d \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be continuously differentiable on a neighborhood of $(\overline{p},\overline{x})$ and such that $f(\overline{p},\overline{x})=0$. Let's define the partial Jacobian of f with respect to x at $(\overline{p},\overline{x})$, $D_x f(\overline{p},\overline{x})$. The following statements hold.*

1. *If $D_x f(\overline{p},\overline{x})$ is invertible, then there exist open sets $U \subseteq \mathbb{R}^d,\ V \subseteq \mathbb{R}^n$ with $\overline{p} \in U,\ \overline{x} \in V$ and a function $s\colon U \longrightarrow V$ such that:*

    (a) the function s is continuously differentiable on U;

    (b) for all $p \in U$ we have $f(p,s(p)) = 0$;

    (c) the Jacobian of s, $D_p s$, satisfies

    $$D_p s(p) = -D_x f(p,s(p))^{-1} D_p f(p,s(p)) \text{ for all } p \in U.$$

2. *Conversely, if there exist open sets $U \subseteq \mathbb{R}^d,\ V \subseteq \mathbb{R}^n$ with $\overline{p} \in U,\ \overline{x} \in V$ and a continuously differentiable $s\colon U \longrightarrow V$ such that $f(p,s(p)) = 0$ for all $p \in U$, then $D_x f(\overline{p},\overline{x})$ is invertible.*

For the proof, see Dontchev and Rockafellar (2009, Theorem 1.B.1 and Theorem 1.B.9).

The first part of the theorem can be extended in two directions. First, if f is k times continuously differentiable around $(\overline{p},\overline{x})$ then the implicit function s is k times continuously differentiable, by Proposition 1.B.5 in Dontchev and Rockafellar (2009). Second, $s(p)$ is the unique solution on V to the equation $f(p,x) = 0$, for all $p \in U$, by Theorem 9.4 in Loomis and Sternberg (2014).

The implicit function theorem with weaker assumptions

With weaker assumptions, we retain the local differentiability of the implicit function. (We state the uniqueness result explicitly, differently than for the standard result.)

Theorem 2 (Theorem 1 in Hurwicz and Richter (2003))**.** *Let $f\colon \mathbb{R}^d \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be differentiable at $(\overline{p}, \overline{x})$ with invertible partial Jacobian of f with respect to x at $(\overline{p}, \overline{x})$, $D_x f(\overline{p}, \overline{x})$, and satisfy $f(\overline{p}, \overline{x}) = 0$. The following statements hold.*

1. *There exist open sets $U \subseteq \mathbb{R}^d$, $V \subseteq \mathbb{R}^n$ with $\overline{p} \in U$, $\overline{x} \in V$, and a function $s\colon U \longrightarrow V$ such that:*
   - *(a) for all $p \in U$ we have $f(p, s(p)) = 0$;*
   - *(b) the function $t\colon U \longrightarrow V$ is differentiable at $\overline{x}$ if t satisfies $f(p, t(p)) = 0$ for all $p \in U$;*
   - *(c) the Jacobian of t, $D_p t$, satisfies*
   $$D_p t(\overline{p}) = -D_x f(\overline{p}, \overline{x})^{-1} D_p f(\overline{p}, \overline{x})$$
   if t satisfies $f(p, t(p)) = 0$ for all $p \in U$.
2. *If, in addition, the differentiability and invertibility hypotheses hold globally, then U, V can be chosen as above such that:*
   - *(a) for all $p \in U$ we have that $s(p)$ is the unique solution on V to the equation $f(p, x) = 0$;*
   - *(b) the Jacobian of s, $D_p s$, satisfies*
   $$D_p s(p) = -D_x f(p, s(p))^{-1} D_p f(p, s(p)) \text{ for all } p \in U.$$

With even weaker assumptions, we retain the continuity of the implicit function.

Theorem 3 (Theorem 2 in Hurwicz and Richter (2003))**.** *Let $f\colon \mathbb{R}^d \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be continuous, satisfy $f(\overline{p}, \overline{x}) = 0$, and let $f(\overline{p}, \cdot)$ be differentiable at $\overline{x}$ with invertible partial Jacobian of f with respect to x at $(\overline{p}, \overline{x})$, $D_x f(\overline{p}, \overline{x})$. The following statements hold.*

1. *There exist open sets $U \subseteq \mathbb{R}^d$, $V \subseteq \mathbb{R}^n$ with $\overline{p} \in U$, $\overline{x} \in V$, and a function $s\colon U \longrightarrow V$ such that:*

(a) *for all* $p \in U$ *we have* $f(p, s(p)) = 0$*;*

(b) *the function* $t \colon U \longrightarrow V$ *is continuous at* $\overline{x}$ *if* t *satisfies* $f(p, t(p)) = 0$ *for all* $p \in U$*.*

2. *If, in addition, the differentiability property of* f *holds globally (with respect to* x*,) then* U *and* V *can be chosen as above so that a unique such function* s *exists.*

References

Dontchev, Asen L. and R. Tyrrell Rockafellar (2009), *Implicit Functions and Solution Mappings A View from Variational Analysis*. Springer New York, NY.

Hurwicz, Leonid and Marcel K. Richter (2003), *Implicit functions and diffeomorphisms without C1*, 65–96. Springer Japan, Tokyo.

Loomis, Lynn Harold and Shlomo Sternberg (2014), *Advanced Calculus*, revised edition. World Scientific.