Coordination in complex environments

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January 13, 2025

[Latest Version]

This paper introduces a framework to study coordination in highly uncertain environments. Coordination is an important aspect of innovative contexts, where: the more innovative a course of action, the more uncertain its outcome. To study the interplay of coordination and informational complexity, this paper embeds a beauty-contest game into a complex environment. I uncover a new conformity phenomenon. The new effect may push towards exploration of unknown alternatives, or constitute a status quo bias, depending on the network structure of the players' interaction. Applications of the model include oligopoly pricing and centralization in organizations.

Keywords: Coordination, Conformity, Complexity, Status Quo.
J.E.L. Codes: D80, D23, D43.

Contents

*I am indebted to Mehmet Ekmekci, Laurent Mathevet, Utku Ünver, and Bumin Yenmez for their advice. I am grateful for helpful comments from Zeinab Aboutalebi, Georgy Artemov, Giacomo Calzolari, Krishna Dasaratha, Marco Errico, Marcelo Fernandez, Hideo Konishi, Maciej Kotowski, Alessandro Lavia, Lucas Maestri, Chiara Margaria, Andrea Mattozzi, Konuray Mutluer, Parth Parihar, Luigi Pollio, Sven Rady, Giacomo Rubbini, Tom Rutter, Junze Sun, Yu Fu Wong, and audiences at CSEF, Notre Dame, Bonn, Tor Vergata, EAYE (PSE), ASSET, AMASES, GRASS Workshop, 1st BC-BU Theory Conference, 2nd BC-Brown-BU Theory Workshop, Warwick PhD Conference, and Brown Theory/Experimental Lunch. I thank the EUI, where part of this work was conducted, for the hospitality. Contact: pietro.dallara@unina.it.

1 Introduction

Coordination poses challenges in highly uncertain environments. Consider retailers that share the same manufacturer and choose marketing campaigns, as in co-op advertising (Jørgensen and Zaccour, 2014). Innovative advertisement comes with uncertainty about the brand image of the manufacturer. Moreover, retailers need to coordinate their advertisements and succeed in distinct markets. Does uncertainty lead to a unified brand image, and do the campaigns align with the manufacturer's interests? Coordination is also an important aspect of technological innovation, for example, hospitals find it advantageous to adopt popular electronic medical records in the U.S. (Lin, 2023). Does uncertainty lead to innovation? This paper studies coordination problems in the face of "incremental" uncertainty, referred to as *complexity*, such that: the more innovative a decision, the more uncertain its outcome.

This paper introduces a model of coordination in a complex environment. In the model, every player wants the outcome of her action to be close to a target. The target of a player combines her fixed favorite outcome with the individual outcomes of the opponents, leading to a coordination-adaptation tradeoff. A given network of players determines how much a target weighs each individual outcome. Analogous coordination motives arise in oligopolies, organizations, and labor markets (Topkis, 1998; Marschak and Radner, 1972; Diamond, 1982).

The complexity is modeled as uncertainty about how actions translate into outcomes. This approach captures the idea that more innovative actions lead to more volatile outcomes. This type of uncertainty involves a status quo and a *covariance structure*. The status quo is an action leading to relatively low uncertainty. The covariance structure describes the likelihood that two actions yield similar outcomes. For example, complexity is relevant when deciding about the adoption of novel pricing strategies and how boldly to innovate in new technologies. Players simultaneously choose policies and there is an *outcome function* determining the outcome of every policy. Players know that the outcome function is the realized path of a Brownian motion. The initial point of the Brownian motion represents the status quo: a known outcome corresponds to the initial policy. Instead, different policies than the status quo lead to outcomes known only up to a noise. The more an outcome differs in expectation from the status-quo outcome, the higher its variance; this approach to modeling complex environments is introduced by Callander (2011a).

I show that the interplay of coordination and complexity leads to a novel conformity phenomenon. In particular, expected outcomes are closer across players than in an environment without complexity, in all equilibria when the network is complete. This conformity occurs in addition to the status-quo bias identified by Callander and the conformity merely due to coordination motives. To separate the new conformity from previously studied phenomena, I decompose equilibrium expected outcomes in terms of three elements: the equilibrium outcomes in a non-complex environment, the status-quo bias absent strategic interaction, and a new conformity element due to the interplay of complexity and coordination.

The new element in the equilibrium characterization arises from an endogenous leader-follower relationship among players introduced by the covariance structure. Consider the two ways in which the policy of a player influences the incentives of her opponents. First, policies enter into the expected targets of players. Second, the policy of a player determines the correlation between her outcome and her opponents' outcomes. The *follower* in a pair of players is the one with the closest policy to the status quo. Given a pair of players with different policies, the only player whose policy directly affects the covariance is the follower, not the leader.[1] As a result, the follower has an extra incentive to explore by choosing a policy in the direction of the leader. The new incentive of the follower is the source of conformity. The leader-follower relationship induces an asymmetry among players that interacts with the network of connections

Conformity has a rich interaction with the network of players. A player may exert substantial influence on a follower player. This influence can be so strong that it steers the follower away from a third player. In this case, "counter-formity" emerges, leading to expected outcomes that are more distant than in the no-complexity case. In general, the leader-follower relationship is determined in equilibrium. The equilibrium decomposition serves to verify that a certain leader-follower structure can be sustained.

To illustrate the conformity phenomenon, I study applications of the model. In oligopolistic competition, coordination motives arise from strategic complementarities whenever the incentives to raise prices increase with the prices of competitors. Moreover, a pricing algorithm may rely on data not available when algorithmic pricing is adopted (Brown and MacKay, 2023). Hence, complexity arises when innovative pricing rules are associated with high uncertainty. In this case, conformity takes the form of concentrated expected prices across firms. The presence of conformity suggests a downward bias when firm heterogeneity is estimated from price data and the analyst does not control for complexity.[2] The equilibrium decomposition provides a tool for isolating the new conformity effect.

I show that conformity increases in the complexity of the environment, whenever two players exist who are the leader and the follower for each of their opponents. The measure of complexity is the additional uncertainty implied by a change in expected outcome away from the status quo.[3] The intuition for the comparative statics follows

[1] This property is due to independent increments, a reasonable assumption in innovative contexts owing to a maximum-uncertainty principle (Jovanovic and Rob, 1990). Other Gaussian processes have covariance structures that lead to leader–follower-like relationships, e.g., the Ornstein-Uhlenbeck covariance between two "outcomes" is increasing only in one of the two "policies".

[2] Since Bresnahan (1987), a common empirical exercise is to infer the cost parameters from data.

[3] Letting μ and ω be the drift and variance parameters of the Brownian motion, respectively, the measure of complexity is $\omega/(2|\mu|)$.

from the "first-order" effect of an increase in complexity. In particular, matching the outcome of a leader becomes more "cost-effective" for a follower, relative to targeting a favorite outcome. This comparative statics is consistent with findings in social psychology. Since Asch (1951), psychologists observe that conformity "is far greater on difficult items than on easy ones." The "difficulty" is typically obtained by asking experimental subjects about their "certainty of judgement" (Krech et al., 1962).

New coordination problems arise in complex environments. The source of equilibrium multiplicity is the presence of endogenous "kinks" in payoffs. Intuitively, there is a premium to choosing the same policy as another player because two individual outcomes are the same if the policies are the same. Hence, coordination problems are linked to the leader-follower relationship: by choosing the same policy as an opponent, a player nullifies the asymmetry. The location of kinks is determined in equilibrium: a player's payoff has a kink at every policy of an opponent. The game admits a "potential" with a unique maximizer, which acts as an equilibrium selection (Monderer and Shapley, 1996).[4] I leverage the characterization of the potential-maximizer equilibrium to study welfare, select among multiple equilibria, and for comparison with the no-complexity case (without complexity, the unique equilibrium maximizes the potential.) In a two-type network, a decrease in inter-group heterogeneity below a cutoff triggers coordination problems: every player faces an interval of policies sustainable in equilibrium. This result is important for policy interventions that change favorite outcomes of players (Galeotti et al., 2020): certain interventions may bring about coordination problems. Moreover, for sufficiently high complexity, extreme conformity prevails: all players choose the same policy. The equilibrium selection allows to retrieve the heterogeneity between groups given such homogeneous behavior. In particular, extreme conformity is observationally equivalent to the optimal choice of a representative player. The equilibrium selection pins down the weighted average of favorite outcomes that constitutes the "representative" favorite outcome.

Complexity has implications for the management of organizations with decentralized authority, encompassing practices such as co-op advertising and multi-branding. Division managers face a trade-off between coordinating other managers and adapting to idiosyncratic needs. Moreover, communication frictions create uncertainty in

[4]Uniqueness of the potential-maximizer equilibrium obtains jointly with the multiplicity of equilibria because the potential is not smooth. Specific nondifferentiable potentials are studied as counterexamples to the results for smooth potentials (Radner, 1962; Neyman, 1997).

the implementation of managerial instructions. I show that an organization with decentralized authority can implement profit maximization in sufficiently complex environments. Hence, complexity provides a rationale for decentralized organizations.[5]

I separately identify the role played by variance and covariance of the environment in a general model in which players have "correlated" outcome functions. In particular, the interplay between coordination and complexity takes the form of a linear combination of two effects in the decomposition of equilibrium expected outcomes. First, a pure status-quo bias, which arises with uncorrelated outcomes across players. This effect pushes every player towards the status quo, and is magnified by the network of players. Second, a pure experimentation motive that arises only with correlated outcomes. This effect pulls players away from the status quo and it is introduced by the correlation component.

Related Literature I borrow the model of complexity from the literature initiated with Callander (2011a), who studies a dynamic exploration-exploitation tradeoff using a Brownian motion. The main role of the covariance structure in the dynamic interaction is to discipline learning over time. Cetemen et al. (2023) study a similar complex environment in which discoveries are correlated over time and members of a team contribute resources for exploration. I contribute to the complexity literature by studying coordination motives and network games. I also show that the status-quo bias survives the introduction of coordination motives and incomplete information about a heterogeneous status quo. Other work considers strategic interactions and Gaussian processes. In particular, the covariance structure has a direct role in Bardhi and Bobkova (2023) and Bardhi (2023), in which a principal incentivizes agents to provide information about the underlying outcome function. These authors study covariances that are characterized by the "nearest-attribute" property, including the Brownian covariance.[6] I focus on the Brownian covariance because of two characteristics. First, the Brownian covariance preserves the strategic complementarities of the coordination

[5]This result complements the literature that studies informational asymmetries within organizations, see, e.g., Alonso et al. (2008); Rantakari (2008); Dessein and Santos (2006); the present model is biased towards favoring centralization because it abstracts away from division managers' private information.

[6]Other strategic settings include: the dynamic models in Callander and Matouschek (2019), Callander and Hummel (2014), and Garfagnini and Strulovici (2016), which analyze intertemporal interactions; the communication models in Callander (2008), Callander et al. (2021), and Aybas and Callander (2023), in which a sender informs a receiver about the underlying outcome function; and the electoral competition in Callander (2011b). Gaussian processes are used in a similar way as in the complexity literature to study innovation, price rigidity, and in psychology (Jovanovic and Rob, 1990; Ilut and Valchev, 2022; Ilut et al., 2020; Anderson, 1960).

game (Lemma 1); second, this covariance contains the leader-follower asymmetry that explains conformity in a simple way, described in Section 3. Other covariances are "asymmetric" but not supermodular (e.g., squared-exponential covariance), and vice versa (squared-polynomial). Garfagnini (2018) studies the rich welfare properties of the Brownian-motion structure for a network game with a "trivial" covariance structure: the decision-outcome mappings are drawn from player-specific independent Brownian motions. (Section 5 describes the generalization of the model with imperfectly correlated outcome functions.)

The literature on coordination games with quadratic ex-post payoffs includes models of oligopolistic competition, peer effects, and network games (Choné and Linnemer, 2020; Jackson and Zenou, 2015). Complexity introduces coordination problems under the standard upper bound on the strength of coordination motives. Moreover, complexity makes best responses nonlinear. The nonlinearity is due to the kinks in expected payoffs and implies that equilibrium strategies do not have constant slope in the heterogenous–status-quo game, studied in the Appendix. Instead, typical quadratic-payoff beauty contests with incomplete information admit a unique equilibrium, and only linear strategies consitute an equilibrium (Radner, 1962; Morris and Shin, 2002; Angeletos and Pavan, 2007). The study of status-quo heterogeneity does not rely on results valid for linear best-reply games, and heterogeneity is modeled as an interim Bayesian game. Not all the results of Van Zandt and Vives (2007) and Van Zandt (2010) can be applied off the shelf, but the additional structure of preferences leads to measurability of the greatest-best-reply mapping.

Outline Section 2 describes the model and Section 3 studies conformity. Section 4 contains applications. Section 5 discusses generalizations and concludes. The Appendix contains the general model and proofs.

2 Model

2.1 Players and payoffs

Every player $i \in N := \{1 \dots n\}$ has preferences over profiles of outcomes. An outcome profile is a list of individual outcomes, $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbf{R}^n$. The payoff to player i

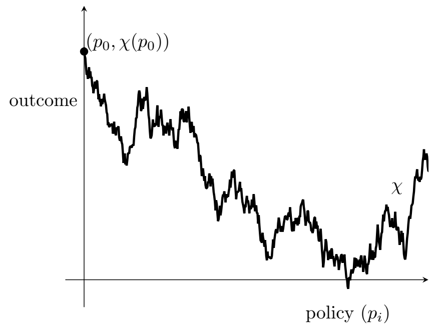


Figure 1: The outcome function χ maps individual policies to individual outcomes and is given by the realized path of a Brownian motion.

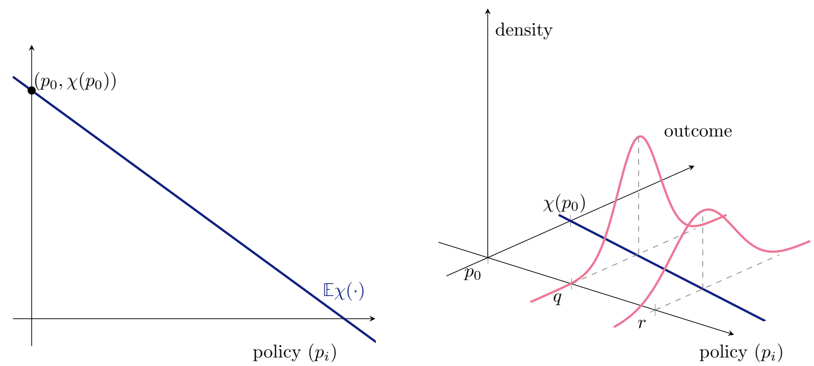
from the outcome profile $\boldsymbol{x}$ is

$$\pi_i(\boldsymbol{x}) = -\left(x_i - (1-\alpha)\delta_i - \alpha \sum_{j \neq i} \gamma^{ij} x_j\right)^2,$$

in which $\alpha \in [0,1)$ measures the strength of coordination motives, δ_i is the favorite outcome of player i, and $\gamma^{ij} \geq 0$ weighs the connection between player j and player i. Connections are symmetric, $\gamma^{ij} = \gamma^{ji}$ for all players $i,\ j \in N$. Payoffs reflect a desire for coordination because $\alpha\gamma^{ij}$ is nonnegative. Similar payoffs are used to model organizations, industries, and peer effects in education (Jackson and Zenou, 2015).

Every player i chooses a policy $p_i \in P := [\underline{p}, \overline{p}]$ simultaneously, for $\underline{p}, \overline{p} \in \mathbf{R}$ with $\underline{p} < \overline{p}$. The outcome corresponding to policy $p \in P$ is given by the *outcome function* $\chi\colon \mathbf{R} \to \mathbf{R}$, evaluated at p. The outcome function is the realized path of a Brownian motion with drift $\mu < 0$, variance parameter $\omega > 0$, and starting point $(p_0, \chi(p_0))$, as in Figure 1 (Karatzas and Shreve, 1998, Definition 1.1 and 5.19). Players know the status-quo policy $p_0 \in (\underline{p}, \overline{p})$, the corresponding status-quo outcome $\chi(p_0) \in \mathbf{R}$, μ, and ω. The Brownian motion disciplines the beliefs of players. Player i believes that $\chi(p)$ and $\chi(q)$ are jointly Gaussian random variables, for all pairs of policies $p, q \in P \setminus \{p_0\}$. This structure of uncertainty captures a complex environment because a player is more certain about the outcome of a policy the closer the policy is to the status quo (Figure 2). This way of modeling a complex environment is introduced by Callander (2011a) and the measure of the complexity is $k := \omega/(2|\mu|)$.

Player i's payoff from the outcomes corresponding to the policy profile $\boldsymbol{p} \in P^n$ is given by $\pi_i(\chi(p_1), \ldots, \chi(p_n))$, which we denote by $\pi_i(\boldsymbol{\chi}(\boldsymbol{p}))$. Player i's expected payoff from the policy profile $\boldsymbol{p}$ is denoted by $\mathbb{E}\pi_i(\boldsymbol{\chi}(\boldsymbol{p}))$.



(a) Expected value of outcomes.

(b) Expected value and variance of outcomes.

Figure 2: Player i believes that outcomes are given by normal random variables with expectations given by the drift line of the Brownian motion (panel (a)). The closer the policy r is to the status-quo policy, the lower the variance of outcome $\chi(r)$, as in panel (b).

2.2 Strategies and equilibrium

The main focus of the paper is the game $G(x_0)$, in which the strategy space of player i is the policy space P and player i's utility is her expected payoff given the status-quo outcome $x_0 \in \mathbf{R}$. In particular, I study the strategic-form game $\langle N, \{P, \mathbb{E}\pi_i(\boldsymbol{\chi}(\cdot))\}_{i\in N}\rangle$ with $\chi(p_0) = x_0$. The policy profile $\boldsymbol{p}$ is an *equilibrium* if:

$$\mathbb{E}\pi_i(\boldsymbol{\chi}(\boldsymbol{p})) \geq \mathbb{E}\pi_i(\ldots, \chi(p_{i-1}), \chi(q_i), \chi(p_{i+1}), \ldots), \text{ for all } q_i \in P \text{ and } i \in N.$$

In the specific case of no complexity, which is the limit game when $\omega = 0$, the policy-outcome mapping is $\psi\colon p_i \mapsto \chi(p_0) + \mu(p_i - p_0)$ and the profile of outcomes corresponding to the policy profile $\boldsymbol{p}$ is $\boldsymbol{\psi}(\boldsymbol{p})$.[7] An *equilibrium without complexity* is a a Nash equilibrium of the strategic-form game $\langle N, \{P, \pi_i(\boldsymbol{\psi}(\cdot))\}_{i\in N}\rangle$.

2.3 Discussion and interpretation

Network of players The matrix of connections, $\boldsymbol{\Gamma} := [\gamma^{ij} : i, j \in N]$, is the adjacency matrix of a network of players, letting $\gamma^{ii} = 0$ for all $i \in N$. I use $\boldsymbol{\delta}$ for the column vector of favorite outcomes, $\boldsymbol{I}$ for the identity matrix and $\boldsymbol{B}(\boldsymbol{M}) := (\boldsymbol{I} - \boldsymbol{M})^{-1}$

[7] In the equilibrium definition, "$\ldots, \chi(p_{i-1}), \chi(q_i), \chi(p_{i+1}), \ldots$" denotes the outcome profile corresponding to $(\chi(q_i), (\chi(p_j))_{j\in N\setminus\{i\}})$. Due to strict concavity of $p_i \mapsto \mathbb{E}\pi_i(\boldsymbol{\chi}(\boldsymbol{p}))$, player i's best response is unique (Appendix, Lemma B.12) and focusing on pure strategies is without loss.

for the Leontief inverse of the n-by-n matrix $\boldsymbol{M}$ if $\boldsymbol{I}-\boldsymbol{M}$ is nonsingular. The *centrality of player* i is the i-th entry of the column vector $\boldsymbol{\beta}$ given by

$$\boldsymbol{\beta} = (1-\alpha)\boldsymbol{B}(\alpha\boldsymbol{\Gamma})\boldsymbol{\delta}.$$

The graph of the network $\langle N, \boldsymbol{\Gamma}\rangle$ provides an interpretation of centrality.[8] The ij-the entry of the Leontief inverse of $\alpha\boldsymbol{\Gamma}$ counts the walks of every length from node i to node j and discounts walks of length ℓ by α^ℓ. So, the centrality of player i counts all "α-discounted" walks starting from i and weighs every walk to player j by $(1-\alpha)\delta_j$.

Complexity The following formulas help to analyze the implications of the Brownian-motion structure of uncertainty (Appendix A.3). The parameters of the distribution of $(\chi(p), \chi(q))$ are denoted by $\mathbb{E}\chi(p)$, $\mathbb{V}\chi(p)$, and $\mathbb{C}(\chi(p), \chi(q))$, with standard notation. For all policies $p, q \in P$, we have

$$\begin{aligned}
\mathbb{E}\chi(p) &= \chi(p_0) + \mu(p - p_0), \\
\mathbb{V}\chi(p) &= |p - p_0|\omega, \\
\mathbb{C}(\chi(p), \chi(q)) &= \begin{cases} \min\{\mathbb{V}\chi(p), \mathbb{V}\chi(q)\} & \text{if } p, q \geq p_0 \text{ or } p_0 \leq p, q, \\ 0 & \text{if } p > p_0 > q \text{ or } q > p_0 > p. \end{cases}
\end{aligned}$$

Large changes in individual expected outcomes are associated with high variance of the corresponding outcomes. The measure of complexity, k, is the additional variance implied by a marginal movement of the expected outcome away from the status quo, scaled by 1/2. The covariance expression is due to the independent-increments property of the Brownian motion, and is determined by the closest policy to the status quo.

No-coordination benchmark If $\alpha = 0$, there isn't any strategic interaction. The model reduces to the static version of Callander (2011a). In that case, player i's optimal policy p_i^* trades off closeness of the expected outcome to δ_i with the variance induced by the distance of p_i^* from the status quo. Hence, player i does not optimally choose the policy p_i° such that $\mathbb{E}\chi(p_i^\circ) = \delta_i$ — except possibly in the knife-edge case in which $\chi(p_0) = \delta_i$. The optimal policy reflects a *status-quo bias*, because p_i^* is closer to the status quo than the policy p_i°. To find the optimal policy, player i does not

[8]The matrix $\boldsymbol{I} - \alpha\boldsymbol{\Gamma}$ is positive definite due to Assumption 1 (next section) so centralities are well-defined and $\boldsymbol{B}(\alpha\boldsymbol{\Gamma}) = \sum_{\ell=0}^{\infty} \alpha^\ell \boldsymbol{\Gamma}^\ell$. Other definitions of Katz-Bonacich centrality do not adjust by $(1-\alpha)$ or do not include the case of $\delta_i \neq 1,\ i \in N$.

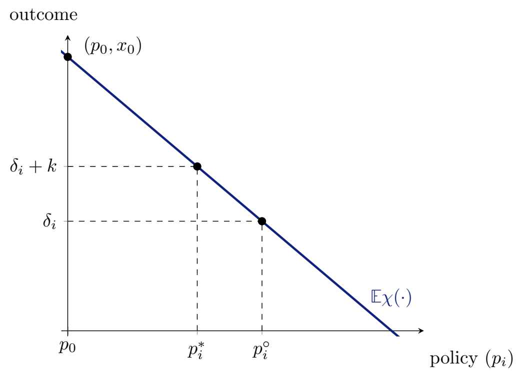


Figure 3: If $\alpha = 0$, player i has a unique optimal policy p_i^*. The policy p_i^* trades off closeness of the expected outcome to δ_i with the variance induced by the distance from the status-quo policy p_0. (For this figure: $\delta_i = 1, \mu = -1/2, \omega = 1/2, \alpha = 0, p_0 = 0 = \underline{p}, \chi(0) = 2.5$, and $\overline{p} \geq 3$.)

consider the correlation between outcomes of distinct policies because only her own outcome is payoff-relevant. In particular, player i's expected payoff is decomposed as follows,

$$\begin{aligned} \mathbb{E}\pi_i(\boldsymbol{\chi}(\boldsymbol{p})) &= -\mathbb{E}(\chi(p_i) - \delta_i)^2, \\ &= -(\mathbb{E}\chi(p_i) - \delta_i)^2 - \mathbb{V}\chi(p_i). \end{aligned}$$

The first equality follows from the definition of π_i and the second from mean-variance decomposition. The variance term is a continuous and piecewise-linear function of player i's policy with a kink at the status-quo policy.[9] The presence of this kink leads to a second form of status-quo bias: for an interval of status-quo outcomes, the optimal policy is the status-quo policy (as in: Callander 2011a; Ilut et al. 2020.)

Coordination and complexity Players take into account the correlation between outcomes of different policies, because the outcomes of opponents are payoff-relevant. A given distance in expected outcome from the status quo is "less expensive", in terms of uncertainty, for player i if is comes with a high covariance between i's own outcome and the outcomes of other players. The interplay of strategic interaction ($\alpha > 0$) and complexity of the environment ($k > 0$) gives rise to *endogenous* kinks in expected payoffs. Player i's expected payoff in the two-player case with $\delta_i = 0$ and

[9] I adopt the convention of calling a function linear if it is affine.

$\gamma^{ij} = 1,\ j \neq i$, is as follows,

$$\begin{aligned}\mathbb{E}\pi_i(\boldsymbol{\chi}(\boldsymbol{p})) &= -\mathbb{E}(\chi(p_i) - \alpha\chi(p_j))^2,\\ &= -(\mathbb{E}\chi(p_i) - \alpha\mathbb{E}\chi(p_j))^2 - \mathbb{V}\chi(p_i) + 2\alpha\mathbb{C}(\chi(p_i), \chi(p_j)) - \alpha^2\,\mathbb{V}\,\chi(p_j).\end{aligned}$$

The mean-variance decomposition is "kinked" due to the presence of covariance terms if $k > 0$ and $\alpha > 0$. The location of the kinks is endogenous: the expected payoff of player i has a kink at the policy of player j. A second type of kink is located at the status-quo policy and it leads to the status-quo bias.

No-complexity benchmark The special case of the model without complexity is essentially equivalent to the linear–best-response game $S := \langle N, \{\mathbf{R}, \pi_i\}_{i\in N}\rangle$, studied in the literature on games played over networks (Ballester et al., 2006). The strategy profile $(\beta_1, \dots, \beta_n)$ is the unique equilibrium of S (Corollary 2). The result holds because the best-reply mapping in S is affine and contractive. Furthermore, Neyman (1997) establishes uniqueness of the correlated equilibrium. With complexity, best responses are not as smooth because of endogenous kinks, and they admit multiple equilibria under the same upper bound on coordination motives.

Notation The set of strategy profiles, P^n, and the set of profiles of opponents' strategies, P^{n-1}, are endowed with the product order. All partial orders are denoted by $\leq$ and $<$ denotes the asymmetric part of $\leq$. For posets S and T, the function $g\colon S \times T \to \mathbf{R}$ exhibits (strictly) increasing differences if $t \mapsto g(s', t) - g(s, t)$ is (increasing) nondecreasing for all $s',\ s \in S$ with $s < s'$. The set of players other than i is denoted by $-i$. Bold notation is used for column vectors and matrices. The Hadamard (element-by-element) product of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is denoted by $\boldsymbol{A} \odot \boldsymbol{B}$.

2.4 Analysis

The following requirement ensures existence and uniqueness of an equilibrium absent complexity and is common in the literature on games played over networks.

Assumption 1. *Let $\lambda(\boldsymbol{\Gamma})$ denote the largest eigenvalue of $\boldsymbol{\Gamma}$, then:*

$$\alpha\lambda(\boldsymbol{\Gamma}) < 1.$$

The requirement bounds the magnitude of overall coordination motives and isolates coordination problems induced by the introduction of complexity.[10] Assumption 1 is maintained in what follows.

The game $G(x_0)$ inherits the strategic-complementarity feature of the no-complexity benchmark S.

Lemma 1. *For every player i, the expected payoff $\mathbb{E}\pi_i(\boldsymbol{\chi}(\boldsymbol{p}))$ exhibits increasing differences in $(p_i, \boldsymbol{p}_{-i})$.*

Intuitively, the returns to choosing higher policies are increasing in the opponents' policies. The key observation in the proof leverages the covariance structure given by the Brownian motion (Section 2.3). When opponents increase their policies, a higher own policy implies: (i) a closer expected outcome to the opponents' expected outcomes, (ii) a change in the volatility of own outcome, and (iii) a change in the covariance between the outcomes of players. Under the no-complexity benchmark, only consequence (i) holds, so the change in expected outcome is consistent with the increasing differences. The willingness to incur volatility stems from variance and covariance elements, and varies with opponent's policies. The covariance between two outcomes is supermodular in the associated policies, as observed in Section 2.3. The reason is that the player with the least-volatile outcome is "controlling" the covariance directly, in every pair of players. Thus, if player i is a *follower* of player j — player i incurs less volatility than player j —, then she has an incentive to adjust her policy towards player j's policy. Moreover, the incentives of the *leader* player — player i — are not affected by player j's policy, except via the target. The different incentives of leaders and followers, in each pair of players, are the key for supermodularity and are studies in detail in Section 3 (Figure 5).

Due to strategic complementarities, the set of equilibria is nonempty and admits an order structure, via known arguments based on Tarski's fixed-point theorem (Milgrom and Roberts, 1990; Vives, 1990).

Proposition 1. *There exist a greatest and least equilibrium.*

The following result characterizes equilibria by decomposing equilibrium expected outcomes.

[10]The square matrix $\boldsymbol{\Gamma}$ is nonnegative, so $\lambda(\boldsymbol{\Gamma})$ is equal to the spectral radius of $\boldsymbol{\Gamma}$ (Horn and Johnson, 2013, Theorem 8.3.1). To see that the assumption bounds the magnitude of coordination motives, note that $\lambda(\boldsymbol{\Gamma})$ is nonnegative and nondecreasing in γ^{ij}, so the upper bound on α is more stringent if players are more interconnected.

Proposition 2. *The profile of policies $\boldsymbol{p} \in (\underline{p}, \overline{p})^n$ is an equilibrium if, and only if:*

$$\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) = \boldsymbol{\beta} + \boldsymbol{b}k + \alpha(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}(\boldsymbol{\Gamma} \odot \boldsymbol{A})\mathbf{1}k,$$

for a matrix $\boldsymbol{A} = [a_{ij} : i,\ j \in N]$ and a vector $\boldsymbol{b}$ such that $a_{ij},\ b_i \in [-1, 1]$ and

$$b_i = \begin{cases} 1 & \text{if } p_i > p_0, \\ -1 & \text{if } p_i < p_0, \end{cases} \text{ and } a_{ij} = \begin{cases} 1 & \text{if } p_i > p_j, \\ -1 & \text{if } p_i < p_j. \end{cases}$$

The decomposition is stated for equilibria in which all players choose interior policies, the complete result is in Appendix C. The decomposition provides a tool to verify whether a policy profile is an equilibrium using the induced expected outcomes. If the expected outcomes satisfy the decomposition for a matrix $\boldsymbol{A}$, which is constrained by the induced location of players in the policy interval, then the policy profile is an equilibrium.

The three summands that constitute equilibrium expected outcomes are labeled in order to study the interplay between coordination and complexity:

$$\mathbb{E}\boldsymbol{\chi}(p) = \underbrace{\boldsymbol{\beta}}_{\substack{\text{equilibrium outcomes} \\ \text{without complexity}}} + \underbrace{\boldsymbol{b}k}_{\substack{\text{status-quo} \\ \text{bias}}} + \underbrace{\alpha(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}(\boldsymbol{\Gamma} \odot \boldsymbol{A})\mathbf{1}k}_{\substack{\text{additional conformity} \\ \text{effect}}}.$$

If $k = 0$, the decomposition characterizes the unique equilibrium without complexity, which is determined by the centrality vector. If $\alpha = 0$, the decomposition characterizes the unique equilibrium without coordination motives, which is determined by the vector of favorite outcomes and a status-quo bias term. The interplay of coordination motives and complexity generates an additional term: the *endogenous* matrix $\boldsymbol{A}$, which keeps track of leader-follower asymmetries in every pair of players.

The decomposition leaves room for multiple equilibria and coordination problems: possibly for multiple policy profiles there exists a matrix $\boldsymbol{A}$ satisfying the decomposition. Figure 4 shows that a two-player game admits an interval of policies that can be sustained in equilibrium. In order to attribute the multiplicity to the interplay between coordination motives and complexity, the following results focus on the particular cases of no complexity and no coordination.

Corollary 1. *Let $\alpha = 0$. There exists a unique equilibrium of $G(x_0)$. Moreover, the*

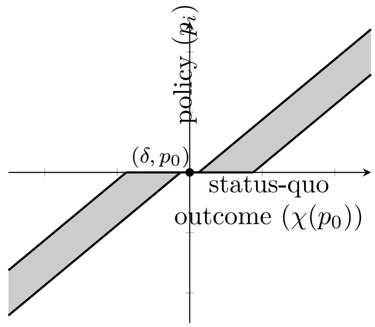


Figure 4: If $n = 2$ and $\delta_1 = \delta_2 =: \delta$, then every equilibrium $\boldsymbol{p}$ is symmetric, i.e., $p_1 = p_2$. The grey area — including black lines and the point (δ, p_0) — illustrates the equilibrium set, represented by player i's policy, for every status-quo outcome. (For this figure: $n = 2, \delta_1 = \delta_2 = 0, \mu = -1/2, \omega = 1/2, \alpha = 1/3$.)

profile of policies $\boldsymbol{p} \in (\underline{p}, \overline{p})^n$ is an equilibrium of $G(x_0)$ if, and only if:

$$\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) = \boldsymbol{\beta} + \boldsymbol{b}k,$$

for a vector $\boldsymbol{b}$ such that $b_i \in [-1, 1]$ and

$$b_i = \begin{cases} 1 & \text{if } p_i > p_0, \\ -1 & \text{if } p_i < p_0. \end{cases}$$

Corollary 2. *There exists a unique equilibrium of the game $G(x_0)$ without complexity. Moreover, the profile of policies $\boldsymbol{p} \in (\underline{p}, \overline{p})^n$ is an equilibrium of $G(x_0)$ without complexity if, and only if:*

$$\boldsymbol{\psi}(\boldsymbol{p}) = \boldsymbol{\beta}.$$

Remark 1. *This remark considers identical players. Coordination problems increase in α: the equilibrium set grows in the inclusion sense as α increases (Appendix, Corollary 3, proofs for this remark as in Appendix, Section D.) Let $\gamma^{ij} = \gamma$ and $\delta_i = 0$ for all players $i,\ j \in N$ with $i \neq j$. In every equilibrium $\boldsymbol{p}$, $p_i = p_j$ for all players $i, j \in N$. Moreover, let $\underline{q}(a)$ and $\overline{q}(a)$ be, respectively, the policies in the least and greatest equilibrium when the degree of coordination motives is $\alpha = a$. If $\alpha_1 < \alpha_2$ and $\underline{q}(\alpha_1), \overline{q}(\alpha_1), \underline{q}(\alpha_2), \overline{q}(\alpha_2) \in (p_0, \overline{p})$, then $\underline{q}(\alpha_2) < \underline{q}(\alpha_1)$ and $\overline{q}(\alpha_2) > \overline{q}(\alpha_1)$. For intuition, suppose the policy space is $[p_0, \overline{p}]$. Then, the least equilibrium decreases in α and the greatest equilibrium increases in α for a complete network. As shown in* *the Appendix, the equilibrium set for a complete network with* $\boldsymbol{\delta} = \mathbf{0}$ *gets larger in set inclusion as* α *increases. As a result, the greatest equilibrium (i.e., the equilibrium with the least volatile outcomes) gets closer to the status quo, and the least equilibrium (i.e., the equilibrium with the most uncertain outcomes) involves more exploration, as* α *increases.*

3 Conformity

This section studies a conformity phenomenon arising because of the interplay between coordination and complexity. Mathematically, the source of conformity is in the decomposition of equilibrium expected outcomes (Proposition 2), in which the endogenous matrix $\boldsymbol{A}$ keeps track of leader-follower relationships.

3.1 Conformity with two players

To develop the intuition, I start with a two-player example. Furthermore, assume that the favorite outcomes are sufficiently distinct, $\delta_1 - \delta_2 > 2k\alpha/(1-\alpha)$. This ensures that the centralities are strictly ordered, $\beta_1 > \beta_2$, there exists a unique equilibrium, $\boldsymbol{p}^\star$, and player 1 is the follower.[11] Because each policy implies a unique expected outcome, I treat expected outcomes as the choice variables.

The best response of player i in the game without complexity is the expected outcome

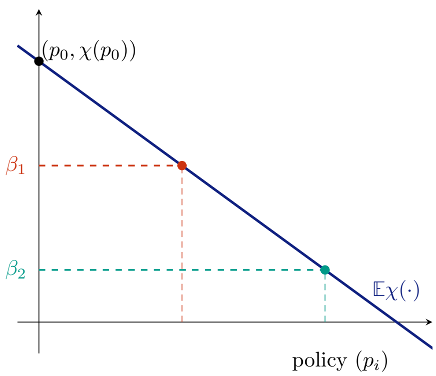
$$(1-\alpha)\delta_i + \alpha \mathbb{E}\chi(p_j), \tag{1}$$

which is a function of the expected outcome of player j. There exists a unique pair of expected outcomes that induces an equilibrium: (β_1, β_2) (Corollary 2 and Panel (a) in Figure 5.)[12] The distance between equilibrium expected outcomes is given by centralities: $\beta_1 - \beta_2$.

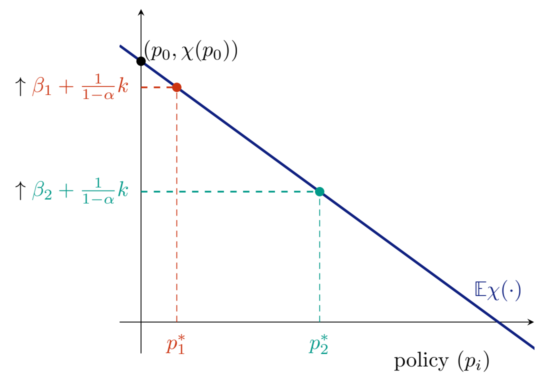
Complexity introduces two elements to the best-response analysis: a status-quo bias and a leader-follower asymmetry, reflecting variance and covariance features of the environment. First, consider a model with noisy and *independent* outcomes (illustrated in panel (b) of Figure 5, see also Section 5.) In this case, the best response

[11] In particular, we have $\overline{p} > p_2^\star > p_1^\star > p_0$ for sufficiently large $\chi(p_0)$ and $\overline{p}$. The remaining cases are considered in Appendix, Section E.

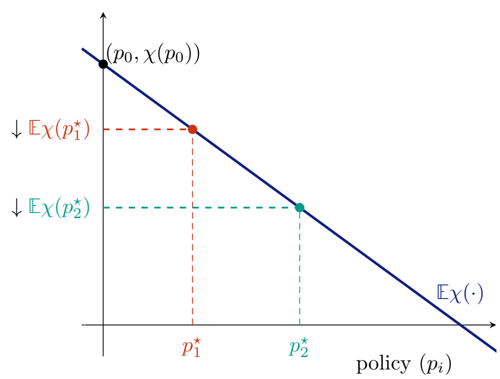
[12] To make the discussion simpler, best responses are restricted on $\big(\mathbb{E}\chi(\overline{p}), \chi(p_0)\big)$.



(a) The equilibrium in the game without complexity. The expected outcomes are given by the centrality of players, (β_1, β_2).



(b) Noisy and independent outcomes. The equilibrium expected outcomes are given by centrality of players and the adjusted status-quo bias, $(\beta_1 + mk, \beta_2 + mk)$. The arrows indicate the equilibrium status-quo bias: expected outcomes are higher than in the game without complexity (panel (a)).



(c) Equilibrium in $G(x_0)$. The expected outcomes are given by the decomposition in Proposition 2, which includes the leader-follower asymmetry, $(\beta_1 + k - k\frac{\alpha}{1+\alpha}, \beta_2 + k + k\frac{\alpha}{1+\alpha})$. The arrows indicate the extra exploration induced by the covariance structure: expected outcomes are lower than in the game without correlation (panel (b)).

Figure 5: Panel (a) illustrates the equilibrium in the game without complexity. Panel (b) illustrates the equilibrium when outcomes are noisy but independent across policies, given $\mathbb{V}\chi(p) = 0.5p$ and $\mathbb{C}(\chi(p), \chi(q)) = 0$, for $p, q > p_0$. Panel (c) illustrates the equilibrium in the game $G(x_0)$ when $\omega = 1/2$. (For the figures: $\delta_1 = 2, \delta_2 = 0, \mu = -1/2, \omega = 1/2, \alpha = 1/3, p_0 = 0 = \underline{p}, \chi(0) = 2.5, \overline{p} > 2.75$.)

of player i is the expected outcome

$$(1-\alpha)\delta_i + \alpha\mathbb{E}\chi(p_j) + k. \tag{2}$$

The best response shifts upwards with respect to the case of no complexity, i.e., expression (1), by the same amount as in the single-player game (Callander, 2011a). An incentive to stay close to the status quo emerges and there is not any leader-follower asymmetry. There exists a unique pair of equilibrium expected outcomes: $\big(\beta_1 + mk, \beta_2 + mk\big)$, in which $m := 1/(1-\alpha)$ is a "social multiplier" (Jackson and Zenou, 2015). The multiplier magnifies the status-quo bias identified by Callander: when player i moves towards the status quo, player j has an incentive to do the same (due to the presence of $\alpha\mathbb{E}\chi(p_i)$ in the best response of player j.) Player 1 is a "follower" in the sense that she incurs less uncertainty than player 2. In equilibrium, the distance between expected outcomes is pinned down by centralities, $\beta_1 - \beta_2$, because best responses shift by the same amount. Hence, an increase in uncertainty alone does not lead to conformity.

Consider the complex environment in the game $G(x_0)$, i.e., with noisy and correlated outcomes. The best response of player 1 is:

$$(1-\alpha)\delta_1 + \alpha\mathbb{E}\chi(p_2) + k - 2\alpha k, \tag{3}$$

while the best response of player 2 is the same as with uncorrelated outcomes, i.e., expression (2). The introduction of correlation makes player 1 willing to explore more. Hence, the follower has an incentive to catch up with the leader, which clashes with the push towards the status quo. This exploration motive is reflected by a downward shift of the best response of player 1 — relative to the uncorrelated-outcomes case of expression (2). There is a unique equilibrium $\boldsymbol{p}^\star$ for the given leader-follower relationship, described by the pair of expected outcomes $(\beta_1 + k - k\frac{\alpha}{1+\alpha}, \beta_2 + k + k\frac{\alpha}{1+\alpha})$. In general, the equilibrium exhibits three features.

(1) Additional conformity arises due to complexity. In particular,

$$\mathbb{E}\chi(p_1^\star) - \mathbb{E}\chi(p_2^\star) - (\beta_1 - \beta_2) < 0.$$

(2) The new conformity increases (locally) in complexity. The difference in

expected outcomes, netting out the no-conformity difference $\beta_1 - \beta_2$, is:

$$\mathbb{E}\chi(p_1^\star) - \mathbb{E}\chi(p_2^\star) - (\beta_1 - \beta_2) = -2\frac{\alpha}{1+\alpha}k.$$

Strict monotonicity is local. If complexity exceeds the cutoff implied by our requirement — i.e., $\delta_1 - \delta_2 > 2k\alpha/(1-\alpha)$) —, then players have the same equilibrium expected outcome.

(3) The leader "pulls" the follower away from the status quo. With the introduction of complexity, the follower faces two new incentives. First, she is pushed towards the status quo, via the status-quo bias that is present also without correlation in outcomes. Second, she is pulled away from the status quo, via the conformity that is introduced by the covariance structure. The interplay between the covariance of the environment and coordination motives leads to an extra exploration incentive, which "controls" for the variance effect that is isolated in the uncorrelated-outcomes case (Figure 5).

In general, conformity is "scaled" by the correlation between outcomes. In particular, suppose two Brownian motions, with same initial points, drift and variance, that are correlated with parameter ρ (see Section 5.) The best response of the leader is identical to the no-correlation case (expression 2), whereas the best response of the follower is

$$(1-\alpha)\delta_1 + \alpha\mathbb{E}\chi(p_2) + k - 2\alpha\rho k,$$

in which the follower's exploration motive is scaled by ρ. Hence, the higher the correlation, the stronger the conformity effect. In particular, $\mathbb{E}\chi(\tilde{p}_1) - \mathbb{E}\chi(\tilde{p}_2) - (\beta_1 - \beta_2) = \rho(-2\frac{\alpha}{1+\alpha}k)$, in an equilibrium $\tilde{\boldsymbol{p}}$. The presence of a nontrivial covariance structure induces players to explore more without sacrificing coordination.

3.2 Pairwise conformity

Under a complete network, complexity unambiguously leads to a strong form of conformity that holds for all pairs of players and equilibria of $G(x_0)$.

Lemma 2. *Let $\gamma^{ij} = \gamma$ for all players $i, j \in N$ with $i \neq j$, and $\boldsymbol{p} \in (\underline{p}, \overline{p})^n$ be an equilibrium. If $p_i < p_j$, then:*

$$\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_j) < \beta_i - \beta_j.$$

The above result compares the expected outcomes of every pair of players in equilibrium to the no-complexity case, across equilibria. The introduction of complexity makes players choose closer policies.

An equilibrium $\boldsymbol{p}$ is *ordered* if it satisfies $p_0 < p_1 < p_2 < \cdots < p_n < \overline{p}$. Equilibria are naturally ordered by the primitives of certain applications. In oligopolistic competition, for instance, demand intercepts and marginal costs order equilibrium prices (Section **??**). For ordered equilibria, conformity (locally) increases with the complexity of the environment.

Lemma 3. *Let $\gamma^{ij} = \gamma$ for all players $i, j \in N$ with $i \neq j$, and $\boldsymbol{p} \in (p_0, \overline{p})^n$ be an ordered equilibrium. Then, for all $i \in \{1, \ldots, n-1\}$,*

$$\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_{i+1}) = \beta_i - \beta_{i+1} - 2\frac{\alpha\gamma}{1+\alpha\gamma}k.$$

The comparative statics holds locally. If the conformity motive is sufficiently strong, the difference in favorite outcomes does not sustain the leader-follower asymmetry. This is the case, for instance, if complexity exceeds the cutoffs implied Lemma 3. In this case, extreme conformity arises: the relevant players choose the same policy.

3.3 Counterformity

Conformity interacts with the network of players. A player may exert substantial network influence on a follower player. If this influence is strong enough, it drives the follower away from a third player. "Counter-formity" emerges when equilibrium expected outcomes in a pair of players are more distant than in a non-complex environment. This situation is illustrated in Figure 6.

In general, conformity has a nuanced interaction with the network of players, which is described intuitively relying on the leader-follower asymmetry. Consider an ordered equilibrium. Player n is a leader for every other player, whereas player 1 is a follower for every opponent. The first term of the infinite sum induced by $\alpha(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}(\boldsymbol{\Gamma} \odot \boldsymbol{A})\boldsymbol{1}k$, i.e., $\alpha k(\boldsymbol{\Gamma} \odot \boldsymbol{A})\boldsymbol{1}k$, represents a "first-order" conformity effect. Player n's opponents are choosing policies closer to the status quo than her, whereas player 1's opponents are incurring more uncertainty than him.[13] Hence, player n has an additional incentive than player 1 to choose a policy close to the status quo. This incentive is an *endogenous* status-quo bias for player n, relative to player 1, because

[13]This configuration of players implies that $a_{nj} = 1$, $j \neq n$, and $a_{1k} = -1$, $k \neq 1$ (Proposition 2).

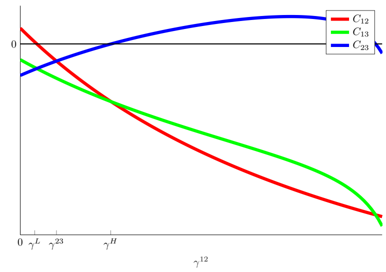


Figure 6: The additional conformity is defined by $C_{ij} = \mathbb{E}\chi(p_i^\star) - \mathbb{E}\chi(p_j^\star) - \beta_i + \beta_j$, for players $i,\ j \in N$ in an equilibrium $\boldsymbol{p}^\star$. Suppose that there exists a "middle" player, player 2. In particular, player 2 is the follower to player 3 and the leader to player 1. If the connection between player 1 and 2 is sufficiently weak ($\gamma^{12} < \gamma^L$), the middle player values the pull of the "global leader" more than the push towards the status quo of the global follower. As a result, counterformity arises between player 1 and 2. A similar phenomenon occurs between player 2 and 3 when γ^{12} is sufficiently large. (For this figure: $n = 3, \gamma^{23} = 0.2, \gamma^{13} = 0, \delta_1 = 1, \delta_2 = 0, \delta_3 = -1, k = 2, \alpha = 0.45, p_0 = 0 = \underline{p}$ and sufficiently large $x_0,\ \overline{p}$.)

it is determined in equilibrium. I tentatively define the "extra status-quo bias" for player n that takes into account the connections among players by averaging the entries in the nth row of $\boldsymbol{A}$, each weighted according to the connection of player n with the corresponding opponent; this average yields

$$\sum_j a_{nj}\gamma^{nj} > 0.$$

The same intuition leads to an "extra exploration motive" for player 1,

$$\sum_j a_{1j}\gamma^{1j} < 0.$$

The vector $\alpha k(\boldsymbol{\Gamma} \odot \boldsymbol{A})\mathbf{1}$ collects these first-order incentives of all players scaled by αk. The complete intuition takes into account how the extra status-quo biases and exploration motives feed into the network of players. The resulting equilibrium conformity effect is

$$(\boldsymbol{\Gamma} \odot \boldsymbol{A})\mathbf{1}\alpha k + \alpha\boldsymbol{\Gamma}(\boldsymbol{\Gamma} \odot \boldsymbol{A})\mathbf{1}\alpha k + (\alpha\boldsymbol{\Gamma})^2(\boldsymbol{\Gamma} \odot \boldsymbol{A})\mathbf{1}\alpha k + \ldots,$$

which yields the vector $\boldsymbol{B}(\alpha\boldsymbol{\Gamma})(\alpha\boldsymbol{\Gamma} \odot \boldsymbol{A})\mathbf{1}k$, present in the decomposition of equilibrium expected outcomes. Thus, player i's conformity effect counts all the discounted walks starting from i and weighs each walk to player j by the endogenous status-quo bias

$\alpha k \sum_\ell a_{j\ell}\gamma^{j\ell}$.

As the next result suggests, heterogeneity in network connections is related to counterformity. We say that $\mathbf{\Gamma}$ is a *line* if: (i) $\gamma^{ii+1} = 1$ for all $i \in \{1, \ldots, n-1\}$, (ii) $\gamma^{ii-1} = 1$ for all $i \in \{2, \ldots, n\}$, and (iii) $\gamma^{ij} = 0$ otherwise. In a line network, conformity emerges pairwise, and it increases in complexity. (In Figure 6, $\mathbf{\Gamma}$ is a line only when $\gamma^{12} = \gamma^{23}$, in which case there is "only" conformity.)

Lemma 4. *Let $\mathbf{\Gamma}$ be a line, $\alpha \leq 1/2$, and $\boldsymbol{p} \in (p_0, \overline{p})^n$ be an ordered equilibrium. Then, for all $i \in \{1, \ldots, n-1\}$,*

$$\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_{i+1}) = \beta_i - \beta_{i+1} - c_i k,$$

for some $c_i > 0$.

Remark 2. *The design of network interventions studies changes in favorite outcomes that induce certain equilibrium behavior of players (Galeotti et al., 2020). Suppose an ordered equilibrium in a complete network or in a line. Moderate changes in favorite outcomes do not affect conformity. Hence, if a policymaker adopts a "small" intervention, the presence of complexity does not lead to unintended consequences: the results about optimal interventions under a "small budget" are robust to a low level of complexity. Substantial interventions, on the other hand, change the leader-follower relationships, and, so, the pattern of conformity.*

4 Applications

Incremental uncertainty and coordination motives are present in many economic environments.

- In social psychology, it is documented that conformity increases in the difficulty of the task and in the "cohesion" of the group (Krech et al., 1962). By the comparative statics results, conformity increases in the strength of coordination motives and the number of players, consistently with the observation that "yielding to the group pressures" is frequent for high "group cohesion" and "group size" Krech et al. (1962).[14]

[14]The main comparative statics is in Lemma 3, a simple corollary is that "overall" conformity increases in the number of players: $\mathbb{E}\chi(p_1) - \mathbb{E}\chi(p_n) = \beta_i - \beta_{i+1} - 2(n-1)\frac{\alpha}{1+\alpha}k$. Similar results follow from Lemma 4.

- Peer recognition is important in scientific research (Partha and David, 1994). In general, coordination motives are present in certain interactions in which exploration of unknown alternatives is important. If society values exploration, conformity may limit learning about the underlying outcome function.[15] The presence of conformity is important for the design of incentives for research and innovation.

- The management of every subsidiary of a holding company coordinates with other subsidiaries and adapts to idiosyncratic circumstances. Communication frictions are a source of noise in the implementation of production processes. This noise may be particularly relevant for the adoption of innovative technologies. In Section 4.3, I show that an organization with decentralized decision-making can implement profit maximization in sufficiently complex environments.

- In oligopolistic competition, firms that rely on algorithmic pricing face uncertainty over their own listed prices. This uncertainty arises because an algorithm conditions prices on data not available when the algorithm is selected (Brown and MacKay, 2023). Price competition exhibits strategic complementarities in many models of oligopoly. Appendix **??** proposes a model in which firms choose pricing policies knowing the resulting listed prices up to some noise, reflecting market uncertainty or the recent introduction of algorithmic pricing. Complexity leads to less dispersed expected prices across products, by leveraging Lemma 3 and a natural ordering property of equilibrium policies. If the net demand intercepts are sufficiently heterogeneous, then every equilibrium is ordered; this situation arises if firms are sufficiently different in their production efficiency.

4.1 Equilibrium selection

In order to study different games in which a similar equilibrium analysis holds, I define an auxiliary utility function of player i over outcomes, $v_i(\boldsymbol{x}) = 2(1-\alpha)\delta_i x_i - x_i^2 + 2\alpha \sum_{j \in N} \gamma^{ij} x_i x_j$. The next result studies the strategic-form game $F(x_0)$, in

[15] In Brownian-motion models, however, learning occurs in two ways: radical and incremental experimentation, given, respectively, by the extreme ($\max\{p_1, \ldots, p_n\}$ and $\min\{p_1, \ldots, p_n\}$) and non-extreme policies that are chosen (similarly to Garfagnini and Strulovici (2016).) If conformity increases, less is known about radical experimentation, but, possibly, more about incremental experimentation.

which players and strategy spaces are the same as in $G(x_0)$ and utility functions are $\mathbb{E}v_1(\boldsymbol{\chi}(\cdot)), \ldots, \mathbb{E}v_n(\boldsymbol{\chi}(\cdot))$.

Lemma 5. *For every player $i \in N$, there exists a function $g_i \colon P^{n-1} \times \mathbf{R} \to \mathbf{R}$ such that:*

$$\mathbb{E}\pi_i(\boldsymbol{\chi}(\boldsymbol{p})) = \mathbb{E}v_i(\boldsymbol{\chi}(\boldsymbol{p})) + g_i(\boldsymbol{p}_{-i}, x_0) \text{ for all } \boldsymbol{p} \in P^n, x_0 \in \mathbf{R}.$$

The game $F(x_0)$ has the same set of equilibria as $G(x_0)$ because the games are Von-Neumann-Morgenstern equivalent (Morris and Ui, 2004). The applications that follow leverage this observation to apply the analysis of Section 2.4. Moreover, the applications makes use of the fact that $G(x_0)$ is a potential game (Monderer and Shapley, 1996).

A game is a common-interest game if all players have the same payoff function. A game is a potential game if it is "best-response equivalent" to an auxiliary game that is a common-interest game (definitions are in the Appendix.) Let's call the *potential* is the function $V \colon P^n \to \mathbf{R}$ given by

$$V(\boldsymbol{p}) = \mathbb{E}\Big[2(1-\alpha)\boldsymbol{\delta}^{\mathsf{T}}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\chi}(\boldsymbol{p})^{\mathsf{T}}(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})\boldsymbol{\chi}(\boldsymbol{p})|\chi(p_0) = x_0\Big].$$

A *potential maximizer* is a policy profile $\boldsymbol{p}^\star$ that maximizes the potential, i.e., $\boldsymbol{p}^\star \in \operatorname{Arg\,max}_{\boldsymbol{p} \in P^n} V(\boldsymbol{p})$.

Proposition 3. *The following properties of the potential maximizer hold.*

(1) If the policy profile $\mathbf{p} \in P^n$ is a potential maximizer, then $\mathbf{p}$ is an equilibrium.

(2) If $P = [p_0, \overline{p}]$, there exists a unique potential maximizer.

For part (1), I establish von-Neumann-Morgenstern equivalence (Morris and Ui, 2004) between the two strategic-form games played in the outcome space with utility functions $\{\pi_1, \pi_2, \ldots, \pi_n\}$ and $\{v, v, \ldots, v\}$, in which $v(\boldsymbol{x}) = 2(1-\alpha)\boldsymbol{\delta}^{\mathsf{T}}\boldsymbol{x} - \boldsymbol{x}^{\mathsf{T}}(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})\boldsymbol{x}$. This result extends to the induced games played in the policy space, and so it establishes that $G(x_0)$ is a potential game, a fortiori.[16] Since a strategy profile that maximizes the potential is necessarily an equilibrium of the potential game (Radner, 1962), part (1) follows. Moreover, the potential for $G(x_0)$ is uniquely defined up to a

[16]In particular, for every player $i \in N$ there exists a function $g_i \colon P^{n-1} \times \mathbf{R} \to \mathbf{R}$ such that: $\mathbb{E}\pi_i(\boldsymbol{\chi}(\boldsymbol{p})) = \mathbb{E}v(\boldsymbol{\chi}(\boldsymbol{p})) + g_i(\boldsymbol{p}_{-i}, x_0)$ for all $\boldsymbol{p} \in P^n$ and $x_0 \in \mathbf{R}$. The last step of the proof verifies that von-Neumann-Morgenstern equivalence is consistent with the definition of a potential game.

constant term.[17] These two observations imply that the potential maximizer provides a valid equilibrium selection for $G(x_0)$. Strict concavity on $[p_0, \overline{p}]$ leads to existence and uniqueness of the potential maximizer. I study the superdifferential of V to characterize the potential maximizer.

Proposition 4. *Let $P = [p_0, \overline{p}]$. The policy profile $\boldsymbol{p} \in (p_0, \overline{p})^n$ is a potential maximizer if, and only if:*

$$\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) = \boldsymbol{\beta} + \mathbf{1}k + \alpha(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}(\boldsymbol{\Gamma} \odot \boldsymbol{A})\mathbf{1}k,$$

for a skew-symmetric matrix $\boldsymbol{A} = [a_{ij} : i, j \in N]$ such that $a_{ij} \in [-1, 1]$ and $a_{ij} = 1$, if $p_i > p_j$.

The decomposition for the potential maximizer has a similar structure as the equilibrium decomposition. The main difference is the skew-symmetry property of the endogenous matrix $\boldsymbol{A}$ that implies the uniqueness result.

The uniqueness and characterization of the potential maximizer allow to make predictions about strategic interactions in complex environments using the potential maximizer as equilibrium selection. The selection is useful precisely due to complexity. If $k > 0$, the strictly concave potential is not smooth and there are multiple equilibria. In particular, the potential V is not differentiable whenever $p_i = p_j$ for a pair of players, due to the covariance structure (sections 2.3 and A.4.) If $k = 0$, the potential is differentiable and there exists a unique equilibrium: the potential maximizer.[18]

I study the welfare in the game $F(x_0)$ using the tools developed for the maximization of the potential of $G(x_0)$. The *utilitarian welfare in* $F(x_0)$ is given by the function $W : \boldsymbol{p} \mapsto \sum_i \mathbb{E}v_i(\boldsymbol{\chi}(\boldsymbol{p}))$. A *welfare maximizer* is a policy profile $\boldsymbol{p}^W$ that maximizes utilitarian welfare in $F(x_0)$, i.e., $\boldsymbol{p}^W \in \operatorname{Arg\,max}_{\boldsymbol{p} \in P^n} W(p)$. The following result characterizes the welfare maximizer.

Proposition 5. *Let $P = [p_0, \overline{p}]$ and $2\alpha\lambda(\boldsymbol{\Gamma}) < 1$. There exists a unique welfare maximizer. Moreover, the policy profile $\boldsymbol{p} \in (p_0, \overline{p})^n$ is a welfare maximizer if, and*

[17] In Appendix C, I establish that V is an exact potential; Monderer and Shapley (1996) introduce the notion of exact potential, a particular case of the weighted potential; Morris and Ui (2004) study the equivalence between weighted potential games and potential games in connection with von-Neumann-Morgenstern equivalence.

[18] To establish this observation, it suffices that: if $\boldsymbol{p}^\circ \in P^n$ satisfies $\boldsymbol{\psi}(\boldsymbol{p}^\circ) = \boldsymbol{\beta}$, then it maximizes $\boldsymbol{p} \mapsto v(\boldsymbol{\psi}(\boldsymbol{p}))$ on P^n. This claim is established by showing that $\boldsymbol{p} \mapsto v(\boldsymbol{\psi}(\boldsymbol{p}))$ is a potential for the game $G(x_0)$ without complexity (Appendix).

only if:

$$\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) = (1-\alpha)\boldsymbol{B}(2\alpha\boldsymbol{\Gamma})\boldsymbol{\delta} + \mathbf{1}k + 2\alpha\boldsymbol{B}(2\alpha\boldsymbol{\Gamma})(\boldsymbol{\Gamma}\odot\boldsymbol{A})\mathbf{1}k,$$

for a matrix $\boldsymbol{A} = [a_{ij} : i, j \in N]$ *such that* $a_{ij} \in [-1, 1]$, $a_{ij} = -a_{ji}$, *and* $a_{ij} = 1$ *if* $p_i > p_j$.

In the proof, I leverage the known observation that utilitarian welfare maximization in $F(x_0)$ is equivalent to maximization of the potential of an auxiliary game in which the magnitude of cost externalities is doubled. The reason is that the game $F(x_0)$ is a coordination game, by the results established for $G(x_0)$ (Lemma 1 and 5), in which players do not internalize all the externality of their policy.

4.2 Application: network of players

I study the game in which every player is part of only one of two groups, A and B, and players in the same group have the same favorite outcomes and connections.

The parameter γ denotes the connection between a player in group A and a player in B, by $\delta_g, \gamma^{gg}, \beta_g$ and n_g, respectively, the favorite outcome, the weight of an intra-group connection, the centrality of a player and the number of players for group $g \in \{A, B\}$. The *two-type network game* is the game $G(x_0)$ with the restriction just described. In every equilibrium of a two-type game, player i chooses the same policy as player j if they are in the same group.[19] Hence, an equilibrium is represented by a pair (p_A, p_B), such that $i \in A$ plays p_A, and $j \in B$ plays p_B. I use $\alpha_A := \frac{\alpha\gamma n_B}{1-\alpha\gamma^{AA}(n_A-1)}$ and $\alpha_B := \frac{\alpha\gamma n_A}{1-\alpha\gamma^{BB}(n_B-1)}$. By Assumption 1, $\alpha_A, \alpha_B \in [0,1]$ and $\frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B} \in [0,1]$ (Section E).

Lemma 6. *Let* $\beta_A \geq \beta_B$ *and* $(p_A, p_B) \in (p_0, \overline{p})^2$ *be the unique potential maximizer of the two-type network game.*

(1) If $\beta_A - \beta_B \geq \frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B}k$, *then* $p_A < p_B$ *and*

$$\mathbb{E}\chi(p_A) - \mathbb{E}\chi(p_B) = \beta_A - \beta_B - \frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B}k.$$

[19] The proof of this result uses the fact that the game $G(x_0)$ is a potential game, and that, for given policies chosen in group g', the "reduced potential" that includes only members of g is "symmetric"; see, e.g., Vives (1999), Chapter 2, Footnote 23.

(2) If $\beta_A - \beta_B \leq \frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B}k$, then $p_A = p_B$ and

$$\mathbb{E}\chi(p_A) = \frac{\alpha_B(1-\alpha_A)\beta_A + \alpha_A(1-\alpha_B)\beta_B}{\alpha_B(1-\alpha_A)+\alpha_A(1-\alpha_B)} + k.$$

For sufficiently high complexity, conformity is extreme: all players choose the same policy. In this case, the expected outcome is the same as if a representative player were choosing an optimal policy, in isolation and with a favorite outcome equal to $\frac{\alpha_B(1-\alpha_A)\beta_A+\alpha_A(1-\alpha_B)\beta_B}{\alpha_B(1-\alpha_A)+\alpha_A(1-\alpha_B)}$, which is a weighted average of centralities in the two groups.

4.3 Application: centralization in organizations

A firm is made of two divisions, each producing a different good. When quantity produced by division i is x_i, the cost of division i is $c_i x_i - g x_1 x_2$, in which the parameter $g > 0$ measures the degree of cost externalities and $c_i > 0$. An increase in the quantity produced by one division reduces the marginal costs of the other division, as in Alonso et al. (2015). The inverse demand function for product i is given by $a_i - \frac{1}{b}x_i$, in which $b > 0$ measures the price elasticity of demand. The profits of division i given the profile of quantities $\boldsymbol{x}$ are

$$\pi_i^O(\boldsymbol{x}) := \Big(a_i - \frac{1}{b}x_i - c_i + g x_j\Big)x_i.$$

The CEO's objective is the maximization of total profits $\pi_1^O + \pi_2^O$. I impose an upper bound on the strength of cost externalities for the CEO's profit maximization to be well-behaved: $bg < 1$.[20] Each division manager chooses a production policy $p_i \in [p_0, \overline{p}]$. The function χ specifies the quantity produced by a division for every production policy. Division i's profits given the pair of policies $\boldsymbol{p}$ are given by $\pi_i^O(\boldsymbol{\chi}(\boldsymbol{p}))$. The division managers set production policies simultaneously and independently in the *production game*, $\Big\langle \{1,2\}, \{\mathbb{E}\pi_i^O(\boldsymbol{\chi}(\cdot)), [p_0, \overline{p}]\}_{i\in\{1,2\}} \Big\rangle$.

The following result investigates the compatibility of managerial incentives with total-profit maximization. The analysis assumes that $a_1 - c_1 = a_2 - c_2 =: \widehat{a}$, so managers choose the same policy in equilibrium and for total-profit maximization (Section E).

Proposition 6. *There exists a unique policy profile $\boldsymbol{p}^O$ that maximizes expected total*

[20] The Hessian of total profits $\pi_i^O + \pi_j^O$ is negative definite iff: $bg < 1$.

profits. Moreover, $\boldsymbol{p}^O$ is an equilibrium of the production game if and only if:

$$\widehat{a}\frac{b}{1-bg} \leq 2k.$$

The result gives conditions under which $\boldsymbol{p}^O$ is in the equilibrium set. First, I show that the CEO's objectives are well-defined by studying the maximization of total profits, which is equivalent to the maximization of utilitarian welfare in the coordination game between the division managers. The maximization of expected total profits is solved using the welfare analysis in Proposition 5 and the equilibrium set is characterized using Proposition 2 and Lemma 5.

The result associates firms with weak cost externalities and operating in complex environments with a more effective implementation of the CEO's optimal production policy. A necessary and sufficient condition to for maximization of total profits to be implemented in equilibrium is that complexity exceeds the threshold $\widehat{a}\frac{b}{2(1-bg)}$. The threshold increases in the net demand intercept $\widehat{a}$ and price sensitivity of demand, reflecting that the interests of division managers move farther apart from the CEO's interests for favorable individual market conditions. The threshold also increases in g, because the "non-internalized" externalities increase in g.

A reason for the presence of noise in the mapping from production processes to quantities is frictions in the command chain. Suppose that each division manager only instructs lower-end division managers about production decisions, who in turn interact with store managers, and so forth. The division manager is unsure about how her instructions are communicated along the chain of command and utlimately implemented. Complexity captures the noise perceived by the division manager; e.g., the longer the chain, the less predictable the outcome of the original instruction. The result suggests that centralized decision-making may be less desirable in the presence of coordination problems. The analysis points to a responsibility of the holding company's management: leveraging the coordination problems induced by the environment and making maximization of the holding's profits a focal point for the management of subsidiaries.

5 Extensions and conclusion

The model can be extended to include player-specific noise sources.[21] Suppose that the outcome function of player 1 is $X^1 = Y^1$, while the outcome function of player 2 is $X^2 = \rho Y^1 + \sqrt{1-\rho^2} Y^2$, for $\rho \in [0,1]$ and a 2-dimensional Brownian motion (Y^1, Y^2) with common drift μ, variance parameter ω, and independent coordinates. The analysis in this paper leads to the following characterization of equilibria for two players: $\boldsymbol{p} \in (p_0, \overline{p})^2$ is an equilibrium if, and only if:

$$\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) = \boldsymbol{\beta} + (\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}(\boldsymbol{I} + \rho\alpha\boldsymbol{\Gamma} \odot \boldsymbol{C})\mathbf{1}k,$$

for a matrix $\boldsymbol{C}$ such that $C_{ij} \in [-1, 0]$, $C_{ij} = 0$ if $p_i > p_j$ and $C_{ij} = -1$ if $p_i < p_j$.

This general model allows for a finer decomposition that separates the two elements of the complexity of the environment: variance of outcomes and covariance of pairs of outcomes. The new term in the decomposition is a linear combination of two effects. First, a *pure status-quo bias*, which arises with independent outcomes across players (i.e., the positive vector $(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}\mathbf{1}k$, discussed in Section 3.) This component pushes every player towards the status quo, and is magnified by the network of players. Second, a *pure experimentation motive*, that arises only with correlated outcomes (i.e., the nonpositive vector $(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}(\rho\alpha\boldsymbol{\Gamma} \odot \boldsymbol{C})\mathbf{1}k$.) This component pulls players away from the status quo.

In strategic interactions with coordination motives, the willingness to take risk is endogenous. The reason is the incentive to make decisions with "correlated" consequences — not just with similar consequences in expectation. The interplay of coordination and complexity manifests itself via a conformity motive and leader-follower asymmetries. Conformity "pulls" certain players away from the status quo and "pushes" others towards it. Future research could explore the impact of conformity for learning. Additionally, this model provides a tool for the characterization of the conformity pattern that emerges in specific networks of innovators, identified by theoretical and empirical work (König et al., 2014; Zacchia, 2020).

[21] The extension captures, e.g., firms buying pricing services from different providers or trained on separate datasets. The model considered in this paragraph is constructed as in Section 2, except that $\boldsymbol{p} \mapsto \pi_i(X^i(p_i), X^j(p_j))$ replaces $\boldsymbol{p} \mapsto \pi_i(\chi(p_i), \chi(p_j))$. This construction generalizes for n players via a suitable linear combination of the coordinates of an n-dimensional standard Brownian motion (Karatzas and Shreve, 1998, Definition 5.1 and 5.19, Chapter 2).

Appendix

A Preliminaries

A.1 Heterogeneous status quo

This section considers an incomplete-information extension of the game $G(x_0)$ introduced in Section 2. In the following sections of the Appendix, the proofs use the notation that is introduced for this model.

Ex-Post payoffs are the same as in Section 2.1. The following description of interim beliefs defines a Bayesian game parametrized by a profile of status-quo policies, $\mathcal{G}(\boldsymbol{p}_0)$, which is defined explicitly in the Appendix (Section C).

Player i believes that the outcome function χ is the realized path of a Brownian motion with drift $\mu < 0$, variance parameter $\omega > 0$ and starting point $(p_0^i, \chi(p_0^i))$. Every player knows the profile of status-quo policies $\boldsymbol{p}_0 = (p_0^1, \dots, p_0^n) \in \mathbf{R}^n$. The status-quo outcome of player i is known to player i and not known to her opponents: $\chi(p_0^i)$ is player i's *type*. Beliefs are consistent with the limit of a common prior over a Brownian motion.[22] I denote by $\mathbb{P}^i$ the probability of an event and by $\mathbb{E}^i$ the expectation operator induce by player i's beliefs at a given type $\chi(p_0^i)$ (see the Appendix, Section A.3, for more details.)

Every player simultaneously chooses a policy. In this section, $P_i = [\underline{p}_i, \overline{p}_i]$ is the policy space of player i, for $\underline{p}_i, \overline{p}_i \in \mathbf{R}$ with $\underline{p}_i \leq p_0^i \leq \overline{p}_i$, and $P = \times_i P_i$ to ease readability, with a slight inconsistency of notation with respect to the previous sections. A strategy for player i is a measurable function $\sigma_i \colon \mathbf{R} \to P_i$. The set of strategies for player i is denoted by Σ_i, the set of strategy profiles by $\Sigma := \times_{i \in N} \Sigma_i$, and the set of profiles of strategies for players other than i by $\Sigma_{-i} = \times_{i \in -i} \Sigma_j$; Σ_i is endowed with the pointwise order to be a lattice, Σ_{-i} and Σ are endowed with the product order. The following notation is used, given a profile of strategies of player i's opponents σ_{-i}:

$$(\chi(p_i), \chi(\sigma_{-i})) = (\dots, \chi(\sigma_{i-1}(\chi(p_0^{i-1}))), \chi(p_i), \chi(\sigma_{i+1}(\chi(p_0^{i+1}))), \dots),$$

The expected payoff of player i, given σ_{-i}, is

$$\Pi_i(p_i, x_0^i; \sigma_{-i}) := \mathbb{E}^i[\pi_i(\chi(p_i), \chi(\sigma_{-i}))]$$

An equilibrium of $\mathcal{G}(\boldsymbol{p}_0)$ is an interim Bayesian Nash equilibrium; the definition uses $\varphi_i(x_0^i; \sigma_{-i}) := \operatorname{Arg\,max}_{p_i \in P_i} \Pi_i(p_i, x_0^i; \sigma_{-i})$.

Definition 1. *The strategy profile $\sigma \in \Sigma$ is an* equilibrium of $\mathcal{G}(\boldsymbol{p}_0)$ *if, and only if:*

$$\sigma_i(x_0^i) \in \varphi_i(x_0^i; \sigma_{-i}), \quad \textit{for all } x_0^i \in \mathbf{R}, i \in N.$$

[22] Given a Brownian motion with starting point $(0, z)$ and realized path denoted by ξ, suppose that each player observes the point $(p_0^i, \xi(p_0^i))$ and a signal about z with Gaussian noise that is i.i.d. across players. As the noise grows, player i's belief about $\xi(q)$ given $\xi(p_0^i) = x_0^i$ converges to her belief in $\mathcal{G}(\boldsymbol{p}_0)$ about $\chi(q)$ when her type is $\chi(p_0^i) = x_0^i$.

Remark 1. Consider the game $\mathcal{G}((p_0, \ldots, p_0))$, in which players have the same status-quo policy p_0. This game is effectively the collection of strategic-form games $\{G(x_0)\}_{x_0 \in \mathbf{R}}$, because the profile of status-quo outcomes is common knowledge. Hence, the game $G(x_0)$ is the subgame of $\mathcal{G}((p_0, \ldots, p_0))$ starting at $\chi(p_0) = x_0$.

Results The assumption that status-quo policies are different across players is maintained in this section.

Assumption 2. *Status-Quo policies are different across players: $p_0^i \neq p_0^j$ for all $i, j \in N$ with $j \neq i$.*

Player i's belief about $\chi(q)$ is nondecreasing in $\chi(p_0^i)$ in the sense of first-order stochastic dominance (FOSD) and satisfies a translation-invariance property studied in Mathevet (2010).[23]

Lemma A.1. *Player i's belief about the outcome of policy q is nondecreasing in $\chi(p_0^i)$ according to first-order stochastic dominance. Moreover, player i's belief satisfies the following* translation invariance *property:*

$$\mathbb{P}^i\{\chi(q) < x | \chi(p_0^i) = x_0^i\} = \mathbb{P}^i\{\chi(q) < x + \Delta | \chi(p_0^i) = x_0^i + \Delta\}, \textit{ for all } \Delta \in \mathbf{R}.$$

FOSD monotonicity is used to establish the single-crossing property of expected payoffs in own policy and type.

A more stringent upper bound on the strength of coordination motives than Assumption 1 is used to establish single-crossing of expected payoffs, which is used for the existence of equilibria in monotone strategies.

Assumption 3. *For every player i,*

$$\alpha \sum_{j \in N} \gamma^{ij} < 1.$$

Assumption 3 implies that $\boldsymbol{I} - \alpha \boldsymbol{\Gamma}$ has strictly dominant diagonal, which is a known sufficient condition for Assumption 1.

The incomplete-information game $\mathcal{G}(\boldsymbol{p}_0)$ exhibits strategic complementarities.

Lemma A.2. *For all $i \in N$, the expected payoff $(\boldsymbol{p}, \chi(p_0^i)) \mapsto \mathbb{E}^i \pi_i(\boldsymbol{\chi}(\boldsymbol{p}))$ exhibits strictly increasing differences in p_i, p_j, $j \in -i$, and in $(p_i, \chi(p_0^i))$.*

The upper bound on coordination motives is key for increasing differences in own policy and type. To establish this property, the right-derivative of $p_i \mapsto \mathbb{E}^i \pi_i(\boldsymbol{\chi}(\boldsymbol{p}))$ is shown to be an affine function of x_0^i, where the coefficient on x_0^i is $1 - \alpha \sum_j \gamma^{ij}$ (Appendix). The upper bound on coordination motives is necessary for the single-crossing property of expected payoffs in (p_i, x_0^i), which associates higher policies to higher types.

The following result establishes existence of Bayesian Nash equilibrium in nondecreasing strategies.

[23] For notational convenience, in the following result I use the symbol "|", even though the beliefs of players do not necessarily arise as conditional probabilities, because $\mathcal{G}(\boldsymbol{p}_0)$ is an interim Bayesian game.

Proposition A.1. *There exist a greatest and a least Bayesian Nash equilibrium, $\overline{\sigma}$ and $\underline{\sigma}$, respectively. Moreover, $\overline{\sigma}$ and $\underline{\sigma}$ are profiles of nondecreasing strategies.*

Because the type spaces are necessarily unbounded, results from the literature on incomplete-information games with strategic complementarities do not apply directly. However, I establish that the expected payoff function $p_i \mapsto \Pi_i(p_i, x_0^i; \sigma_{-i})$ is strictly concave for a profile of nondecreasing strategies σ_{-i}. Given strict concavity of Π_i, compactness of P_i and strategic complementarities, type spaces can be compactified to establish similar results as Van Zandt and Vives (2007). In particular, the greatest-best-reply mapping $x_0^i \mapsto \sup \varphi_i(x_0^i, \sigma_{-i})$ is measurable; see Lemma B.15 in Appendix.)

Remark 2. Let $\alpha = 0$. From the analysis in Callander (2011a) and Corollary 1, it follows that: (i) there exists a unique Bayesian Nash equilibrium, and (ii) in the unique Bayesian Nash equilibrium, the strategy of each player is nondecreasing in her type.

The following result shows a status-quo effect.

Lemma A.3. *For every Bayesian Nash equilibrium in nondecreasing strategies σ and player i, the following holds:*

There exist cutoffs $c_1^i, c_2^i \in \mathbf{R}$ with $c_1^i < c_2^i$ such that: $\sigma_i(x) = p_0^i$ for all $x \in [c_1^i, c_2^i]$, and $\sigma_i(x) \neq p_0^i$ for all $x \in \mathbf{R} \setminus [c_1^i, c_2^i]$.

There are two takeaways. First, the reason why the slope of equilibrium strategies is not constant is the presence of a status quo: if the status-quo outcome of player i is in an interval $[x_1^i, x_2^i]$, player i prefers to stick to the status-quo policy, than to incur the uncertainty implied by a change of expected outcome. This equilibrium behavior is consistent with the optimal strategy in the game without coordination motives (Corollary 1).

Secondly, equilibrium strategies do not have a constant slope, differently from general models of beauty contest under incomplete information. Strategies with constant slope are either the focus or constitute the unique possibility in equilibrium in standard beauty-contest models of incomplete information. In Lambert et al. (2018) — where the environment is "informationally complex" because of the arbitrarily large, though finite, dimensionality of the state and type profile —, the authors establish the existence of an equilibrium in strategies with constant slope.

The following result offers a partial characterization of equilibria in nondecreasing strategies, using χ_j for $\chi(\sigma_j(p_0^j))$, given $\sigma_j \in \Sigma_j$ and $j \in N$.

Lemma A.4. *Let $P_i = [p_0^i, \infty)$ for all $i \in N$. The profile of nondecreasing strategies σ is an equilibrium if, and only if, the following condition holds. For all $i \in N$ and $x_0^i \in \mathbf{R}$ such that $\sigma_i(x_0^i) > p_0^i$, there exists a vector $[a_{ij} : j \in N]$, such that:*

$$\mathbb{E}^i\chi_i - \alpha \sum_{j\in N} \gamma^{ij}\mathbb{E}^i\chi_j = \beta_i - \alpha \sum_{j\in N} \gamma^{ij}\beta_j + k + \alpha k \sum_{j\in N} \gamma^{ij} a_{ij},$$

and $a_{ij} \in \Big[2\mathbb{P}^i\{\sigma_j(\chi(p_0^j)) < \sigma_i(x_0^i) | \chi(p_0^i) = x_0^i\} - 1, 2\mathbb{P}^i\{\sigma_j(\chi(p_0^j)) \leq \sigma_i(x_0^i) | \chi(p_0^i) = x_0^i\} - 1\Big].$

The next result studies the multiplicity of equilibria, letting d denote the sup-norm distance between two strategies for player i.[24]

Proposition A.2. *The following holds:*

$$\max_{i\in N} d(\overline{\sigma}_i, \underline{\sigma}_i) \leq 2k \max_{i\in N} \frac{\alpha \sum_j \gamma^{ij}}{1-\alpha\sum_j \gamma^{ij}} \frac{1}{|\mu|}.$$

By Proposition A.1, all equilibria lie between two extreme strategy profiles, $\overline{\sigma}$ and $\underline{\sigma}$. Therefore, the distance between player i's strategies in any two equilibria is at most the distance between the extremal equilibria, i.e. $d(\overline{\sigma}_i, \underline{\sigma}_i)$, which is upper bounded by the Proposition.

In the Appendix, I study the game with finite policy spaces. With two players and finite policy spaces, there exists a unique equilibrium in nondecreasing strategies. The key step of the proof is the observation that increasing differences — which yield strategic complementarities in $G(x_0)$ and single-crossing in $\mathcal{G}(\boldsymbol{p}_0)$ — are constant in own type. This "constant-type" monotonicity, and the translation invariance and FOSD monotonicity properties of beliefs suffice establish uniqueness by using the results in Mathevet (2010); the author shows that under "translation-monotone" and FOSD-nondecreasing beliefs, a class of coordination games admits a unique equilibrium because the best-response mapping to nondecreasing strategies is a contraction.

In this section, we study the properties of payoffs over outcomes defined in Section 2, the outcome distribution discussed in Section 2.3, and the potential of $G(x_0)$. In Section A.3, we extend the model to study a common-prior model. The analysis maintains Assumption 1.

A.2 Ex-post payoffs

In this section, we study the ex-post payoff functions. Player $i \in N = \{1,\dots,n\}$ has preferences over outcome profiles $x \in \mathbf{R}^n$ that are represented by the *payoff* $u_i\colon \mathbf{R}^n \to \mathbf{R}$, which takes a quadratic-loss form:

$$\pi_i(x_i, x_{-i}) = -\left(x_i - (1-\alpha)\delta_i - \alpha \sum_{j\in N} \gamma^{ij} x_j\right)^2,$$

in which $\delta_i \in \mathbf{R},\ \alpha \in [0,1),\ \gamma^{ij} \geq 0$, and $\gamma^{ii} = 0$.

We note that: $\pi_i(x_i, x_{-i}) = 2(1-\alpha)\delta_i x_i - x_i^2 + 2\alpha \sum_{j\in N} \gamma^{ij} x_i x_j + h_i(x_{-i})$, in which $h_i(x_{-i})$ is constant with respect to x_i. Player i's *effort-game payoff* is: $v_i\colon \mathbf{R}^n \to \mathbf{R}$, with

$$v_i(x_i, x_{-i}) = 2(1-\alpha)\delta_i x_i - x_i^2 + 2\alpha \sum_{j\in N} \gamma^{ij} x_i x_j.$$

We let $\boldsymbol{\delta}$ and $\boldsymbol{\Gamma}$ be, respectively, the column vector of favorite outcomes $(\delta_1,\dots,\delta_n)^\mathsf{T}$ and the interactions matrix $[\gamma^{ij} : i,j \in N]$. We let $\boldsymbol{G} := \alpha\boldsymbol{\Gamma}$, $\boldsymbol{Q} := \boldsymbol{I} - \boldsymbol{G}$, $\boldsymbol{b} := (1-\alpha)\boldsymbol{\delta}$. We define $\boldsymbol{\beta} := \boldsymbol{Q}^{-1}\boldsymbol{b}$. $\mathbf{1}$ and $\boldsymbol{I}$ denote, respectively, a column vector of ones and the $n \times n$ identity matrix.

[24] The sup-norm of a strategy for a player is well-defined because policy spaces are bounded. Moreover, in the Appendix I establish that (i) equilibrium strategies are continuous and (ii) type spaces can be compactified, so that the sup can be replaced by the max in d by Weierstrass' Theorem (Lemmata B.13 and B.14).

For a matrix $\boldsymbol{A}$, we let a_{ij} be the entry in the ith row and jth column of $\boldsymbol{A}$, and $a_{i\bullet}$ be the column vector corresponding to the ith row of $\boldsymbol{A}$.

We let $\boldsymbol{x}$ be the column vector given by the outcome profile $(x_1, \dots x_n)$. We define the *potential* $v\colon \mathbf{R}^n \to \mathbf{R}$, such that

$$v(\boldsymbol{x}) = 2(1-\alpha)\boldsymbol{\delta}^\mathsf{T}\boldsymbol{x} - \boldsymbol{x}^\mathsf{T}(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})\boldsymbol{x}.$$

We note that: $v(\boldsymbol{x}) = -(\boldsymbol{x} - \boldsymbol{\beta})^\mathsf{T}\boldsymbol{Q}(\boldsymbol{x} - \boldsymbol{\beta}) + \boldsymbol{\beta}^\mathsf{T}\boldsymbol{Q}\boldsymbol{\beta}$. The *effort-game utilitarian welfare* is $\sum_{i\in N} v_i$, so that

$$\sum_{i\in N} v_i(\boldsymbol{x}) = 2(1-\alpha)\boldsymbol{\delta}^\mathsf{T}\boldsymbol{x} - \boldsymbol{x}^\mathsf{T}(\boldsymbol{I} - 2\alpha\boldsymbol{\Gamma})\boldsymbol{x}.$$

The following Lemma states that player i's payoff is best-response equivalent to the effort-game payoff and to the potential. In particular, we show that the three strategic-form games $(N, (\pi_i, \mathbf{R})_{i\in N})$, $(N, (v_i, \mathbf{R})_{i\in N})$ and $(N, (v, \mathbf{R})_{i\in N})$ are von Neumann–Morgenstern equivalent (Morris and Ui, 2004). We adopt the following notational conventions: $\boldsymbol{x}$ denotes (x_i, x_{-i}), and $-i := N \setminus \{i\}$, for all $i \in N$.

Lemma A.5 ("V-NM" equivalence)**.** *For all $i \in N$, there exist $h_i, g_i\colon \mathbf{R}^{n-1} \to \mathbf{R}$ such that:*

$$u_i(\boldsymbol{x}) - v_i(\boldsymbol{x}) = h_i(x_{-i}) \text{ and } u_i(\boldsymbol{x}) - v(\boldsymbol{x}) = g_i(x_{-i}) \text{ for all } \boldsymbol{x} \in \mathbf{R}^n.$$

Proof. The second result is a consequence of symmetry of $\boldsymbol{\Gamma}$. In particular, we note that: $\sum_{(i,j)\in N^2} \gamma^{ij} x_i x_j - 2\sum_{j\in N} \gamma^{ij} x_i x_j$ is constant with respect to x_i, and:

$$\begin{aligned} \pi_i(\boldsymbol{x}) - v_i(\boldsymbol{x}) &= -\left((1-\alpha)\delta_i + \alpha \sum_{j\in N} \gamma^{ij} x_j\right)^2, \\ v(\boldsymbol{x}) - v_i(\boldsymbol{x}) &= \sum_{j\in -i} \left(2(1-\alpha)\delta_j x_j - x_j^2\right) + \alpha \sum_{(i,j)\in N^2} \gamma^{ij} x_i x_j - 2\alpha \sum_{j\in N} \gamma^{ij} x_i x_j. \end{aligned}$$

■

A.3 Interim beliefs

In this section, we study player i's beliefs in the game $\mathcal{G}(\boldsymbol{p}_0)$, given $p_0^i \neq p_0^j$, for all $i, j \in N$ with $i \neq j$.

Every player knows the profile of status-quo policies $(p_0^1, \dots, p_0^n) \in \mathbf{R}^n$. Player i privately knows the outcome corresponding to her own status quo policy: $\chi(p_0^i)$. Player i believes that the outcome function $\chi\colon \mathbf{R} \to \mathbf{R}$ is the realized path of a Brownian motion with drift $\mu < 0$, variance parameter $\omega > 0$ and starting point $(p_0^i, \chi(p_0^i))$. This belief structure is consistent with a common prior that is studied in section A.3

Expectation and covariance We define $\mathbb{E}^i, \mathbb{V}^i, \mathbb{C}^i$ as, respectively, the conditional expectation, variance and covariance operators given knowledge of $\chi(p_0^i)$.

Lemma A.6. *The following formulas hold. For all policies $p, q \in \mathbf{R}$ we have:*

$$
\begin{aligned}
\mathbb{E}^i\chi(p) &:= \mathbb{E}\Big[\chi(p) \mid \chi(p_0^i)\Big] = \chi(p_0^i) + \mu(p - p_0^i),\\
\mathbb{V}^i\chi(p) &:= \mathrm{Var}\Big[\chi(p) \mid \chi(p_0^i)\Big] = |p - p_0^i|\omega,\\
\mathbb{C}^i(\chi(p), \chi(q)) &:= \mathrm{Cov}\Big[\chi(p), \chi(q) \mid \chi(p_0^i)\Big]\\
&= \begin{cases} \min\{\mathbb{V}^i\chi(p), \mathbb{V}^i\chi(q)\} & \textit{if } \mathrm{sgn}(p - p_0^i) = \mathrm{sgn}(q - p_0^i),\\ 0 & \textit{if } p > p_0^i > q \textit{ or } q > p_0^i > p. \end{cases}
\end{aligned}
$$

Proof. The formulas for the expectation and the variance operators are known in the experimentation literature (Callander, 2011a). Let's show that the covariance formula is a consequence of the Markov property of beliefs. By the law of iterated expectations:

$$
\begin{aligned}
\mathbb{C}^i(\chi(p), \chi(q)) =& \mathbb{E}\Big[\chi(p)\mathbb{E}\Big[\chi(q) \mid \chi(p), \chi(p_0^i)\Big] \mid \chi(p_0^i)\Big]\\
&- \mathbb{E}^i\chi(p)\mathbb{E}\Big[\mathbb{E}\Big[\chi(q) \mid \chi(p), \chi(p_0^i)\Big] \mid \chi(p_0^i)\Big].
\end{aligned}
$$

Moreover, if $q \geq p \geq p_0^i$, then: $\mathbb{E}[\chi(q) \mid \chi(p), \chi(p_0^i)] = \mathbb{E}[\chi(q) \mid \chi(p)]$, by the Markov property, so the covariance expression simplifies to

$$
\begin{aligned}
\mathbb{C}^i(\chi(p), \chi(q)) =& \mathbb{E}\Big[\chi(p)\mathbb{E}[\chi(q) \mid \chi(p)] \mid \chi(p_0^i)\Big] - \mathbb{E}^i\chi(p)\mathbb{E}\Big[\mathbb{E}[\chi(q) \mid \chi(p)] \mid \chi(p_0^i)\Big]\\
=& \mathbb{E}\Big[\chi(p)(\chi(p) + \mu(q - p)) \mid \chi(p_0^i)\Big]\\
&- \mathbb{E}^i\chi(p)\mathbb{E}\Big[\chi(p) + \mu(q - p) \mid \chi(p_0^i)\Big]\\
=& \mathbb{V}^i\chi(p),
\end{aligned}
$$

in which the second equality uses $\mathbb{E}[\chi(q) \mid \chi(p)] = \chi(p) + \mu(q - p)$. Instead, if $q > p_0^i > p$, then: $\mathbb{E}[\chi(q) \mid \chi(p), \chi(p_0^i)] = \mathbb{E}[\chi(q) \mid \chi(p_0^i)]$, by the Markov property, so the covariance expression simplifies to

$$
\begin{aligned}
\mathbb{C}^i(\chi(p), \chi(q)) &= \mathbb{E}\Big[\chi(p)\mathbb{E}\Big[\chi(q) \mid \chi(p_0^i)\Big] \mid \chi(p_0^i)\Big] - \mathbb{E}^i\chi(p)\mathbb{E}\Big[\chi(q) \mid \chi(p_0^i)\Big]\\
&= 0.
\end{aligned}
$$

Thus, the covariance formula holds. ■

The Brownian motion structure implies that the conditional distribution of $\chi(p)$ and $\chi(q)$ given $\chi(p_0^i)$ is jointly Gaussian, for all $p, q \in \mathbf{R} \setminus \{p_0^i\}$. The CDF of $\chi(q) \mid \chi(p_0^i)$ is denoted by $F(\cdot, q|\chi(p_0^i), p_0^i)$. The next result states that beliefs are monotone in status-quo outcome and admit an invariance property.

Proof of Lemma A.1

Lemma A.7 (FOSD and Translation Invariance of beliefs.)**.** *For all $y, y' \in \mathbf{R}$ such that $y \geq y'$,* *we have:*

$$F(\cdot, q|y, p_0^i) \leq F(\cdot, q|y', p_0^i) \quad \textit{pointwise for all } q, p_0^i \in \mathbf{R},$$

moreover: $F(x+\Delta, q|y+\Delta, p_0^i) = F(\cdot, q|y', p_0^i)$ *for all* $\Delta, x, y, q, p_0^i \in \mathbf{R}$.

Proof. Letting Φ be the CDF of a standard Gaussian random variable, we observe that $F(x', q|y', p_0^i) = \Phi\Big(\frac{x'-y'-\mu(q-p_0^i)}{\sqrt{|q-p_0^i|\omega}}\Big)$. ■

Derivatives of variance and covariance terms We define the left and right derivatives of $\mathbb{V}^i\chi(p)$ and $\mathbb{C}^i(\chi(p), \chi(q))$ with respect to p, using Iverson brackets ($[Y] = 1$ if Y is true, and $[Y] = 0$ otherwise). First, let's observe that:

$$\mathbb{C}^i(\chi(p), \chi(q)) = \begin{cases} (q-p_0^i)_+\omega & \text{if } q < p \text{ and } p \geq p_0^i, \\ (p-p_0^i)\omega & \text{if } p \leq q \text{ and } p \geq p_0^i, \\ (p_0^i-p)\omega & \text{if } q < p \text{ and } p < p_0^i, \\ (p_0^i-q)_+\omega & \text{if } p < q \text{ and } p < p_0^i, \end{cases}$$

from which it follows that:

$$\partial_-\mathbb{V}^i\chi(p) = \begin{cases} -\omega & p \leq p_0^i, \\ \omega & p > p_0^i, \end{cases} \qquad \partial_+\mathbb{V}^i\chi(p) = \begin{cases} -\omega & p < p_0^i, \\ \omega & p \geq p_0^i, \end{cases}$$
$$\partial_-\mathbb{C}^i(\chi(p), \chi(q)) = \begin{cases} [p \leq q]\omega & p > p_0^i, \\ -[p > q]\omega & p \leq p_0^i, \end{cases} \qquad \partial_+\mathbb{C}^i(\chi(p), \chi(q)) = \begin{cases} [p < q]\omega & p \geq p_0^i, \\ -[p \geq q]\omega & p < p_0^i. \end{cases}$$

In particular, we have that:

$$\begin{aligned} \partial\mathbb{C}^i(\chi(p), \chi(q)) &= \begin{cases} \partial_p(\min\{p, q\} - p_0^i)\omega & \text{if } p \geq p_0^i, \\ -\partial_p(\max\{p, q\} - p_0^i)\omega & \text{if } p < p_0^i. \end{cases} \\ &= \begin{cases} \Big(\frac{1}{2} - \frac{1}{2}\partial_p|p-q|\Big)\omega & \text{if } p \geq p_0^i, \\ \Big(-\frac{1}{2} - \frac{1}{2}\partial_p|p-q|\Big)\omega & \text{if } p < p_0^i. \end{cases} \\ &= \frac{1}{2}\Big(1 - 2[p < p_0^i] - \partial_p|p-q|\Big)\omega \\ \partial\mathbb{V}^i\chi(p) &= 1 - 2[p < p_0^i] \end{aligned}$$

Lemma A.8 (Concavity of "VCV"). *The function* $p_i \mapsto \sum_{(i,j)\in N^2} q^{ij} \operatorname{Cov}[\chi(p_i), \chi(p_j) \mid \chi(p_0^\star) =$

$x_0^\star]$ *is convex on* $\mathbf{R}$ *for all* $i \in N$ *and* $p_0^\star \in \mathbf{R}$*, and*

$$\begin{aligned} g_i(p_i, p_{-i}) &:= \partial_{+p_i} \sum_{(i,j)\in N^2} q^{ij} \operatorname{Cov}[\chi(p_i), \chi(p_j) \mid \chi(p_0^\star) = x_0^\star] \\ &= \boldsymbol{q}_{i\bullet}^\intercal \mathbf{1} - 2\boldsymbol{q}_{i\bullet}^\intercal \mathbf{1}[p_i < p_0^i] + \alpha \sum_{j\in N} g_{ij} \partial_{+p_i} |p_i - p_j| \\ &= \boldsymbol{q}_{i\bullet}^\intercal \mathbf{1} - 2\boldsymbol{q}_{i\bullet}^\intercal \mathbf{1}[p_i < p_0^i] + \alpha \sum_{j\in N} g_{ij} ([p_i \geq p_j] - [p_i < p_j]), \end{aligned}$$

and $g_i(p_i, \cdot)$ *is nonincreasing on* $\mathbf{R}^{n-1}$*. Moreover, the function* $\boldsymbol{p} \mapsto \sum_{(i,j)\in N^2} q^{ij} \operatorname{Cov}[\chi(p_i), \chi(p_j) \mid \chi(p_0^\star) = x_0^\star]^n$ *is convex on* $[p_0^\star, \overline{p}]^n$*.*

Proof. First, we show that the function $f\colon p_i \mapsto \sum_{(i,j)\in N^2} q^{ij} \operatorname{Cov}[\chi(p_i), \chi(p_j) \mid \chi(p_0^\star) = x_0^\star]$ is convex. By definition of $\boldsymbol{Q}$, we have that:

$$f(p_i) = \sum_{i\in N} \operatorname{Var}[\chi(p_i) \mid \chi(p_0^\star) = x_0^\star] - \sum_{(i,j)\in N^2} g_{ij} \operatorname{Cov}[\chi(p_i), \chi(p_j) \mid \chi(p_0^\star) = x_0^\star]$$

Thus, for all $i \in N$, assuming $\omega = 1$ without loss of generality, we have:

$$\begin{aligned} \partial_{+p_i} f(p_i) &= 1 - 2[p_i < p_0^i] - \alpha \sum_{j\in N} g_{ij} (-\partial_{+p_i} |p_i - p_j| - 2[p_i < p_0^i]) - \alpha \sum_{j\in N} g_{ij} \\ &= \boldsymbol{q}_{i\bullet}^\intercal \mathbf{1} - 2\boldsymbol{q}_{i\bullet}^\intercal \mathbf{1}[p_i < p_0^i] + \alpha \sum_{j\in N} g_{ij} \partial_{+p_i} |p_i - p_j|. \end{aligned}$$

Thus, $\partial_{+p_i} f$ is a nondecreasing function and so$f$ is convex on $\mathbf{R}$ (Rockafellar, 1970).

Let's show the second part of the lemma. Let's observe that:

$$\boldsymbol{p} \in [p_0^\star, \overline{p}]^n \implies f(p_i) = \sum_{(i,j)\in N^2} q^{ij} \min\{p_i - p_0^\star, p_j - p_0^\star\}\omega.$$

Joint convexity follows. ■

Common prior In this section, we define a common prior over the outcome function, parametrized by the amount of noise about the initial value of the Brownian motion. As the noise grows unboundedly large, the interim beliefs converge to the beliefs of the heterogeneous status quo game introduced in Section A.1 and analyzed in Section B.

Timeline Let's describe a timeline of a game. Every player knows the profile of status-quo policies $(p_0^1, \ldots, p_0^n) \in \mathbf{R}^n$.

(1) Nature draws the initial value $X(0)$ from a normal distribution with mean 0 and variance $\sigma_0^2 \geq 0$.

(2) Nature draws the outcome function $X\colon \mathbf{R} \to \mathbf{R}$ from a Brownian motion with drift $\mu < 0$, variance parameter $\omega > 0$ and starting point $(0, X(0))$.

(3) Player i privately observes the realization of signal S_i about $X(0)$ and the outcome corresponding to her own status quo policy: $X(p_0^i)$.

After (3), players update their beliefs using Bayes' rule, and then simultaneously choose real-valued policies. i's payoff from the policy profile $\boldsymbol{p}$ is $u_i(X(p_1), \ldots, X(p_n))$. We assume that $S_i = X(0) + \sigma\varepsilon_i$, for $\sigma \geq 0$ and a standard Gaussian random variable ε_i, and that for all pairs of players $i \neq k$, ε_i is independent from ε_k and from $X(0)$. To ease on notation, we assume that $\omega = 1$. In the limit as $\sigma_0 \to \infty$ and $\sigma \to \infty$, Bayes' rule for jointly Gaussian random variables gives us

$$\begin{aligned} \mathbb{E}[X(0) \mid I] &\to X(p_0^i), \\ \mathrm{Var}[X(0) \mid I] &\to p_0^i. \end{aligned}$$

To verify the second formula, let's observe that $X(p_0^i) - \mu p_0^i$ is an unbiased signal about $X(0)$, with precision $1/p_0^i$. In particular, for a Wiener process $W(\cdot)$, we have that:

$$X(p_0^i) - \mu p_0^i = X(0) + \omega(W(p_0^i) - W(0)),$$

and $W(p_0^i) - W(0)$ is Gaussian, centered at 0, with variance p_0^i. $W(p_0^i) - W(0)$ is independent of $X(0)$ and $(\varepsilon_i)_{i \in N}$ by our hypotheses.

Interim beliefs The information structure is parametrized by (σ_0, σ). In this section, we derive interim beliefs as a function of (σ_0, σ) and study the behavior as $(\sigma_0, \sigma) \to (\infty, \infty)$. Beliefs are described by Gaussian random variables, thus we study the expectation, variance and covariance terms of the outcomes $X(p), X(q)$ given the realization of $(S_i, X(p_0^i)) = I$, for $(p, q) \in \mathbf{R}^2$, with $q \leq p$. We claim that $\mathbb{E}[X(p) \mid I] \to \mathbb{E}[X(p) \mid X(p_0^i)]$ and $\mathrm{Cov}[X(p), X(q) \mid I] \to \mathrm{Cov}[X(p), X(q) \mid X(p_0^i)]$ for all $(p, q) \in \mathbf{R}^2$.

Case 1: $0 \leq p_0^i \leq q$. By the Markov property: $\mathbb{E}[X(p) \mid I] = \mathbb{E}[X(p) \mid X(p_0^i)]$, $\mathbb{E}[X(q) \mid I] = \mathbb{E}[X(q) \mid X(p_0^i)]$, and $\mathrm{Cov}[X(p), X(q) \mid I] = \mathrm{Cov}[X(p), X(q) \mid X(p_0^i)]$.

Case 2: $0 \leq q \leq p_0^i \leq p$. By the Brownian bridge properties, $\mathbb{E}[X(q) \mid I] = \mathbb{E}[X(q) \mid X(p_0^i)]$, using $\mathbb{E}[X(0) \mid I] = X(p_0^i)$. By the Markov property: $\mathbb{E}[X(p) \mid I] = \mathbb{E}[X(p) \mid X(p_0^i)]$. By the law of iterated covariance:

$$\begin{aligned} \mathrm{Cov}[X(p), X(q) \mid I] = {} & \mathbb{E}[\mathrm{Cov}[X(p), X(q) \mid X(0), I] | I] \\ & + \mathrm{Cov}[\mathbb{E}[X(p) \mid X(0), I], \mathbb{E}[X(q) \mid X(0), I] \mid I], \end{aligned}$$

By the Markov property, both terms on the right-hand side are 0.

Case 3: $0 \leq q \leq p \leq p_0^i$. By the Brownian bridge properties, $\mathbb{E}[X(q) \mid I] \to \mathbb{E}[X(q) \mid X(p_0^i)]$, using the formula for $\mathbb{E}[X(0) \mid I]$. Similarly, we obtain that $\mathbb{E}[X(p) \mid I] \to \mathbb{E}[X(p) \mid X(p_0^i)]$. Towards using the law of iterated covariance, we observe that, by the Brownian bridge properties

$$\mathrm{Cov}\Big[X(p), X(q) \mid X(p_0^i), X(0)\Big] = \frac{(p_0^i - p)q}{p_0^i}.$$

Moreover, for a, b, c, d given by the Brownian bridge properties

$$\begin{aligned}\operatorname{Cov}\Big[\mathbb{E}\Big[X(p) \mid X(0), X(p_0^i)\Big], \mathbb{E}\Big[X(q) \mid X(0), X(p_0^i)\Big] \mid X(p_0^i), S_i\Big] = \\ \operatorname{Cov}\Big[aX(0) + bX(p_0^i), cX(0) + dX(p_0^i) \mid X(p_0^i), S_i\Big],\end{aligned}$$

from which it follows that:

$$\operatorname{Cov}\Big[\mathbb{E}\Big[X(p) \mid X(0), X(p_0^i)\Big], \mathbb{E}\Big[X(q) \mid X(0), X(p_0^i)\Big] \mid X(p_0^i), S_i\Big] = ab\operatorname{Var}[X(0) \mid I].$$

By the Brownian bridge properties $ab = \frac{p_0^i - p}{p_0^i}\frac{p_0^i - q}{p_0^i}$. Using the law of iterated covariance and the formula for $\operatorname{Var}[X(0) \mid I]$, we observe that

$$\begin{aligned}\operatorname{Cov}[X(p), X(q) \mid I] &\to \frac{(p_0^i - p)q}{p_0^i} + \frac{p_0^i - p}{p_0^i}(p_0^i - q) \\ &\to p_0^i - p.\end{aligned}$$

The remaining cases are dealt with similarly.

A.4 Potential

For a profile of policies $\boldsymbol{p} \in P$, we denote the corresponding column vector of outcomes as $\boldsymbol{\chi}(\boldsymbol{p})$, or $\boldsymbol{\chi}$ if the policy profile is unambiguous. In this section, we study the following function:

$$\begin{aligned}V(\cdot, x_0)\colon P &\to \mathbf{R} \\ \boldsymbol{p} &\mapsto \mathbb{E}\{v(\boldsymbol{\chi}(\boldsymbol{p}))|\chi(p_0) = x_0\},\end{aligned}$$

under the assumption that $P_i = [p_0, \overline{p}]$ for all $i \in N$, for given $p_0, x_0 \in \mathbf{R}$.

Definition 2. *Let $x_0 \in \mathbf{R}$. An element of* $\operatorname{Arg\,max}_{\boldsymbol{p}\in[p_0,\overline{p}]^n} V(\boldsymbol{p}, x_0)$ *is called the* potential maximizer *given x_0.*

It will be useful to study $f(\boldsymbol{p}, x_0) = -V(\boldsymbol{p}, x_0)$, and also to omit the dependence on x_0 when it leads to no confusion. Moreover, we let $\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) = \mathbb{E}[\boldsymbol{\chi}(\boldsymbol{p})|\chi(p_0) = x_0]$.

Lemma A.9. *$f\colon \boldsymbol{p} \to -V(\boldsymbol{p}, x_0)$ is a strictly convex function on $\mathbf{R}^n$, and*

$$f(\boldsymbol{p}) = (\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta})^\mathsf{T}\boldsymbol{Q}(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta}) + \omega\boldsymbol{p}^\mathsf{T}\boldsymbol{Q}\mathbf{1} + \omega\sum_{(i,j)\in N^2} g_{ij}\frac{|p_i - p_j|}{2} - \boldsymbol{\beta}^\mathsf{T}\boldsymbol{Q}\boldsymbol{\beta}.$$

Proof. First, we observe that v is a quadratic function of the outcome profile. So, we have the next chain of equalities:

$$
\begin{aligned}
V(\boldsymbol{p}) &= -(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta})^{\mathsf{T}}\boldsymbol{Q}(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta}) - \sum_{(i,j)\in N^2} q_{ij}(\min\{p_i, p_j\} - p_0)\omega + \boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{Q}\boldsymbol{\beta} \\
&= -(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta})^{\mathsf{T}}\boldsymbol{Q}(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta}) - \sum_{(i,j)\in N^2} q_{ij}(p_i/2 + p_j/2)\omega + \\
&+ \sum_{(i,j)\in N^2} q_{ij}|p_i - p_j|\omega/2 + \boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{Q}\boldsymbol{\beta} \\
&= -(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta})^{\mathsf{T}}\boldsymbol{Q}(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta}) + \\
&+ \sum_{i\in N}(1 - \boldsymbol{g}_{i\bullet}^{\mathsf{T}}\mathbf{1})p_i\omega - \sum_{(i,j)\in N^2} g_{ij}|p_i - p_j|\omega/2 + \boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{Q}\boldsymbol{\beta}.
\end{aligned}
$$

The second equality expresses $\min\{p_i, p_j\} = \frac{p_i + p_j - |p_i - p_j|}{2}$, and the third uses the definition of $\boldsymbol{Q}$. ■

Towards finding the potential maximizer, we find the subdifferential of f, and ∂ denotes the subdifferential operator with respect to the vector of policies $\boldsymbol{p}$. By the above Lemma, we have that:

$$
\begin{aligned}
\partial f(\boldsymbol{p}) &= 2\mu\boldsymbol{Q}(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta}) + \boldsymbol{Q}\mathbf{1}\omega + \frac{\omega}{2}\partial \sum_{(i,j)\in N^2} g_{ij}|p_i - p_j| \\
\frac{\partial f(p)}{2\mu} &= \boldsymbol{Q}(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta} - \mathbf{1}k) - \frac{k}{2}\partial \sum_{(i,j)\in N^2} g_{ij}|p_i - p_j|.
\end{aligned}
$$

The subdifferential of f is

$$
\begin{aligned}
\partial f(\boldsymbol{p}) = \Big\{\boldsymbol{y} \in \mathbf{R}^n : \frac{\boldsymbol{y}}{2\mu} = \boldsymbol{Q}(\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) - \boldsymbol{\beta} - \mathbf{1}k) - (\boldsymbol{G} \odot \boldsymbol{A})\mathbf{1}k,\ &\text{for some } \boldsymbol{A} \text{ such that} \\
a_{ij} = -a_{ji}, p_i > p_j \implies a_{ij} = 1, p_i = p_j \implies a_{ij} &\in [-1, 1]\Big\}.
\end{aligned}
$$

Let $\mathbf{0}$ be a column of zeroes and $I_S\colon \mathbf{R}^n \to \mathbf{R}$ be the characteristic function of $S \subseteq \mathbf{R}^n$. By strict convexity of f and convexity of P, standard results in convex analysis (Rockafellar, 1970) imply that the potential maximizer is the unique $\boldsymbol{p} \in P$ such that:

$$
\mathbf{0} \in \partial f(\boldsymbol{p}) + \partial I_P(\boldsymbol{p}).
$$

Lemma A.10. *There exists a unique potential maximizer given $x_0 \in \mathbf{R}$. Moreover, $\boldsymbol{p} \in (p_0, \overline{p})^n$ is the unique potential maximizer given $x_0 \in \mathbf{R}$ if, and only if:*

$$
\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) = \boldsymbol{\beta} + \mathbf{1}k + (\boldsymbol{I} - \boldsymbol{G})^{-1}(\boldsymbol{G} \odot \boldsymbol{A})\mathbf{1}k,
$$

for some skew-symmetric $\boldsymbol{A} = [a_{ij} : i, j \in N]$ such that:

$$
a_{ij} = 1 \text{ if } p_i > p_j, \text{ and } a_{ij} \in [-1, 1] \text{ if } p_i = p_j, \text{ for all } i, j \in N.
$$

Proof. For interior $\boldsymbol{p}$, it is necessary and sufficient that $\mathbf{0} \in \partial f(\boldsymbol{p})$. The result follows from the preceding derivation. ■

B Proofs for Appendix A.1

B.1 General model

In this section, we study the heterogeneous-status-quo game. We formulate it as a Bayesian game and study its Bayesian Nash equilibria. The definition of the Bayesian game and of Bayesian Nash equilibria are in terms of interim beliefs, and follow closely the respective definitions in Van Zandt and Vives (2007). The following definitions depend on a vector of status-quo policies $\boldsymbol{p}_0$ such that: $p_0^i \neq p_0^j$, for all players i, j with $i \neq j$. Thus, the heterogeneous status-quo game given $\boldsymbol{p}_0$ is $\mathcal{G}(\boldsymbol{p}_0)$. In this section, we maintain Assumption 3.

Components of the game

(1) The set of players is N.

(2) The type space of player i is $(\mathbf{R}, \mathcal{B})$, in which $\mathcal{B}$ is the Borel sigma-algebra; the typical type of player i is denoted by x_0^i.

(3) Player i's type-dependent beliefs are represented by an $n-1$-dimensional Gaussian random vector $(\chi(p_0^j))_{j\in -i}$ with expectation and variance-covariance that are functions of i's type. Let $j, k \in N \setminus \{i\}$, and x_0^i be i's type, then: the expectation and variance-covariance of $(\chi(p_0^j))_{j\in -i}$ are given, respectively, by $\mathbb{E}\Big[\chi(p_0^j) \mid \chi(p_0^i) = x_0^i\Big]$ and $\mathrm{Cov}\Big[\chi(p_0^j), \chi(p_0^k) \mid \chi(p_0^i) = x_0^i\Big]$, which are defined in Section A.3. Let $f_i(\cdot|x_0^i)\colon \mathbf{R}^{n-1} \to \mathbf{R}$ be the PDF of the Gaussian random vector $(\chi(p_0^j))_{j\in -i}$ with mean and variance-covariance as above. We note that f_i is well-defined because $p_0^i \neq p_0^j$, for all players i, j with $i \neq j$. Thus, player i's type-dependent belief is such that: for every measurable $A \subseteq \mathbf{R}^{n-1}$ and type $x_0^i \in \mathbf{R}$, we have the following formula for the probability of A:

$$\mathbb{P}((\chi(p_0^j))_{j\in -i} \in A | x_0^i) = \int_A f(x_0^{-i}|x_0^i)\, \mathrm{d}x_0^{-i}.$$

In particular, let's define $p_i(x_0^i)$ as the probability measure on $\mathbf{R}^{n-1}$ induced by the set-valued mapping $A \mapsto \int_A f(x_0^{-i}|x_0^i)\, \mathrm{d}x_0^{-i}$. The function $x_0^i \mapsto p_i(x_0^i)$ gives player i's interim beliefs.

(4) The action set of player i is $P_i = [\underline{p}_i, \overline{p}_i]$, for $\underline{p}_i < \overline{p}_i$, and $\underline{p}_i, \overline{p}_i \in \mathbf{R}$; we let $P := \times_{i\in N} P_i$ and $P_{-i} := \times_{j\in -i} P_j$.

(5) The payoff of player i is $u_i\colon P \times \mathbf{R} \to \mathbf{R}$, such that:

$$u_i(\boldsymbol{p}, x_0^i) = \mathbb{E}\Big[\pi_i(\chi(p_1), \dots \chi(p_n)) \mid \chi(p_0^i) = x_0^i\Big].$$

Properties of the components of the game In this section, the superdifferential operator ∂ refers to differentiation with respect to i's policy p_i.

Lemma B.11 (Best-response equivalence)**.** *For all $i \in N$, there exist $h_i, g_i \colon P_{-i} \times \mathbf{R} \to \mathbf{R}$ such that, letting $\boldsymbol{\chi} = (\chi(p_1), \ldots, \chi(p_n))$:*

$$\mathbb{E}\Big[\pi_i(\boldsymbol{\chi}) \mid \chi(p_0^i) = x_0^i\Big] - \mathbb{E}\Big[v_i(\boldsymbol{\chi}) \mid \chi(p_0^i) = x_0^i\Big] = h_i(p_{-i}, x_0^i)$$

$$\textit{and } \mathbb{E}\Big[\pi_i(\boldsymbol{\chi}) \mid \chi(p_0^i) = x_0^i\Big] - \mathbb{E}\Big[v(\boldsymbol{\chi}) \mid \chi(p_0^i) = x_0^i\Big] = g_i(p_{-i}, x_0^i) \textit{ for all } \boldsymbol{p} \in P, x_0^i \in \mathbf{R}.$$

Proof. Follows from VNM Equivalence established in Lemma A.5. ■

Proof of Lemma A.2

Lemma B.12 (Lemma A.2)**.** *The function $u_i(\cdot, x_0^i)$ exhibits strictly increasing differences in (p_i, p_{-i}) for all $x_0^i \in \mathbf{R}$, and the function $u_i((\cdot, p_{-i}), \cdot)$ exhibits strictly increasing differences in (p_i, x_0^i) for all $p_{-i} \in P_{-i}$. Moreover, $u_i((\cdot, p_{-i}), x_0^i)$ is strictly concave.*

Proof. First, we establish strict concavity of $u_i((\cdot, p_{-i}), x_0^i)$. For a profile of policies of i's opponents p_{-i} and $x_0^i \in \mathbf{R}$, we study the function

$$\begin{aligned} \boldsymbol{p} \mapsto -(x_0^i \mathbf{1} + \mu(\boldsymbol{p} - p_0^i \mathbf{1}) - \boldsymbol{\beta})^{\mathsf{T}} \boldsymbol{Q} (x_0^i \mathbf{1} + \mu(\boldsymbol{p} - p_0^i \mathbf{1}) - \boldsymbol{\beta}) \\ - \sum_{(i,j) \in N^2} q^{ij} \operatorname{Cov}[\chi(p_i), \chi(p_j) \mid \chi(p_0^i) = x_0^i] \end{aligned}$$

First, we observe that $\boldsymbol{p} \mapsto -(x_0^i \mathbf{1} + \mu(\boldsymbol{p} - p_0^i \mathbf{1}) - \boldsymbol{\beta})^{\mathsf{T}} \boldsymbol{Q} (x_0^i \mathbf{1} + \mu(\boldsymbol{p} - p_0^i \mathbf{1}) - \boldsymbol{\beta})$ is strictly concave on $\mathbf{R}^n$ because $\boldsymbol{Q}$ is positive definite. Strict concavity follows from previous results and Best-Response Equivalence.

Let's establish strictly increasing differences in (p_i, x_0^i). By absolute continuity of the concave function $u_i((\cdot, p_{-i}), x_0^i)$:

$$u_i((r_i, p_{-i}), x_0^i) - u_i((q_i, p_{-i}), x_0^i) = \int_{q_i}^{r_i} \partial_- u_i((p_i, p_{-i}), x_0^i) \, \mathrm{d}p_i.$$

By the formulas from Lemma A.8

$$\begin{aligned} \partial_- u_i(p_i, p_{-i}, x_0^i) = -2\mu \boldsymbol{q}_{i\bullet}^{\mathsf{T}} (\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta}) \\ - \boldsymbol{q}_{i\bullet}^{\mathsf{T}} \mathbf{1} \omega + 2 \boldsymbol{q}_{i\bullet}^{\mathsf{T}} \mathbf{1} [p_i < p_0^i] \omega - \alpha \sum_{j \in N} g_{ij} \partial_{-p_i} |p_i - p_j| \omega. \end{aligned}$$

We observe that: (i) monotonicity of $F(\cdot, p_j; t_i, p_0^i)$ in i's own type (Lemma A.7) and (ii) strict diagonal dominance of $\boldsymbol{Q}$ jointly imply that $\partial_- u_i(p_i, p_{-i}, x_0^i)$ is strictly increasing in x_0^i, thus the function $u_i((\cdot, p_{-i}), \cdot)$ has strictly increasing differences in (p_i, x_0^i) for all $p_{-i} \in P_{-i}$.

Similarly, we establish that the function $u_i(\cdot, x_0^i)$ has strictly increasing differences in (p_i, p_{-i}) for all $x_0^i \in \mathbf{R}$ by monotonicity of $\partial_- u_i(p_i, p_{-i}, x_0^i)$ with respect to p_{-i}, established in Lemma A.8. ■

Given the strategic complementarities established in Lemma B.12, we draw on the toolset developed by the literature on incomplete-information games with complementarities to show that a greatest and a least equilibria exist and are in monotone strategies. Since payoffs in $\mathcal{G}(\boldsymbol{p}_0)$ are not necessarily bounded, we leverage strict concavity of expected payoffs in own action and compactness of action spaces to establish similar results to (Van Zandt and Vives, 2007).

Remark B.1. *Let's observe that: "Assumption 1.", "Assumption 2.", "Assumption 3.", "Part (1) of Assumption 4.", and "Part (2) of Assumption 4." from Van Zandt and Vives (2007) hold. Assumption 1. holds because we endow the type space of player i, $\mathbf{R}$, with the usual order. Assumption 2. holds because P_i is a compact interval of the real line, and we endow P_i with the usual metric, so P_i is a lattice. Let's show that Assumption 3. holds by verifying that $x_0^i \to \int_A f(x_0^{-i}|x_0^i)\,\mathrm{d}x_0^{-i}$ is measurable. Measurability holds because f is a well-defined and a continuous real-valued function of x_0^i on $\mathbf{R}$. In particular, x_0^i enters f only through the expected value of $(\chi(p_0^j))_{j\in -i}$. $u_i(\boldsymbol{p},\cdot)$ is a real-valued continuous function on $\mathbf{R}$ for all $\boldsymbol{p}\in P$, and $u_i(\cdot, x_0^i)$ defines a real-valued continuous function on $\mathbf{R}^n$ by concavity of $u_i(\cdot, x_0^i)$; thus, parts (1) and (2) of Assumption 4. hold.*

Strategies and equilibrium A strategy for player i is a measurable function $\sigma_i\colon \mathbf{R}\to P_i$. Let Σ_i denote the set of strategies for player i. Let $\Sigma := \times_{i\in N}\Sigma_i$ denote the set of strategy profiles, and let $\Sigma_{-i} = \times_{i\in -i}\Sigma_j$ denote the set of profiles of strategies for players other than i. Σ_i is endowed with the pointwise order to be a lattice, Σ_{-i} and Σ are endowed with the product order and $\leq$ denotes every partial order

We use the following shorthand notation given a profile of strategies of i's opponents $\sigma_{-i} = (\dots,\sigma_{i-1},\sigma_{i+1},\dots)$:

$$\begin{aligned}\chi_{-i} = \chi(\sigma_{-i}) &= (\dots,\chi(\sigma_{i-1}(\chi(p_0^{i-1}))),\chi(\sigma_{i+1}(\chi(p_0^{i+1}))),\dots)\\ (\chi_i,\chi_{-i}) = (\chi(p_i),\chi(\sigma_{-i})) &= (\dots,\chi(\sigma_{i-1}(\chi(p_0^{i-1}))),\chi(p_i),\chi(\sigma_{i+1}(\chi(p_0^{i+1}))),\dots),\end{aligned}$$

and $\boldsymbol{\chi}$ is the column vector of outcomes corresponding to (χ_i,χ_{-i}).

The expected payoff of player i, given σ_{-i}, is

$$U_i(p_i, x_0^i;\sigma_{-i}) := \mathbb{E}\{u_i(\chi(p_i),\chi(\sigma_{-i}))|\chi(p_0^i) = x_0^i\},\ x_0^i, p_i\in\mathbf{R}.$$

We use $U_i(p_i, x_0^i;\sigma_{-i},\boldsymbol{p}_0)$ when the particular status-quo policy profile is important; we note that $U_i(p_i, x_0^i;\sigma_{-i},\boldsymbol{p}_0)$ depends on p_0^j through $F(\cdot, p_0^j; x_0^i, p_0^i)$ if $j\neq i$. Let $\varphi_i(x_0^i;\sigma_{-i})$ be the set of policies that maximize $U_i(p_i, x_0^i;\sigma_{-i})$,

$$\varphi_i(x_0^i;\sigma_{-i}) := \operatorname*{Arg\,max}_{p_i\in P_i} U_i(p_i, x_0^i;\sigma_{-i}).$$

Then, we have that $\sigma\in\Sigma$ is a Bayesian Nash equilibrium if, and only if:

$$\sigma_i(x_0^i)\in\varphi_i(x_0^i;\sigma_{-i}),\quad \text{for all } x_0^i\in\mathbf{R},\ i\in N.$$

Let $\beta_i \colon \Sigma_{-i} \to \Sigma_i$ denote player i's best-response correspondence

$$\beta_i(\sigma_{-i}) := \{\sigma_i \in \Sigma_i : \sigma_i(x_0^i) \in \varphi_i(x_0^i; \sigma_{-i}) \text{ for all } x_0^i \in \mathbf{R}\}.$$

Lemma B.13. *The expected payoff to player i is, up to a term that is constant with respect to i's policy p_i:*

$$\begin{aligned} U_i(p_i, x_0^i; \sigma_{-i}) = &- (\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta})^\mathsf{T} \boldsymbol{Q} (\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta}) \\ &- \mathbb{V}[\chi(p_i) \mid \chi(p_0^i) = x_0^i] \\ &- 2 \sum_{j \in -i} q^{ij} \int_{x_0^j \in \mathbf{R}} \operatorname{Cov}[\chi(p_i), \chi(s_j(x_0^j)) \mid \chi(p_0^i) = x_0^i] \,\mathrm{d}F(x_0^j, p_0^j; x_0^i, p_0^i). \end{aligned}$$

Moreover:

(1) $U_i(p_i, x_0^i; \sigma_{-i})$ is strictly concave in p_i.

(2) $U_i(p_i, x_0^i; \sigma_{-i})$ exhibits strictly increasing differences in (p_i, x_0^i) if σ_{-i} is a profile of nondecreasing strategies.

Proof. First, we establish strict concavity using a result in Radner (1962) ("Lemma", p. 863) and Lemma B.12.

Let's establish strictly increasing differences in (p_i, x_0^i). By absolute continuity of the concave function $U_i(\cdot, x_0^i; \sigma_{-i})$, we have

$$U_i(r_i, x_0^i; \sigma_{-i}) - U_i(q_i, x_0^i; \sigma_{-i}) = \int_{q_i}^{r_i} \partial_- U_i(p_i, x_0^i; \sigma_{-i}) \,\mathrm{d}p_i.$$

We inspect monotonicity of $\partial_- U_i(p_i, x_0^i; \sigma_{-i})$ with respect to t_i, using the formulas in Lemma B.12 and Lemma A.8. Our proof is complete given: (i) monotonicity of $F(\cdot, p_0^j; x_0^i, p_0^i)$ in the sense of FOSD with respect to x_0^i (Lemma A.7), and (ii) strict diagonal dominance of $\boldsymbol{Q}$. ■

Remark B.2. *Item (2) in Lemma B.13 implies that the Single Crossing Condition for games of incomplete information (Athey, 2001) is satisfied in $\mathcal{G}(\boldsymbol{p}_0)$. The reason is that strictly increasing differences imply the Milgrom-Shannon single-crossing property of incremental returns.*

The following result restricts the type spaces to compact sets.

Lemma B.14 (Compact type spaces)**.** *For all i, there exist types $\underline{x}_0^i, \overline{x}_0^i \in \mathbf{R}$, such that:*

$$\begin{aligned} x_0^i > \overline{x}_0^i &\implies \varphi_i(x_0^i, \sigma_{-i}) = \{\overline{p}_i\}, \text{ for all } \sigma_{-i} \in \Sigma_{-i} \\ \text{and } x_0^i < \underline{x}_0^i &\implies \varphi_i(x_0^i, \sigma_{-i}) = \{\underline{p}_i\}, \text{ for all } \sigma_{-i} \in \Sigma_{-i}. \end{aligned}$$

Proof. We establish the first claim. Let $\underline{\sigma}_{-i}$ be the least element in Σ_{-i}, which is given by a profile of constant functions. Let $\overline{x}_0^i$ be such that: $\overline{p}_i \in \varphi_i(\overline{x}_0^i, \underline{\sigma}_{-i})$. $\overline{x}_0^i$ is well-defined by an application of Topkis' Theorem, because (i) $\varphi_i(\cdot, \underline{\sigma}_{-i})$ is nonempty-valued and continuous correspondence (by strict concavity of $U_i(p_i, x_0^i; \underline{\sigma}_{-i})$ as a function of p_i and Berge's Theorem, respectively), and (ii) $U_i(p_i, x_0^i; \underline{\sigma}_{-i})$ exhibits strictly increasing differences in p_i, x_0^i on $P_i \times \mathbf{R}$. $U_i(p_i, x_0^i; \underline{\sigma}_{-i})$ exhibits increasing differences in (p_i, x_0^i) (Lemma B.13), thus $x_0^i > \overline{x}_0^i \implies \varphi_i(x_0^i, \underline{\sigma}_{-i}) = \{\overline{p}_i\}$. The first follows because $U_i(p_i, x_0^i; \sigma_{-i})$ exhibits increasing differences in (p_i, σ_{-i}). The second claim is established analogously. ■

Lemma B.15 (Measurability of "GBR")**.** *The mapping $x_0^i \to \sup \varphi_i(x_0^i; \sigma_{-i})$ is measurable.*

Proof. By strict concavity of $U_i(\cdot, x_0^i; \sigma_{-i})$, its maximizer on P_i exists and is unique, so

$$\sup \varphi_i(x_0^i; \sigma_{-i}) = \varphi_i(x_0^i; \sigma_{-i})$$

. $U_i(p_i, \cdot; \sigma_{-i})$ is continuous, so by Berge's maximum theorem the unique selection from $\varphi_i(\cdot; \sigma_{-i})$ is a real-valued continuous function on $\mathbf{R}$. The claim follows from Corollary 4.26 in Aliprantis and Border (2006). ■

Remark B.3. *Lemma B.15 admits a different proof that is similar to the apprach taken by Van Zandt and Vives (2007). Let's observe that $U_i(p_i, \cdot; \sigma_{-i})$ is a continuous real-valued function on $\mathbf{R}$ by Lemma B.13. Let's observe that $U_i(\cdot; \sigma_{-i})$ is continuous in i's own policy, and measurable in i's own type. Thus, $U_i(\cdot; \sigma_{-i})$ is a Carathéodory function. Therefore, the Measurable Maximum Theorem (Aliprantis and Border (2006), Theorem 18.19) holds.*

If σ_i is a nondecreasing function, by Lemma B.14 its generalized inverse σ_i^{-1} is well-defined:

$$\sigma_i^{-1}(p_i) = \inf\Big\{x_0^i \in \mathbf{R} \ : \ p_i \le \sigma_i(x_0^i)\Big\}, \quad p_i \in P_i.$$

Moreover, if σ_i is nondecreasing, σ_i^{-1} is nondecreasing, left-continuous and admits a limit from the right at each point given Lemma B.14. We define σ_i^{-} to be the generalized inverse of σ_i extended by continuity to be a correspondence:

$$\begin{aligned} \sigma_i^{-} &: P_i \rightrightarrows \mathbf{R} \\ & p_i \mapsto [\sigma_i^{-1}(p_i), \lim_{p_i' \to p_i^+} \sigma_i^{-1}(p_i')] =: [\sigma_{i1}^{-}(p_i), \sigma_{i2}^{-}(p_i)]. \end{aligned}$$

Proof of Lemma A.3 The result is a consequence of the following Lemma.

Lemma B.16. *If σ is a Bayesian Nash equilibrium, the left and right derivatives of $U_i(p_i, x_0^i; \sigma_{-i})$ with respect to p_i and evaluated at $p_i = \sigma_i(x_0^i)$ are, respectively:*

$$\partial_{-}U_i(p_i, x_0^i; \sigma_{-i}) = \begin{cases} -2\mu \boldsymbol{q}_{i\bullet}^{\intercal}(\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta}) - \boldsymbol{q}_{i\bullet}^{\intercal}\mathbf{1}\omega - & \\ \sum_j g_{ij}[2F(\sigma_{j1}^{-1}(p_i), p_0^j; x_0^i, p_0^i) - 1]\omega & \text{if } p_i > p_0^i, \\ -2\mu \boldsymbol{q}_{i\bullet}^{\intercal}(\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta}) + \boldsymbol{q}_{i\bullet}^{\intercal}\mathbf{1}\omega - & \\ \sum_{j \in -i} g_{ij}[2F(\sigma_{j1}^{-1}(p_i), p_0^j; x_0^i, p_0^i) - 1]\omega & \text{if } p_i \le p_0^i, \end{cases}$$

$$\partial_{+}U_i(p_i, x_0^i; \sigma_{-i}) = \begin{cases} -2\mu \boldsymbol{q}_{i\bullet}^{\intercal}(\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta}) - \boldsymbol{q}_{i\bullet}^{\intercal}\mathbf{1}\omega - & \\ \sum_j g_{ij}[2F(\sigma_{j2}^{-1}(p_i), p_0^j; x_0^i, p_0^i) - 1]\omega & \text{if } p_i \ge p_0^i, \\ -2\mu \boldsymbol{q}_{i\bullet}^{\intercal}(\mathbb{E}[\boldsymbol{\chi} \mid \chi(p_0^i) = x_0^i] - \boldsymbol{\beta}) + \boldsymbol{q}_{i\bullet}^{\intercal}\mathbf{1}\omega - & \\ \sum_j g_{ij}[2F(\sigma_{j2}^{-1}(p_i), p_0^j; x_0^i, p_0^i) - 1]\omega & \text{if } p_i < p_0^i. \end{cases}$$

Proof. The result follows from Lemma B.13 and the expression for the covariance in Section A.3. ■

Lemma B.17. *Let σ_{-i} be a profile of nondecreasing strategies of i's opponents. Then: $\varphi_i(\cdot;\sigma_{-i})$ is nonempty-valued, uniquely-valued, continuous and nondecreasing in the strong set order.*

Proof. $\varphi_i(\cdot;s_{-i})$ is nonempty-valued, uniquely-valued and continuous by Berge's Theorem, since: (i) P_i is nonempty and compact, and (ii) $U_i(\cdot,x_0^i;\sigma_{-i})$ is strictly concave (Lemma B.13), and $U_i(p_i,x_0^i;\sigma_{-i})$ is a continuous function of x_0 (Lemma B.13, noting that $U_i(p_i,x_0;s_{-i})$ is a strictly concave function of x_0).

$\varphi_i(\cdot;\sigma_{-i})$ is nondecreasing by Topkis' Theorem (Topkis (1978), Theorem 6.3), because $U_i(p_i,x_0^i;\sigma_{-i})$ exhibits strictly increasing differences in (p_i,x_0^i) (Lemma B.13). ■

Lemma B.18. *The strategy profile of nondecreasing strategies σ is a Bayesian Nash equilibrium if, and only if, the following conditions are satisfied for all $i \in N$, $x_0^i \in \mathbf{R}$.*

$$
\begin{aligned}
&k\sum_{j\in N} g_{ij}[2F(\sigma_{j2}^{-1}(\sigma_i(x_0^i)),p_0^j;x_0^i,p_0^i)-1] \geq \boldsymbol{q}_{i\bullet}^{\mathsf{T}}(\mathbb{E}[\boldsymbol{\chi}\mid\chi(p_0^i)=x_0^i]-\boldsymbol{\beta}-\mathbf{1}k)\\
&\qquad\geq k\sum_{j\in N} g_{ij}[2F(\sigma_{j1}^{-1}(\sigma_i(x_0^i)),p_0^j;x_0^i,p_0^i)-1] \text{ if } \sigma_i(x_0^i)>p_0^i,\\
&k\sum_{j\in N} g_{ij}[2F(\sigma_{j2}^{-1}(\sigma_i(x_0^i)),p_0^j;x_0^i,p_0^i)-1] \geq \boldsymbol{q}_{i\bullet}^{\mathsf{T}}(\mathbb{E}[\boldsymbol{\chi}\mid\chi(p_0^i)=x_0^i]-\boldsymbol{\beta}+\mathbf{1}k)\\
&\qquad\geq k\sum_{j\in N} g_{ij}[2F(\sigma_{j1}^{-1}(\sigma_i(x_0^i)),p_0^j;x_0^i,p_0^i)-1] \text{ if } \sigma_i(x_0^i)<p_0^i,\\
&k\boldsymbol{q}_{i\bullet}^{\mathsf{T}}\mathbf{1}+k\sum_{j\in N} g_{ij}[2F(\sigma_{j2}^{-1}(\sigma_i(x_0^i)),p_0^j;x_0^i,p_0^i)-1] \geq \boldsymbol{q}_{i\bullet}^{\mathsf{T}}(\mathbb{E}[\boldsymbol{\chi}\mid\chi(p_0^i)=x_0^i]-\boldsymbol{\beta})\\
&\qquad\geq -k\boldsymbol{q}_{i\bullet}^{\mathsf{T}}\mathbf{1}+k\sum_{j\in N} g_{ij}[2F(\sigma_{j1}^{-1}(\sigma_i(x_0^i)),p_0^j;x_0^i,p_0^i)-1] \text{ if } \sigma_i(x_0^i)=p_0^i.
\end{aligned}
$$

Proof of Lemma A.4

Proof. The result is a consequence of the above Lemma. Assuming $\underline{p}_i = p_0^i$, the strategy profile of nondecreasing strategies σ is a Bayesian Nash equilibrium if, and only if, the following condition is satisfied. For all $i \in N$ and $x_0^i \in \mathbf{R}$ such that $\sigma_i(x_0^i) > p_0^i$, there exists a matrix $\boldsymbol{A} = [a_{ij}]$, such that:

$$\mathbb{E}[\boldsymbol{\chi}\mid\chi(p_0^i)=x_0^i] = \boldsymbol{\beta}+\mathbf{1}k+\boldsymbol{Q}^{-1}\boldsymbol{G}\odot\boldsymbol{A}\mathbf{1}k,$$

and $a_{ij} \in [2F(\sigma_{j1}^{-1}(\sigma_i(x_0^i)),p_0^j;x_0^i,p_0^i)-1, 2F(\sigma_{j2}^{-1}(\sigma_i(x_0^i)),p_0^j;x_0^i,p_0^i)-1]$. ■

Existence of Bayesian Nash equilibria

Lemma B.19 (Properties of GBR mapping)**.** *The following hold.*

(1) $\beta_i(\sigma_{-i})$ has a greatest element, which we call $\overline{\beta}_i(\sigma_{-i})$, for all $\sigma_{-i} \in \Sigma_{-i}$.

(2) For $\sigma'_{-i},\sigma_{-i} \in \Sigma_{-i}$ such that $\sigma'_{-i} \geq \sigma_{-i}$, we have that $\beta_i(\sigma'_{-i}) \geq \beta_i(\sigma_{-i})$.

(3) *If the strategies in σ_{-i} are nondecreasing, then the unique strategy given by $\overline{\beta}_i(\sigma_{-i})$ is nondecreasing (in i's type).*

Proof. $U_i(p_i, x_0^i; \sigma_{-i})$ is continuous as a function of p_i and has increasing differences in p_i, σ_{-i} because increasing differences are preserved by integration. Thus, by "Lemma 7" in Van Zandt and Vives (2007), $\varphi_i(x_0^i; \sigma_{-i})$ is a nonempty complete lattice, and (2) holds.

(3) is established in Lemma B.17.

(1) is a consequence of Lemma B.15. ■

Proof of Proposition A.1

Lemma B.20 (Proposition A.1)**.** *There exist a greatest and a least Bayesian Nash equilibrium, and they are in nondecreasing strategies.*

Proof. $u_i(\cdot, x_0^i)$ is a continuous real-valued function on the compact set P, so $u_i(\cdot, x_0^i)$ is bounded. Given Lemma B.19, the proof follows from the same argument as that of "Lemma 6" in Van Zandt and Vives (2007). ■

Proof of Proposition A.2 The result is a consequence of the following result, which upper bounds the distance between two equilibrium strategies of any player, in the sense of the sup norm.

Lemma B.21. *If $\overline{\sigma}_i(x_0^i) - \underline{\sigma}_i(x_0^i) > c$, for $i \in N, x_0^i \in \mathbf{R}, c > 0$, then:*

$$\omega > \mu^2 \frac{1}{\frac{\alpha \sum_j g_{ij}}{1-\alpha \sum_j g_{ij}}} c$$

Equivalently, if $\omega \leq v$, then: $\max_{i \in N} |\overline{\sigma}_i - \underline{\sigma}_i| \leq v \frac{\alpha \sum_j g_{ij}}{1-\alpha \sum_j g_{ij}} / (\mu^2)$.

Proof. Let $\overline{\sigma}, \underline{\sigma} \in \Sigma$ be, respectively, the greatest and least Bayesian Nash equilibria, and suppose that they are distinct elements of Σ. Let $i \in N$ be such that: $i \in \operatorname{Arg\,max}_{i' \in N} \max_{x_0^{i'} \in \mathbf{R}} \overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'})$. First, we verify that i is well defined. By hypothesis, $\overline{\sigma}_{i'} \geq \underline{\sigma}_{i'}$ pointwise. Thus, $x_0^{i'} \mapsto \overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'})$ is bounded below pointwise by a constant function that takes value 0, and bounded above pointwise by a constant function that takes value $\max_{j \in N} \overline{p}_j - \underline{p}_j > 0$. It follows that $\sup\{\overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'}) : x_0^{i'} \in [\underline{x}_0^{i'}, \overline{x}_0^{i'}]\}$ is well defined, and $\sup\{\overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'}) : x_0^{i'} \in [\underline{x}_0^{i'}, \overline{x}_0^{i'}]\} = \max\{\overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'}) : x_0^{i'} \in [\underline{x}_0^{i'}, \overline{x}_0^{i'}]\}$ because $\underline{\sigma}_{i'}, \overline{\sigma}_{i'}$ are continuous by Berge's Theorem (Lemma B.17). By Lemma B.14 result, $\max\{\overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'}) : x_0^{i'} \in [\underline{x}_0^{i'}, \overline{x}_0^{i'}]\} \geq \max\{\overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'}) : x_0^{i'} \notin [\underline{x}_0^{i'}, \overline{x}_0^{i'}]\} = \{0\}$. It follows that $\operatorname{Arg\,max}_{x_0^{i'} \in \mathbf{R}} \overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'}) \subseteq [\underline{x}_0^{i'}, \overline{x}_0^{i'}]$. Thus, $\max_{i' \in N} \max_{x_0^{i'} \in \mathbf{R}} \overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'})$ has a solution. It follows that i is well defined.

Let $y_{i'} \in \operatorname{Arg\,max}_{x_0^{i'} \in \mathbf{R}} \overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'})$, for all $i' \in N$. The problem $\max_{i' \in N} \overline{\sigma}_{i'}(y_{i'}) - \underline{\sigma}_{i'}(y_{i'})$ has a solution, which we denote by j, and we define $t := y_j$. By definition of $y_{i'}, i' \in N$, we have that $\max_{i' \in N} \overline{\sigma}_{i'}(y_{i'}) - \underline{\sigma}_{i'}(y_{i'}) \geq \max_{i' \in N} \max_{x_0^{i'} \in \mathbf{R}} \overline{\sigma}_{i'}(x_0^{i'}) - \underline{\sigma}_{i'}(x_0^{i'})$. Therefore, $i = j$ and t is the type (of player i) for which $\overline{\sigma}_j(x_0^j) - \underline{\sigma}_j(x_0^j)$ is maximized across players (j) and types (x_0^j). By the definition of Bayesian Nash equilibrium, we have

$$\partial_{+p_i} U_i(\overline{\sigma}_i(t), t; \overline{\sigma}_{-i}) \geq 0 \text{ and } \partial_{-p_i} U_i(\underline{\sigma}_i(t), t; \underline{\sigma}_{-i}) \leq 0.$$

Therefore:

$$\partial_{+p_i} U_i(\overline{\sigma}_i(t), t; \overline{\sigma}_{-i}) - \partial_{-p_i} U_i(\underline{\sigma}_i(t), t; \underline{\sigma}_{-i}) \geq 0.$$

Let's verify that:

$$\begin{aligned} A := -2\mu\Big(\mathbb{E}\Big[\chi(\overline{\sigma}_i(t))|\chi(p_0^i) = t\Big] - \mathbb{E}\Big[\chi(\underline{\sigma}_i(t))|\chi(p_0^i) = t\Big] \\ - \sum_j g_{ij}\mathbb{E}\Big[\chi(\overline{\sigma}_j(\chi(p_0^j)))|\chi(p_0^i) = t\Big] - g_{ij}\mathbb{E}\Big[\chi(\underline{\sigma}_j(\chi(p_0^j)))|\chi(p_0^i) = t\Big]\Big) < -2\mu^2 c \boldsymbol{q}_{i\bullet}^{\mathsf{T}}\mathbf{1}. \end{aligned}$$

The claim follows from the next inequality,

$$\begin{aligned} A &= -2\mu^2(\overline{\sigma}_i(t) - \underline{\sigma}_i(t)) + 2\mu^2 \sum_j g_{ij}\mathbb{E}\Big[\chi(\overline{\sigma}_j(\chi(p_0^j)))|\chi(p_0^i) = t\Big] - \mathbb{E}\Big[\chi(\underline{\sigma}_j(\chi(p_0^j)))|\chi(p_0^i) = t\Big] \\ &\leq -2\mu^2(\overline{\sigma}_i(t) - \underline{\sigma}_i(t))\boldsymbol{q}_{i\bullet}^{\mathsf{T}}\mathbf{1}, \end{aligned}$$

which holds by definition of i and t.

We have that:

$$\begin{gathered} \partial_{+p_i} U_i(\overline{\sigma}_i(t), t; \overline{\sigma}_{-i}) - \partial_{-p_i} U_i(\underline{\sigma}_i(t), t; \underline{\sigma}_{-i}) = \\ A + B - [\overline{\sigma}_i(t) > p_0^i] 2\boldsymbol{q}_{i\bullet}^{\mathsf{T}}\mathbf{1}\omega + [\underline{\sigma}_i(t) > p_0^i] 2\boldsymbol{q}_{i\bullet}^{\mathsf{T}}\mathbf{1}\omega. \end{gathered}$$

With:

$$B := -2\omega \sum_{j\in N} g_{ij} F(\overline{\sigma}_{j2}^{-1}(\overline{\sigma}_i(t)), p_0^j; t, p_0^i) - g_{ij} F(\underline{\sigma}_{j1}^{-1}(\underline{\sigma}_i(t)), p_0^j; t, p_0^i) \in [-2\omega(1 - \boldsymbol{q}_{i\bullet}^{\mathsf{T}}\mathbf{1}), 2\omega(1 - \boldsymbol{q}_{i\bullet}^{\mathsf{T}}\mathbf{1})]$$

Then:

$$\begin{aligned} A + B - [\overline{\sigma}_i(t) > p_0^i] 2\boldsymbol{q}_{i\bullet}^{\mathsf{T}}\mathbf{1}\omega + [\underline{\sigma}_i(t) > p_0^i] 2\boldsymbol{q}_{i\bullet}^{\mathsf{T}}\mathbf{1}\omega &> 0 \\ B &> -A \\ 2\omega(1 - \boldsymbol{q}_{i\bullet}^{\mathsf{T}}\mathbf{1}) &> 2\mu^2 c \boldsymbol{q}_{i\bullet}^{\mathsf{T}}\mathbf{1} \\ \omega \frac{\alpha \sum_j g_{ij}}{1 - \alpha \sum_j g_{ij}} &> \mu^2 c \end{aligned}$$

■

B.2 Finite Policy Spaces

Auxiliary results The expected payoff of player i given symmetric information, σ_{-i}, and a profile of status quo outcomes $(x_0^1, \dots, x_0^n)^\intercal = \boldsymbol{x}_0 \in \mathbf{R}^n$ is

$$U_i(p_i, \boldsymbol{x}_0; \sigma_{-i}) := \mathbb{E}\{u_i(\chi(p_i), \chi(\sigma_{-i})) | \chi(p_0^1) = x_0^1, \dots, \chi(p_0^n) = x_0^n\},$$

for all $p_i \in \mathbf{R}$. We use $U_i(p_i, \boldsymbol{x}_0; \sigma_{-i}, \boldsymbol{p}_0)$ when the status-quo policy profile is important.

We derive a second expression for the right and left derivatives of expected payoffs, based on v_i. For given policy p and nondecreasing strategy s_j:

$$\mathbb{C}^i(\chi(p), \chi(s_j)) = \begin{cases} \omega \int_{\left(-\infty, s_{j1}^-(p)\right)} s_j(x_0^j) - p_0^i \,\mathrm{d}F^i(x_0^j) + \omega\Big[1 - F^i(s_{j1}^-(p))\Big](p - p_0^i) & , p > p_0^i, \\ 0 & , p = p_0^i, \\ \omega F^i(s_{j2}^-(p))(p - p_0^i) - \omega \int_{\left(s_{j2}^-(p), \infty\right)} s_j(x_0^j) - p_0^i \,\mathrm{d}F^i(x_0^j) & , p < p_0^i. \end{cases}$$

Thus, we have

$$\partial\mathbb{C}^i(\chi(p_i), \chi(s_j)) = \begin{cases} \omega\Big[1 - F^i(s_j^-(p^i))\Big] & , p^i > p_0^i, \\ \Big[-\omega F^i(s_{j2}^-(p_0^i)), \omega - \omega F^i(s_{j1}^-(p_0^i))\Big] & , p^i = p_0^i \\ -\omega F^i(s_j^-(p^i)) & , p^i < p_0^i. \end{cases}$$

We express the left and right derivatives of the conditional expected payoff at $p_i \neq p_0^i$ as follows.

$$\begin{aligned} \partial_- U_i(p_i, x_0^i; s_{-i}) \propto \mathbb{E}^i\chi(p_i) - \delta_i - \alpha \sum_j \gamma^{ij} \mathbb{E}^i\chi(s_j) - \frac{1}{-2\mu}\frac{\partial}{\partial p_i}\mathbb{V}^i\chi(p_i) + \\ +2\alpha\frac{1}{-2\mu}\sum_j \gamma^{ij}\partial_-\mathbb{C}^i(\chi(p_i), \chi(s_j)) \end{aligned}$$

$$\begin{aligned} \partial_+ U_i(p_i, x_0^i; s_{-i}) \propto \mathbb{E}^i\chi(p_i) - \delta_i - \alpha \sum_j \gamma^{ij} \mathbb{E}^i\chi(s_j) - \frac{1}{-2\mu}\frac{\partial}{\partial p_i}\mathbb{V}^i\chi(p_i) + \\ +2\alpha\frac{1}{-2\mu}\sum_j \gamma^{ij}\partial_+\mathbb{C}^i(\chi(p_i), \chi(s_j)), \end{aligned}$$

in which -2μ is the proportionality constant.

Lemma B.22 (Continuity)**.** *Let s be a strategy profile and $x_0 := (x_0^1, \dots, x_0^n)$ be the profile of status-quo outcomes corresponding to status-quo policies $p_0 := (p_0^1, \dots, p_0^n)$. Then:*

(1) $\overline{U}_i(p_i, x_0; s_{-i}, p_0)$ is a continuous function of $(\dots, s_{i-1}(x_0^{i-1}), p_i, s_{i+1}(x_0^{i+1}), \dots)$.

(2) If $\dots < p_0^{\ell-1} < p_0^\ell < p_0^{\ell+1} < \dots$, then: $\overline{U}_i(p_i, x_0; s_{-i}, p_0)$ is a continuous function of p_0^ℓ on $(p_0^{\ell-1}, p_0^{\ell+1})$, $\ell \in N$.

Proof. We prove (1) first. We have:

$$\overline{U}_i(p_i, x_0; s_{-i}, p_0) = \int \cdots \int u_i(\ldots, \chi(s_{i-1}(x_0^{i-1})), \chi(p_i), \chi(s_{i+1}(x_0^{i+1})), \ldots) \\ \mathrm{d}m(\ldots, \chi(s_{i-1}(x_0^{i-1})), \chi(p_i), \chi(s_{i+1}(x_0^{i+1})), \ldots),$$

Where m is the distribution of a random vector that we describe in what follows. Because u_i is quadratic, the mean vector and the variance-covariance matrix of the random vector described by G determine $\overline{U}_i(p_i, x_0; s_{-i}, p_0)$. Thus, we prove (1) by means of the next two claims:

$\mathbb{E}[\chi(q)|\chi(p_0^1) = x_0^1, \ldots, \chi(p_0^n) = x_0^n]$ *is a continuous function of* q. By the properties of Brownian bridges:

$$\mathbb{E}\Big[\chi(q)|\chi(p_0^1) = x_0^1, \ldots, \chi(p_0^N) = x_0^n\Big] =$$
$$\begin{cases} \chi(p_1) + \frac{\chi(p_2)-\chi(p_1)}{p_2-p_1}(q - p_1) & p_1 \leq q \leq p_2, p_1 = \max\{p_0^i : p_0^i \leq q\}, p_2 = \min\{p_0^i : p_0^i \geq q\} \\ \chi(\max p_0) + \mu(q - \max p_0) & q \geq \max p_0 \\ \chi(\min p_0) + \mu(q - \min p_0) & q \leq \min p_0 \end{cases}$$

$\mathrm{Cov}[\chi(q), \chi(q')|\chi(p_0^1) = x_0^1, \ldots, \chi(p_0^n) = x_0^n]$ *is a continuous function of* q, q'. Let $q \leq q'$:

$$\mathbb{C}\mathrm{ov}\Big[\chi(q), \chi(q')|\chi(p_0^1) = x_0^1, \ldots, \chi(p_0^n) = x_0^n\Big] =$$
$$\begin{cases} \omega\frac{(p_2-q')(q-p_1)}{p_2-p_1} & p_1 \leq q \leq p_2, p_1 = \max\{p_0^i : p_0^i \leq q\}, p_2 = \min\{p_0^i : p_0^i \geq q\} \\ \mathbb{C}\mathrm{ov}[\chi(q'), \chi(q) \mid \chi(\max p_0)] & q' \geq q \geq \max p_0 \\ \mathbb{C}\mathrm{ov}[\chi(q'), \chi(q) \mid \chi(\min p_0)] & q \leq q' \leq \min p_0 \\ 0 & \text{else.} \end{cases}$$

Let's establish (2). Let $p_0^1 < p_0^2 < \ldots$. The expressions above show that mean and covariance terms of the pair of random variables $\chi(q), \chi(q') \mid \chi(p_0^1), \ldots, \chi(p_0^n)$ are locally continuous in $p_0^1, \ldots, p_0^n$. ∎

Definitions and assumptions We consider the same interim Bayesian game as the heterogeneous status quo game, except that the policy space of every agent is a finite nonempty set and that $n = 2$. In particular, we consider the two-player heterogeneous status quo game $\mathcal{F}$, for fixed status quo policy profile $\boldsymbol{p}_0 \in \mathbf{R}^2$ and the finite policy spaces defined in what follows, under the maintained assumption that $p_0^1 \neq p_0^2$.

Let $A_i = \{a_{i,1}, \ldots, a_{i,M_i}\}$, for given $M_i \in \mathbf{N}$ and every $i \in N$. We define the following payoff differences, towards studying strategic complementarities

$$du_i(a_i, a_i', a_{-i}, x_0^i) = \int_{a_i'}^{a_i} u_i(p_i, a_{-i}, x_0^i)\,\mathrm{d}p_i$$
$$\delta_i(a_i, a_i', a_{-i}, a_{-i}', x_0^i) = du_i(a_i, a_i', a_{-i}, x_0^i) - du_i(a_i, a_i', a_{-i}', x_0^i).$$

Lemma B.23 (Dominance Region)**.** *There exists* $\underline{x}, \overline{x} \in \mathbf{R}$ *such that:* $\underline{x} < \overline{x}$ *and, for all*

$i \in N,\ a_{-i} \in A_{-i}$ *it holds that*

$$du_i(a_{i,M_i}, a_i', a_{-i}, x_0^i) > 0 \text{ if } a_i \neq a_{i,M_i} \text{ and } x_0^i > \overline{x},$$
$$\text{and } du_i(a_{i,1}, a_i', a_{-i}, x_0^i) > 0 \text{ if } a_i \neq a_{i,1} \text{ and } x_0^i < \underline{x}.$$

Proof. The result follows from Lemma B.14. In particular, in the notation of the aforementioned result, we define

$$\overline{x} := \max\{\overline{x}_1, \overline{x}_2\}$$
$$\underline{x} := \max\{\underline{x}_1, \underline{x}_2\}.$$

■

Lemma B.24 (Strategic Complementarities)**.** *The function $u_i(\cdot, x_0^i)$ exhibits increasing differences in (a_i, a_{-i}), for all $i \in N$ and $x_0^i \in \mathbf{R}$.*

Proof. The result follows from Lemma B.12. ■

Lemma B.25 (Type Monotonicity)**.** *The function $u_i(\cdot, a_{-i}, x_0^i)$ exhibits strictly increasing differences in (a_i, x_0^i), for all $i \in N$ and $a_{-i} \in A_{-i}$.*

Proof. The result follows from Lemma B.12. ■

Lemma B.26 (Constant Type Monotonicity)**.** *For all $i \in N,\ a_i'', a_i' \in A_i$ with $a_i'' > a_i'$, and all $a_{-i}'', a_{-i}' \in A_{-i}$ with $a_{-i}'' > a_{-i}'$, the function $\delta_i(a_i'', a_i', a_{-i}'', a_{-i}', \cdot)$ is constant on $\mathbf{R}$.*

Proof. In the proof of Lemma B.12, we show an expression for $du_i(a_i, a_i', a_{-i}, x_0^i)$, which we use to write:

$$\delta_i(a_i'', a_i', a_{-i}'', a_{-i}', x_0^i) =$$
$$\int_{a_i'}^{a_i''} -2\mu(-g_{i-i})\mu(a_{-i}'' - a_{-i}') - g_{i-i}(\partial_-|p_i - a_{-i}''| - \partial_-|p_i - a_{-i}'|)\omega \, \mathrm{d}p_i.$$

The result follows. ■

Lemma B.27 (Existence of Cutoffs)**.** *For all $i \in N,\ a_i'', a_i' \in A_i$ and all $a_{-i} \in A_{-i}$, there exists $\widetilde{x} \in \mathbf{R}$ such that*

$$du_i(a_i'', a_i', a_{-i}, \widetilde{x}) = 0.$$

Proof. In the proof of Lemma B.12, we show that u_i is strictly concave in i's policy. The result follows. ■

Lemma B.28 (Payoff Continuity)**.** *For all $i \in N,\ a_i \in A_i$ and $a_{-i} \in A_{-i}$, the function $u_i(a_i, a_{-i}, \cdot)$ is continuous on $\mathbf{R}$.*

Proof. $u_i(a_i, a_{-i}, \cdot)$ is a strictly concave function (of the column vector $(\mathbb{E}[\chi(a_j)|\chi(p_0^i) = x_0^i], \mathbb{E}[\chi(a_{-j})|\chi(p_0^i) = x_0^i])^\mathsf{T}$, for a given $j \in N$,) by positive definiteness of $\boldsymbol{Q}$. The result follows since $u_i(a_i, a_{-i}, x_0^i)$ is a function of x_0^i only through $(\mathbb{E}[\chi(a_j)|\chi(p_0^i) = x_0^i], \mathbb{E}[\chi(a_{-j})|\chi(p_0^i) = x_0^i])^\mathsf{T}$, and the function $x_0^i \mapsto (\mathbb{E}[\chi(a_j)|\chi(p_0^i) = x_0^i], \mathbb{E}[\chi(a_{-j})|\chi(p_0^i) = x_0^i])^\mathsf{T}$ is affine. ■

In $\mathcal{F}$, a strategy for player i is a function $\alpha_i \colon \mathbf{R} \to A_i$. We study Bayesian Nash equilibria of $\mathcal{F}$ defined at the interim stage.

Existence of Bayesian Nash equilibria The proof is an adaptation of the one in Athey (2001). For simplicity of exposition, we prove the theorem in the where $A := A_1 = A_2$ and $M_1 - 1 =: M$, so that we may relabel policies as in $A = \{a_0, \dots, a_M\}$. We say that strategy α_i' improves upon strategy α_i given α_{-i} if: $U_i(\alpha_i(x_0^i), x_0^i; \alpha_{-i}) \leq U_i(\alpha_i'(x_0^i), x_0^i; \alpha_{-i})$ for all x_0^i.

We define the set of i's cutoffs as

$$\widehat{\Sigma}_i := \{(x_1, \dots, x_M) \in (\mathbf{R} \cup \{-\infty, \infty\})^M : x_1 \leq x_2 \leq \cdots \leq x_M\},$$

$\widehat{\Sigma} = \times_{i \in N} \widehat{\Sigma}_i$, and $\widehat{\Sigma} = \times_{j \in -i} \widehat{\Sigma}_j$, We say that a strategy α_i has finite cutoffs if $a_0, a_M \in \alpha_i(\mathbf{R})$.

Lemma B.29 (Finite Cutoffs)**.** *Let's fix $i \in N$. If α_i does not have finite cutoffs, there exists strategy α_i' that has finite cutoffs and improves upon α_i given some nondecreasing strategy profile of i's opponents.*

Proof. Let's suppose $a_0 \in \alpha_i(\mathbf{R})$ and $a_M \notin \alpha_i(\mathbf{R})$. Let's define $b = \inf\{x_0^i \in \mathbf{R} : \alpha_i(x_0^i) = \max \alpha_i(\mathbf{R})\}$. There exists $k > 0$ such that $\partial\_U_i(A_M, b + k; \alpha_{-i}) > 0$, because $\partial\_U_i'(A_M, \cdot; \alpha_{-i})$ is increasing for nondecreasing α_{-i} Let's define the strategy α_i' for player i as follows:

$$\alpha_i' : y \mapsto \begin{cases} \alpha_i(y) & , y \leq b + k \\ a_M & , y > b + k \end{cases}$$

The other cases can be dealt with similarly. ■

Definition 3. *(i) Given a nondecreasing strategy α_i, $x \in \widehat{\Sigma}_i$* represents α_i *if the following holds for all $m \in \{0, \dots, M\}$.*

$x_m = \infty$ if $a_m > \max \alpha_i(\mathbf{R})$, $x_m = -\infty$ if $a_m < \min \alpha_i(\mathbf{R})$, and:

$$x_m = \inf\{x_0^i \in \mathbb{R} \ : \ \alpha_i(x_0^i) \geq a_m\}, \text{ otherwise.}$$

(ii) Given a vector $x \in \widehat{\Sigma}_i$, strategy α_i is consistent with x *if:*

$$\alpha_i(x_0^i) = \begin{cases} a_0 & , x_0^i \leq x_1 \\ a_1 & , x_1 < x_0^i \leq x_2 \\ \vdots & \\ a_M & , x_M < x_0^i. \end{cases}$$

For fixed cutoff profile of i's opponents, $X^{-i} = (x^j)_{j\in -i} \in \widehat{\Sigma}_{-i}$, we denote i's expected payoff from policy p as her expected payoff from $(\chi(p), \chi(\alpha_{-i}(x_0^{-i})))$, in which α_j is consistent with x^j, $j \in -i$; thus, we have

$$\widehat{U}_i(p, x_0^i; X^{-i}) := U_i(p, x_0^i; \alpha_{-i}).$$

We define the best response to X^{-i} of i as:

$$\widehat{a}_i^{BR}(x_0^i, X^{-i}) = \operatorname*{Arg\,max}_{a\in\mathcal{A}_i} \widehat{U}_i(a, x_0^i; X^{-i})$$

Lemma B.30 (Bounds of best-response cutoffs)**.** *There exists $\underline{t}, \overline{t}$ such that the following holds. For every $i \in N, X^{-i} \in \widehat{\Sigma}_{-i}$, nondecreasing selection ζ from $\widehat{a}_i^{BR}(x_0^i, X^{-i})$ and cutoffs $x^i \in \widehat{\Sigma}_i$ representing ζ, we have:*

$$-\infty < \underline{t} \le x_1^i \le \cdots \le x_M^i \le \overline{t} < \infty.$$

Proof. The result follows from Lemma B.14. ∎

Proposition B.3 (Existence in Discrete Game)**.** *In the game $\mathcal{F}$, there exists an equilibrium in nondecreasing strategies.*

Proof. We apply Kakutani's theorem to the following correspondence. Let's define the set of cutoff vectors that represent best response strategies to the profile X:

$$\Gamma_i(X^{-i}) = \{y \in \widehat{\Sigma}_i : \text{there exists a strategy for } i \text{ consistent with } y \text{ that is a selection from } a_i^{BR}(\cdot, X^{-i})\}.$$

We claim that there exists a fixed point of the correspondence $(\Gamma_1, \ldots, \Gamma_I) : \Sigma \to \Sigma$, where:

$$\Sigma := \times_{i\in N}\Sigma_i \quad \text{and} \quad \Sigma_i := \{x \in [\underline{t}, \overline{t}]^M \ : \ x_1 \le x_2 \le \cdots \le x_M\}.$$

Σ_i is compact, convex subset of $\mathbf{R}^{nM}$. Γ is nonempty-valued because action spaces are finite and the Single Crossing Condition for games of incomplete information holds. Γ is convex-valued due to "Lemma 2" in Athey (2001), and the Single Crossing Condition for games of incomplete information. Γ has closed graph, as established in the proof of "Lemma 3" in Athey (2001). Thus, by Kakutani's theorem, there exists a fixed point of Γ.

Next, we claim that a fixed point of Γ is an equilibrium of $\mathcal{F}$. It follows from Lemma B.30, because if a strategy is a best-response against X^{-i}, than it admits a representation with finite uniformly bounded cutoffs. ∎

Remark B.4. *We note that the proof of existence of Bayesian Nash equilibria in $\mathcal{F}$ does not rely on the assumption that $n = 2$. Thus, it also establishes existence with finite policy spaces and n players.*

Remark B.5 (Existence in $\mathcal{G}(\boldsymbol{p}_0)$)**.** *Following the approach in Athey (2001), there is a second existence proof for nondecreasing strategy equilibria in $\mathcal{G}(\boldsymbol{p}_0)$, which uses a purification argument given existence of an equilibrium in nondecreasing strategies in $\mathcal{F}$.*

Lemma B.31. *In $\mathcal{G}(\boldsymbol{p}_0)$, there exists an equilibrium in which every player's strategy is nondecreasing.*

Proof. For each player i, let's consider a sequence of action spaces $P_i^\bullet$, in which

$$P_i^k = \left\{ \underline{p}_i + \frac{m}{10^k}(\overline{p}_i - \underline{p}_i) \ : \ m = 0, \ldots, 10^k \right\} \quad , k \in \mathbf{N}.$$

For every k, the game where finite action spaces $P_1^k,\ P_2^k, \ldots$ replace $A_1,\ A_2, \ldots$ has an equilibrium, by Lemma B.3. Let's fix a sequence of equilibria in nondecreasing strategies, $s^\bullet$. Because action spaces $P_1^k,\ P_2^k, \ldots$ are bounded by $\min \underline{p}_i$ and $\max \overline{p}_i$, $s^\bullet$ is a sequence of uniformly bounded nondecreasing functions. By Helly's selection theorem, $s^\bullet$ admits a pointwise convergent subsequence, so we define $s^\star := \lim s^\bullet$. Because s^k is an equilibrium, it holds that $U_i(s_i^k(x_0^i), x_0^i; s_{-i}^k) \geq U_i(p, x_0^i; s_{-i}^k)$, for all k and $p \in P_i^k$. $U_i(p, \boldsymbol{x}_0; s_{-i}^k)$ is a continuous function of $(\ldots, s_{i-1}^k(x_0^{i-1}), s_{i+1}^k(x_0^{i+1}), \ldots)$, by lemma B.22. Thus, $U_i(p, x_0^i; s_{-i}^k)$, which is the expectation of $U_i(p, \boldsymbol{x}_0; s_{-i}^k)$, converges as $k \to \infty$. Therefore: it holds that $U_i(s_i^\star(t_i), x_0^i; s_{-i}^\star) \geq U_i(p, x_0^i; s_{-i}^\star)$, for all $p \in P_i$. $s^\star$ is an equilibrium of the game $\mathcal{G}(p_0^1, \ldots, p_0^N)$. ■

Uniqueness of Bayesian Nash equilibria with two players First, we establish two properties of beliefs in $\mathcal{F}$, which we leverage to establish uniqueness of non-decreasing strategy equilibrium.

Let C_i denote the space of nondecreasing strategies for player $i \in N$, in which a nondecreasing strategy is identified by its finite sequence of "real cutoffs" (Mathevet, 2010). For $k > 1$, let's compute the probability that i attaches to her opponent playing strictly less than $g = a_{-i,k} \in A_{-i}$, given that i's type is x_0^i and $-i$'s strategy is α_{-i}:

$$\Phi\left(\frac{\alpha_{-i1}^-(g) - x_0^i - \mu(p_0^{-i} - p_0^i)}{\sqrt{\omega |p_0^i - p_0^{-i}|}} \right),$$

in which $\alpha_{-i1}^-(g)$ is the real cutoff between $a_{-i,k-1}$ and $a_{-i,k}$ implied by α_{-i}. For $k = 1$, that probability is 0.

Towards a definition of the above probability as a function of real cutoffs, we make the following definitions. Given a policy $g \in A_{-i}$, we let $k_{-i}(g)$ be such that: $g = a_{-i,k_{-i}(g)}$. A real cutoff between $a_{-i,k}$ and $a_{-i,k+1}$ is denoted by $c_{-i,k}^r$, for $k \in \{1, \ldots, M_i - 1\}$ (the interpretation for $c_{-i,k}^r$ is that types below $c_{-i,k}^r$ play $a_{-i,k}$ and types above $c_{-i,k}^r$ play $a_{-i,k+1}$).

Given a nondecreasing strategy $c_{-i} \in C_{-i}, g \in A_{-i}, x_0^i \in \mathbf{R}$, we define:

$$\Lambda_i(g | c_{-i}, x_0^i) = \begin{cases} \Phi\left(\frac{c_{-i,k_{-i}(g)-1}^r - x_0^i - \mu(p_0^{-i} - p_0^i)}{\sqrt{\omega |p_0^i - p_0^{-i}|}} \right) & \text{if } k_{-i}(g) > 1, \\ 0 & \text{if } k_{-i}(g) = 1. \end{cases}.$$

Lemma B.32 (FOSD and Translation Invariance)**.** *For all $i \in N$, and $y_0^i, x_0^i \in \mathbf{R}$ with $y_0^i > x_0^i$, we have:*

$$\Phi\left(\frac{s - y_0^i - \mu(p_0^{-i} - p_0^i)}{\sqrt{\omega|p_0^i - p_0^{-i}|}}\right) < \Phi\left(\frac{s - x_0^i - \mu(p_0^{-i} - p_0^i)}{\sqrt{\omega|p_0^i - p_0^{-i}|}}\right).$$

Moreover, let c_{-i} be a column vector real cutoffs with M_{-i} columns corresponding to an element of C_{-i}, we have that

$$\Lambda_i(g|c_{-i} + \Delta\mathbf{1}, x_0^i + \Delta) = \Lambda_i(g|c_{-i}, x_0^i),$$

for all $i \in N$, $g \in A_{-i}$ and $\Delta \in [0, \overline{x} - \underline{x}]$.

Proof. The first part follows from Lemma A.7. The second part follows from the definition of Λ_i. ∎

Proposition B.4. *In the game $\mathcal{F}$, there exists a unique equilibrium in nondecreasing strategies.*

Proof. Given that we established existence of an equilibrium in nondecreasing strategies, it suffices to establish that there exists at most one equilibrium in nondecreasing strategies. The proof uses the same argument as "Proposition 2" and "Theorem 1" in Mathevet (2010). In particular, Lemmata B.23 through B.28 imply "Assumptions 1, 2, 3, 4, 5, 6" in Mathevet (2010), and beliefs in $\mathcal{F}$ satisfy FOSD and Translation Invariance. ∎

Remark B.6. *This remark explains why the results for $\mathcal{G}(\boldsymbol{p}_0)$, either for existence and for the characterization of extremal equilibria, are not used in $\mathcal{F}$. This remark is informed by the approach taken in Mathevet (2010) to establish uniqueness. For notational convenience, our next definition is valid under the assumption that $A_i \subseteq P_i$ for all $i \in N$,*

$$\varphi_i^F(x_0^i, \alpha_{-i}) = \operatorname*{Arg\,max}_{p_i \in A_i} U_i(p_i, x_0^i; \alpha_{-i}).$$

We note that φ_i^F differs from φ_i because the respective optimization problems have different feasible sets: A_i and P_i, respectively. If the mapping $x_0^i \to \sup \varphi_i^F(x_0^i, \alpha_{-i})$ is measurable, then there exists a unique equilibrium in $\mathcal{F}$.[25] *However, $\varphi_i^F(x_0^i, \alpha_{-i})$ is not necessarily single-valued, so the Caratheodory-function argument used in $\mathcal{G}(\boldsymbol{p}_0)$ does not hold in $\mathcal{F}$.*

C Proofs for Section 2

Proof of Lemma 1.

[25] Here is the reason. Let's order individual strategies and strategy profiles in $\mathcal{F}$ as in the heterogeneous status quo game. To establish uniqueness, by Proposition B.4, it suffices to establish that there exists a largest and a smallest equilibrium, and that they are in nondecreasing strategies. Once we establish that the "GBR" mapping is measurable — ie, the equivalent in $\mathcal{F}$ of Lemma B.15 in $\mathcal{G}(\boldsymbol{p}_0)$ —, the same argument that we adopt to establish Proposition B.20 in $\mathcal{G}(\boldsymbol{p}_0)$ is valid in $\mathcal{F}$.

Proof. By strict concavity of expected payoff in own policy (Lemma B.12), it is enough to verify that, up to a positive proportionality constant of -2μ, the right derivative of expected payoff in own policy is:

$$\partial_{p_i+}\mathbb{E}\pi_i(\boldsymbol{\chi}(p)) \propto \mathbb{E}\chi(p_i) - \beta_i - \alpha\sum_j \gamma^{ij}(\mathbb{E}\chi(p_j) - \beta_j + \partial_{p_i+}|p_i - p_j|k)$$
$$-\left(1 - \alpha\sum_j \gamma^{ij}\right)\frac{1}{-2\mu}\partial_{p_i+}\mathbb{V}\chi(p_i),$$

which follows from the independent Proposition E.5. The result follows because $p_{-i} \mapsto \partial_{p_i+}\mathbb{E}\pi_i(\boldsymbol{\chi}(p))$ is increasing (this step is shown explicitely in the proof of B.12, and it is omitted here for the sake of brevity.) ∎

Proof of Proposition 1.

Proof. In G_0, strategy spaces are compact intervals and player i's payoff function is continuous in p_i for all p_{-i} (Lemma B.12) and strictly supermodular in (p_i, p_{-i}) (Lemma A.2). The result follows from Tarski's fixed point theorem, and the argument is known in the literature on supermodular games (Milgrom and Roberts, 1990; Vives, 1990). ∎

Proof of Proposition 2.

Proof. Without loss of generality, we set $p_0 = 0$ to ease on notation. By right and left differentiation of the strictly concave expected payoff of player i in own payoff (Lemma B.12), at policy profile p, and by the best-response equivalence established in Lemma E.5, the best response constraints for i are equivalent to the following pair of inequalities:

$$\mathbb{E}\chi(p_i) - \beta_i - \alpha\sum_j \gamma^{ij}(\mathbb{E}\chi(p_j) - \beta_j) \leq ([p_i \geq 0] - [p_i < 0])k$$
$$+\alpha\sum_j \gamma^{ij}([p_i \geq p_i] - [p_i < p_j])k$$
$$\text{and } ([p_i < 0] - [p_i \leq 0])k + \alpha\sum_j \gamma^{ij}([p_i < p_i] - [p_i \geq p_j])k \geq \mathbb{E}\chi(p_i) - \beta_i$$
$$-\alpha\sum_j \gamma^{ij}(\mathbb{E}\chi(p_j) - \beta_j),$$

which are found by left and right differentiation of the strictly concave potential, *separately* in each individual policy (i.e. for all p_i's). The result follows from rearranging the above inequalities in matrix notation. ∎

Proof of Lemma 2.

Proof. The result follows directly from the results in Belhaj et al. (2014), and also the analysis in Ballester et al. (2006). ∎

Proof of Corollary 1.

Proof. The result follows from the analysis of Callander (2011a), or the same arguments leading to Lemma 1 and Proposition 2. ■

D Proofs for Section 3

Proofs of Section 3

Proof of Lemma 2.

Proof. The present proof uses the notation described in Section A. By the equilibrium decomposition:

$$\boldsymbol{Q}\mathbb{E}\boldsymbol{\chi} = \boldsymbol{b} + \boldsymbol{Q}\mathbf{1}k + (\boldsymbol{G} \odot \boldsymbol{A})\mathbf{1}k$$

Thus:

$$\boldsymbol{q}_{i\bullet}^{\intercal}\mathbb{E}\boldsymbol{\chi} = b_i + \boldsymbol{q}_{i\bullet}^{\intercal}\mathbf{1}k + \sum_j g_{ij}a_{ij}k$$

So, by symmetry of $\boldsymbol{G}$

$$\begin{aligned}(\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_j))(1 + g_{ij}) &= b_i - b_j + \boldsymbol{q}_{i\bullet}^{\intercal}\mathbf{1}k - \boldsymbol{q}_{j\bullet}^{\intercal}\mathbf{1}k + \\ + \sum_{\ell\notin\{i,j\}}(g_{i\ell} - g_{j\ell})\mathbb{E}\chi_\ell &+ \sum_{\ell\notin\{i,j\}}(g_{i\ell}a_{i\ell} - g_{j\ell}a_{j\ell})k + g_{ij}(a_{ij} - a_{ji})k\end{aligned}$$

Which simplifies to:

$$(\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_j))(1 + g_{ij}) = b_i - b_j + \alpha\gamma \sum_{\ell\notin\{i,j\}}(a_{i\ell} - a_{j\ell})k - 2gk$$

From the equilibrium decomposition, it holds that: (i) $a_{i\ell} - a_{j\ell} \in [-2, 0]$ if $p_i < p_j$, and (ii) $a_{i\ell} - a_{j\ell} = 0$ only if: $p_\ell \in \{p_i, p_j\}$ or $p_\ell \in P_i \setminus [p_i, p_j]$. The result follows. ■

Proof of Lemma 3

Proof. We use the notation developed in Section A. We have that, for all $i, m \in N$

$$\begin{aligned}\mathbb{E}\chi(p_i) &= \beta_i + k + (\boldsymbol{I} - \boldsymbol{G})^{-1}_{ii}\sum_{\ell\in N} g_{i\ell}a_{i\ell}k + \\ &+ \sum_{j\in N\setminus\{i,m\}}(\boldsymbol{I} - \boldsymbol{G})^{-1}_{ij}\sum_{\ell\in N} g_{j\ell}a_{j\ell}k + (\boldsymbol{I} - \boldsymbol{G})^{-1}_{im}\sum_{\ell\in N} g_{m\ell}a_{m\ell}k.\end{aligned}$$

Thus:

$$\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_m) = \beta_i - \beta_m + \left[(\boldsymbol{I} - \boldsymbol{G})^{-1}_{ii} - (\boldsymbol{I} - \boldsymbol{G})^{-1}_{mi}\right]\left(\sum_{\ell\in N} g_{i\ell}a_{i\ell} - \sum_{\ell\in N} g_{m\ell}a_{m\ell}\right)k$$

Letting $g := \alpha\gamma$, by computation of $(\boldsymbol{I}-\boldsymbol{G})^{-1}$, we have that the diagonal element is $\frac{1-g(n-1)+g}{(1-g(n-1))(1+g)}$ and the off-diagonal element is: $\frac{g}{(1-g(n-1))(1+g)}$, so that:

$$(\boldsymbol{I}-\boldsymbol{G})^{-1}_{ii} - (\boldsymbol{I}-\boldsymbol{G})^{-1}_{im} = \frac{1}{1+g}.$$

Thus, by the preceding equality we have:

$$\begin{aligned}\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_m) &= \beta_i - \beta_m + \frac{g}{1+g}\left(\sum_{\ell\in N} a_{i\ell} - \sum_{\ell\in N} a_{m\ell}\right)k \\ &= \beta_i - \beta_m - \frac{g}{1+g}2k + \frac{g}{1+g}\left(\sum_{\ell\in N\setminus\{i,m\}} a_{i\ell} - a_{m\ell}\right)k.\end{aligned}$$

The result follows from the equilibrium decomposition in Proposition 2 and the hypotheses on p. ■

Proof of Lemma 5.

Proof. The result follows from Lemma A.5. ■

Towards the proof of Lemma 3, we establish an auxiliary result. We say that $\boldsymbol{\Gamma}$ is *complete* if: $\gamma^{ij} = 1$ for all $j \in N \setminus \{i\}$ and $\gamma^{ii} = 0$ for all $i \in N$. We say that the equilibrium $\boldsymbol{p}$ is ordered if: $p_1 < p_2 < \cdots < p_n$, and a the equilibrium $\boldsymbol{p}$ is interior if: $p_i \in (p_0, \overline{p}), i \in N$.

Lemma D.33. *Let $\boldsymbol{\Gamma}$ be complete. Then, Assumption 1 is satisfied if, and only if: $\alpha < 1/(n-1)$. Moreover, if $\boldsymbol{p} \in (p_0, \overline{p})^n$ is an ordered equilibrium and $i \in \{1, \ldots, n-1\}$, then*

$$\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_{i+\ell}) = \beta_i - \beta_{i+\ell} - 2\ell\frac{\alpha}{1+\alpha}k, \ell \in \{1, \ldots, n-i\}.$$

Furthermore, if $\delta_i - \delta_{i+1} > 2\frac{\alpha}{1-\alpha}k$, then: every interior equilibrium is ordered, and there exists at most one ordered interior equilibrium.

Proof. **Assumption 1 is satisfied if, and only if: $\alpha < 1/(n-1)$.** The result follows from the largest eigenvalue of $\boldsymbol{\Gamma}$ being $\lambda(\boldsymbol{\Gamma}) = n-1$.

"Moreover" part. By the Decomposition of equilibrium expected outcomes, $p_i < p_j$ implies

$$\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_j) = \beta_i - \beta_j + \frac{\alpha}{1+\alpha}\sum_{\ell\in N\setminus\{i,j\}}(a_{i\ell} - a_{j\ell})k - 2\frac{\alpha}{1+\alpha}k,$$

in which $a_{i\ell}, a_{j\ell}$ are elements of the matrix A in the decomposition, and we used the properties of the complete $\boldsymbol{\Gamma}$. The formula for $\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_{i+\ell})$ in the Lemma follows from the properties of A stated in the decomposition given that $\boldsymbol{p}$ is ordered.

It remains to verify that $\mathbb{E}\chi(p_i) \geq \beta_i$. We set $\widehat{\alpha} = \alpha(n-1)$ for $\widehat{\alpha} \in (0,1)$ — if $\widehat{\alpha} = 0$, then $\mathbb{E}\chi(p_i) = \beta_i + k \geq \beta_i$. After computation of the Leontieff inverse $\boldsymbol{B}$, it is established that:

$$1 + \widehat{\alpha} \sum_{j \in N} B_{ij} a_{ij} = 1 - \frac{(n-1)(1-\widehat{\alpha}) + \widehat{\alpha}}{(n-1+\widehat{\alpha})(1-\widehat{\alpha})} \widehat{\alpha} + \frac{\widehat{\alpha}}{(n-1+\widehat{\alpha})(1-\widehat{\alpha})} \widehat{\alpha}(n-1),$$

using the properties of the matrix $\boldsymbol{A}$ for an interior ordered equilibrium $\boldsymbol{p}$ (and the entries of $\boldsymbol{B}$, described in the proof of Proposition **??**).

We verify that

$$1 + \widehat{\alpha} \sum_{j \in N} B_{ij} a_{ij} \leq 0 \iff (n-1)(1-\alpha) + \alpha + 2\alpha^2(n-2) \leq 0$$

Since the left-hand side of the above inequality is always positive, the result follows.

"Furthermore" part. This result is established in the proof of Proposition **??**. ■

Proof of Lemma 3.

Proof. The result is an implication of Lemma D.33 ■

Towards the proof of Lemma 4, we establish an auxiliary result. We say that $\boldsymbol{\Gamma}$ is a *line* if: (i) $\gamma^{ii+1} = 1$ for all $i \in \{1, \ldots, n-1\}$, (ii) $\gamma^{ii-1} = 1$ for all $i \in \{2, \ldots, n\}$, and (iii) $\gamma^{ij} = 0$ otherwise. We say that the equilibrium $\boldsymbol{p}$ is ordered if: $p_1 < p_2 < \cdots < p_n$.

Lemma D.34. *Let $\boldsymbol{\Gamma}$ be a line and $0 < \alpha < 1/2$. Then, Assumption 1 is satisfied. Moreover, if $\boldsymbol{p} \in (p_0, \overline{p})^n$ is an ordered equilibrium and $i \in \{1, \ldots, n-1\}$, then*

$$\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_{i+\ell}) = \beta_i - \beta_{i+\ell} - a(i, \ell, n, \alpha)k, \ell \in \{1, \ldots, n-i\},$$

for some $a(i, \ell, n, \alpha) > 0$.

Furthermore, $\mathbb{E}\chi(p_i) \geq \beta_i$.

Proof. **(1) Characterization of the inverse of $\boldsymbol{I} - \alpha\boldsymbol{\Gamma}$ using Toeplitz matrices.**

We have that $\boldsymbol{I} - \alpha\boldsymbol{\Gamma} =: \boldsymbol{S} = [S^{ij} : i, j \in N]$ in which (i) $S^{ii+1} = -\alpha$ for all $i \in \{1, \ldots, n-1\}$, (ii) $S^{ii-1} = -\alpha$ for all $i \in \{2, \ldots, n\}$, (iii) $S^{ij} = 1$ and (iv) $S^{ij} = 0$ otherwise. This matrix $\boldsymbol{S}$ Toeplitz becase it is constant on each diagonal. We study the following transformation $\boldsymbol{T}$ of $\boldsymbol{S}$.

$$\boldsymbol{T} = \frac{1}{\alpha}\boldsymbol{S},$$

so that $\boldsymbol{T}$ in which (i) $T^{ii+1} = -1$ for all $i \in \{1, \ldots, n-1\}$, (ii) $T^{ii-1} = -1$ for all $i \in \{2, \ldots, n\}$, (iii) $T^{ij} = a := 1/\alpha$ and (iv) $T^{ij} = 0$ otherwise. $\boldsymbol{T}$ is Toepliz, and the entries of its inverse can be characterized starting from the two solutions to $r^2 - ar + 1 = 0$. If $0 < \alpha < 1/2$, there exists two distinct roots, defined as:

$$
\begin{aligned}
r_- &:= \frac{1-\sqrt{(1+2\alpha)(1-2\alpha)}}{2\alpha} \\
r_+ &:= \frac{1+\sqrt{(1+2\alpha)(1-2\alpha)}}{2\alpha}.
\end{aligned}
$$

It is straightforward to establish that $0 < r_- < 1 < 1/\alpha < r_+ < 1/\alpha + 1$. By the characterization of inverse of Toeplitz matrices (e.g., Theorem 2.8 in Meurant (1992)), we have: $\boldsymbol{T}^{-1} = [T^{-1}_{ij} : i, j \in N]$ and

$$
T^{-1}_{ij} = \frac{(r^i_+ - r^i_-)(r^{n-j+1}_+ - r^{n-j+1}_-)}{(r_+ - r_-)(r^{n+1}_+ - r^{n+1}_-)}, j \geq i.
$$

(2) Characterization of vector $\alpha\boldsymbol{\Gamma} \odot \boldsymbol{A}\mathbf{1}k$, given an ordered equilibrium. We have that:

$$
\alpha\boldsymbol{\Gamma} \odot \boldsymbol{A}\mathbf{1}k = \alpha \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} k.
$$

(3) Characterization of vector $\boldsymbol{e} := (\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}\alpha\boldsymbol{\Gamma} \odot \boldsymbol{A}\mathbf{1}k$, given an ordered equilibrium. By using the definition of $\boldsymbol{T}^{-1}$, and $\boldsymbol{e} = [e_i : i \in N]$ we have that:

$$
\begin{aligned}
e_i &= -k\left(-\frac{r^i_+ - r^i_-}{r^{n+1}_+ - r^{n+1}_-} + \frac{r^{n-i+1}_+ - r^{n-i+1}_-}{r^{n+1}_+ - r^{n+1}_-} \right) \\
&= -\frac{r^{n-i+1}_+ - r^i_+ - r^{n-i+1}_- + r^i_-}{r^{n+1}_+ - r^{n+1}_-} k.
\end{aligned}
$$

It follows that

$$
e_i - e_{i+\ell} \propto -\left(r^{n-i+1}_+ - r^{n-i-\ell+1}_+ - r^i_+ + r^{i+\ell}_+ - r^{n-i+1}_- + r^{n-i-\ell+1}_- + r^i_- - r^{i+\ell}_- \right),
$$

which is a positive number. We take:

$$
a(i, \ell, n, \alpha) = \frac{r^{n-i+1}_+ - r^{n-i-\ell+1}_+ - r^i_+ + r^{i+\ell}_+ - r^{n-i+1}_- + r^{n-i-\ell+1}_- + r^i_- - r^{i+\ell}_-}{r^{n+1}_+ - r^{n+1}_-}.
$$

(4) Largest Eigenvalue of $\boldsymbol{\Gamma}$. The adjacency matrix $\boldsymbol{\Gamma}$ is Toeplitz. By known results (Theorem 2.2 in Kulkarni et al., 1999), the largest eigenvalue is

$$
\lambda(\boldsymbol{\Gamma}) = -2\cos(\pi n/(n+1)) \in [0, 2).
$$

“Furthermore” Part. We verify that $e_i \geq -k$. In particular,

$$\begin{aligned} -e_i/k > 1 &\iff r_+^{n-i+1} - r_+^i - r_-^{n-i+1} + r_-^i > r_+^{n+1} - r_-^{n+1} \\ &\iff -r_+^i + r_-^i > r_+^{n+1}(1 - r_+^{-i}) - r_-^{n+1}(1 - r_-^{-i}). \end{aligned}$$

The right-hand side of the above inequality is positive and the left-hand side is negative, by definition of r_+, r_- and $\alpha \in (0, 1/2), i \in N$. Thus, it holds that $-e_i/k \leq 1$. ■

Proof of Lemma 3.

Proof. The result is an implication of Lemma D.34. ■

Proofs for section ??

A representative consumer has quasi-linear preferences over bundles of $n+1$ goods, which are represented by the quadratic utility function U such that

$$U(q_1, \ldots, q_n, z) = \sum_i \widehat{a}_i q_i - \frac{1}{2} b \sum_i q_i^2 - \frac{1}{2} c \sum_{i,j:j \neq i} q_i q_j + z,$$

in which r denotes the numéraire good. Let $\boldsymbol{B} = c\boldsymbol{1}\boldsymbol{1}^\mathsf{T} + (b-c)\boldsymbol{I}$ be the matrix with b on the main diagonal and c in off-diagonal entries.

Lemma D.35. *Let $b > c > 0$. Then: $\boldsymbol{B}$ is a symmetric and positive definite matrix. Its inverse $\boldsymbol{B}^{-1}$ is symmetric, positive definite, its entries given by $\frac{b-c+(n-1)c}{(b-c)[(n-1)c+b]}$ on the main diagonal, and $-\frac{c}{(b-c)([(n-1)c+b])}$ in off-diagonal entries.*

Proof. $\boldsymbol{B}$ is symmetric. The eigenvalues of $\frac{1}{b}\boldsymbol{B}$ are $1 - c/b$ and $1 + \frac{n-1}{b}c$, so $\boldsymbol{B}$ is positive definite. Then, $\boldsymbol{B}^{-1}$ is well-defined, positive definite and has eigenvalues $(b-c)^{-1}$ and $(b+(n-1)c)^{-1}$.

We verify that $\boldsymbol{B}^{-1} = r\boldsymbol{1}\boldsymbol{1}^\mathsf{T} + \frac{1}{b-c}\boldsymbol{I}$, for $r = -\frac{c}{(b-c)((n-1)c+b)}$. Let's observe that $\boldsymbol{1}\boldsymbol{1}^\mathsf{T}\boldsymbol{1}\boldsymbol{1}^\mathsf{T} = n\boldsymbol{1}\boldsymbol{1}^\mathsf{T}$, and:

$$\begin{aligned} \boldsymbol{B}\boldsymbol{B}^{-1} = \boldsymbol{I} &\iff r\boldsymbol{1}\boldsymbol{1}^\mathsf{T} c\boldsymbol{1}\boldsymbol{1}^\mathsf{T} + \boldsymbol{I} + r(b-c)\boldsymbol{1}\boldsymbol{1}^\mathsf{T}\boldsymbol{I} + \frac{c}{b-c}\boldsymbol{1}\boldsymbol{1}^\mathsf{T}\boldsymbol{I} = \boldsymbol{I} \\ &\iff rcn\boldsymbol{1}\boldsymbol{1}^\mathsf{T} + \Big[r(b-c) + \frac{c}{b-c}\Big]\boldsymbol{1}\boldsymbol{1}^\mathsf{T} = \boldsymbol{I} - \boldsymbol{I} \\ &\iff r = -\frac{c}{(b-c)((n-1)c+b)}. \end{aligned}$$

■

By normalizing the main-diagonal entries of $\boldsymbol{B}^{-1}$ to 1, the off-diagonal elements are $1 - \frac{1}{b-c}$. We note that $1 - \frac{1}{b-c} < 0 \iff 1 - (b-c) > 0$. Thus, in what follows we assume $1 > b - c$. Moreover, we assume that $\zeta := \frac{1-(b-c)}{b-c} < \frac{2}{n-1}$. Our parameter assumptions are summarized as follows

Assumption 4 (Demand System 2)**.** *We assume that*

(1) Goods are utility-substitute and U is strictly concave, which is equivalent to what is assumed in the main body of the text.

(2) Own-price coefficients of demand are all equal to -1 and that the degree of utility substitutability c is bounded above by $b - \frac{n-1}{n+1}$.

The two assumptions are jointly represented by:

$$c \geq 0 \text{ and } 1 > b - c > \frac{n-1}{n+1}.$$

$b > c \geq 0$ is equivalent to requiring that the following two conditions jointly hold: (i) goods are utility-substitute (U is submodular) and (ii) U is strictly concave. The requirement $1 > b - c$ is needed following the normalization that own-price coefficient of demand is -1, and $\frac{1-(b-c)}{b-c} < \frac{2}{n-1}$ is the content of Assumption 1 in the current setup after the normalization (we note that $\frac{1-(b-c)}{b-c} < \frac{2}{n-1} \iff b-c > \frac{n-1}{n+1}$). In the following remark, we verify that the additional assumptions can be dispensed of, which justifies that in the main text we only assume $b > c \geq 0$.

Remark D.7 (Comparison of Assumption 4 with the model of oligopoly in Section 3)**.** *Under our assumptions, goods are mutually direct substitutes (Weinstein, 2022), substitutes in the sense of Hedgeworth and Marshallian demand satisfies the Law of Demand (Amir et al., 2017). Moreover, for a positive price vector $\boldsymbol{v}$ and sufficiently large income, demand for the goods excluding the numeraire is given by $\boldsymbol{B}^{-1}(\widehat{\boldsymbol{a}} - \boldsymbol{x})$.*

Let's show that under $b > c \geq 0$ the analysis goes through without the extra content in Assumption 4. First, let's observe that the concavity assumption on demand — positive definiteness of $\boldsymbol{B}$ following from $b > c \geq 0$ according to Lemma D.35 — guarantees positive definiteness of $\boldsymbol{B}^{-1}$, and induces a contractive property on the best-response mapping of the game $\left\langle N, \{\pi_i^B, \mathbf{R}\}_{i \in N}\right\rangle$. Letting $\mathrm{Diag}(\boldsymbol{M})$ return an $n \times n$ diagonal matrix whose entries are the n elements in the main diagonal of matrix $\boldsymbol{M}$, such best-response mapping follows form first-order conditions and is given by:

$$\begin{aligned}
\mathrm{BR}(\boldsymbol{x}) &= -2\,\mathrm{Diag}(\boldsymbol{B}^{-1})\boldsymbol{x} + \left[\mathrm{Diag}(\boldsymbol{B}^{-1}) - \boldsymbol{B}^{-1}\right]\boldsymbol{x} + \boldsymbol{B}^{-1}\widehat{\boldsymbol{a}} + \mathrm{Diag}(\boldsymbol{B}^{-1})\widehat{\boldsymbol{x}} \\
&= -\left[\mathrm{Diag}(\boldsymbol{B}^{-1}) + \boldsymbol{B}^{-1}\right]\boldsymbol{x} + \boldsymbol{B}^{-1}\widehat{\boldsymbol{a}} + \mathrm{Diag}(\boldsymbol{B}^{-1})\widehat{\boldsymbol{x}}.
\end{aligned}$$

The Jacobian of $\mathrm{BR}(\boldsymbol{x})$ is given by $-[\mathrm{Diag}(\boldsymbol{B}^{-1}) + \boldsymbol{B}^{-1}]$, which is negative definite iff $\mathrm{Diag}(\boldsymbol{B}^{-1}) + \boldsymbol{B}^{-1}$ is positive definite. The diagonal entries of $\boldsymbol{B}^{-1}$ are positive (Lemma D.35). Thus, the best-reply mapping is a contraction.

Secondly, to establish that the normalization on demand coefficients is innocuous, we show that the coefficients of $\boldsymbol{B}^{-1}$ are negative, shown in Lemma D.35.

We assume that each of the prices of n goods is set by one of n firms that compete in prices. Each of n firms has constant marginal costs and no fixed costs. Let $\boldsymbol{D} := -\boldsymbol{B}^{-1} = [D_{ij} : i, j \in N]$ be the matrix of demand coefficients. Given a profile of prices $\widehat{x}$ and marginal costs $\widehat{m}$, profits of firm i are:

$$\begin{aligned}\pi_i^B(\widehat{x}) &:= (\widehat{x}_i - \widehat{m}_i)\left[\sum_{j\in N} D_{ij}(\widehat{x}_j - \widehat{a}_j)\right] \\ &= \left(\widehat{m}_i + \widehat{a}_i - \sum_{j\in -i} D_{ij}a_j\right)\widehat{x}_i - \widehat{x}_i^2 + \sum_{j\in -i} D_{ij}\widehat{x}_i\widehat{x}_j + F,\end{aligned}$$

for a term $F = -\widehat{m}_i\Big(\widehat{a}_i - \sum_{j\in -i} D_{ij}\widehat{a}_j\Big) - \widehat{m}_i \sum_{j\in -i} D_{ij}\widehat{x}_j$ that is constant with respect to $\widehat{x}_i$. We can equivalently express profits in terms of markups, $x := \widehat{x} - \widehat{m}$, letting $a := \widehat{a} - \widehat{m}$, to write

$$\pi_i^B(x) := \left(a_i - \zeta \sum_{j\in -i} a_j\right)x_i - x_i^2 + \zeta \sum_{j\in -i} x_j x_i,$$

for $\zeta = \frac{1-(b-c)}{b-c}$. In particular, we note that we may set:

$$\begin{aligned}2\alpha\gamma^{ij} &= \zeta \\ 2(1-\alpha)\delta_i &= \left(a_i - \zeta \sum_{j\in -i} a_j\right).\end{aligned}$$

So that the largest eigenvalue of $\mathbf{\Gamma}$ is $\frac{\zeta}{2}(n-1)$ and the content of Assumption 4 is justified in light of Assumption 1.

Proof of Proposition **??**.

Proof. First, the pricing game has the same set of equilibria as the particular case of $G(x_0)$ in which: $\underline{p} - p_0$, the favorite outcome of i is $\widehat{a}_i/[2(1-\alpha)]$, coordination motives are $\zeta/2$ and Γ is the adjacency matrix of a network in which $\gamma^{ij} = 1, i \in N, j \in -i$, which we refer to as a *complete* network for the present proof. This result follows from Lemma 5. This observation implies the first part of the proposition via Lemma 3.

Second, let's establish a property of equilibria. Let $\boldsymbol{p}$ be an equilibrium. By the decomposition in Proposition 2, if the network is complete and $p_i = p_j$, then

$$(1+\alpha)[\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_j)] = (1-\alpha)(\delta_i - \delta_j) - m\alpha k,$$

for $m \in [0, M]$, in which- $M = |\{\ell \in N : p_\ell \in [p_i, p_j]\}|$. In particular, a similar derivation is described in the proof of Lemma 2, and it is omitted in the present proof. From the above equality it follows that: $p_i = p_j$ implies that $m\alpha k \geq (1-\alpha)(\delta_i - \delta_j)$. In the pricing game, then, $p_i = p_j$ implies that

$$m\zeta k \geq \widehat{a}_i - \widehat{a}_j. \tag{4}$$

Third, we establish that: if $\min_{i\in N, j\in -i} |\widehat{a}_i - \widehat{a}_j| > 2\zeta k$, the no two players choose the same policy in equilibrium. In what follows, we fix an equilibrium $\boldsymbol{p}^\star \in (p_0, \overline{p})$, and a policy $p \in (p_1^\star, \ldots, p_n^\star)$ that is played in equilibrium by a number of players $m \in \{2, n\}$. For fixed number of players $m \in \{2, \ldots, n\}$ who play the same policy p in equilibrium $\boldsymbol{p}^\star$, there exist players i', j' who play p and with

$$\widehat{a}_{i'} - \widehat{a}_{j'} > (m-1) \min_{i \in N, j \in -i} |\widehat{a}_i - \widehat{a}_j| \tag{5}$$

In particular, this observation holds by taking i', j' to be the players choosing, respectively, $\min\{p_1^\star, \ldots, p_n^\star\}$ and $\max\{p_1^\star, \ldots, p_n^\star\}$. Let's observe that: if $\min_{i \in N, j \in -i} |\widehat{a}_i - \widehat{a}_j| > 2\zeta k$, then $\min_{i \in N, j \in -i} |\widehat{a}_i - \widehat{a}_j| > \frac{m'}{m'-1}\zeta k$ for all $m' \in \{2, \ldots, n\}$, so:

$$(m-1) \min_{i \in N, j \in -i} |\widehat{a}_i - \widehat{a}_j| > m\zeta k.$$

Hence, if $\min_{i \in N, j \in -i} |\widehat{a}_i - \widehat{a}_j| > 2\zeta k$, inequality 4 contradicts inequality 5.

Fourth, we show that the only interior equilibrium in which no two players choose the same policy is $p_1 <, \ldots, < p_n$ if $\min_{i \in N, j \in -i} |\widehat{a}_i - \widehat{a}_j| > 2\zeta k$. By the proof of Lemma 2, if the network is complete and $\boldsymbol{p} \in (p_0, \overline{p})^n$ is an equilibrium with $p_0 < p_1 < \cdots < p_n < \overline{p}$, then

$$(1+\alpha)[\mathbb{E}\chi(p_i) - \mathbb{E}\chi(p_j)] = (1-\alpha)(\delta_i - \delta_j) - 2\alpha k,$$

whenever $p_i < p_j$. We note that $\alpha < 1$ under a complete network, by Assumption 1.

Hence, if $\min_{i \in N, j \in -i} |\widehat{a}_i - \widehat{a}_j| > 2\zeta k$ and $\boldsymbol{p} \in (p_0, \overline{p})^n$ is an equilibrium of the pricing game, then $p_0 < p_1 < \cdots < p_n < \overline{p}$ up to a permutation of players. Moreover, by the decomposition in Proposition 2, if $\min_{i \in N, j \in -i} |\widehat{a}_i - \widehat{a}_j| > 2\zeta k$ there exists at most one interior equilibrium.

■

Proofs for Remark 1 We say that players have the *same unweighted centrality* if $\boldsymbol{u} := (\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}\boldsymbol{1}$ is such that $u_i = u_j$ for all players $i, j \in N$. An equilibrium $\boldsymbol{p} \in P^n$ is *symmetric* if $p_i = p_j$ for all players $i, j \in N$.

Lemma D.36. *Let players have the same centrality, same unweighted centrality, and $\underline{p} = p_0$. If $\chi(p_0)$ and $\overline{p}$ are sufficiently large, there exist a greatest and a least symmetric equilibrium, respectively $\boldsymbol{q}$ and $\boldsymbol{s}$. Moreover:*

$$\begin{aligned}\mathbb{E}\boldsymbol{\chi}(\boldsymbol{q}) &= \boldsymbol{\beta} + \boldsymbol{1}k \\ \mathbb{E}\boldsymbol{\chi}(\boldsymbol{s}) &= \boldsymbol{\beta} + 2\boldsymbol{1}k - \boldsymbol{u}k.\end{aligned}$$

Proof. **Application of the Decomposition of Equilibrium Expected Outcomes**
Let $([p_i < p_j],\ i, j \in N)$ and $([p_i \le p_j],\ i, j \in N)$ be two n-by-n matrices, in which $[Y]$ is the Iverson bracket of the statement Y, so $[Y] = 1$ if the statement Y is true, and $[Y] = 0$ otherwise. We define $\boldsymbol{\Gamma}_+(p) = \boldsymbol{\Gamma} \odot ([p_i < p_j],\ i, j \in N)$ and $\boldsymbol{\Gamma}_-(p) = \boldsymbol{\Gamma} \odot ([p_i \le p_j],\ i, j \in N)$. By the decomposition in Proposition 2, $p \in (p_0, \overline{p})^n$ is an interior equilibrium if, and only if:

$$k(\boldsymbol{I} - 2\alpha\boldsymbol{\Gamma}_-(p))\boldsymbol{1} \le (\boldsymbol{I} - \alpha\boldsymbol{\Gamma})(\mathbb{E}\boldsymbol{\chi}(p) - \boldsymbol{\beta}) \le k(\boldsymbol{I} - 2\alpha\boldsymbol{\Gamma}_+(p))\boldsymbol{1}.$$

Implications of symmetric equilibria
If $p \in (p_0, \overline{p})^n$, then:

$$\boldsymbol{\beta} + (\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}(\boldsymbol{I} - 2\alpha\boldsymbol{\Gamma}_{-}(p))\mathbf{1}k = \boldsymbol{\beta} + 2\mathbf{1}k - \boldsymbol{u}k$$
$$\boldsymbol{\beta} + (\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}(\boldsymbol{I} - 2\alpha\boldsymbol{\Gamma}_{+}(p))\mathbf{1}k = \boldsymbol{\beta} + \mathbf{1}k.$$

(The first equality follows from the definition of $\boldsymbol{B}$.)

The result follows. ■

Corollary 3. *Let $\delta_i = 0$ for all $i \in N$ and players have the same unweighted centrality. Then, $\boldsymbol{p} \in (p_0, \overline{p})^n$ is an equilibrium if, and only if:*

$$\mathbb{E}\boldsymbol{\chi}(\boldsymbol{p}) \in [(2\mathbf{1} - \boldsymbol{u})k, \boldsymbol{u}k].$$

Moreover: $\boldsymbol{u}k$ is increasing in α and k, $(2\mathbf{1} - \boldsymbol{u})k$ is decreasing in α, and $(2 - u_i)k$ is increasing in k iff $u_i < 2$.

Proof of Lemma 1

Proof. The first part of the proof is a consequence of an observation made in Vives (1999), Chapter 2, Footnote 23, and the potential structure of the game (Proposition B.11.) The second part follows from Corollary 3, after noting that players have the same unweighted centralities under a complete network. ■

E Proofs for Section 4.1

Proofs for Section 4.1

Towards the proof of Proposition 3, introduce a definitions and several lemmata.

Definition 4 (Monderer and Shapley (1996))**.** *The game in strategic form $\langle I, \{S_i, u_i\}_{i\in I}\rangle$ is a* potential game *if there exists a function $U\colon \times_i S_i \to \mathbf{R}$ such that, for all $i \in I$, $s_{-i} \in \times_{j\neq i} S_j$ and $s_i, s_i' \in S_i$:*

$$u_i((s_i, s_{-i})) > u_i((s_i', s_{-i})) \text{ iff } U((s_i, s_{-i})) > U((s_i', s_{-i}));$$

the function U is called a potential *for the game.*

Towards the study of a selection rule for equilibria of $G(x_0)$, we introduce a function that is related to the potential of the game without complexity. The no-complexity potential is the function $v\colon \mathbf{R}^n \to \mathbf{R}$ given by

$$v(\boldsymbol{x}) = 2(1-\alpha)\boldsymbol{\delta}^{\mathsf{T}}\boldsymbol{x} - \boldsymbol{x}^{\mathsf{T}}(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})\boldsymbol{x}.$$

And the expected no-complexity potential $V\colon P^n \to \mathbf{R}$ is given by

$$V(p) = \mathbb{E}v(\boldsymbol{\chi}(p)), \text{ for all } p \in P^n.$$

The expected no-complexity potential, or *potential*, provides a potential for the game $G(x_0)$, as established by the next results. The function v is the potential of the game S defined in Section 2.3; this result is a corollary to Proposition E.5 and is known (Jackson and Zenou, 2015).

Lemma E.37. *The game $G(x_0)$ is a potential game. Moreover, for every player $i \in N$ there exists a function $g_i\colon P^{n-1} \times \mathbf{R} \to \mathbf{R}$ such that:*

$$\mathbb{E}\pi_i(\boldsymbol{\chi}(p)) = \mathbb{E}v(\boldsymbol{\chi}(p)) + g_i(p_{-i}, x_0^i) \text{ for all } p \in P^n \text{ and } x_0 \in \mathbf{R},$$

and a potential for $G(x_0)$ is the expected no-complexity potential $V\colon p \mapsto \mathbb{E}v(\boldsymbol{\chi}(p))$ given the status-quo outcome x_0.

Proof of Lemma E.37.

Proof. We first establish von-Neumann-Morgenstern equivalence (Morris and Ui, 2004) between the two strategic-form games S and $\langle N, \{P, v\}_{i\in N}\rangle$. Thus, we show that: for all $i \in N$, there exists a function $h_i\colon \mathbf{R}^{n-1} \to \mathbf{R}$ such that

$$\pi_i(\boldsymbol{x}) - v(\boldsymbol{x}) = h_i(x_{-i}) \text{ for all } \boldsymbol{x} \in \mathbf{R}^n.$$

The claim is a consequence of $\mathbf{\Gamma}$ being a symmetric matrix. In particular, we note that $\sum_{(i,j)\in N^2} \gamma^{ij} x_i x_j - 2\sum_{j\in N} \gamma^{ij} x_i x_j$ is constant with respect to x_i, and:

$$v(\boldsymbol{x}) - v_i(\boldsymbol{x}) = \sum_{j\in -i}\Big(2(1-\alpha)\delta_j x_j - x_j^2\Big) + \alpha \sum_{(i,j)\in N^2} \gamma^{ij} x_i x_j - 2\alpha \sum_{j\in N} \gamma^{ij} x_i x_j.$$

The second part of the Lemma follows, by observing that $v_i(\boldsymbol{x}) - \pi_i(\boldsymbol{x})$ is constant in x_{-i}, as shown in Section A, and taking expectations given the status-quo outcome.

It remains to establish that von-Neumann-Morgenstern equivalence between G_0 and $\langle N, \{P, \mathbb{E}v(\boldsymbol{\chi}(p))\}_{i\in N}\rangle$ implies that G_0 is a potential game according to the definition in Monderer and Shapley (1996). We prove a stronger statement: V is a w-potential for $G(x_0)$ with $w_i = 1$ for all $i \in N$, that is, $G(x_0)$ is an weighted and exact potential game, and V is a weighted and exact potential. The intuition for the observation is the same underlining Lemma 1 in Morris and Ui (2004), we include a proof solely because the authors assume finite strategy spaces.

Let $\Pi_i(q_i, p_{-i}) := \mathbb{E}\pi_i(\chi(p_1), \ldots, \chi(q_i), \chi(p_{i+1}), \ldots)$. By the definitions of Monderer and Shapley (1996), pages 127-128, V is an exact potential for $G(x_0)$ if $\Pi_i(p_i, \cdot) - \Pi_i(p_i', \cdot) = V((p_i, \cdot)) - V((p_i', \cdot))$ for all $p_i, p_i' \in P$. By our preceding results:

$$\Pi_i(p_i, p_{-i}) - V((p_i, p_{-i})) = g_i(p_{-i}, x_0) \text{ and } \Pi_i(p_i', p_{-i}) - V((p_i', p_{-i})) = g_i(p_{-i}, x_0).$$

Thus, we have

$$\Pi_i(p_i, p_{-i}) - V((p_i, p_{-i})) = \Pi_i(p_i', p_{-i}) - V((p_i', p_{-i})),$$

which we rearrange to write:

$$\Pi_i(p_i, p_{-i}) - \Pi_i(p_i', p_{-i}) = V((p_i, p_{-i})) - V((p_i', p_{-i})).$$

■

Lemma E.38. *If U is a potential for the game $G(x_0)$, there exists a constant $c \in \mathbf{R}$ such that*

$$U(p) = V(p) + c, \text{ for all } p \in P^n.$$

Moreover, if p is a potential maximizer, then p is an equilibrium of $G(x_0)$.

Proof of Lemma E.38.

Proof. Let $p \in P^n$ be a potential maximizer and $i \in N, q_i \in P$ such that

$$\mathbb{E}\pi_i(\boldsymbol{\chi}(p)) < \mathbb{E}\pi_i(\dots, \chi(p_{i-1}), \chi(q_i), \dots).$$

By Lemma E.37, we have

$$\mathbb{E}v(\boldsymbol{\chi}(p)) < \mathbb{E}v(\dots, \chi(p_{i-1}), \chi(q_i), \dots),$$

Which contradicts the definition of p.

The second part of the Lemma follows from Lemma 2.7 in Monderer and Shapley (1996) if $G(x_0)$ is an exact potential game, using a definition in Monderer and Shapley (1996), pages 127-128. In the proof of Lemma E.37, we establish that $G(x_0)$ is an exact potential game when we show that V is an exact potential for $G(x_0)$. ■

Proposition E.5. *The game $G(x_0)$ is a potential game and $V\colon P^n \to \mathbf{R}$ is a potential for $G(x_0)$. Moreover,*

(1) If $U\colon P^n \to \mathbf{R}$ is a potential for $G(x_0)$, there exists a constant $c \in \mathbf{R}$ such that

$$U(p) = V(p) + c, \text{ for all } p \in P^n.$$

(2) If the policy profile $p \in P^n$ maximizes V, then p is an equilibrium of $G(x_0)$.

Proof of Proposition E.5

Proof. The Proposition follows directly from Lemmata E.37 and E.38. ■

We establish an auxiliary Lemma towards the proof of Proposition 4. Towards a characterization of the potential maximizer, we note that the no-complexity potential can be expressed as $v(\boldsymbol{x}) = -(\boldsymbol{x}-\boldsymbol{\beta})^\mathsf{T}(\boldsymbol{I}-\alpha\boldsymbol{\Gamma})(\boldsymbol{x}-\boldsymbol{\beta}) + \boldsymbol{\beta}^\mathsf{T}(\boldsymbol{I}-\alpha\boldsymbol{\Gamma})\boldsymbol{\beta}$, which directly implies the following expression for V.

Lemma E.39. *For all policy profiles $p \in P^n$, we have that*

$$V(p) = -(\mathbb{E}\boldsymbol{\chi}(p) - \boldsymbol{\beta})^\mathsf{T}(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})(\mathbb{E}\boldsymbol{\chi}(p) - \boldsymbol{\beta}) - \sum_{i \in N} \mathbb{V}\chi(p_i) + \alpha \sum_{i,j \in N} \gamma^{ij}\mathbb{C}[\chi(p_i), \chi(p_j)],$$

up to a term that is constant in p.

Proof of Lemma E.39.

Proof. We observe that the potential function v is a quadratic form, so $V(p) = -(\mathbb{E}\boldsymbol{\chi}(p) - \boldsymbol{\beta})^\mathsf{T}(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})(\mathbb{E}\boldsymbol{\chi}(p) - \boldsymbol{\beta}) - \mathrm{tr}((\boldsymbol{I} - \alpha\boldsymbol{\Gamma})\boldsymbol{\Omega}) + \boldsymbol{\beta}^\mathsf{T}(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})\boldsymbol{\beta}$, in which $\boldsymbol{\Omega}$ is the variance-covariance matrix of $\boldsymbol{\chi}(p)$ given $\chi(p_0) = x_0$, which is well-defined by joint Gaussianity of outcomes and $\omega > 0$. ∎

Proposition E.6 (Potential maximizer)**.** *Let $P = [p_0, \overline{p}]$. There exists a unique potential maximizer. Moreover, the policy profile $p \in (p_0, \overline{p})^n$ is a potential maximizer if, and only if:*

$$\mathbb{E}\boldsymbol{\chi}(p) = \boldsymbol{\beta} + \mathbf{1}k + \alpha(\boldsymbol{I} - \alpha\boldsymbol{\Gamma})^{-1}(\boldsymbol{\Gamma} \odot \boldsymbol{A})\mathbf{1}k,$$

for a skew-symmetric matrix $\boldsymbol{A} = [a_{ij} : i, j \in N]$ such that $a_{ij} \in [-1, 1]$ and $a_{ij} = 1$, if $p_i > p_j$.

Proof of Proposition E.6.

Proof. The first part of the result is a consequence of standard tools in convex analysis. First, we claim that there exists at most one potential maximizer. This follows from strict concavity of V, proved in Section A.4. For existence given strict convexity of $-V$ see, e.g., Proposition 9.3.2, part (iv), in Briceño-Arias and Combettes (2013), stated in a game-theoretic environment.

The characterization of the potential maximizer is established in Lemma A.10. ∎

Proof of Proposition 3.

Proof. Part (1) follows from Proposition E.5. Part (2) follows from Proposition E.6. ∎

Proof of Proposition 4.

Proof. The result follows from Proposition E.6. ∎

Proof of Proposition 5.

Proof. We use the notation developed in Section A, in which we define v_i as the "effort-game ex-post payoff", defined over outcome profiles. It holds that:

$$v(\boldsymbol{x}) = \sum_i v_i(\boldsymbol{x}) - \alpha\boldsymbol{x}^\mathsf{T}\boldsymbol{\Gamma}\boldsymbol{x}.$$

Thus, we have that:

$$W(p) = \mathbb{E}\Big[v(\boldsymbol{\chi}(p)) + \alpha\boldsymbol{\chi}(p)^\mathsf{T}\boldsymbol{\Gamma}\boldsymbol{\chi}(p) | \chi(p_0) = x_0\Big].$$

Strict concavity of W on $[p_0, \overline{p}]^n$ follows from the same argument as Lemma A.8. Thus, the superdifferential of W is well-defined. By standard subgradient calculus (Rockafellar, 1970), we write the following expression for ∂W, using $+$ for (Minkowski) set addition,

$$\partial W(p) = \partial V(p) + \partial \mathbb{E}\Big[\alpha \boldsymbol{\chi}(p)^\intercal \boldsymbol{\Gamma} \boldsymbol{\chi}(p) | \chi(p_0) = x_0\Big].$$

Using the decomposition of expectation of quadratic forms, we have:

$$\partial W(p) = \partial V(p) + 2\alpha \boldsymbol{\Gamma} \partial \mathbb{E}[\boldsymbol{\chi}(p)] + \alpha \partial \sum_{(i,j)\in N^2} \gamma^{ij}\, \mathbb{C}(\chi(p_i), \chi(p_j)),$$

for which we also apply symmetry of $\boldsymbol{\Gamma}$. The result follows from the characterization of $\partial V(p)$ in Lemma A.10, in which we also characterize $\partial \sum_{(i,j)\in N^2} \gamma^{ij}\, \mathbb{C}(\chi(p_i), \chi(p_j))$. ■

Proofs for Section 4.2 and Section 4.3

In this section, we assume that $P = [p_0, \overline{p}]$.

Lemma E.40. *Let $|a_1 - c_1 - a_2 + c_2| \leq -gk$. For sufficiently large $\chi(p_0)$, total profits are maximized by*

$$\mathbb{E}\chi(p_i) = \min\Bigg\{b\frac{a - c_1 + a - c_2}{4(1+gb)} + k, \chi(p_0)\Bigg\}.$$

The maximization of total profits is implemented in equilibrium if, and only if: $a - c_1 + a - c_2 \leq \frac{1+bg}{b}2k$.

Proof. By Lemma 5, we find the set of equilibria using Proposition 2. By Proposition 5 and Lemma 5, we find the maximizer of total profits by using E.41 and $2g$ in place of g. ■

Dyad We assume that $N = 2$, and we use $\widehat{\alpha} := \alpha\gamma^{12}$. We use $\chi_i := \chi(p_i)$, $\boldsymbol{\chi}$ for the column vector of outomes $(\chi(p_1), \chi(p_2))'$, and ∂_{p_i} for the subdifferential with respect to p_i. The expectation operators are conditional on $\chi(p_0) = x_0$. Let $y_+ := \max\{\beta_1, \beta_2\} + k\Big(1 - \frac{\widehat{\alpha}}{1+\widehat{\alpha}}\Big)$, $y_- := \min\{\beta_1, \beta_2\} + k\Big(1 + \frac{\widehat{\alpha}}{1+\widehat{\alpha}}\Big)$.

Lemma E.41 (Dyad)**.** *Let $y_+ \geq x_0$ and $\mathbb{E}\chi(\overline{p}) \geq y_-$. The following hold.*

(1) If $(1-\alpha)(\delta_2 - \delta_1) \geq 2\widehat{\alpha}k$, then there exists a unique equilibrium in $G|_{x_0}$. Moreover, in equilibrium:

$$\mathbb{E}\chi_1 = \beta_1 + k\Big(1 + \frac{\widehat{\alpha}}{1+\widehat{\alpha}}\Big)$$
$$\mathbb{E}\chi_2 = \beta_2 + k\Big(1 - \frac{\widehat{\alpha}}{1+\widehat{\alpha}}\Big),$$

which imply

$$\mathbb{E}\chi_2 - \mathbb{E}\chi_1 = \beta_2 - \beta_1 - 2\frac{\widehat{\alpha}}{1+\widehat{\alpha}}k.$$

.

(2) If $(1-\alpha)(\delta_2-\delta_1) < 2\widehat{\alpha}k$*, then there exist multiple equilibria in* $G|_{x_0}$*. Moreover, in equilibrium:*

$$\begin{aligned}(1-\alpha)(\delta_2-\delta_1) = \widehat{\alpha}(d_1-d_2),\ \textit{for some } d_2, d_1 \in [-1,1]\\ \mathbb{E}\chi_1 = \mathbb{E}\chi_2 = \frac{\beta_1+\beta_2}{2} + k + \frac{\widehat{\alpha}}{1-\widehat{\alpha}}\frac{d_1+d_2}{2}k\\ \in \left[\frac{\beta_1+\beta_2}{2} + k - \frac{\widehat{\alpha}}{1-\widehat{\alpha}}k, \frac{\beta_1+\beta_2}{2} + k + \frac{\widehat{\alpha}}{1-\widehat{\alpha}}k\right].\end{aligned}$$

(3) If $0 \le (1-\alpha)(\delta_2-\delta_1) < 2\widehat{\alpha}k$*, then there exists a unique potential maximizer in* $G|_{x_0}$*. Moreover, in the potential maximizer:* $(1-\alpha)(\delta_2-\delta_1) = 2\widehat{\alpha}d_1k$*,* $d_1 \in [0,1)$*, and:*

$$\begin{aligned}\mathbb{E}\chi_1 = \beta_1 + k\Big(1 + \frac{\widehat{\alpha}}{1+\widehat{\alpha}}d_1\Big)\\ \mathbb{E}\chi_2 = \beta_2 + k\Big(1 - \frac{\widehat{\alpha}}{1+\widehat{\alpha}}d_1\Big),\end{aligned}$$

which imply

$$\mathbb{E}\chi_1 = (\beta_1+\beta_2)/2 + k.$$

Proof. The expected effort-game payoff to player i is:

$$\mathbb{E}v_i(\chi_i,\chi_j) = 2(1-\alpha)\delta_i\mathbb{E}\chi_i - (\mathbb{E}\chi_i)^2 + 2\widehat{\alpha}\mathbb{E}\chi_i\mathbb{E}\chi_j - \mathbb{V}\chi_i + 2\widehat{\alpha}\mathbb{C}\chi_i\chi_j,$$

up to a term that is constant with respect to p_i. The superdifferential of $\mathbb{E}v_i(\chi_i,\chi_j)$ with respect to p_i is:

$$2\mu(1-\alpha)\delta_i - 2\mu\mathbb{E}\chi_i + 2\mu\widehat{\alpha}\mathbb{E}\chi_j - \omega + \widehat{\alpha}\omega - \widehat{\alpha}\omega\partial_{p_i}|p_i-p_j|.$$

In any interior equilibrium p:

$$0 \in \begin{pmatrix}1 & -\widehat{\alpha}\\ -\widehat{\alpha} & 1\end{pmatrix}\mathbb{E}\boldsymbol{\chi} - (1-\alpha)\boldsymbol{\delta} - \begin{pmatrix}1 & -\widehat{\alpha}\\ -\widehat{\alpha} & 1\end{pmatrix}\mathbf{1}k - \widehat{\alpha}\begin{pmatrix}\partial_{p_1}|p_1-p_2|\\ \partial_{p_2}|p_2-p_1|\end{pmatrix}k$$

Thus, we obtain the following interior equilibrium condition. $p \in (p_0,\overline{p})$ is an equilibrium if, and only if:

$$\mathbb{E}\boldsymbol{\chi} \in \frac{1-\alpha}{1-\widehat{\alpha}^2}\begin{pmatrix}1 & \widehat{\alpha}\\ \widehat{\alpha} & 1\end{pmatrix}\delta + k\mathbf{1} + \frac{\widehat{\alpha}}{1-\widehat{\alpha}^2}\begin{pmatrix}1 & \widehat{\alpha}\\ \widehat{\alpha} & 1\end{pmatrix}\begin{pmatrix}\partial_{p_1}|p_1-p_2|\\ \partial_{p_2}|p_2-p_1|\end{pmatrix}k,$$

and in an equilibrium in which $p_1 > p_2$ the last term simplifies to a singleton:

$$\frac{\widehat{\alpha}}{1-\widehat{\alpha}^2}\begin{pmatrix}1 & \widehat{\alpha}\\ \widehat{\alpha} & 1\end{pmatrix}\begin{pmatrix}\partial_{p_1}|p_1-p_2|\\ \partial_{p_2}|p_2-p_1|\end{pmatrix}k = \left\{\frac{\widehat{\alpha}}{1+\widehat{\alpha}}\begin{pmatrix}1\\ -1\end{pmatrix}k\right\}.$$

In an equilibrium p in which $p_1 = p_2$, the last term can be written as:

$$\frac{\widehat{\alpha}}{1-\widehat{\alpha}^2}\begin{pmatrix}1 & \widehat{\alpha}\\ \widehat{\alpha} & 1\end{pmatrix}\begin{pmatrix}\partial_{p_1}|p_1-p_2|\\ \partial_{p_2}|p_2-p_1|\end{pmatrix}k = \frac{\widehat{\alpha}}{1-\widehat{\alpha}^2}\begin{pmatrix}\partial_{p_1}|p_1-p_2| + \widehat{\alpha}\partial_{p_2}|p_2-p_1|\\ \partial_{p_2}|p_2-p_1| + \widehat{\alpha}\partial_{p_1}|p_1-p_2|\end{pmatrix}k.$$

In the potential maximizer p, we have that: $\partial_{p_1}|p_1-p_2| = -\partial_{p_2}|p_2-p_1|$, and so the last term simplifies to:

$$\frac{\widehat{\alpha}}{1-\widehat{\alpha}^2}\begin{pmatrix}1 & \widehat{\alpha}\\ \widehat{\alpha} & 1\end{pmatrix}\begin{pmatrix}\partial_{p_1}|p_1-p_2|\\ \partial_{p_2}|p_2-p_1|\end{pmatrix}k = \frac{\widehat{\alpha}}{1+\widehat{\alpha}}\begin{pmatrix}1\\ -1\end{pmatrix}\partial_{p_1}|p_1-p_2|k.$$

■

Two-type network We assume that there are two groups of players, A and B, such that: $N = A \cup B$, and $A \cap B = \emptyset$. We let $n_G := |G|$, $G \in \{A, B\}$, and $G(\ell), -G(\ell)$ denote, respectively, the group of player ℓ and the other group. Moreover, we assume that: $\delta_\ell = \delta_{G(\ell)}$, and

$$\gamma^{\ell k} = \gamma^{G(\ell)G(k)}, \text{ for all } \ell, k \in N.$$

We note that, by our maintained assumptions: $\gamma^{AB} = \gamma^{BA}$, and: $\gamma^{GF} = o(n)$, because $n_F\gamma^{GF} + (n_G-1)\gamma^{GG} \leq 1$, for all $G, F \in \{A, B\}, G \neq F$.

The potential function is such that

$$G(i) = G(j) \text{ implies } v(x_1, \dots x_i, \dots x_j, \dots x_n) = v(x_1, \dots x_j, \dots x_i, \dots x_n),$$

so every equilibrium is represented by a pair (p_A, p_B), such that $i \in A$ plays p_A, and $j \in B$ plays p_B. We let $\mathbb{E}\chi_{G(i)} = \mathbb{E}\chi_i$ in the potential maximizer p. We use $\alpha_A := \frac{\alpha\gamma^{AB}n_B}{1-\alpha\gamma^{AA}(n_A-1)}$ and $\alpha_B := \frac{\alpha\gamma^{BA}n_A}{1-\alpha\gamma^{BB}(n_B-1)}$. We note that: $\alpha_A \leq \frac{\alpha\gamma^{AB}n_B}{\alpha\gamma^{AB}n_B+\alpha\gamma^{AA}(n_A-1)-\alpha\gamma^{AA}(n_A-1)} = 1$, and, similarly, $\alpha_B \leq 1$.

We note that $\frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B} \in [0,1]$, because:

$$\alpha_A + \alpha_B - 2\alpha_A\alpha_B > 0 \iff \frac{\alpha_A}{1-\alpha_A} + \frac{\alpha_B}{1-\alpha_B} > 0,$$

and

$$\frac{\alpha_A + \alpha_B - 2\alpha_A\alpha_B}{1-\alpha_A\alpha_B} = 1 - \frac{(1-\alpha_A)(1-\alpha_B)}{1-\alpha_A\alpha_B}.$$

Also, we note that $\frac{\partial}{\partial\alpha_{G(i)}}\frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B} = \left(\frac{1-\alpha_{-G(i)}}{1-\alpha_A\alpha_B}\right)^2$.

Lemma E.42. *Let Γ be a two-type network, such that: $\beta_A \geq \beta_B$, and let $x_0 \geq \beta_A + k - \alpha_A(1-\alpha_B)\frac{1}{1-\alpha_A\alpha_B}k$ and $\beta_B + k + \alpha_B(1-\alpha_A)\frac{1}{1-\alpha_A\alpha_B}k \geq \mathbb{E}\chi(\overline{p})$.*

(1) If $\beta_A - \beta_B \geq \frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B}k$, then $p_A \leq p_B$ in the unique interior potential maximizer. Moreover:

$$\mathbb{E}\chi_A = \beta_A + k - \alpha_A(1-\alpha_B)\frac{1}{1-\alpha_A\alpha_B}k$$
$$\mathbb{E}\chi_B = \beta_B + k + \alpha_B(1-\alpha_A)\frac{1}{1-\alpha_A\alpha_B}k,$$

which imply:

$$\mathbb{E}\chi_A - \mathbb{E}\chi_B = \beta_A - \beta_B - \frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_N}k.$$

(2) If $\beta_A - \beta_B < \frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B}k$, then $p_A = p_B$ in the unique interior potential maximizer. Moreover:

$$\mathbb{E}\chi_A = \beta_A + k - \frac{\alpha_A(1-\alpha_B)}{1-\alpha_A\alpha_B}dk$$
$$\mathbb{E}\chi_B = \beta_B + k + \frac{\alpha_B(1-\alpha_A)}{1-\alpha_A\alpha_B}dk,\ d \in [0,1].$$

and $\beta_A - \beta_B = \frac{\alpha_A+\alpha_B-2\alpha_A\alpha_B}{1-\alpha_A\alpha_B}dk$, which imply:

$$\mathbb{E}\chi_A = \frac{\alpha_B(1-\alpha_A)\beta_A + \alpha_A(1-\alpha_B)\beta_B}{\alpha_B(1-\alpha_A)+\alpha_A(1-\alpha_B)} + k.$$

Proof. The superdifferential of $\mathbb{E}v_i(\chi_1,\dots,\chi_n)$ with respect to p_i, $i \in A$, evaluated at an equilibrium, is:

$$2\mu(1-\alpha)\delta_A - 2\mu\mathbb{E}\chi_i + 2\mu\alpha\gamma^{AA}(n_A-1)\mathbb{E}\chi_A + 2\mu\alpha\gamma^{AB}n_B\mathbb{E}\chi_B +$$
$$-\omega + \alpha\gamma^{AA}(n_A-1)\omega + \alpha\gamma^{AB}(n_B)\omega - \alpha\gamma^{AA}(n_A-1)\partial_{p_i}|p_i-p_A|\omega - \alpha\gamma^{AB}n_B\partial_{p_i}|p_i-p_B|.$$

If p is the potential maximizer, then: $p_i = p_{G(i)}$, and:

$$\begin{aligned}
0 \in\ & 2\mu(1-\alpha)\delta_A - 2\mu\mathbb{E}\chi_A + 2\mu\alpha\gamma^{AA}(n_A-1)\mathbb{E}\chi_A + 2\mu\alpha\gamma^{AB}n_B\mathbb{E}\chi_B + \\
& -\omega + \alpha\gamma^{AA}(n_A-1)\omega + \alpha\gamma^{AB}(n_B)\omega - \alpha\gamma^{AB}n_B\partial_{p_A}|p_A-p_B| \\
0 \in\ & 2\mu(1-\alpha)\delta_B - 2\mu\mathbb{E}\chi_B + 2\mu\alpha\gamma^{BB}(n_B-1)\mathbb{E}\chi_B + 2\mu\alpha\gamma^{BA}n_A\mathbb{E}\chi_A + \\
& -\omega + \alpha\gamma^{BB}(n_B-1)\omega + \alpha\gamma^{BA}(n_A)\omega - \alpha\gamma^{BA}n_A\partial_{p_B}|p_B-p_A|.
\end{aligned}$$

We use $\alpha_A := \frac{\alpha\gamma^{AB}n_B}{1-\alpha\gamma^{AA}(n_A-1)}$ and $\alpha_B := \frac{\alpha\gamma^{BA}n_A}{1-\alpha\gamma^{BB}(n_B-1)}$. We note that: $\alpha_A \leq \frac{\alpha\gamma^{AB}n_B}{\alpha\gamma^{AB}n_B+\alpha\gamma^{AA}(n_A-1)-\alpha\gamma^{AA}(n_A-1)}$, so $\alpha_A \leq 1$, and, similarly, $\alpha_B \leq 1$. Thus, if p is the potential maximizer, then $p_i = p_{G(i)}$, and, for some $d \in \partial_{p_A}|p_A - p_B|$:

$$0 = 2\mu(1-\alpha)\begin{pmatrix} \frac{\delta_A}{1-\alpha\gamma^{AA}(n_A-1)} \\ \frac{\delta_B}{1-\alpha\gamma^{BB}(n_B-1)} \end{pmatrix} - 2\mu\begin{pmatrix} 1 & -\alpha_1 \\ -\alpha_2 & 1 \end{pmatrix}\begin{pmatrix} \mathbb{E}\chi_A \\ \mathbb{E}\chi_B \end{pmatrix} - \begin{pmatrix} 1 & -\alpha_1 \\ -\alpha_2 & 1 \end{pmatrix}\mathbf{1}\omega + \begin{pmatrix} \alpha_A \\ -\alpha_B \end{pmatrix}\omega d.$$

Thus, $p \in (p_0, \overline{p})^n$ is the unique potential maximizer if, and only if: $p_i = p_{G(i)}$, $i \in N$, and:

$$\begin{pmatrix} \mathbb{E}\chi_A \\ \mathbb{E}\chi_B \end{pmatrix} = \begin{pmatrix} \beta_A \\ \beta_B \end{pmatrix} + k\mathbf{1} + \begin{pmatrix} 1 & -\alpha_A \\ -\alpha_B & 1 \end{pmatrix}^{-1}\begin{pmatrix} -\alpha_A \\ \alpha_B \end{pmatrix}kd, \ d \in \partial_{p_A}|p_A - p_B|.$$

In the unique potential maximizer for $p_A < p_B$, we have:

$$\begin{pmatrix} 1 & -\alpha_A \\ -\alpha_B & 1 \end{pmatrix}^{-1}\begin{pmatrix} -\alpha_A \\ \alpha_B \end{pmatrix}kd = \frac{1}{1-\alpha_A\alpha_B}\begin{pmatrix} -(1-\alpha_B)\alpha_A \\ (1-\alpha_A)\alpha_B \end{pmatrix}k,$$

and:

$$\mathbb{E}\chi_A - \mathbb{E}\chi_B = \beta_A - \beta_B - \frac{\alpha_A + \alpha_B - 2\alpha_A\alpha_B}{1-\alpha_A\alpha_B}k.$$

■

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