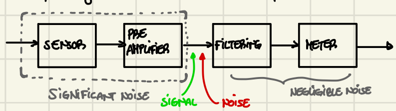
SIGNAL DESCRIPTION

INTRODUCTION TO SIGNALS

DEF: we define 'SIGNAL' (deterministic) something that varies over time, carrying a certain AMOUNT OF INFORMATION.

it comes from a MEASUREMENT made in the real world, by translating the physical quantity of interest into an ELECTRICAL QUANTITY.

tipically the CHAIN OF ACQUISITION of a SIGNAL is:



CONVOLUTION

DEF: the convolution of a signal can be defined as:

$$\boxed{y(t) = x(\alpha) * g(\alpha) = \int_{-\infty}^{+\infty} x(\alpha)\, g(t-\alpha)\, d\alpha}$$

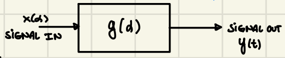
CONVOLUTION

- it can be seen as the SUM (INTEGRAL) of the MULTIPLICATIONS between the input signal $x(\alpha)$ and the corresponding PULSE RESPONSE $g(\alpha)$.
- take the δ-response. then, REVERSE and SHIFT. EVERY SHIFT, MULTIPLICATE and TAKE THE AREA, which gives exactly the value of the output for that time-shift $y(t=T)$.

it's very useful in CONSTANT PARAMETER LINEAR FILTERS:

TIME INVARIANT $\Rightarrow g[x(\alpha-\tau)](\alpha) = g[x(\alpha)](\alpha-\tau)$

NO SWITCHES, NO TIME-VARIANT COMPONENTS.



where

$x(\alpha)$: LINEAR SUPERPOSITION OF δ PULSES OF AMPLITUDE $x(\alpha)d\alpha$

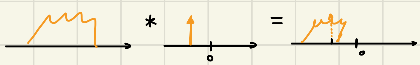
therefore

$y(t)$: LINEAR SUPERPOSITION (SUM) OF ELEMENTARY δ-PULSE RESPONSES $x(\alpha)d\alpha\, g(t-\alpha)$

CONVOLUTION

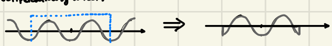
IMPORTANT EXAMPLE.

the convolution of a signal $x(t)$ for a δ is the REPLICA of $x(t)$ centred in correspondence of that δ



TRUNCATED SIGNALS

REMEMBER: since we cannot make an 'infinite-time' measurement, we always deal with TRUNCATED SIGNALS, (multiplication by a rect):



SIMILARITY OF TWO FUNCTIONS, AUTO and CROSS-CORRELATION

let's suppose we have two functions $f(t)$ and $g(t)$, we want a NUMBER to CHARACTERIZE HOW MUCH THESE TWO FUNCTIONS RESEMBLE EACH OTHER.

→ WE NEED AN OPERATION that in some way COMPARES THE VALUES OF THE TWO FUNCTION AT EACH TIME t, AND TELL US HOW MUCH ARE SIMILAR.
HOWEVER, TO MAKE THIS WORK, WE NEED TO MAKE THIS COMPARISON FOR EVERY POSSIBLE TIME SHIFT.
this avoids the problem of having equal shapes seem different only because they're shifted.

$$\boxed{K_{fg}(\tau) = \int_{-\infty}^{+\infty} f(\alpha)\, g(\alpha+\tau)\, d\alpha}$$

CROSS-CORRELATION

where

τ: RELATIVE TIME-SHIFT applied to g before comparison.

IF $f(\alpha) = g(\alpha)$ it's called... AUTOCORRELATION which tell us the AMOUNT OF TIME for WHICH SIGNAL IS COHERENT WITH ITSELF.

LOOKING AT AUTO CORR. WE CAN UNDERSTAND WHERE THE INFORMATION IS LOCATED

- PROPERTY:

$$\boxed{k_{fg}(\tau) = [f(-t) * g(t)]\,(\tau)}$$

CROSSCORRELATION = CONVOLUTION WITH f FLIPPED IN TIME

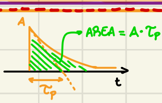
- we can also demonstrate that:

$$\boxed{k_{xx}(\tau) \leq k_{xx}(0) \quad \text{and} \quad k_{fg}(\tau) \leq \sqrt{k_{ff}(0)\, k_{gg}(0)}}$$

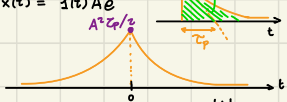
intuitively we expect a lower value since the correlation operation gives the degree of similarity. so we e

EXAMPLE:

$$x(t) = 1(t)\, A e^{-t/\tau_p}$$



$$x(t)\, x(t+\tau) = A^2 e^{-\frac{\tau}{\tau_p}}\, e^{-\frac{2t}{\tau_p}}$$



$$k_{xx}(\tau) = A^2 \frac{\tau_p}{2}\, e^{-\frac{|t|}{\tau_p}}$$

FOURIER TRANSFORM

it's the most important mathematical tool to describe TIME DOMAIN and FREQUENCY DOMAIN.

↳ the signal can be seen as a SUPERPOSITION OF SINUSOIDS.

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df$$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

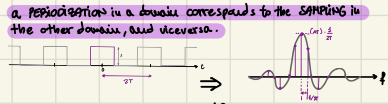
FOURIER TRANSFORM

Here are some PROPERTIES:

A) the value in zero of one domain is the area in the other domain.

$$\boxed{X(0) = \int_{-\infty}^{+\infty} x(t)\, dt \qquad x(0) = \int_{-\infty}^{+\infty} X(f)\, df}$$

B) a PERIODIZATION in a domain corresponds to the SAMPLING in the other domain, and viceversa.



if periodic every x
↳ samples every $1/x$ and multiply by $1/x$ the amplitude

if sample every x
↳ periodic every $1/x$ and divide by x the amplitude

C) the CONVOLUTION in a domain corresponds to the multiplication in the other domain:

$$\boxed{\begin{aligned} y(t) = x(t) * g(t) &\xrightarrow{\mathcal{F}} Y(f) = X(f)\, G(f) \\ Y(f) = X(f) * G(f) &\xrightarrow{\mathcal{F}^{-1}} y(t) = x(t)\cdot g(t) \end{aligned}}$$

D) other mathematical properties are.

TABLE 7.2 Fourier Transform Operations

Operation	$x(t)$	$X(\omega)$
Scalar multiplication	$kx(t)$	$kX(\omega)$
Addition	$x_1(t) + x_2(t)$	$X_1(\omega) + X_2(\omega)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Scaling (a real)	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{\omega}{a}\right)$
Time shifting	$x(t-t_0)$	$X(\omega)e^{-j\omega t_0}$
Frequency shifting (ω_0 real)	$x(t)e^{j\omega_0 t}$	$X(\omega-\omega_0)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$
Time differentiation	$\frac{d^n x}{dt^n}$	$(j\omega)^n X(\omega)$
Time integration	$\int_{-\infty}^{t} x(u)\, du$	$\frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$

E) the Parseval theorem:

$$\boxed{\int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} |X(f)|^2\, df}$$

FOURIER TRANSFORM COMMON PAIRS:

Table of Common Functions and their Fourier Transforms

Function name	Function in the time domain	Fourier Transform (in the frequency domain)
	$w(t)$	$\hat{W}(f)$
Dirac delta	$\delta(t)$	1
Constant	1	$\delta(f)$
Cosine	$\cos(2\pi f_0 t)$	$\frac{\delta(f-f_0)+\delta(f+f_0)}{2}$
Sine	$\sin(2\pi f_0 t)$	$\frac{\delta(f-f_0)-\delta(f+f_0)}{2j}$
Unit step function	$u(t) = \begin{cases} 0, \text{if } t < 0 \\ 1, \text{if } t \geq 0 \end{cases}$	$\frac{1}{j\omega}$ (for $\omega = 2\pi f$)
Decaying exponential (for t > 0)	$e^{-\alpha t}u(t)$,	$\frac{1}{\alpha + j2\pi f}$, $\alpha > 0$
Box or rectangle function	$\mathrm{rect}(at) = \begin{cases} 0, \text{if } \lvert at\rvert > \frac{1}{2} \\ 1, \text{if } \lvert at\rvert \leq \frac{1}{2} \end{cases}$	$\frac{1}{\lvert a\rvert}\mathrm{sinc}\left(\frac{f}{a}\right) = \frac{\sin(\pi f/a)}{\pi f/a}$
Sinc function	$\mathrm{sinc}(at) = \frac{\sin(\pi a t)}{\pi a t}$	$\frac{1}{\lvert a\rvert}\mathrm{rect}\left(\frac{f}{a}\right)$
Comb function	$\sum_{n=-\infty}^{\infty} \delta(t-nT)$	$\frac{1}{T}\sum_{k=-\infty}^{\infty} \delta(f-\frac{k}{T})$

ENERGY SIGNALS

the ENERGY is one of the main properties of a signal. It's always positive; it's zero only when the quantity is zero and it's proportional to the magnitude squared

$$\boxed{E[x(t)] = \int_{-\infty}^{+\infty} x(t)^2 \, dt}$$

ENERGY OF A SIGNAL

$\Longrightarrow$ signals for which this quantity exist in a finite number are called energy signals.

↳ PROPERTY:

- by recalling the definition of AUTOCORRELATION and CONVOLUTION we obtain:

$$\boxed{E[x(t)] = K_{xx}(0) = [x(-t) * x(t)] \ (\tau = 0)}$$

energy =
- AUTOCORR. in ZERO
- CONVOLUTION in ZERO between $x(t)$ and its flipped shape

- there is also a FREQUENCY INTERPRETATION, by exploiting PARSEVAL THEOREM:

$$\boxed{E[x(t)] = \int_{-\infty}^{+\infty} x(t)^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df}$$

where $|X(f)|^2$ is called ENERGY SPECTRAL DENSITY $S_x(f)$ of the signal.

↳ since we take the area

then, we obtain:

$$\boxed{K_{xx}(0) = E[x(t)] = \int_{-\infty}^{+\infty} S_x(f) \, df} \quad \text{and} \quad \boxed{\mathcal{F}[K_{xx}(\tau)] = |X(f)|^2 = S_x(f)}$$

POWER SIGNALS

how to deal with INFINITE ENERGY SIGNALS? Is it physically possible? If we look at an infinite energy signal for a finite amount of time T, obviously it will have a finite energy.

↳ So, we are interested in a FLOW OF ENERGY PER UNIT TIME:

$\Rightarrow$ AVERAGE OF THE ENERGY OVER TIME

$$\boxed{P[x(t)] = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)^2 \, dt}$$

signals with finite power are called POWER SIGNALS. A more compact definition can be obtained using TRUNCATED SIGNALS $x_T(t)$:

$$x_T(t) = \begin{cases} x(t) & \text{for } |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Longrightarrow \boxed{P[x(t)] = \lim_{T\to\infty} \frac{1}{T} E[x_T(t)]}$$

since TRUNCATED SIGNALS are FINITE ENERGY SIGNALS, we can apply all energy theory already developed. Let's suppose $x(t)$ non-periodic signal, and $x_T(t)$ the truncated signal. Then:

$$E[x_T(t)] = \int_{-\infty}^{+\infty} |X_T(f)|^2 df$$

$$\boxed{P[x(t)] = \lim_{T\to\infty} \frac{1}{T} E[x_T(t)] = \int_{-\infty}^{+\infty} \lim_{T\to\infty} \frac{1}{T} |X_T(f)|^2 df}$$

So we can symmetrically define the POWER SPECTRAL DENSITY as:

$$\boxed{S_x(f) = \lim_{T\to\infty} \frac{1}{T} |X_T(f)|^2} \qquad \boxed{P[x(t)] = \int_{-\infty}^{+\infty} S_x(f) \, df}$$

since the area of $S_x(f)$ over frequency axis give us the power, it's a density.

for $T \to \infty$ then $x_T(t) \longrightarrow x(t)$

↳ PROPERTY:

- we can extend the concept of CROSS-CORRELATION between POWER SIGNALS:

$$\boxed{\begin{aligned} K_{f,g}(\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(\alpha) g(t+\alpha) \, d\alpha \\ K_{f,g}(\tau) &= \langle f(\alpha) g(t+\alpha) \rangle \end{aligned}}$$

CROSS-CORRELATION

$\langle \cdot \rangle$ time average

- if $f = g$ we can notice that:

$$\boxed{K_{xx}(0) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(\alpha) \, d\alpha = P[x(t)]}$$

VALUE IN ZERO OF AUTOCORR. is POWER

- it's possible to demonstrate that

$$\boxed{\mathcal{F}[K_{xx}(\tau)](f) = S_x(f)}$$

FOURIER TRANSFORM OF AUTOCORRELATION
=
POWER SPECTRUM

NOISE DESCRIPTION

INTRODUCTION TO NOISE

At the pre-amplifier output we tipically have a significant noise level superimposed to the signal, which is due to sensor intrinsic noise and to noise limitations of the gain stage. That's why, after the pre-amp, we typically find a FILTERING BLOCK, which task is to shape signal and noise to improve the SIGNAL-to-NOISE RATIO (SNR)

DEF: Noise can be defined as RANDOM DISTURBANCES, SUPER-IMPOSED TO A USEFUL SIGNAL, which is degradated by the noise itself. It's generated by the devices themselves, and may limit the overall RESOLUTION of the system.

→ ACQUIRING A SIGNAL = RECOVER INFORMATION FROM NOISE.

COMPLETE NOISE DESCRIPTION WITH PROBABILITY DISTRIBUTION

• The MARGINAL PROBABILITY is INSUFFICIENT. We need also the JOINT PROBABILITY $p_j(x_1, x_2, t_1, t_2)dx_1 dx_2$ OF HAVING A VALUE x_1 AT t_1 and x_2 AT t_2. It can be seen also as:

$$p_j(x_1, x_2, t_1, t_2) = p_j(x_1, x_2, t_1, t_1 + \tau)$$

IF THE NOISE IS STATIONARY, p_j DEPENDS ONLY ON TIME-DISTANCE τ BETWEEN SAMPLES and NOT ON THE POSITION OF t_1.

• Let's also distinguish between

A) TIME-AVERAGING:

$$\boxed{\langle x \rangle = \lim_{T\to\infty} \int_{-T/2}^{+T/2} \frac{x(t)}{T}\, dt}$$

on TIME AXIS

B) AVERAGE OVER THE ENSEMBLE:

$$\boxed{\overline{x} = \int_{-\infty}^{+\infty} x\, p(x)\, dx}$$

on ENSEMBLE AXIS (diff. measurements)

NOISE DESCRIPTION WITH 2ND ORDER MOMENTS

DEF: given two statistical variables x and y, we define MOMENTS OF A MARGINAL PROBABILITY DISTRIBUTION $p(x)$:

$$\boxed{m_n = \overline{x^n} = \int_{-\infty}^{+\infty} x^n p(x)\, dx}$$ MARGINAL

and MOMENTS OF A JOINT PROBABILITY:

$$\boxed{m_{jk} = \overline{x^j y^k} = \int_{-\infty}^{+\infty} x^j y^k p(x,y)\, dx\, dy}$$ JOINT

the higher the order n or j+k the more detailed is the information

⇓

By limiting to the 2ND ORDER:

$$\boxed{m_2 = \overline{x^2} = \int_{-\infty}^{+\infty} x^2 p(x)\, dx = \sigma_x^2}$$ MEAN SQUARE VALUE or VARIANCE

$$\boxed{m_{11} = \overline{xy} = \int_{-\infty}^{+\infty} xy\, p(x,y)\, dx\, dy = \sigma_{xy}^2}$$ MEAN PRODUCT VALUE or COVARIANCE

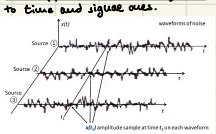
By applying it to NOISE:

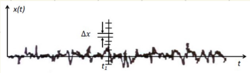
A) MEAN SQUARE VALUE: $\overline{x^2(t_1)} = \sigma_x^2(t_1) \underset{\text{if STATIONARY}}{=} \sigma_x^2$

B) MEAN PRODUCT VALUE: $\overline{x(t_1)x(t_2)} = \overline{x(t_1)\, x(t_1+\tau)}$ if stationary depends only on τ

STATISTICS OF NOISE and PROBABILITY DISTRIBUTION

Performing measurements on many identical devices we will end-up with different values every time. This mean that we should add another axis to time and signal ones.





By sampling a value $x(t_1)$, we can compare it to a scale of discrete values x_k, spaced by constant interval Δx (choose the nearest x_k).
We can define:

$$\boxed{\Delta f_k = \frac{\Delta N_k}{N}}$$ with ΔN_k: number of samples at x_k; N: number of noise waveforms

STATISTICAL FREQUENCY

Then, we can build an istogram of the number of samples for each x_i:

Moving to differentials:

$\Delta x \to dx$; $\Delta N_k \to dN_k = n(x_k)dx$

$\Delta f_k \to df_k = \frac{dN_k}{N} = \frac{n(x_k)}{N} dx$ for $N \to \infty$: $df_k = \frac{n(x_k)}{N} dx = p(x)dx$

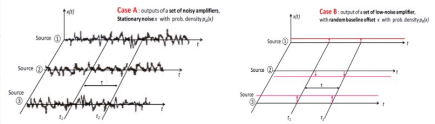
where

$$\boxed{p(x) = \frac{n(x_k)}{N}}$$ PROBABILITY DENSITY (MARGINAL)

which can be:

- STATIONARY: $p = p(x)$
- NON-STATIONARY: $p = p(x,t)$

• However, this does NOT GIVE US a complete mathematical description of noise behavior.



In the example, the probability density is the same, but the TIME-DEPENDENCE of the two values in the two cases is completely different.

DEF: a TIME CONSTANT VALUE AT THE OUTPUT is CALLED OFFSET OR BASELINE

NOISE AUTOCORRELATION FUNCTION and NOISE POWER

DEF: the NOISE AUTOCORRELATION FUNCTION is defined as:

$$\boxed{R_{xx}(t_1, t_1+\tau) = \overline{x(t_1)x(t_1+\tau)} = \sigma_{xx}^2}$$

it's always function of the interval τ between the two instants t_1 and t_2. If it's NON-STATIONARY also on t_1.

• NOTE THAT:

- the AUTOCORRELATION FUNCTION OF NOISE x, CALLED $R_{xx}(\tau)$ is an ENSEMBLE AVERAGE.
- the AUTOCORRELATION FUNCTION OF SIGNAL $K_{xx}(\tau)$ is a TIME AVERAGE.

DEF: the NOISE MEAN SQUARE VALUE is also called NOISE POWER and corresponds to...

$$\boxed{\sigma_x^2 = \overline{x^2(t)} = R_{xx}(t,0) \underset{\text{if STATIONARY}}{=} R_{xx}(0)}$$

NOISE POWER SPECTRUM

we can extend the POWER SIGNAL THEORY to NOISE by AVERAGING ON ENSEMBLE:

$$P = \overline{\lim_{T\to\infty} \int_{-T}^{+T} \frac{x^2(\alpha)}{2T} d\alpha} \Longrightarrow P = \overline{\lim_{T\to\infty} \int_{-\infty}^{+\infty} \frac{x_T(\alpha)}{2T} d\alpha} = \overline{\lim_{T\to\infty} \int_{-\infty}^{+\infty} \frac{|X_T^2(f)|}{2T} df} = \int_{-\infty}^{+\infty} \overline{\lim_{T\to\infty} \frac{|X_T^2(f)|}{2T}} df$$

let's pass to TRUNCATED SIGNAL to exploit PARSEVAL THEOREM (integral should be from $-\infty$ to $+\infty$)

we define:

$$\boxed{S_x(f) = \overline{\lim_{T\to\infty} \frac{|X_T^2(f)|}{2T}}}$$

POWER SPECTRUM of the NOISE

• PROPERTY: (truncated!)

$$\boxed{S_x(f) = \lim_{T\to\infty} \frac{\mathcal{F}[\overline{K_{xx,T}(\tau)}]}{2T}} \qquad \boxed{P = \int_{-\infty}^{+\infty} S_x(f)\,df = \overline{K_{xx}(0)}}$$

• NOTE THAT:
- the $S_x(f)$ in the previous formula is called BILATERAL SPECTRAL DENSITY $S_{xB}(f)$ since it's from $-\infty$ to $+\infty$.
- Since $S_{xB}(f)$ is SYMMETRICAL, we can also use the UNILATERAL SPECTRAL DENSITY $S_{xU}(f)$:

$$\begin{cases} P = 2\int_0^{+\infty} S_{xB}(f)\,df \\ \boxed{S_{xU}(f) = 2S_{xB}(f)} \end{cases} \Rightarrow \boxed{P = \int_0^{+\infty} S_{xU}(f)\,df}$$

We have:

$$\boxed{S_x(f) = \mathcal{F}\left[\overline{K_{xx}(\tau)}\right]} \qquad K_{xx}(\tau) = \langle x(\tau)\,x(t+\tau)\rangle$$

then there are two averages (over TIME and over ENSEMBLE) It can be demonstrated that THE ORDER OF AVERAGES CAN BE EXCHANGED.

By replacing K_{xx} with R_{xx}:

$$\boxed{S_x(f) = \mathcal{F}\left[\langle R_{xx}(t, t+\tau)\rangle\right]} \qquad \text{with } R_{xx}(t, t+\tau) = \overline{x(t)\,x(t+\tau)}, \quad \langle R_{xx}\rangle = \overline{K_{xx}}$$

and finally, for STATIONARY NOISE, NO NEED OF TIME-AVERAGING:

$$\boxed{\langle R_{xx}(t, t+\tau)\rangle = R_{xx}(\tau) \qquad S_x(f) = \mathcal{F}\left[R_{xx}(\tau)\right]}$$

if STATIONARY NO NEED OF TIME AVERAGE

NOISE TYPES AND SOURCES

INTRODUCTION TO NOISE TYPES

Our final goal is to find the rms value of the noise, in order to obtain the MINIMUM DETECTABLE SIGNAL OF THE SYSTEM:

$$V_{P_{min}} = SNR_{min} \cdot \sigma_n$$

the minimum usable SNR is to us in choiche. Usually it's about 3, but for sake of semplicity also 1 can be chosen.

↳ TO OBTAIN THE VARIANCE OF NOISE we will need BOTH POWER SPECTRAL DENSITY AND AUTOCORRELATION FUNCTION OF THE NOISE.

SHOT NOISE

DEF: It's present when we have RANDOM FLUCTUATIONS OF THE CURRENT VALUE DUE TO THE DISCRETE VARIATIONS OF THE CARRIERS MOVING UNDER THE EFFECT OF THE ELECTRIC FIELD.

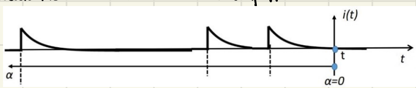
For example, considering a reverse biased p-n junction, one carrier reaching the depletion layer will cross this region under the effect of the electric field, and it will move in an almost straight line due to the high electric field; the number of carriers crossing the junction could have a constant average value but it has also an intrinsic variation due to the discrete nature of the carriers.

We can study the TOTAL CURRENT in our device studying the behavior of EACH SINGLE PULSE DUE TO A SINGLE CARRIER. Let's use an exponential decay for the pulse: The integral of a CURRENT PULSE $f(t)$ in TIME is the CHARGE OF THE CARRIER, q.

$$f(t) = q \cdot h(t) \quad \text{with} \quad \int_{-\infty}^{+\infty} h(t)\,dt = 1$$

The sequence of pulses CROSSING THE DEPLETION REGION is A POISSON STATISTICAL PROCESS, since every pulse is independent from the other. The probability of having a pulse between t and $t+dt$ is $p \cdot dt$ where p is constant.

• Let's calculate the SHOT NOISE MEAN: let's use a flipped α axis:



↳ A pulse which starts at time α contributes a current $q\,h(\alpha)$ at time t. The probability for a pulse to start in $\alpha \to \alpha + d\alpha$ is $p\,d\alpha$.

$$\overline{i(t)} = I = \int_0^{\infty} q\,h(\alpha)\,p\,d\alpha = pq\int_0^{\infty} h(\alpha)\,d\alpha = p \cdot q$$

(intuitive also because p is pulse time rate · pulse current contribution)

∘ Now, to obtain MEAN SQUARE VALUE, we consider two different pulses:

$$\overline{i^2} = \left[q\,h(\alpha) + q\,h(\beta)\right]^2 = \underbrace{q^2 h^2(\alpha) + q^2 h^2(\beta)}_{\substack{\text{the PROBABILITY of the SQUARED PULSE is the same of the}\\ \text{PROBABILITY of THE PULSE ITSELF}}} + \underbrace{2q^2 h(\alpha)h(\beta)}_{\substack{\text{the PROBABILITY of the CROSS-PRODUCT is the JOINT PROBABILITY}\\ \text{OF THE TWO PULSES, because BOTH PULSES MUST OCCUR AT THE SAME TIME.}}}$$

since the two pulses are independent the joint probability is just the product of the two probabilities

$$\overline{i^2(t)} = \int_0^{\infty} q^2 h^2(\alpha) \cdot p\,d\alpha + \iint_0^{\infty} q\,h(\alpha)\,p\,d\alpha\; q\,h(\beta)\,p\,d\beta = pq^2\int_0^{\infty} h^2(\alpha)\,d\alpha + \left(pq\int_0^{\infty} h(\alpha)\,d\alpha\right)\left(pq\int_0^{\infty} h(\alpha)\,d\alpha\right) \Rightarrow$$

$$\ldots = pq^2\int_0^{\infty} h^2(\alpha)\,d\alpha + (pq)^2 = pq^2\int_0^{\infty} h^2(\alpha)\,d\alpha + \left(\overline{i(t)}\right)^2 = \underbrace{pq^2\int_0^{\infty} h^2(\alpha)\,d\alpha}_{\text{FLUCTUATIONS}} + \underbrace{I^2}_{\text{AVERAGE VALUE}}$$

we are interested in those!

then, let's consider only fluctuations:

$$\boxed{\overline{h_i^2}} = pq^2\int_0^{\infty} h^2(\alpha)\,d\alpha = \boxed{qI\int_0^{\infty} h^2(\alpha)\,d\alpha}$$

CAMPBELL THEOREM

↳ direct relationship between the power of noise and shape of delta response.

• Now, we can obtain the POWER SPECTRUM: since $h(\alpha) = 0$ for $\alpha < 0$ ...

$$\begin{cases} \overline{h_i^2} = qI\int_{-\infty}^{+\infty} h^2(\alpha)\,d\alpha \underset{\text{Parseval}}{=} qI\int_{-\infty}^{+\infty} |H(f)|^2 df \quad H(f) = \mathcal{F}[h(t)] \\ \overline{h_i^2} = \int_{-\infty}^{+\infty} S_n(f)\,df \quad \text{(from noise theory)} \end{cases} \Rightarrow \boxed{S_{n}(f) = qI|H(f)|^2}$$

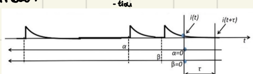
POWER SPECTRUM

a lot of function can match that integral area ...

... to fully demonstrate it we need to compute autocorrelation

• Let's compute the AUTOCORRELATION of the current:

$$R_{ii}(\tau) = \overline{i(t)\,i(t+\tau)}$$



with

$$i(t) = q\,h(\alpha) + q\,h(\beta)$$

$$i(t+\tau) = q\,h(\alpha+\tau) + q\,h(\beta+\tau)$$

after some calculations we end up with

AUTOCORRELATION

$$\boxed{R_{mm}(\tau) = qI\,K_{hh}(\tau)} \qquad R_{mm}(0)$$

and:

$$\boxed{S_m(f) = \mathcal{F}\left[R_{mm}(\tau)\right]}$$

$$S_m(f) = qI\,|H(f)|^2$$

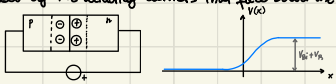
MEAN SQUARE VALUE

$$\boxed{\int_{-\infty}^{+\infty} S_m(f)\,df = \overline{h_i^2} = pq^2 K_{hh}(0) = qI\int_0^{\infty} h(\alpha)^2\,d\alpha}$$

COMPUTATION OF POWER OF THE NOISE ALSO IN TIME-DOMAIN.

NOISE IN DIODES

REVERSE BIASED DIODES: we have a higher voltage drop across the depletion region. The only contribution to the current is given by the minority carriers that fall down the potential barrier and cross the junction:



↳ Then we can model the current as a sequence of elementary pulses: the pulse width will be related to the transit time in the junction (from ratio between length 0.1 μm to 100 μm and speed. (about ps-1ns))

↳ For most of applications we are interested in frequencies up to the MHz range, so a CORRELATION TIME of μs. Since the shape of the pulse is much shorter it can be considered a δ for our studies.

$$h(t) \approx \delta(t) \quad \text{and} \quad |H(f)| \approx 1$$

with this approximation, the NOISE POWER SPECTRAL DENSITY is:

$$\boxed{S_{mB}(f) = qI_s|H(f)|^2 \approx qI_s \qquad S_{mU}(f) = 2qI_s|H(f)|^2 \approx 2qI_s}$$

Therefore, the corresponding AUTOCORRELATION FUNCTION is:

$$\boxed{R_{nn}(\tau) = qI_s k_{hh}(\tau) \approx qI_s\delta(\tau)}$$

FORWARD BIASED DIODE: the current flowing is composed by two contributions:
- the majority carriers that cross the potential barrier, and the minority carriers that fall down the potential barrier.
- since we are reducing the potential barrier the number of holes and electrons able to jump over increases exponentially:

$$I = I_s\left(e^{qV/kT} - 1\right) = I_s e^{qV/kT} - I_s$$

the MEAN CURRENT is the difference of the two components, but the fluctuations are quadratically added.

↳ STATISTICS FLUCTUATIONS can be either positive or negative with the same probability. In order to take into account variations, we square them and then sum the contributions:

$$S_{mU}(f) = 2qI_s e^{qV/kT} + 2qI_s = 2q(I+I_s) + 2qI_s = 2qI + 4qI_s$$

in FORWARDING BIAS:

REDUCED BARRIER ⇒ $I \gg I_s \longrightarrow \boxed{S_{mU}(f) \approx 2qI}$

at zero BIAS $I=0$ and MAJORITY CURRENT EQUAL MINORITY ONE

$$\Rightarrow \boxed{S_{mU}(f) \approx 4qI_s} \quad \text{POWER SPECTRUM}$$

NOISE IN RESISTORS — JOHNSON-NYQUIST NOISE

also known as THERMAL NOISE, it's generated by the BROWNIAN MOTION of the carriers. It's mainly present in RESISTOR and TRANSISTORS.

↳ if we analyse a floating resistor with both terminals open, we see that the voltage difference on the terminal is NOT zero, only the AVERAGE is zero.

THE VOLTAGE VARIES DUE TO RANDOM MOVEMENT OF ELECTRONS. THE ELECTRONS INEQUALITY BETWEEN TERMINALS GENERATES VOLTAGE VARIATIONS (with a TIME WINDOW $t_c \sim 10\,fs$: COLLISION TIME OF CARRIERS)

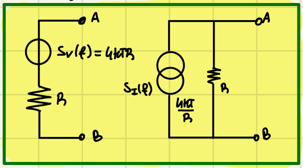
⇒ THEREFORE, WE CAN CONSIDER NOISE TO BE FLAT UP TO A FREQUENCY $1/t_c \sim$ THz

- Voltage across terminals is a STATISTICAL VARIABLE WITH A GAUSSIAN PROBABILITY DISTRIBUTION.

$$\boxed{\begin{aligned} S_{VB}(f) &= 2kTR \quad \text{(VOLTAGE)} \\ S_{IB}(f) &= \frac{S_{VB}}{R^2} = \frac{2kT}{R} \quad \text{(CURRENT)} \end{aligned}}$$

POWER SPECTRAL DENSITY (Bilateral)

which can be used as VOLTAGE/CURRENT POWER GENERATORS coupled with (theoretically) NOISELESS RESISTORS:



EQUIVALENT CIRCUIT MODEL for NOISE IN RESISTORS

DIFFERENCES BETWEEN STATIONARY AND NON-STATIONARY NOISE

In the case of NON-STATIONARY WHITE NOISE we still have a δ AUTOCORRELATION FUNCTION, BUT THE POWER IS DEPENDING ON TIME OF MEASUREMENT t_m (so S_b value varies with t_m)

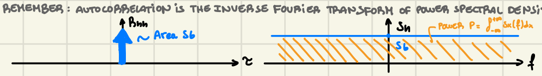
- for the PROPERTY OF FOURIER TRANSFORM: the value in the origin of the power spectral density is the area of the autocorrelation function. Therefore, THE AREA OF THE AUTOCORRELATION FUNCTION CHANGES WITH TIME FOR NON-STATIONARY WHITE NOISE!

AUTOCORRELATION OF NOISE $\quad \boxed{R_{nn}(t, t+\tau) = S_b(t)\ \delta(\tau)}$

WHITE NOISE

DEF: we call WHITE NOISE, a noise with a CONSTANT POWER SPECTRAL DENSITY. Therefore, the AUTOCORRELATION FUNCTION OF THE WHITE NOISE IS A δ IN THE ORIGIN

(REMEMBER: AUTOCORRELATION IS THE INVERSE FOURIER TRANSFORM OF POWER SPECTRAL DENSITY)



Naturally, in the real world, WHITE NOISE DOES NOT HAVE 'INFINITE' BAND (which would give an infinite RMS NOISE VALUE, and S/N = 0. Also, the value in zero of autocorrelation is the area of the spectrum ⇒ then, for IDEAL WHITE NOISE, it's consistent to have a δ as autocorr. with an infinite value, related to infinite area of the spectrum). THERE IS ALWAYS A HIGH. FREQ. CUT-OFF.

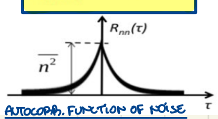
for EXAMPLE: taking SHOT NOISE, AUTOCORRELATION is related to the current pulse duration $\sim 100\,ps \Rightarrow$ cut-off at ~ 10 GHz. Up to a decade before the cut-off we can treat the SHOT NOISE as an IDEAL WHITE NOISE.
→ same for JOHNSON-NYQUIST NOISE, cut-off at ~THz

⇒ TO HAVE A GOOD APPROXIMATION FOR IDEAL WHITE NOISE also THE TIME WIDTH OF NOISE AUTOCORRELATION SHOULD BE MUCH SHORTER COMPARED TO TIME SCALE WE ARE WORKING WITH.

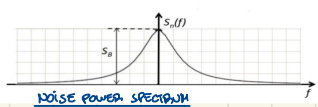
The 'real' WHITE NOISE, with the cut-off band limitation is called WIDE BAND NOISE

↳ a typical spectrum is the LORENTZIAN SPECTRUM, limited by a single pole with time constant T_p ($f_p = 1/2\pi T_p$)

$$\boxed{R_{nn}(\tau) = \bar{n}^2 e^{-|\tau|/T_p}} \qquad \boxed{S_n(f) = \frac{S_b}{1 + (2\pi f T_p)^2}}$$

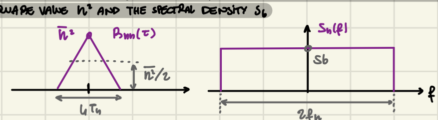


AUTOCORR. FUNCTION OF NOISE



NOISE POWER SPECTRUM

We can make some semplifications for WIDE-BAND NOISE, useful especially when FILTER'S AUTOCORRELATION WIDTH IS COMPARABLE WITH THE NOISE ONE. WE NEED TO PRESERVE IN THE APPROX. ARE THE MEAN SQUARE VALUE $\bar{n}^2$ AND THE SPECTRAL DENSITY S_b



Consistent:

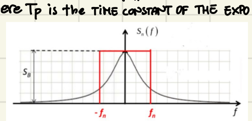
$$\bar{n}^2 = S_b \cdot \underbrace{2f_n}_{\Delta T} \qquad S_b = \frac{1}{2}\underbrace{4T_n}_{\Delta f} \cdot \bar{n}^2$$

$$\Rightarrow 2T_n \cdot 2f_n = 1 \quad \Rightarrow f_{eq} = \frac{1}{4T_n}$$

NOTE THAT $T_n = T_p$ where T_p is the TIME CONSTANT OF THE EXPONENTIAL.

$$\boxed{f_{eq} = \frac{1}{4T_p} = \frac{\pi}{2}f_p}$$

EQUIVALENT WIDE-BAND NOISE BW



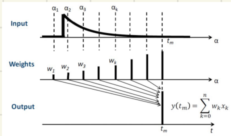
FILTERING SIGNALS

DISCRETE TIME and CONTINOUS TIME FILTERING

If the FILTER is LINEAR, the elements inside it are described by linear equations, and the output is a linear combination of the inputs. THE OUTPUT is the weighted sum of the input values taken at various times t:

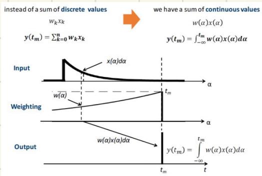
$$\boxed{y(t_m) = \sum_{k=0}^{n} w_k\, x(\alpha_k) = \sum_{k=0}^{n} w_k x_k}$$

IN TIME VARIANTS FILTERS, WEIGHTS CAN CHANGE OVER TIME, AND THE SHAPE OF THE FILTER WITH THEM



example of ADAPTIVE FILTERING, MOST RECENT VALUES, MORE WEIGHT. IN THIS CASE IS WRONG!! WE WANT HIGH WEIGHTS FOR HIGH SIGNALS

↳ we can move to CONTINOUS FILTERING:



we can find the WEIGHTING FUNCTION $w(\alpha)$ to each element $x(\alpha)d\alpha$. It's also called MEMORY FUNCTION.

$$y(t_m) = \int_{-\infty}^{t_m} w(\alpha)\, x(\alpha)\, d\alpha$$

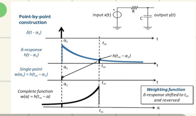
CONSTANT PARAMETER FILTERS (CPF)

Note that OUTPUT of CPF is INVARIANT WITH RESPECT TO TRANSLATION ALONG THE TIME AXIS. If the same signal is applied at t_m and $t_m + \alpha$ the output is the same but translated in time of $\Delta T = \alpha$.

Now, HOW TO CALCULATE THE WEIGHTING FUNCTION OF A CPF?

1) COMPUTE THE IMPULSE RESPONSE OF THE FILTER

if we make the measurement at $t = t_m$ the weight given to a pulse happened $\Delta t = \alpha$ before is the value of the pulse response at that distance $h(t_m - \alpha)$



2) FLIP AND OBTAIN WEIGHTING FUNCTION:

$$\boxed{w(\alpha) = h(t_m - \alpha)}$$

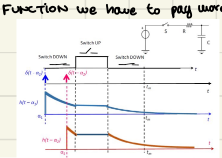
WEIGHTING FUNCTION

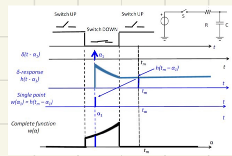
WEIGHTS OF ANY δ DEPENDS ONLY ON TIME-DISTANCE BETWEEN MEASUREMENT AND δ ITSELF.

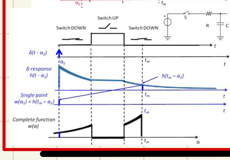
can be demonstrated from $y(t_m) = x(t) * h(t)\big|_{t_m}$

TIME-VARIANT LINEAR FILTERS

a TIME-VARIANT PARAMETER FILTER have parameters whose are not constant over time (for example due to a SWITCH in the CIRCUIT). To calculate its WEIGHTING FUNCTION we have to pay more attention:







NOTE THAT, WHEN THE SWITCH IS OPEN, THE WEIGHTING FUNCTION $w(\alpha)$ IS ZERO

WEIGHTING IN THE FREQUENCY DOMAIN

sometimes can be convenient to switch to the FREQUENCY DOMAIN to make calculations:

↳ the idea is to consider $y(t)$ as weighted sum of components, and extend this in frequency:

PARSEVAL THEOREM

$$\int_{-\infty}^{+\infty} a^2(t)\,dt = \int_{-\infty}^{+\infty} A(f)\cdot A^*(f)\,df = \int_{-\infty}^{+\infty} A(f)\,A(-f)\,df$$

by extending to the product of two functions:

$$\boxed{y(t_m) = \int_{-\infty}^{+\infty} x(\alpha)\,w(\alpha)\,d\alpha = \int_{-\infty}^{+\infty} X(f)\cdot W(-f)\,df}$$

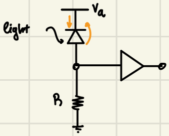
FILTERING NOISE

FOUNDATIONS OF NOISE FILTERING

Noise can be seen as SUPERPOSITION OF ELEMENTARY PULSES, each one with its own WAVEFORM (and FOURIER TRANSFORM). AUTOCORR. is determinated by the type of noise.

⇒ THE FILTERING ACTION OF CONSTANT PARAMETER FILTERS MODIFIES THE ELEMENTARY PULSE SHAPE

EXAMPLE: photodiode connected to an amplifier:



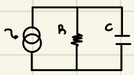
$C = \varepsilon_0 \varepsilon_{Si} \dfrac{A}{W_d}$ (CAPACITANCE OF DIODE)

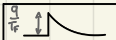
A: area

W_d: width of depletion layer

current generator
↳ superposition of elementary pulses.
If T_h of pulse << RC
pulse ∼ $q\,\delta(t)$

Current flowing in RESISTOR:
↝ elem. pulses with exponential shape (RC LPF)

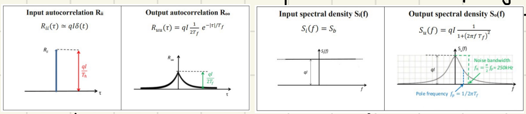




$f(t) = \dfrac{1}{T_F} e^{-t/T_F}\, 1(t)$

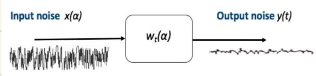
$F(f) = 1/(1 + j 2\pi f T_F)$

It's obtained:



time-domain   freq. domain

KEY PARAMETERS



- A key parameter is the AUTOCORRELATION FUNCTION of BOTH INPUT and OUTPUT:

$$R_{xx}(\alpha, \alpha+\gamma) = \overline{x(\alpha)\, x(\alpha+\gamma)} \quad \text{(INPUT)}$$

$$R_{yy}(t, t+\tau) = \overline{y(t)\, y(t+\tau)} \quad \text{(OUTPUT)}$$

$$R_{yy}(t_1, t_2) = \overline{y(t_1)\, y(t_2)} = \int_{-\infty}^{+\infty} x(\alpha) w_1(\alpha)\, d\alpha \cdot \int_{-\infty}^{+\infty} x(\beta) w_2(\beta)\, d\beta = \iint_{-\infty}^{+\infty} \overline{x(\alpha)x(\beta)} \cdot w_1(\alpha) w_2(\beta)\, d\alpha\, d\beta = \iint_{-\infty}^{+\infty} R_{xx}(\alpha,\beta)\, w_1(\alpha)\, w_2(\beta)\, d\alpha\, d\beta$$

$\gamma = \beta - \alpha$

$\tau = t_2 - t_1$

AUTOCORRELATION FUNCTION

$$\boxed{R_{yy}(t_1, t_1+\tau) = \iint_{-\infty}^{+\infty} R_{xx}(\alpha, \alpha+\gamma)\, w_1(\alpha)\, w_2(\alpha+\gamma)\, d\alpha\, d\gamma}$$

$$\boxed{\overline{y^2(t_1)} = R_{yy}(t_1, t_1) = \iint_{-\infty}^{+\infty} R_{xx}(\alpha, \alpha+\gamma)\, w_1(\alpha)\, w_1(\alpha+\gamma)\, d\alpha\, d\gamma}$$

MEAN SQUARE NOISE

NOTE: those are valid for all cases of noise and linear filtering, including NON-STATIONARY INPUT and TIME-VARIANT FILTERS.

sono Sara Gasparini } ENJAMBEMENT A
e mi piacciono i dolcini A
qui vicino a me c'è Pie B
che per poco non s'è spezzato le di due B

STATIONARY NOISE

We can simplify; autocorrelation depends only on TIME INTERVAL $\gamma = \beta - \alpha$:

$$R_{xx}(\alpha, \alpha+\gamma) = R_{xx}(\gamma) \quad \text{(INPUT)}$$

$$\Rightarrow R_{yy}(t_1, t_1+\tau) = \int_{-\infty}^{+\infty} R_{xx}(\gamma) \int_{-\infty}^{+\infty} w_1(\alpha) w_2(\alpha+\gamma)\, d\alpha\, d\gamma$$

REMEMBER: if we have stationary noise at the input, the output noise is guaranteed stationary only if the filter is constant parameter.

We can further simplify:

$$\boxed{k_{12w}(\gamma) = \int_{-\infty}^{+\infty} w_1(\alpha) w_2(\alpha+\gamma)\, d\alpha}$$

CROSS-CORRELATION with weighting functions at t_1 and t_2

OUTPUT AUTOCORRELATION FUNCTION

$$\boxed{R_{yy}(t_1, t_1+\tau) = \int_{-\infty}^{+\infty} R_{xx}(\gamma)\, k_{12w}(\gamma)\, d\gamma}$$

MEAN SQUARE NOISE

$$\boxed{\overline{y^2(t_1)} = R_{yy}(t_1, t_1) = \int_{-\infty}^{+\infty} R_{xx}(\gamma)\, k_{11w}(\gamma)\, d\gamma}$$

(time)

FOR ALL FILTERS (CONSTANT PARAMETER and TIME INVARIANT) AND STATIONARY NOISE

By recalling:

$$\boxed{\mathcal{F}\left[k_{11w}(\gamma)\right] = \left|W_1(f)\right|^2}$$

FOURIER TRANSFORM OF WEIGHTING FUNCTION

MEAN SQUARE NOISE

$$\boxed{\overline{y^2(t_1)} = \int_{-\infty}^{+\infty} S_x(f) \left|W_1(f)\right|^2 df}$$

(freq)

FILTERING WHITE NOISE (STATIONARY)

The study of output noise can be further simplified if we have a WHITE STATIONARY NOISE with a constant intensity (power):

↳ the AUTOCORRELATION FUNCTION CAN BE WRITTEN AS:

$$R_{xx}(\alpha, \alpha+\gamma) = R_{xx}(\gamma) = S_b\, \delta(\gamma)$$

$$R_{yy}(t_1, t_1+\tau) = S_b \int_{-\infty}^{+\infty} w_1(\alpha) w_2(\alpha)\, d\alpha = S_b\, k_{12w}(0)$$

↳ the OUTPUT MEAN SQUARE VALUE:

$$\boxed{\overline{y^2(t_1)} = S_b\, k_{11w}(0) = S_b \int_{-\infty}^{+\infty} w_1^2(\alpha)\, d\alpha \overset{\text{parseval}}{=} S_b \int_{-\infty}^{+\infty} \left|W_1(f)\right|^2 df}$$

OUTPUT MEAN SQUARE VALUE for WHITE STATIONARY NOISE

we could have obtained this result also from:

$$\mathcal{F}\left[k_{11w}(\gamma)\right] = \left|W_1(f)\right|^2 \Rightarrow k_{11w}(0) = \int_{-\infty}^{+\infty} \left|W_1(f)\right|^2 df$$

FILTERING NOISE WITH CONSTANT-PARAMETER FILTER

a CONSTANT PARAMETER FILTER is completely characterized by the δ-RESPONSE $h(t)$ in the TIME DOMAIN and by the TRANSFER FUNCTION $H(f) = \mathcal{F}[h(t)]$ in the frequency domain.

↳ PROPERTY: WEIGHTING FUNCTION and FILTER δ-RESPONSE are STRICTLY RELATED:

$$\boxed{w_m(\alpha) = h(t-\alpha) \qquad \left|W_m(f)\right|^2 = \left|H(f)\right|^2}$$

By applying these relations to the autocorrelation function of the output noise:

$$R_{yy}(t_1, t_2) = \iint_{-\infty}^{+\infty} R_{xx}(\alpha,\beta)\, w_1(\alpha) w_2(\beta)\, d\alpha\, d\beta = \iint_{-\infty}^{+\infty} R_{xx}(\alpha,\beta) \cdot h(t_1-\alpha)\, h(t_2-\beta)\, d\alpha\, d\beta = \int_{-\infty}^{+\infty} h(t_1-\alpha)\, d\alpha \int_{-\infty}^{+\infty} R_{xx}(\alpha,\beta)\, h(t_2-\beta)\, d\beta = R_{xx}(\alpha,\beta) * h(\beta) * h(\alpha)$$

taking into account that:
A) INPUT AUTOCORR. depends only on the interval $\gamma = \beta - \alpha$
B) $d\beta = d\gamma$
C) $d\alpha = -d\gamma$

⇒ AUTOCORRELATION is ALSO STATIONARY and depends only on the interval τ:

$$\boxed{R_{yy}(\tau)} = R_{xx}(\gamma) * h(\gamma) * h(-\gamma) = \boxed{R_{xx}(\gamma) * k_{hh}(\gamma)}$$

AUTOCORRELATION OF the OUTPUT (stationary)

and therefore:

$$\boxed{S_y(f) = S_x(f) \cdot \left|H(f)\right|^2}$$

OUTPUT NOISE SPECTRAL DENSITY

⇒ we can calculate output both in time domain and freq. domain

$$\boxed{\begin{aligned} \overline{y^2} &= R_{yy}(0) = \int_{-\infty}^{+\infty} R_{xx}(\gamma)\, k_{hh}(\gamma)\, d\gamma \\ \overline{y^2} &= \int_{-\infty}^{+\infty} S_x(f) \left|H(f)\right|^2 df \end{aligned}}$$

OUTPUT MEAN SQUARE NOISE

since $R_{xx}(\gamma) = S_b\, \delta(\gamma)$ (white noise)

$$\boxed{\overline{y^2} = S_b\, k_{hh}(0) \qquad \overline{y^2} = S_b \int_{-\infty}^{+\infty} \left|H(f)\right|^2 df}$$

OUTPUT MEAN SQUARE NOISE with WHITE NOISE

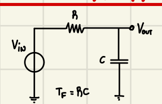
LOW PASS FILTERS

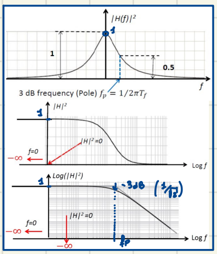
The information of interest is carried out by the signal, which unfortunately is super-imposed to noise.
↳ WE NEED FILTERS ABLE TO EXPLOIT DIFFERENCES BETWEEN SIGNAL AND NOISE, TO 'RECOVER' THE SIGNAL.

⟹ a common situation can be a LOW-FREQUENCY SIGNAL ACCOMPANIED BY A WIDE BAND NOISE. THEREFORE, WE WANT A FILTER TO CANCEL OUT EVERYTHING ABOVE THE MAXIMUM BANDWIDTH OF THE SIGNAL AND LEAVING EVERYTHING ELSE UNTOUCHED.
↳ THE FILTERING WEIGHT IS CONCENTRATED IN A RELATIVELY NARROW FREQUENCY BAND FROM ZERO TO A LIMITED FREQUENCY WHILE ABOVE THE BAND-LIMIT IT FALLS TO NEGLIGIBLE VALUE.
CORRESPONDINGLY, IN TIME DOMAIN, THE WEIGHTING FUNCTION HAS RELATIVELY WIDE TIME-WIDTH, AS ITS AUTOCORRELATION.
⟹ SO, IN TIME DOMAIN, THE ACTION OF LPF IS TO PRODUCE A WEIGHTED TIME-AVERAGE OF THE INPUT OVER A CERTAIN TIME-INTERVAL, DEFINED BY THE WIDTH OF THE WEIGHTING FUNCTION.

CONSTANT PARAMETER LPF

RC INTEGRATOR





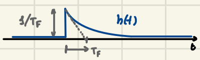
RC FREQUENCY RESPONSE

- THE EFFECT ON SIGNAL IS:

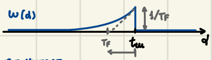
$$V_{out}(s) = \frac{1}{1+sCR} = \frac{1}{1+sT_F}$$

$$\Rightarrow \boxed{H(f) = \frac{1}{1+j2\pi f T_F} = \frac{V_{out}(s)}{V_{in}(s)} \quad \Big\downarrow \mathcal{F} \quad h(t) = \frac{1}{RC} e^{-t/RC} \cdot 1(t)}$$

RC TRANSFER FUNCTION and δ-RESPONSE



δ-RESPONSE h(t)

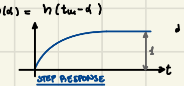


WEIGHTING FUNCTION $w(\alpha) = h(t_m - \alpha)$

makes the average of the input, where weights decrease exponentially. It makes sense to give lower weights to more distant samples
- WE WILL APPROXIMATE AS AN AVERAGE FROM t_m TO $2T_F$ DISTANCE

OUTPUT VOLTAGE

$$\boxed{V_{out}(t) = e^{-t/RC}\left[V_{out}(t_0) + \int_{t_0}^{t} \frac{V_{in}(t)}{RC} e^{t/RC} dt\right]} \Rightarrow$$



STEP RESPONSE

- WHAT ABOUT THE EFFECT ON THE NOISE? We can study both in time domain and in frequency domain:

A) TIME DOMAIN: let's calculate the AUTOCORRELATION OF WEIGHTING FUNCTION

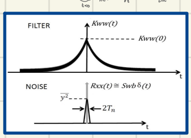
$$K_{ww}(t) = K_{hh}(t) = \int_{-\infty}^{+\infty} h(\tau)h(t+\tau)d\tau = \frac{1}{(RC)^2}\int_{-\infty}^{+\infty} e^{-\frac{\tau}{RC}} sca(\tau)\, e^{-\frac{\tau+t}{RC}} sca(t)\, d\tau = \frac{1}{RC^2} e^{-t/RC}\int_{-\infty}^{+\infty} e^{-\frac{2\tau}{RC}} sca(\tau)\, sca(\tau+t)\, d\tau$$

$$t>0: \quad = \frac{1}{RC^2} e^{-t/RC}\int_0^{+\infty} e^{-\frac{2\tau}{RC}} d\tau = \frac{1}{2RC} e^{-t/RC}$$

$$t<0: \quad = \frac{1}{RC^2} e^{-t/RC}\int_t^{+\infty} e^{-\frac{2\tau}{RC}} d\tau = \frac{1}{2RC} e^{-t/RC} e^{+\frac{2t}{RC}} = \frac{1}{2RC} e^{t/RC}$$

AUTOCORRELATION WEIGHTING FUNCTION

$$\boxed{k_{ww}(t) = \frac{1}{2RC} e^{-|t|/RC}}$$



from NOISE THEORY $R_{xx}(t) \cong S_{wb}\,\delta(t)$

$$\Rightarrow \boxed{\overline{y^2} = \int_{-\infty}^{+\infty} k_{ww}(t)\, \underbrace{R_{xx}(t)}_{\text{AUTOCORR. OF NOISE}}\, dt}$$

MEAN SQUARE NOISE

since $T_n << T_F = RC$, NOISE AUTOCORR ≃ δ(t)

$$\simeq k_{ww}(0)\, S_{wb} = S_{wb} \cdot \frac{1}{2T_F}$$

In order to make comparison is useful to define the NOISE EQUIVALENT BW with reference to a white noise input so as the bandwidth value to be employed for computing a multiplication.

NOISE EQ. BW

$$\overline{y^2} = S_{wb}\, 2 f_n = S_{wb}\, k_{ww}(0) \Rightarrow \boxed{f_n = \frac{k_{ww}(0)}{2}} = \frac{1}{4T_F} = \frac{\pi}{2} f_p$$

B) FREQUENCY DOMAIN:

$$\boxed{\overline{y^2} = \int_{-\infty}^{+\infty} S_{wb}\, |H(f)|^2 df} = \dots = S_{wb} \cdot (1/2T_F)$$

REMEMBER:

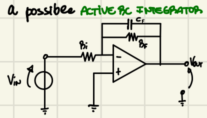
$$\boxed{k_{ww}(0) = \int_{-\infty}^{+\infty} |W(f)|^2 df = \int_{-\infty}^{+\infty} |H(f)|^2 df}$$

ACTIVE RC INTEGRATOR

the RC INTEGRATOR has an UNITARY DC GAIN. This could be an issue whenever we want to combine FILTERING ACTION with GAIN STAGE.

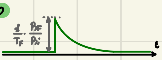
NOTE THAT:
- we want an high gain in order to make the noise of following stages negligible.
- the GAIN of the (FIRST) stage does NOT IMPROVE S/N (also RMS value is multiplied by GAIN)

a possible ACTIVE RC INTEGRATOR could be:



$G(0) = -R_F/R_i$ (INVERTING STAGE) $\quad G(\infty) = 0$

$T_F = R_F C_F \Rightarrow f_p = \frac{1}{2\pi \cdot R_F C_F}$



we obtain:

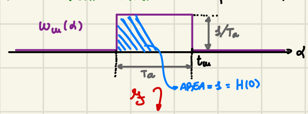
$$\Rightarrow \boxed{H(s) = -\frac{R_F}{R_i}\cdot\frac{1}{1+sT_F} \qquad h(t) = -\frac{R_F}{R_i} e^{-\frac{t}{R_F C_F}} 1(t)}$$

TRANSFER FUNCTION and δ-RESPONSE

$$\boxed{w_m(\alpha) = h(t_m - \alpha) \qquad |W_m(f)|^2 = |H(f)|^2 \qquad |W_m(0)| = R_F/R_i}$$ WEIGHTING FUNCTION

MOBILE MEAN FILTER (MMF)

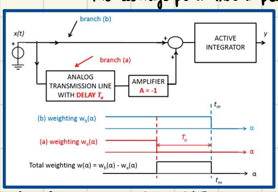
the MOBILE MEAN FILTER (MMF) is a filter with a WEIGHTING FUNCTION.



$\mathcal{F}$ ↓

- the weights are CONSTANT for all the duration of the AVERAGING OPERATION. Basically, it is the MEAN VALUE OF THE INPUTS.
- NOTE THAT THE MMF IS A CONSTANT-PARAMETER FILTER since the weighting function is the same for any readout time t_m

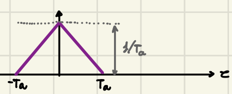
sinc function ⟹ LPF confirmed.
↳ IT HAVE SOME ZEROS. CAN BE EXPLOITED TO GET RID OF SOME DISTURBANCES AT CERTAIN WELL KNOWN FREQS.
(can be explained in time domain: there are some sinusoids at certain freqs, which if integrated over the average period have a perfectly zero integral.)



A POSSIBLE IMPLEMENTATION OF MMF

PRATICAL PROBLEM: obtaining large DELAYS T_a is difficult. even with LONG TRASMISSION LINES WE REACH few of ms and LIMITED PERFORMANCES.

The AUTOCORRELATION FUNCTION OF MMF is:



if we want to compare with RC INTEGRATOR, to have the same OUTPUT RMS:

$$\overline{y^2} = S_{bB} \cdot k_{w_m w_m}(0) = S_{bB} \cdot \int_{-\infty}^{+\infty} w_m^2(\alpha) d\alpha \quad \text{only if} \quad T_a = 1/2T_F$$

↳ ***NOTE***: WE IDEALLY WANT A RECTANGULAR SHAPED FILTER. THIS IS NOT POSSIBLE. IN FACT IT WOULD REQUIRE A δ-RESPONSE = SINC WHICH IS ANTICAUSAL.

CASCADING ELEMENTARY CELLS

We can easily obtain HIGH-ORDER LPF by CASCADING configurations upperboard.
For a 2ND ORDER RC we would have:

$$h(t) = \frac{t}{T_F^2} e^{-t/T_F} \cdot 1(t)$$

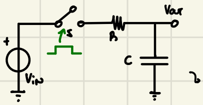
and an RMS value at the output:

$$\overline{\sigma_{out}^2} = S_{wb}\, k_{hh}(0) = S_{wb} \int_{-\infty}^{+\infty} \left(\frac{t}{T_F^2} e^{-t/T_F}\right)^2 dt = \dots = 1/4T_F$$

$$\overline{\sigma_{out}^2} = 2 f_n S_{bB} \;\rightsquigarrow\; f_n = 1/8T_F \;)\; \text{eq. noise BW}$$

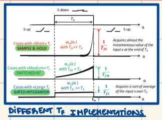
SWITCHED PARAMETER LPF

RC NETWORK + SWITCH



- if the switch S is closed the circuit behaves like a CONSTANT LPF.
- if the switch S is open NO CURRENT FLOWS. CIRCUIT ON HOLD.

↳ usually we want to close the switch only when the signal is present, in order to do not acquire extra noise

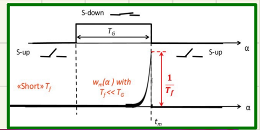


DIFFERENT T_F IMPLEMENTATIONS

SAMPLE & HOLD

Used in ANALOG to DIGITAL CONVERSIONS.

The WEIGHTING FUNCTION is identical to CONSTANT PARAMETER RC FILTER, with a UNITY DC GAIN:



$$W_m(0) = \int_0^\infty w_m(\alpha)d\alpha = 1 \qquad w(\alpha) = \frac{1}{T_F} e^{-\frac{(t_m-\alpha)}{T_F}} 1(t_m-\alpha)$$

- if we suppose $S_b = 2kTR$ due to a RESISTOR R:

$$\overline{y_B^2} = \int_{-\infty}^{+\infty} R_{nn}(\tau) k_{ww}(\tau)d\tau = S_b k_{ww}(0) = 2kTR \cdot \frac{1}{2RC} = \frac{kT}{C}$$

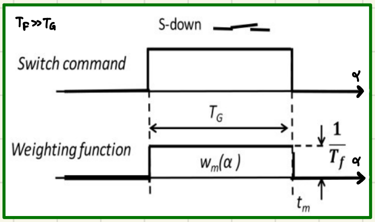
if $T_F \downarrow$ then NOISE $\overline{y_B^2} \uparrow$; IT'S ALSO INDEPENDENT ON R!

GATED INTEGRATOR (GI)

It's obtained by choosing $T_F = RC \gg T_G$.
It's useful with RELATIVELY SLOW SIGNALS (mostly CONSTANT over INTEGRATION WINDOW) to be recovered in presence of strong wide band noise.

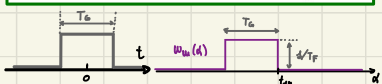
- It has DC GAIN << 1 :

$$G = W_m(0) = \int_0^\infty w_m(\alpha)d\alpha = \frac{T_G}{T_F} << 1$$



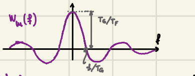
- The WEIGHTING FUNCTION can be approx. with:

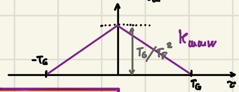
$$W_m(\alpha) = \frac{1}{T_F} \cdot rect_{T_G}\left(\alpha - t_m + \frac{T_G}{2}\right)$$



$\mathcal{F}$

$$W_m(f) = \frac{T_G}{T_F} \, sinc_{T_G}(f)$$





- the AUTOCORR. OF WEIGHTING FUNCTION :

$$k_{ffw} = W_m(\alpha) * W_m^*(-\alpha) = \frac{1}{T_F} rect_{T_G}(t) * \frac{1}{T_F} rect_{T_G}(t) = T_G\left(\frac{1}{T_F^2}\right) tri_{T_G}(t)$$

Differently from S&H the GI has a QUITE BROAD $w(\alpha)$ and a NARROW response in the freq. domain.

↳ G.I. can be an effective option to reduce white noise

Let's consider a CONSTANT INPUT SIGNAL over the INTEGRATION TIME T_G, and a BROADBAND INPUT NOISE POWER limited by a single pole, with $T_F << T_n$:

⇒ we can consider the NOISE as WHITE:

$$\overline{x_n^2} = S_b 2 f_n = S_b / 2T_n$$

SIGNAL OUTPUT

$$y_s = x_s \frac{T_G}{T_F} = x_s \cdot G \quad \text{with GAIN } G = \frac{T_G}{T_F} = 1$$

⇒ the output noise power is then:

$$\overline{y_n^2} = \int_{-\infty}^{+\infty} R_{nn}(\tau) k_{ffw}(\tau)d\tau = S_b k_{ffw}(0) =$$

OUTPUT NOISE POWER

$$= S_b \frac{T_G}{T_F^2} = \frac{S_b}{T_G}\left(\frac{T_G}{T_F}\right)^2 = \frac{S_b}{T_G} G^2 = \frac{S_b}{2T_n} \cdot \frac{2T_n}{T_G} \cdot G^2 = \overline{x_n^2} \frac{2T_n}{T_G} G^2$$

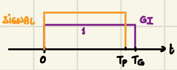
⇒ finally we can calculate the S/N IMPROVEMENT.

$$\left(\frac{S}{N}\right)_y = \frac{y_s}{\sqrt{\overline{y_n^2}}} = \frac{x_s}{\sqrt{\overline{x_n^2}}} \sqrt{\frac{T_G}{2T_n}} = \left(\frac{S}{N}\right)_x \sqrt{\frac{T_G}{2T_n}}$$

also, the LARGER the difference between the FILTER TIME CONSTANT and the AUTOCORR. WIDTH, the larger is the GAIN. This is intuitive from freq. P.O.V. since we are increasing the part of the NOISE SPECTRUM we are removing with the filter.

Can we make an OPTIMIZATION OF THE GI ?

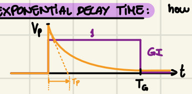
A) RECT-PULSE SIGNAL: let's consider a pulse with duration $T_P < T_G$ integration period.



$$y_s = \int_{-\infty}^{t_m} V_p dt = \int_0^{T_G} V_p dt = V_p T_G \quad (\text{max for } T_G = T_P) \quad [\text{signal}]$$

$$\overline{n_y^2} = S_b k_{ww}(0) = S_b T_G \quad \leadsto \quad SNR = \frac{V_P T_G}{\sqrt{T_G \cdot \frac{S_{bu}}{2}}} = \frac{V_p}{\sqrt{\frac{S_{bu}}{2}}} \cdot \sqrt{T_G}$$

B) EXPONENTIAL DECAY TIME: how do we choose optimally T_G?



$$S_y = \int_0^{T_G} A V_p e^{-t/T_P} dt = A V_p T_P$$

$$SNR = \frac{V_P T_P}{\sqrt{\frac{S_{bu}}{2} T_P}} \cdot \frac{(1-e^{-T_G/T_P})}{\sqrt{T_G/T_P}}$$

can be maximized for $T_G/T_P = 1.25$

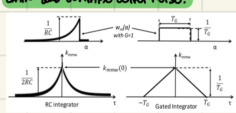
GI vs HMF

Theoretically, GI and HMF SHARE THE SAME WEIGHTING FUNCTION which translates into an identical S/N IMPROVEMENT:

- the HMF is a CONSTANT PARAMETER FILTER. The output is the average of the input signal for every istant.
  the GI is NOT a CONSTANT PARAMETER FILTER, the output is not a waveform in time but just a value representing the signal integral over the filter window.
- the shape of $W(\alpha)$ of GI does not depend on a delay line, and can be modified by acting on the switch.

GI vs OTHER FILTERS

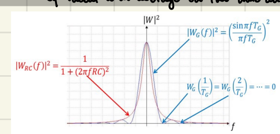
TO COMPARE DIFFERENT FILTERS, USUALLY WE NEED TO FIX THEM TO HAVE THE SAME DC UNITY GAIN and COMPARE OUTPUT NOISE:



They produce the same output noise power if

$$T_G = 2RC$$

That's why we are used to say that the RC LPF in time domain is equivalent to an average on two-time constants.



However, the GI can be preferred because it has zeros t.f. at freqs. $f_k = k/T_G$ that can be exploited to to cancel out specific disturbances.

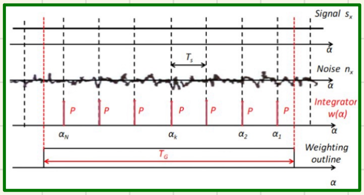
The trend nowadays are DIGITAL FILTERS whose have many advantages with respect to analog filters:

A) THEY DON'T HAVE ANY INTERNAL NOISE SOURCES
B) THEY DO NOT SUFFER FROM ZERO LEVEL DRIFTING
C) THEY DO NOT SUFFER FROM VARIABLE SATURATION (in analog filter power supply limitation)
D) THEY DO NOT HAVE INTRINSIC NONLINEARITIES
E) THEY DO NOT SUFFER FROM LONG HOLDING TIME LEAKAGE

On the other hand, ANALOG FILTERS can PERFORM REAL-TIME PROCESSING OF WAVEFORMS, ON FASTER TIME SCALES

DISCRETE-TIME INTEGRATOR (DTI)

It's the DISCRETE EQUIVALENT of GI.
↳ each sample is multiplied by weight P and summed.



let's suppose:

- **INPUT**: DC-signal S_x
  WIDE BAND STATIONARY noise n_x
  with AUTOCORR. WIDTH $2T_n \ll T_s$
- **PARAMETERS**:
  SAMPLING FREQ. $f_s = 1/T_s$
  NUMBER OF SAMPLES $N = \frac{T_G}{T_s}$
- **OUTPUT**:
  $S_y = (N \cdot P) \cdot S_x$ $\quad [\text{GAIN} = NP] \Rightarrow$ then $\boxed{G = 1 \text{ if } P = 1/N}$ and it's called DISCRETE-TIME AVERAGER
- **NOISE**: SUM OF NOISE SAMPLES $\longrightarrow n_y = \sum_{k=1}^{N} P \cdot n_{x,k} \Longrightarrow \overline{n_y^2} = P^2\left(\overline{n_{x,1}^2 + n_{x,2}^2 + \dots + n_{x,1} n_{x,2} + \dots}\right)$
  $= P^2\left(\overline{n_{x,1}^2} + \overline{n_{x,2}^2} + \dots + \overline{n_{x,1} n_{x,2}} + \dots\right)$
  - IF WE CONSIDER NOISE AS WHITE ($2T_n \ll T_s$) $\quad \overline{n_{x,1} n_{x,2}} = 0$
    SAMPLES ARE UNCORRELATED
  - IF NOISE IS STATIONARY $\quad \overline{n_{x,1}^2} = \overline{n_{x,2}^2} = \dots = \overline{n_x^2}$

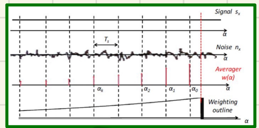
  $\Rightarrow \overline{n_y^2} = N \cdot P^2 \, \overline{n_x^2}$
- **SNR**:

$$\left(\frac{S}{N}\right)_y = \frac{S_y}{\sqrt{\overline{n_y^2}}} = \frac{N \cdot P \cdot S_x}{\sqrt{N \cdot P^2 \, \overline{n_x^2}}} = \sqrt{N} \cdot \left(\frac{S}{N}\right)_x$$

SIGNAL-TO-NOISE RATIO of DTI

DISCRETE-TIME EXPONENTIAL AVERAGER

By looking at $w(\alpha)$ we can say that it's the DISCRETE-TIME EQUIVALENT of an RC-INTEGRATOR:



- **INPUT**: DC-signal S_x
  WIDE BAND STATIONARY noise n_x
  with AUTOCORR. WIDTH $2T_n \ll T_s$
- **PARAMETERS**:
  SAMPLING FREQ. $f_s = 1/T_s$
  NUMBER OF SAMPLES $N = \frac{T_G}{T_s}$
  WEIGHTS FOLLOW AN EXPONENTIAL SHAPE $\boxed{w_k = P \cdot r^k \text{ with } r < 1}$
  also $(1-r) \ll 1$
- **OUTPUT SIGNAL**:
  $S_y = S_x \cdot P \cdot \sum_{k=0}^{\infty} r^k \overset{\text{geometric series}}{=} S_x P \frac{1}{1-r}$ with DC GAIN $G = P \frac{1}{1-r}$
- **OUTPUT NOISE**:
  $\overline{n_y^2} = P^2\left(\overline{n_{x0}^2} + r^2 \cdot \overline{n_{x1}^2} + \dots + r^{2k} \cdot \overline{n_{xk}^2} + \dots + r^k r^j \, \overline{n_{xk} n_{xj}} + \dots\right)$
  - if WHITE NOISE $\Longrightarrow$ SAMPLES UNCORRELATED $\overline{n_{xk} n_{xj}} = 0$
  - if STATIONARY NOISE $\Longrightarrow \overline{n_{x0}^2} = \overline{n_{x1}^2} = \dots = \overline{n_x^2}$

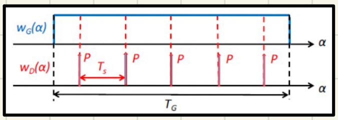
  $\Rightarrow \overline{n_y^2} = \overline{n_x^2} \cdot P^2 \left(1 + r^2 + \dots + r^{2k} + \dots\right) = \overline{n_x^2} \cdot P^2 \frac{1}{1-r^2}$
- **SNR**:

$$\left(\frac{S}{N}\right)_y = \frac{S_y}{\sqrt{\overline{n_y^2}}} = \frac{P S_x}{1-r} \cdot \frac{1}{\sqrt{\frac{\overline{n_x^2} P^2}{1-r^2}}} = \left(\frac{S}{N}\right)_x \sqrt{\frac{1+r}{1-r}} \simeq \left(\frac{S}{N}\right)_x \sqrt{\frac{2}{1-r}}$$

$(1-r) \ll 1 \rightarrow 1+r \simeq 2$ $\quad$ SNR of DTEI

DISCRETE TIME INTEGRATOR vs GATED INTEGRATOR:

We suppose both a UNITARY DC GAIN



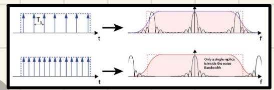
if we compare the SNR:

$$\left(\frac{S}{N}\right)_y \bigg|_{GI} = \left(\frac{S}{N}\right)_x \sqrt{\frac{T_G}{2T_n}}$$

$$\left(\frac{S}{N}\right)_y \bigg|_{DTI} = \sqrt{N} \left(\frac{S}{N}\right)_x$$

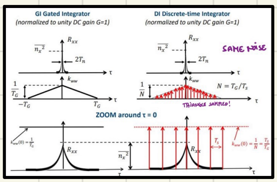
SO, WHY CAN'T WE JUST INCREASE N TO GET A BETTER SNR?
↳ WE CANNOT! because $N = T_G/T_s$. To have $N > T_G/2T_n \Rightarrow T_s < 2T_n$

- samples would be CORRELATED
- improvement is no more $\sqrt{N}$
- it's valid $(S/N)_{DTI} < (S/N)_{GI}$
  with $(S/N)_{DTI} \rightarrow (S/N)_{GI}$



tend to a single sinc in noise BW (which is G.I. in time domain)

WHAT HAPPEN IN TIME DOMAIN?



Output noise is

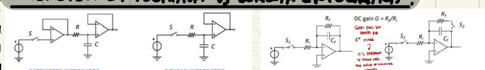
$$\overline{n_y^2} = R_{yy}(0) = \int_{-\infty}^{+\infty} R_{xx}(\tau) k_{ww}(\tau) d\tau$$

for GI, if $T_G \gg T_n$: $\overline{n_y^2} = \frac{1}{T_G}$ (Area of R_{xx})

for DTI: $\overline{n_y^2} = \frac{1}{T_G}$ (area of scaloid that approx. R_{xx})

So if $N \rightarrow \infty$, area of scaloid $\rightarrow$ area of R_{xx}
$\rightarrow$ GI result

RATEMETER INTEGRATOR vs BOXCAR INTEGRATOR:



RATEMETER INTEGRATOR
- Switch S acts as gate on the input source
- Switch S is decoupled from the RC passive filter by the voltage buffer
- The RC integrator is unaffected by S, it has constant parameters, it does NOT have a HOLD state
- The sample average is done on a given time, defined by the RC value
  ↳ 'CONSTANT' DISCHARGE NUMBER OF SAMPLES MATTERS

BOXCAR INTEGRATOR
- Switch S acts as gate on the input source
- Switch S acts also on the RC passive filter (changes the resistor value)
- The time constant T_F of the integrator filter is switched from finite RC (S-down) to infinite (S-up, HOLD state)
- The sample average is done on a given number of samples, defined by the T_F/T_G value
  ↳ BECAUSE WE DISCHARGE C ONLY WHEN WE MAKE SAMPLES

RATEMETER INTEGRATOR
- Switch S_1 acts as gate on the input
- Switch S_1 is decoupled from the active RC integrator by the buffer action of the OP-AMP virtual ground
- The $R_F C_F$ integrator is unaffected by S_1; it has constant parameters, it does NOT have a HOLD state
- The sample average is done on a given time, defined by the $R_F C_F$ value

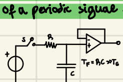
BOXCAR INTEGRATOR
- Switch S_1 acts as gate on the input
- Switch S_1 is decoupled from the active RC integrator by the buffer action of the OP-AMP virtual ground
- A second switch S_2 is required for switching the time constant T_F of the integrator from finite $R_F C_F$ (S_2-down) to infinite (S_2-up, HOLD state)
- The sample average is done on a given number of samples, defined by the T_F/T_G value

BOXCAR INTEGRATOR (BI)

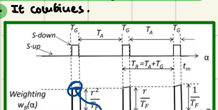
let's suppose a PERIODIC INPUT SIGNAL. With a GI we would use only one replica. The BI try to exploit all replicas of a periodic signal. It combines:

1) SAMPLE ACQUISITION by GI
2) EXPONENTIAL AVERAGING OF SAMPLES

$\underbrace{\qquad\qquad}$ the CAPACITOR IS NOT RESET BETWEEN ACQUISITIONS

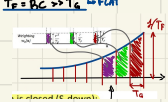


- WHEN [switch open] THE CIRCUIT IS IN HOLD AND THERE IS NO DISCHARGE
- WHEN [switch closed] WE HAVE DISCHARGE (DURING SAMPLES)



$T_F = RC \gg T_G$ → FLAT

↳ • during T_A the C is in HOLD, NO MEMORY LOSS OR CHARGE INPUT
  • during T_G the DISCHARGE OF C happen, with REDUCTION FACTOR OF $r = e^{-T_G/T_F}$



↳ The DC GAIN is 1 (area of $w(\alpha)$) $\longrightarrow$ BI is an AVERAGER

↳ Autocorr. of BI and RC-int are different but $k_{wwB}(0) = k_{wwRC}(0) = \frac{1}{2RC} = \frac{1}{2T_F}$

- **INPUT NOISE**: $\overline{n_x^2} = S_0 \cdot (1/2T_n)$
- **OUTPUT NOISE**: $\overline{n_y^2} = S_0 \, k_{ww_B}(0) = S_0 (1/2T_F) = \overline{n_x^2} \frac{T_n}{T_F}$

$\overset{G=1}{\Longrightarrow} \left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \sqrt{\frac{T_F}{T_n}}$

DOES NOT DEPEND ON THE RATE OF THE SAMPLES (IT'S AN AVERAGE OVER A GIVEN NUMBER. ALSO THE EXPONENTIAL MAKES ONLY FIRST SAMPLES REALLY COUNT)

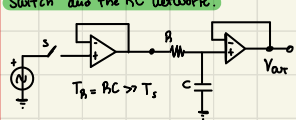
⇓ ⇓ ⇓ ⇓

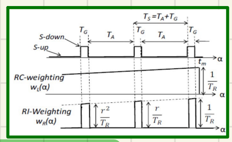
the BI is the equivalent of the cascade of a GI with same T_G and T_F and an EXPONENTIAL AVERAGING.

$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \cdot \underbrace{\sqrt{\frac{T_G}{2T_n}}}_{GI} \cdot \underbrace{\sqrt{\frac{2}{1-r}}}_{\text{EXP.AV (RC)}} \text{ with } r = e^{-T_G/T_F} \simeq 1 - T_G/T_F$$

RATEMETER INTEGRATOR (RI)

It's just like a BOXCAR INTEGRATOR, but with a BUFFER between the switch and the RC network.





Now the capacitor, even when S is open, is still DISCHARGING:

- the WEIGHT REDUCTION from sample to sample is $\longrightarrow \boxed{r = e^{-(T_G+T_A)/T_F} = e^{-T_s/T_F}}$
  depends on SAMPLE RATE (BI does not!)
- during T_G (S-down) the input is integrated in C. during T_A (S-up) the input is not allowed.
- the GAIN < 1 because the area of total exp is 1, but we clip it.
  if $T_R \gg T_s$: $\boxed{G = \int_{-\infty}^{+\infty} W_R(\alpha) d\alpha \simeq \frac{T_G}{T_s} \int_{-\infty}^{+\infty} W_c(\alpha) d\alpha \simeq \frac{T_G}{T_s} = f_s T_G}$ GAIN
- with constant input, produce a quasi DC-output proportional to f_s.

The RI is equivalent to a CASCADE of (i) a GI with same T_G and $RC = T_F$ and an (ii) exponential averager with $r = e^{-T_s/T_R} \simeq 1 - T_s/T_R$

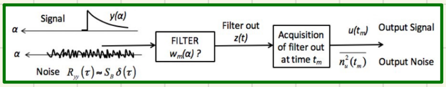
$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_G}{2T_n}} \underbrace{\sqrt{\frac{1+r}{1-r}}}_{\sqrt{\frac{2}{1-r}}} = \left(\frac{S}{N}\right)_x \underbrace{\sqrt{\frac{T_G}{2T_n}}}_{G.I.} \underbrace{\sqrt{\frac{2T_R}{T_s}}}_{\text{exp. AV.}} = \left(\frac{S}{N}\right)_x \sqrt{f_s T_G \frac{T_R}{T_n}}$$

SNR

OPTIMUM FILTERS

OPTIMUM FILTERING

DOES EXIST AN OPTIMAL WEIGHTING FUNCTION THAT MAXIMIZE THE SNR?



let's suppose:
- INPUT SIGNAL: $y(t) = A \cdot b(t)$ with $\int_{-\infty}^{+\infty} b(t) = 1$
- INPUT NOISE: WHITE STATIONARY NOISE

we want to find the $w(\alpha)$ that maximize $\left(\frac{S}{N}\right) = \frac{u(t_m)}{\sqrt{\overline{n_u^2}(t_m)}}$

let's compute this:

$$u(t_m) = \int_{-\infty}^{+\infty} y(\alpha)\, w_m(\alpha)\, d\alpha = A \cdot \int_{-\infty}^{+\infty} b(\alpha)\, w_m(\alpha)\, d\alpha = A\, k_{bw}(0)$$

$$\overline{n_u^2}(t_m) = \int_{-\infty}^{+\infty} R_{yy}(\alpha)\, k_{ww}(\alpha)\, d\alpha = S_b \int_{-\infty}^{+\infty} w_m^2(\alpha)\, d\alpha = S_b\, k_{ww}(0)$$

$$\Rightarrow \boxed{\left(\frac{S}{N}\right)^2 = \frac{u^2(t_m)}{\overline{n_u^2}(t_m)} = \frac{A^2}{S_b} \cdot \frac{k^2_{bw}(0)}{k_{ww}(0)}}$$

by EXPLOITING the Schwartz's inequality:

$$k_{wb}^2(0) \leq k_{bb}(0) \cdot k_{ww}(0) \implies \frac{k^2_{bw}(0)}{k_{ww}(0)} \leq k_{bb}(0)$$

$$\Rightarrow \max\left[\frac{k^2_{wb}(0)}{k_{ww}(0)}\right] = k_{bb}(0) \implies \boxed{w(\alpha) = b(\alpha)}$$

MATCHED FILTER

CONDITION FOR MAXIMUM SNR

which is the best result for the MEASUREMENT OF AMPLITUDE

It follows:

$$\boxed{\left(\frac{S}{N}\right)^2_{opt} = \frac{A^2}{S_b}\, k_{bb}(0) = \frac{A^2}{S_b}\int_{-\infty}^{+\infty} b^2(\alpha)\, d\alpha = \frac{E_b}{S_b}}$$

OPTIMUM SNR

E_b → SIGNAL ENERGY

S_b → NOISE POWER DENSITY (BILATERAL)

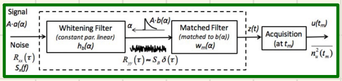
the minimum measurable output is for $S/N = 1$:

$$\boxed{A_{min} = \frac{\sqrt{S_b}}{\sqrt{E_b}} = \frac{\sqrt{S_b}}{\sqrt{\int_{-\infty}^{+\infty} b^2(\alpha)\, d\alpha}} = \frac{\sqrt{S_b}}{\sqrt{\int_{-\infty}^{+\infty} b^2(\alpha) d\alpha}}}$$

MINIMUM MEASURABLE OUTPUT

we can extend the result for any STATIONARY NOISE:
- we need a WHITENING FILTER, to make the noise white. It's possible since always exist a CONSTANT PARAMETER FILTER that do this, with:

$$\boxed{|H_b(f)|^2 \propto \frac{1}{S_x(f)}}$$



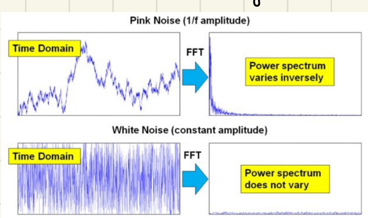
- NOTE THAT THE WHITENING FILTER MODIFY THE SHAPE OF THE SIGNAL, SO THE FOLLOWING MATCHED FILTERS IS RELATED TO THE NEW SHAPE!
- THE WHITENING + MATCHED IS NOT THE ONLY STRUCTURE. IT MAY BE DIFFICULT TO FIND A WHITENING FILTER
- in time domain, the WHITENING FIL. corresponds to have an autocorr. $k_{bb}(\tau)$ such that $k_{bb}(\tau) * R_{xx}(\tau)$ produce a δ-like $R_{yy}(\tau)$ at the output.

1/f NOISE

INTRODUCTION TO 1/f NOISE

the basic distinction with WHITE NOISE is the TIME SPAN of INTERDEPENDENCE BETWEEN SAMPLES

- for WHITE NOISE: samples are UNCORRELATED even at SHORT TIME DISTANCE
- for 1/f NOISE: samples are STRONGLY CORRELATED even at long time distance.



- The real observed POWER DENSITY at low freq. is not $1/f$ but

$$\propto \frac{1}{|f|^\alpha} \quad \text{with} \quad 0{,}8 < \alpha < 1{,}2$$

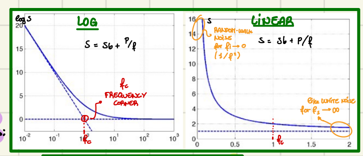
- 1/f NOISE ARISES FROM PHYSICAL PROCESS THAT GENERATE A RANDOM SUPERPOSITION OF ELEMENTARY PULSES WITH RANDOM PULSE DURATION (from very short to very long)

1/f NOISE DESCRIPTION

Typical values for low-noise voltage amplifiers :
- S_B a few 10^{-18} V²/Hz → $\sqrt{S_B}$ a few $\frac{nV}{\sqrt{Hz}}$
- f_c 10Hz to 10kHz, that is
- P a few 10^{-17} to a few 10^{-14} V² → $\sqrt{P}$ from a few nV to a few 100 nV

the NOISE POWER SPECTRAL DENSITY is:

$$\boxed{S_f(f) = P/f} \Rightarrow \overline{n_F^2} = \int_0^{\infty} \frac{P}{f}\, df \longrightarrow \infty$$



HOWEVER, A REAL 1/f NOISE IS LIMITED FROM BOTH SIDES AND DOES NOT DIVERGE. IF THERE IS A WIDE SPACING BETWEEN LOW FREQ. f_i AND HIGH FREQ. f_s OF BW LIMITATIONS:

$$\boxed{\overline{n_F^2} \approx \int_{f_i}^{f_s} \frac{P}{f}\, df = P \ln\left(\frac{f_s}{f_i}\right) = S_B f_c \ln\left(\frac{f_s}{f_i}\right)}$$

$$\boxed{P/|f_c| = S_B \Rightarrow P = S_B \cdot f_c}$$

FREQUENCY CORNER

tell us where the 1/f noise is dominant.

↳ it's slowly divergent for $f_i \to 0$ or $f_s \to \infty$. ⇒ we don't need to perfectly know where f_i and f_s really are. REASONABLE APPROXIMATIONS will be ok!

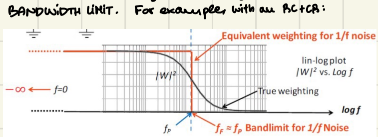
1/f NOISE FILTERING

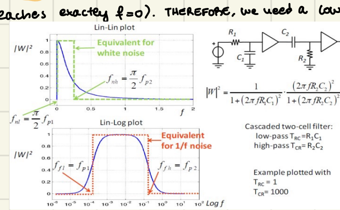
Filtering 1/f noise can be more easily understood by changing variable from f to $\ln(f)$.

$$\boxed{\overline{n_f^2} = S_B f_c \int_0^{\infty} |W(f)|^2 \frac{df}{f} = S_B f_c \int_{-\infty}^{+\infty} |W(\ln f)|^2\, d(\ln f)}$$

where $\overline{n_f^2} \propto$ area of $|W|^2$ in LOG-LIN PLOT

the area is theorically ∞ since $\log(0) \to -\infty$ (it never reaches exactly $f = 0$). THEREFORE, we need a LOWER BANDWIDTH LIMIT. For example, with an RC+CR:





HIGH PASS FILTERS

INTRINSIC HIGH PASS FILTERING by CORRELATE DOUBLE SAMPLING (CDS)

In all real cases, even with DC coupled electronics, weighting is inherently NOT extended down to zero frequency, because AN INTRINSIC HIGH-PASS FILTERING is present in any real operation.

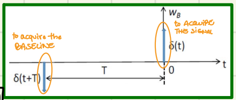
$\Longrightarrow$ INTRINSIC FILTERING ACTION happen because:

a) operation is started at some time before the ACQUISITION

b) operation is started from zero value.

for AMPLITUDE MEASUREMENT the ZERO-SETTING is mandatory: the BASELINE VOLTAGE is preliminarly adjusted to zero, or measured, recorded and subtracted from the measured signal. This corresponds to a WEIGHTING FUNCTION:

$$W_B(\omega) = \mathcal{F}[w_B(t)] = 1 - e^{i\omega T} = 1 - \cos(\omega T) - i\sin(\omega T)$$



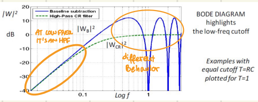
we obtain

$$|W_B(\omega)|^2 = [1-\cos\omega T]^2 + \sin^2\omega T = 2[1-\cos(\omega T)]$$

$$\boxed{|W_B(\omega)|^2 = 4\sin^2\left(\frac{\omega T}{2}\right)} \longrightarrow$$ at $\omega T \ll 1$ a low freq. cut-off is produced $|W_B(\omega)|^2 \approx \omega^2 T^2$

if we compare those result with a CR filter, we end up with the conclusion that a CDS is intrinsectly an HPF:

Baseline subtraction with delay T	High-Pass CR filter (differentiator)
$\lvert W_B(\omega)\rvert^2 = 4\sin^2\left(\frac{\omega T}{2}\right)$	$\lvert W_{CR}(\omega)\rvert^2 = \frac{\omega^2 R^2 C^2}{1+\omega^2 R^2 C^2}$
at low-frequency $\omega \ll 1/T$	at low-frequency $\omega \ll 1/RC$
$\lvert W_B(\omega)\rvert^2 \approx \omega^2 T^2$	$\lvert W_{CR}(\omega)\rvert^2 \approx \omega^2 R^2 C^2$



CDS with FILTERED BASELINE

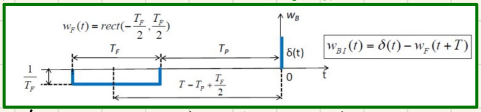
How to avoid NOISE DOUBLING? The instant sampling of BASELINE acquire all frequency components in HIGH and LOW FREQS.

$\hookrightarrow$ MODIFY BASELINE TO ACQUIRE LOW FREQS ONLY. THE SAMPLING HAPPEN WITH A LOW-PASS WEIGHTING FUNCTION $w_F(t)$ WITH BAND-LIMIT f_{Fm} WHICH INCLUDES ONLY THE FREQS TO BE SUBTRACTED.

for example, we can acquire the BASELINE with a GI, with narrower filtering band $f_{Fm} \ll f_s$ ($f_{Fm} = 1/2T_F$, with gate duration T_F):

(f_s is UPPER BANDLIMIT OF NOISE)

WEIGHTING FUNCTION



(we can still consider with long interval $T_P \gg T_F$)

we find:

$$W_{BI}(\omega) = 1 - e^{i\omega T} W_F(\omega) \Longrightarrow |W_{BI}(\omega)|^2 = 1 + W_F^2(\omega) - 2W_F(\omega)\cos\omega T$$

- for $f \ll 1/T$ it's $W_F(f) \approx 1$ and W_{BI} has an high-pass cut-off equivalent to a CR differentiator with $T=RC$:

CUT-OFF FREQ.

$$\boxed{|W_B(\omega)|^2 \approx \omega^2 T^2 = \omega^2\left(T_P + \frac{T_F}{2}\right)^2} \qquad \boxed{f_{if} \approx \frac{1}{2\pi T} = \frac{1}{2\pi\left(T_P + T_F/2\right)}}$$

- At HIGH FREQ above GI low-pass cut-off ($f > f_h = 1/2\pi T$) it's $|W_F(f)| \approx 0$ so that $|W_{BI}(f)|^2 \approx 1$
- in the INTERMEDIATE RANGE (DOUBLE NOISE RANGE) ($1/T \ll f \ll 1/2T_F$) it's roughly $W_F(f) \approx 1$ so, $|W_{BI}(f)|^2 = 2(1-\cos 2\pi f T)$ with average $= 2$
- We can calculate the NOISE.

$$\boxed{\overline{n^2} = \int_0^{f_s} S(f)|W_{BI}(\omega)|^2 df = \int_0^{f_s} S(f)\left[1 + W_F^2 - 2W_F\cos 2\pi f T\right] df}$$

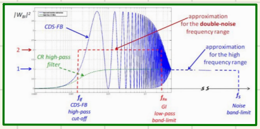
making W_{BI} approximations, we find:

ⓐ 1/f NOISE:

$$\boxed{\overline{n^2_{f_{BI}}} \approx S_B f_c \ln\left(\frac{f_s}{f_{if}}\right) + S_B f_c \ln\left(\frac{f_{Fm}}{f_{if}}\right)}$$

ⓑ WHITE NOISE:

$$\boxed{\overline{n^2_{B,BI}} \approx S_B(f_s - f_i) + S_B(f_{Fm} - f_i) \approx S_B f_s + S_B f_{Fm}}$$



NOISE DOUBLING OCCURS ONLY FROM LOW-FREQ. CUT-OFF TO GI BAND LIMIT

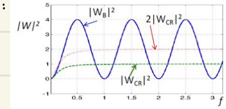
if $\underbrace{f_s}_{\text{NOISE BAND-LIMIT}} \gg \underbrace{f_{Fm}}_{\text{GI BAND-LIMIT}}$:

$$\boxed{\overline{n^2_{f,BI}} \approx S_B f_c \ln\left(\frac{f_s}{f_{if}}\right) \qquad \overline{n^2_{B,BI}} \approx S_B f_s}$$

EFFECTS ON NOISE (CDS vs CR)

To study the effects on white noise we rely on LIN-LIN plot: if f_s is the LPF BW limitation:

$$\overline{n_B^2} = S_B \int_0^{f_s} |W(f)|^2 df$$



CDS: $|W_B|^2$ oscillates around 2 | is the same as a constant $|W_B|^2 = 2$

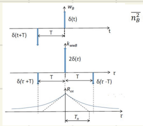
CR: has a cut-off at low freq. $f < f_i = \frac{1}{4RC}$ at HF is $|W_{CR}|^2 \approx 1$

With Baseline sampling & subtraction	With CR high-pass filter
$\overline{n_B^2} = S_B \int_0^{f_S} 2\cdot[1-\cos\omega T]\, df$	$\overline{n_B^2} = S_B(f_S - f_i)$
that is	and since $f_s \gg f_i$
$\overline{n_B^2} = 2\, S_B f_S$	$\overline{n_B^2} \approx S_B f_S$

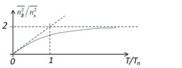
$\Longrightarrow$ THEREFORE THE OUTPUT NOISE WITH CDS IS DOUBLED. What does it happen in TIME-DOMAIN?

- for CDS:

$$R_{xx}(\tau) = \overline{n_x^2}\, e^{-\frac{|\tau|}{T_n}}$$



$$\overline{n_B^2} = S_B \int_{-\infty}^{+\infty} R_{xx}(\tau)\, k_{wwB}(\tau)\, d\tau = 2\overline{n_x^2} - R_{xx}(T) - R_{xx}(-T) = 2\overline{n_x^2}\cdot\left(1 - e^{\frac{-T}{T_n}}\right)$$

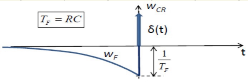


a) NOISE WITH VERY SHORT CORRELATION TIME (HIGH BAND-LIMIT) IS DOUBLED. if $T_n \ll T$ we have $\overline{n_B^2} \approx 2\overline{n_x^2}$

b) NOISE WITH LONG CORRELATION TIME IS STRONGLY ATTENUATED. if $T_n \gg T$ we have $\overline{n_B^2} \approx \overline{n_x^2} \cdot 2\frac{T}{T_n} \ll \overline{n_x^2}$

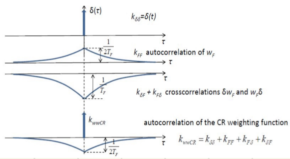
- for CR:

$$w_{CR}(t) = \delta(t) - w_F(t) = \delta(t) - \frac{1}{T_F} e^{-\frac{|t|}{T_F}}$$



$$k_{ww_{CR}}(\tau) = \delta(t) - \frac{1}{2} w_F(|\tau|) = \delta(t) - \frac{1}{2T_F} e^{-\frac{|t|}{T_F}}$$

$$\overline{n_B^2} = \int_{-\infty}^{+\infty} S_B R_{xx}(\tau)\, k_{ww_{CR}}(\tau)\, d\tau = R_{xx}(0) - 2\int_0^{+\infty} R_{xx}(\tau)\frac{1}{2}w_F(|\tau|)d\tau = \overline{n_x^2} - \overline{n_x^2}\cdot\int_0^{\infty}\frac{1}{T_F}\exp\left[-\tau\left(\frac{1}{T_n}+\frac{1}{T_F}\right)\right]d\tau =$$



$$\Rightarrow \overline{n_B^2} = \overline{n_x^2}\left(1 - \frac{T_n}{T_n + T_F}\right) = \overline{n_x^2}\,\frac{T_F}{T_n + T_F}$$

a) for SHORT CORRELATION TIME (HIGH BAND LIMIT) is simply PASSED as it. NOT DOUBLED

if $T_n \ll T$: $\overline{n_B^2} \approx \overline{n_x^2}$

b) for LONG CORRELATION TIME (very LOW BAND-LIMIT) is strongly ATTENUATED at half level of the CDS

if $T_n \gg T$: $\overline{n_B^2} \approx \overline{n_x^2}\cdot\frac{T_F}{T_n} \ll \overline{n_x^2}$

if we want to compare CDS with CR:

- at LOW FREQ. $f \ll 1/T$ the $|W_B|^2$ and the $|W_{CR}|^2$ have the same CUT-OFF with $T=RC$
- at HIGH FREQ. $|W_{CR}|^2 \approx 1$ whereas $|W_B|$ oscillates around mean value 2

$$\boxed{\overline{n_B^2} = \int_0^{f_s} |W_B(f)|^2 d(\ln f) \approx 2\int_0^{f_s} |W_{CR}(f)|^2 d(\ln f)}$$

Therefore:

$$\overline{n^2_{f,B}} \approx 2\,\overline{n^2_{f,CR}} \Longrightarrow$$ the 1/f NOISE POWER OUTPUT OF CDS is approx. double with respect to a CR high-pass with equal cut-off ($T=RC$)

- for CR the HIGH-PASS BAND LIMIT for 1/f NOISE is:

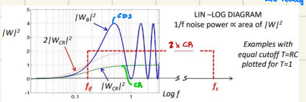
$$\boxed{f_{if} \approx f_p = \frac{1}{2\pi RC} \quad \text{and} \quad \overline{n^2_{f,CR}} = S_B f_c \ln\left(\frac{f_s}{f_{if}}\right)}$$

- for CDS then:

$$\boxed{f_{if} \approx \frac{1}{2\pi T} \quad \text{and} \quad \overline{n^2_{f,B}} \approx 2 S_B f_c \ln\left(\frac{f_s}{f_{if}}\right)}$$

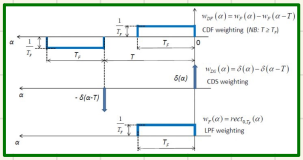
T can be also minutes $\rightarrow$ low f_{if} $\Rightarrow$ DRAWBACK: THE 1/f NOISE POWER IS LIMITED JUST TO A MODERATELY LOW LEVEL, WHICH MAY BE HIGHER THAN THAT OF WHITE NOISE

DRAWBACK: noise is doubled because in baseline sampling all freq. are acquired but in the subtraction, only those with $f \ll 1/T$ are really effective for reducing 1/f noise.



CORRELATED DOUBLE FILTERING (CDF)

In various cases of PULSE AMPLITUDE MEASUREMENT, FILTERING by GI is quite efficient for white noise component, but not for 1/f component. We can improve using a GI not only for BASELINE but also for SIGNAL ACQUISITION.



we have:

$$w_{DF}(\alpha) = w_{DS}(\alpha) * w_F(\alpha)$$
$$W_{DF}(\omega) = W_{DS}(\omega) \cdot W_F(\omega)$$
$$|W_{DF}|^2 = |W_{DS}|^2 \cdot |W_F|^2 \quad \text{(for noise computation)}$$
$$|W_{DS}|^2 = 2(1-\cos\omega T) = 4\sin^2(\omega T/2)$$

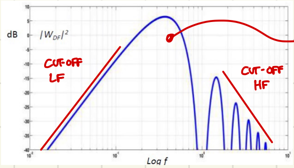
$$\Rightarrow \boxed{|W_{DF}|^2 = 2(1-\cos\omega T)\cdot|W_F|^2 = 4\sin^2\left(\frac{\omega T}{2}\right)\cdot|W_F|^2}$$

it's a COMBINATION of CDS and LPF.

1. the LPF cuts the NOISE at HIGH FREQ. (with f_F)
2. the CDS cuts the NOISE at LOW FREQ. (with $f_{iD} \approx 1/2\pi T$)
3. the CDS enhances the noise in the passband between the band limits (factor 2)

recalling: $|W_F(\omega)| = \left|\text{sinc}\left(\frac{\omega T_F}{2}\right)\right| = \left|\frac{\sin\left(\frac{\omega T_F}{2}\right)}{\frac{\omega T_F}{2}}\right|$

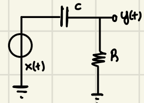
$$\Rightarrow \boxed{|W_{DF}|^2 = 2(1-\cos\omega T)\cdot\frac{\sin^2(\omega T_F/2)}{(\omega T_F/2)^2} = 4\frac{\sin^4\left(\frac{\omega T_F}{2}\right)}{\left(\frac{\omega T_F}{2}\right)^2} = 4\,\text{sinc}^2\left(\frac{\omega T_F}{2}\right)}$$



ONLY HERE NOISE DOUBLING

↳ the region is reduced a lot!

CR DIFFERENTIATOR

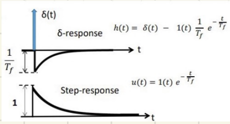


$T_F = RC$

$f_p = \frac{1}{2\pi T_F}$

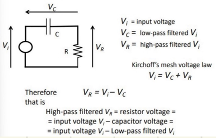
$H(f) = \frac{j2\pi f T_F}{1+j2\pi f T_F}$

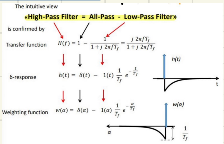
$|H(f)|^2 = \frac{(2\pi f T_F)^2}{1+(2\pi f T_F)^2}$



The CR can always be seen as:

$$\boxed{\text{HPF} = \text{ALL-PASS} - \text{LPF}}$$

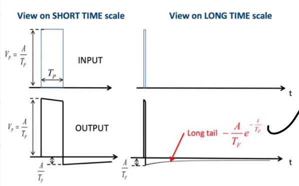




the HIGH-PASS BAND LIMIT is:

- FOR WHITE NOISE: $f_{i_{CR}} = f_{hRC} = \frac{1}{4RC}$
- FOR 1/f NOISE: for $f_s \gg f_p$: $f_{if} \approx f_p = \frac{1}{2\pi RC}$

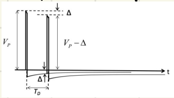
WHAT ARE THE PERFORMANCES FOR PULSES?



DC transfer is zero therefore time area should be zero too.

⇓

if there is another pulse before $5T_F$ it's a problem:

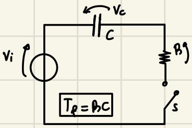


NEXT PULSES START from a LOWER BASELINE!

↳ NOT A PROBLEM WITH PERIOD SIGNALS (deterministic stationary shift). BUT IT'S A PROBLEM WITH RANDOM PULSES!

BASELINE RESTORER

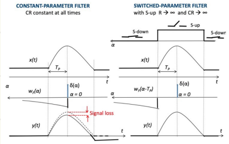
it's a SWITCHED CR: we want HIGH PASS FILTERING action on the NOISE, not the signal.



$CR \to \infty$ when signal is present

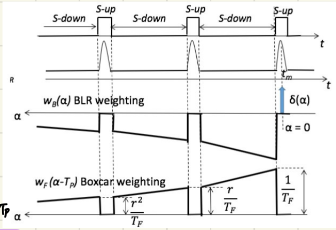
$CR = T_F$ when NO signal is present

$w_F(\alpha - T_P) = 1(\alpha - T_P)\frac{1}{T_F}e^{-\frac{\alpha - T_P}{T_F}}$



to find the WEIGHTIN FUNCTION, again:

$$w_B(\alpha) = \underbrace{}_{\text{HPF}} \underbrace{\delta(\alpha)}_{\text{ALL-PASS}} - \underbrace{w_F(\alpha - T_P)}_{\text{LPF BOXCAR}}$$



$$W_B(\omega) = 1 - e^{-j\omega T_P}W_F(\omega)$$
$$W_F(\omega) = Re_F(\omega) + j\,Im_F(\omega)$$
$$\Rightarrow W_B(\omega) = \left[1 - Re_F\cos\omega T_P - Im_F\sin\omega T_P\right] - j\left[Im_F\cos\omega T_P - Re_F\sin(\omega T_P)\right]$$
$$= 1 + |W_F|^2 - 2Re_F\cos\omega T_P - 2Im_F\sin\omega T_P$$

if we consider interval between pulses $\gg T_F$ then:

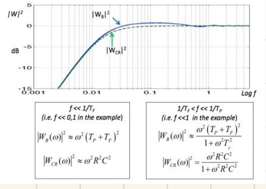
$$w_F(\alpha) = 1(\alpha)\frac{1}{T_F}e^{-\alpha/T_F}$$
$$W_F(\omega) = \frac{1}{1+j\omega T_F}$$

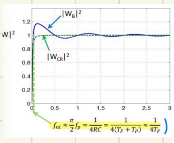
$$\Rightarrow |W_B(\omega)|^2 = 1 + \frac{1}{1+\omega^2T_F^2} - 2\frac{1}{1+\omega^2T_F^2}\cos\omega T_P + 2\omega T_F\cdot\frac{1}{1+\omega^2T_F^2}\sin\omega T_P$$

- in LOW FREQ.:

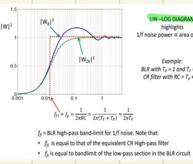
$\omega \ll 1/T_P$: $|W_B(\omega)|^2 = \frac{\omega^2T_F^2}{1+\omega^2T_F^2}\left(1+\frac{T_P}{T_F}\right)^2$

$\omega \ll \frac{1}{T_F} \ll \frac{1}{T_P}$: $|W_B(\omega)|^2 \approx \omega^2(T_P+T_F)^2$ (same of CR)





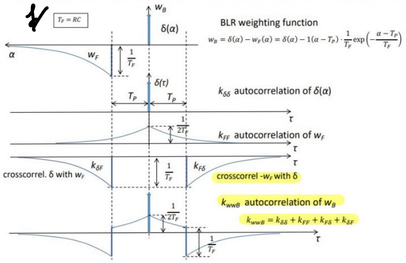
f_{ni} = BLR high-pass band-limit for white noise. Note that:
- f_{ni} is equal to that of the equivalent CR High-pass filter
- f_{ni} is equal to bandlimit of the low-pass section in the BLR circuit



f_p = BLR high-pass band-limit for 1/f noise. Note that:
- f_p is equal to that of the equivalent CR High-pass filter
- f_p is equal to bandlimit of the low-pass section in the BLR circuit

TIME DOMAIN ANALYSIS:

1/



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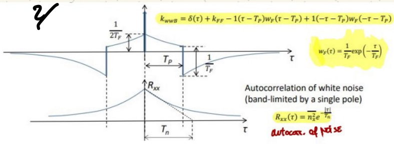


$$\overline{n_B^2} = \int_{-\infty}^{\infty} R_{xx}(\tau)k_{wwB}(\tau)d\tau = R_{xx}(0) + 2\int_0^{\infty} R_{xx}(\tau)\frac{1}{2}w_F(\tau)d\tau - 2\int_T^{\infty} R_{xx}(\tau)w_F(\tau - T_P)d\tau =$$

$$= R_{xx}(0) + \int_{-\infty}^{\infty} R_{xx}(\tau)\frac{1}{2}w_F(\tau)d\tau - 2\int_0^{\infty} R_{xx}(\beta + T_P)w_F(\beta)d\beta$$

Denoting $r_{xx}(\tau) = \frac{R_{xx}(\tau)}{R_{xx}(0)} = \frac{R_{xx}(\tau)}{\overline{n_x^2}}$

We have

$$\overline{n_B^2} = \overline{n_x^2}\left\{1 + \int_{-\infty}^{\infty} r_{xx}(\tau)\frac{1}{T_F}e^{-\frac{\tau}{T_F}}d\tau - 2\int_0^{\infty} r_{xx}(\beta + T_P)\frac{1}{T_F}e^{-\frac{\beta}{T_F}}d\beta\right\}$$

3/

$$\overline{n_B^2} = \overline{n_x^2}\left\{1 + \int_{-\infty}^{\infty} r_{xx}(\tau)\cdot\frac{1}{T_F}e^{-\frac{\tau}{T_F}}d\tau - 2\int_0^{\infty} r_{xx}(\beta + T_P)\cdot\frac{1}{T_F}e^{-\frac{\beta}{T_F}}d\beta\right\}$$

$$= \overline{n_x^2}\left\{1 + \frac{1}{T_F}\int_{-\infty}^{\infty} e^{-\tau\left(\frac{1}{T_F}+\frac{1}{T_n}\right)}d\tau - 2e^{-\frac{T_P}{T_n}}\frac{1}{T_F}\int_0^{\infty} e^{-\beta\left(\frac{1}{T_F}+\frac{1}{T_n}\right)}d\beta\right\} =$$

$$= \overline{n_x^2}\left[1 + \frac{T_n}{T_n + T_F} - 2e^{-\frac{T_P}{T_n}}\frac{T_n}{T_n + T_F}\right]$$

and finally

$$\boxed{\overline{n_B^2} = \overline{n_x^2}\left[1 + \frac{T_n}{T_n + T_F}\left(1 - 2e^{-\frac{T_P}{T_n}}\right)\right]}$$

OVERALL NOISE

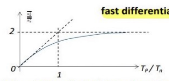
With **fast differentiation**, i.e. with $T_F \ll T_n$, it is quantitatively confirmed that the BLR acts like a CDS with $T = T_P$

$$\overline{n_B^2} \approx 2\overline{n_x^2}\cdot\left(1 - e^{-\frac{T_P}{T_n}}\right)$$

exactly same expression of CDS

BLR Filtering with fast differentiation

4/



fast differentiation with $T_F \ll T_n$

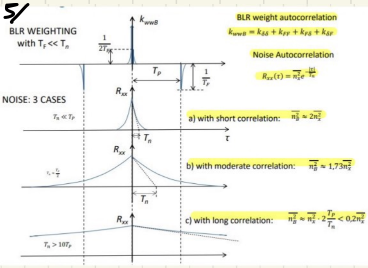
$$\boxed{\overline{n_B^2} \approx 2\overline{n_x^2}\cdot\left(1 - e^{-\frac{T_P}{T_n}}\right)}$$

With $T_F \ll T_n$ the effect of BLR on **band-limited white noise** depends on how long is the correlation time T_n with respect to the delay T_P

- with **short correlation time** (wide band) the noise is **doubled**:
  with $T_n < \frac{T_P}{5}$ it is $\overline{n_B^2} \approx 2\overline{n_x^2}$
- with **moderate correlation time** (moderately wide band) the noise is enhanced:
  with $T_n \approx \frac{T_P}{2}$ it is $\overline{n_B^2} \approx 1{,}73\overline{n_x^2}$
- only with **long correlation time** (low-frequency band) the noise is **attenuated***:
  with $T_n > 10T_P$ it is $\overline{n_B^2} \approx \overline{n_x^2}\cdot 2\frac{T_P}{T_n} < 0{,}2\overline{n_x^2}$

** note that anyway the level is **double** of that given by a simple CR filter with equal cutoff, that is with $T_F = RC = T_P$*

5/



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With T_F **NOT negligible with respect to T_n**, the effect on white noise depends also **on the size of T_F compared to T_n and T_P**. A long T_F can limit the white noise enhancement

$$\boxed{\overline{n_B^2} = \overline{n_x^2}\left[1 + \frac{T_n}{T_n + T_F}\left(1 - 2e^{-\frac{T_P}{T_n}}\right)\right]}$$

fixed

only FREE PARAMETER to be optimized for each noise

study fixed noise but fixed

Let's evaluate how long must be T_F in the various cases of noise correlation

- with **short correlation time $T_n \approx T_P/10$** it is (CDS)

$$\overline{n_B^2} \approx \overline{n_x^2}\left(1 + \frac{T_n}{T_n + T_F}\right)$$

  for keeping $\overline{n_B^2} < 1{,}05\,\overline{n_x^2}$ we need $T_F > 20\,T_n \approx 2\,T_P$

- with **moderate correlation time $T_n \approx T_P/2$** it is

$$\overline{n_B^2} \approx \overline{n_x^2}\left[1 + \frac{T_n}{T_n + T_F}\left(1 - \frac{2}{e^2}\right)\right] = \overline{n_x^2}\left[1 + 0{,}73\frac{T_n}{T_n + T_F}\right]$$

  for keeping $\overline{n_B^2} < 1{,}05\,\overline{n_x^2}$ in this case we need $\boldsymbol{T_F > 7T_n = 3{,}5T_P}$

7/

- with **long correlation time** $T_n > 10\,T_P$ it is

$$\overline{n_B^2} \approx \overline{n_x^2}\left[1 - \frac{T_n}{T_n + T_F}\right] = \overline{n_x^2}\frac{T_F}{T_n + T_F}$$

No problem with such a low-frequency noise: it is attenuted by the BLR just as by a CR constant-parameter filter (with equal time constant $T_F = RC$)

8/

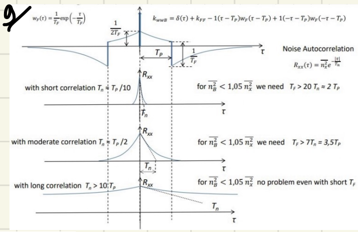
The most interesting case for us is noise with moderate T_n. In fact, when the BLR works on the output of an optimum (or approximate-optimum) filter for wideband noise, the correlation time T_n and delay T_p are comparable, since they are both closely related to the band-limit of the signal pulse.

- We conclude that for avoiding enhancement of the white noise it is necessary to select a fairly slow BLR differentiation, i.e. a fairly long T_F

$$\boxed{T_F \geq 5T_P}$$

- This approach is satisfactory also for filtering the 1/f noise, notwithstanding that making T_F longer than T_P shifts down the BLR cutoff frequency, hence reduces the attenuation of 1/f noise. This is counterbalanced by the fact that the enhancement of 1/f noise at frequencies above the cutoff is limited by the low-pass filtering in the baseline subtraction, whereas with short T_F it is remarkable.

9/



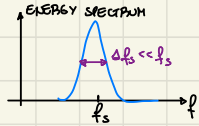
10/

- The BLR is a high-pass filter that acts on noise and disturbances without affecting the pulse signal
- The BLR is a switched-parameter filter: the low-pass section within the high-pass filter structure is a boxcar integrator that acquires the baseline only in the intervals free from pulses
- The BLR can thus establish a high-pass band-limit at a high value (suitable for reducing efficiently the 1/f noise output power) without causing the signal loss suffered with a constant-parameter high-pass filter having the same band-limit
- The high-pass band-limit enforced by the BLR is given (with good approximation) by the low-pass bandlimit of the low-pass section in the BLR circuit structure
- The combination of: (1) optimum filter designed for the case of pulse signal in presence of wideband noise only (i.e. without 1/f noise) and (2) BLR specifically designed (for reducing the actual 1/f noise without worsening the wide-band noise) provides in most cases a quasi-optimum filtering solution.

BAND-PASS FILTERS

PASS-BAND FILTERS are usually used because often signals does not have a DC component, so we need to eliminate both high and low frequencies.

Used for NARROW-BAND SIGNALS, which concentrate their power in a small part of the spectrum (and this allows filtering out most of the noise, reaching an high SNR with respect to BROADBAND SIGNAL with same power)



- ideally it cut-off all unwanted freqs. and have 0dB gain and zero-shift phase but this would require vertical transitions. So we speak about ATTENUATION in the STOP-BAND

ROLL-OFF: steepness of transitions [dB/decade]

FRACTIONAL BW s: $s = \Delta f / f_0$

$\boxed{Q = 1/s = f_0/\Delta f}$ QUALITY FACTOR

INTRODUCTION TO BAND-PASS FILTERING

- for a NARROW-BAND SIGNAL plugged in WHITE NOISE ($f_s > f_c$ of 1/f noise) a BAND-PASS FILTER MATCHED TO SIGNAL BANDWIDTH is very efficient, and make possible to recover even small signals mixed with noise.
- if the signal is plunged in 1/f NOISE, it is still quite efficient, and in many cases it's possible to recover the signal. However, the more dominant is 1/f noise (lower f) the lower the probability of having a good (S/N).

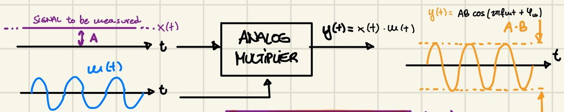
two open questions:

A) HOW TO DESIGN and IMPLEMENT NARROW BW BAND-PASS?

B) CAN WE MOVE A DC-INFORMATION SIGNAL TO HIGH FREQS IN ORDER TO AVOID 1/f NOISE? HOW?

NOTE:
I) DO THE MOVE BEFORE SIGNAL MIXES WITH 1/f
II) FREQS TRANSFER MODULE HAVE THEIR NOISE, AND THEY ARE LIMITED TO MODERATELY HIGH FREQ.
III) SIGNAL MUST BE HIGHER THAN THE NOISE REFERRED TO THE INPUT OF THE FREQUENCY-TRANSFER STAGE. IF IT'S NOT ENOUGH, PREAMPLYFING IS NOT ADVISABLE, BECAUSE IT BRINGS 1/f NOISE.

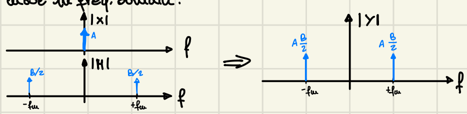
SINUSOIDAL SIGNAL MODULATION



for sinusoidal modulation: $\boxed{m(t) = B\cos(2\pi f_m t)}$ SINUSOIDAL REFERENCE

$$\mathcal{F}[m(t)] = \frac{B}{2}\delta(f - f_m) + \frac{B}{2}\delta(f + f_m)$$

let's move in freq. domain:

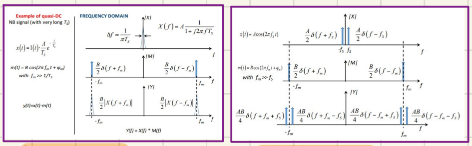


NOTE THAT $|Y(f)| \neq |X(f)| * |M(f)|$ BECAUSE WE FIRST NEED TO COMPUTE Real AND Imaginary PART OF Y(f).

HOWEVER IF X(f) IS CONFINED IN NARROW BW Δf_s and M(f) HAS $f_m >> \Delta f_s$, WE CAN NEGLECT AND USE:

$|Y(f)| \approx |X(f)| * |M(f)|$ and THE SIGNAL IS SHIFTED IN FREQ. BY $-f_m$ and $+f_m$ AND IN PHASE BY $+\varphi_m$ and $-\varphi_m$

EXAMPLE OF MODULATION



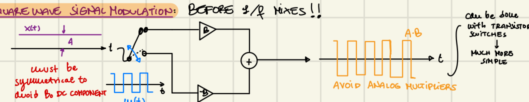
QUASI DC-SIGNAL

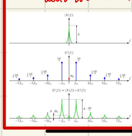
SINUSOIDAL SIGNAL

$x(t) = A\cos(2\pi f_s t)$ $m(t) = B\cos(2\pi f_m t + \varphi_m)$

$y(t) = x(t)\cdot m(t) = \frac{AB}{2}\cos[2\pi(f_s - f_m)t - \varphi_m] + \frac{AB}{2}\cos[2\pi(f_s + f_m) + \varphi_m]$

SQUARE WAVE SIGNAL MODULATION: BEFORE 1/f MIXES!!

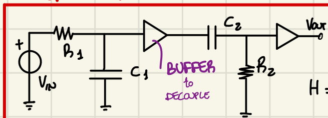




$M(f) = B_0\delta(f) + \sum_{h=1}^{\infty}(-1)^{h+1}\frac{2B}{\pi(2h-1)}\left[\delta(f - (2h-1)f_m) + \delta(f + (2h-1)f_m)\right]$

with $f_m = 1/T_m$ T_m: PERIOD ; B_0: DC VALUE (MEAN VALUE)

RC LPF + CR HPF



$T_1 = R_1 C_1$ $\quad f_{p1} = 1/2\pi T_1$

$T_2 = R_2 C_2$ $\quad f_{p2} = 1/2\pi T_2$

$$H = \frac{V_{OUT}}{V_{IN}} = \frac{1}{1 + \frac{jf}{f_{p1}}}\cdot\frac{jf/f_{p2}}{1 + jf/f_{p2}}$$

$$|H(f)| = \frac{f/f_p}{1 + (f/f_p)^2} \qquad |H(f_p)| = 1/2 \qquad \measuredangle = 0$$

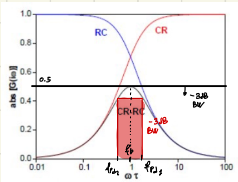
$f_{p1} = f_{p2} = f_p$

The BW is defined by -3dB limit (lower than $1/\sqrt{2}$):

$|H(f_{pd_1})| = |H(f_{pd_2})| = \frac{|H(f_p)|}{\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{x}{1+x^2}$ $\quad x_{1,2} = \sqrt{2} \pm 1$

$\Delta f_{pd} = f_{pd_1} - f_{pd_2} = 2f_p$

$\frac{\Delta f_{pd}}{f_p} = 2 \Rightarrow$ NOT NARROW BAND!



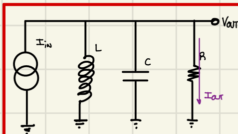
- if we compute WHITE NOISE:

$$\overline{n_B^2} = S_B\cdot|H(f_p)|^2\cdot\Delta f_m = S_B\cdot\frac{1}{4}\cdot\Delta f_m$$

$$\overline{n_B^2} = S_B\int_{-\infty}^{+\infty}|H(f)|^2 df_m = S_B f_p \frac{\pi}{4}$$

EQUIVALENT NOISE BW: $\Delta f_m = \pi f_p = \frac{1}{2T} = \frac{\pi}{2}\Delta f_p$

LCR RESONANT FILTER



$$Z = \frac{V_{OUT}}{V_{IN}} = \frac{1}{j\omega C + \frac{1}{j\omega L} + \frac{1}{R}} = R\frac{j\omega\frac{L}{R}}{(1 - \omega^2 LC) + j\omega\frac{L}{R}}$$

$$H(\omega) = \frac{I_{OUT}}{I_{IN}} = \frac{1}{R}\frac{V_{OUT}}{I_{IN}} = \frac{j\omega\frac{1}{RC}}{\left(\frac{1}{LC} - \omega^2\right) + j\omega\frac{1}{RC}}$$

$$Z = R\cdot H(\omega)$$

- at RESONANCE FREQUENCY $\omega_0 = 1/\sqrt{LC}$ THE REACTIVE IMPEDANCES CANCEL EACH OTHER, SO THAT THE IMPEDANCE IS PURELY RESISTIVE $Z(\omega_0) = R$, which is $H(\omega_0) = 1$ $\measuredangle H(\omega_0) = 0$
- another basic parameter is CHARACTERISTIC RESISTANCE R_0, WHICH FOR THE OSCILLATION AT $\omega = \omega_0$ REPRESENTS THE RATIO:

(AMPLITUDE OF VOLTAGE ON C)/(AMPLITUDE OF CURRENT IN L) $\quad R_0 = \sqrt{\frac{L}{C}}$

- THE POLES ARE:

$$S_{1,2} = -\alpha_0 \pm \sqrt{\alpha_0^2 - \omega_0^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}} \qquad \alpha_0 = \frac{1}{2RC}$$

- we can study the BEHAVIOR of its S-RESPONSE:
  - DAMPED (REAL POLES): $\omega_0^2 < \alpha_0^2$ $\quad R^2 < \left(\frac{R_0}{2}\right)^2$
  - CRITICALLY DAMPED (COINCIDENT REAL POLES): $\omega_0^2 = \alpha_0^2$ that is $R^2 = \left(\frac{R_0}{2}\right)^2$
  - OSCILLATORY (COMPLEX POLES): $\omega_0^2 > \alpha_0^2$ $\quad R^2 > \left(\frac{R_0}{2}\right)^2$

THE HIGHER is R WITH RESPECT TO R_0, THE LOWER is THE DISSIPATION AND THE SLOWER THE DAMPING OF THE OSCILLATION.

. . RESONATOR

The ENERGY E stored in the circuit oscillates from L to C and back while it decay exponentially due to dissipation in R.

↳ THE LOWER is the LOSS RATE, THE HIGHER is THE RESONATOR QUALITY. THE QUALITY FACTOR is THE RECIPROCAL OF LOSS RATE.

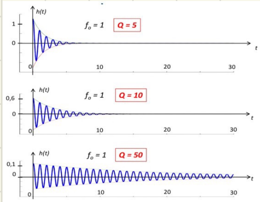
$$-\frac{1}{E}\cdot\frac{dE}{d\theta} = \frac{1}{Q} \longrightarrow \boxed{Q = \frac{\omega_0}{2\alpha} = \frac{R}{R_0}}$$ QUALITY FACTOR

$Q \to +\infty$, $R \to +\infty$

the TRANSFER FUNCTION can be expressed in terms of RESONANCE FREQ. ω_0 and QUALITY FACTOR:

$$\boxed{H(\omega) = \frac{j\omega \frac{\omega_0}{Q}}{(\omega_0^2-\omega^2) + j\omega\frac{\omega_0}{Q}}}$$

$$\boxed{h(t) = 1(t)\,\frac{2\pi f_0}{Q}\, e^{-\frac{\pi f_0}{Q}t}\cos(\omega_0 t)}$$



• ABOUT THE PHASE:

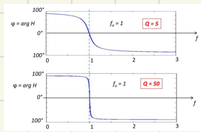
$$\varphi = \arg H(\omega) = \arctan\left[\frac{Q}{\omega\omega_0}(\omega_0+\omega)(\omega_0-\omega)\right]$$

$\omega \to +\infty$ $|H| \to 0$ $\varphi \to -\pi/2$

$\omega \to -\infty$ $|H| \to 0$ $\varphi \to +\pi/2$

$\omega \to \omega_0$ $|H| \to 1$ $\varphi \to 0$

$$\left(d\varphi/d\omega\right)_{\omega=\omega_0} = -2\,Q/\omega_0$$ STEEPNESS OF PHASE



APPROXIMATIONS:

$\omega \ll \omega_0$: $|H(\omega)| \approx |H_L(\omega)| = \frac{\omega}{\omega_0}\cdot\frac{1}{Q}$

$\omega \gg \omega_0$: $|H(\omega)| \approx |H_H(\omega)| = \frac{\omega_0}{\omega}\cdot\frac{1}{Q}$

$\omega = \omega_0$: $|\omega-\omega_0| \ll \omega_0$ $\quad |H_c(\omega)|^2 = \frac{1}{1+4Q^2\left(\frac{\omega-\omega_0}{\omega_0}\right)^2}$

BANDWIDTH for SIGNALS:

if $Q \ll 1$: $|H_c(\omega_d)|^2 = \frac{1}{1+4Q^2\left(\frac{\omega_d-\omega_0}{\omega_0}\right)^2} = \frac{1}{2}$ $\quad \Delta\omega_s = \frac{\omega_0}{Q}$

SIGNAL BANDWIDTH $\boxed{\frac{\Delta\omega_s}{\omega_0} = \frac{\Delta f_s}{f_s} = \frac{1}{Q}}$

WITH RESPECT TO CR-RC:
- NO SIGNAL ATTENUATION
- NARROW FILTERING BANDWIDTH (even with moderately high Q values)

BANDWIDTH for WHITE NOISE:

$$\boxed{\Delta f_m} = \int_0^\infty |H(f)|^2 df = \boxed{\frac{\pi}{2}\frac{f_0}{Q} = \frac{\pi}{2}\Delta f_s}$$ NOISE EQ. BANDWIDTH

if $Q \ll 1$: $|H_c(f)|$ for central lobe approx.

$$\int_0^\infty |H(f)|^2 df \approx 2\int_{f_0}^\infty |H_c(f)|^2 df$$

$$\Delta f_m = 2\int_{f_0}^\infty \frac{1}{1+4Q^2\left(\frac{f-f_0}{f_0}\right)^2}df = \frac{f_0}{Q}\int_0^\infty \frac{1}{1+x^2}dx = \frac{\pi f_0}{2Q}$$

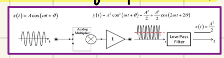
PRO and CONS of REAL TUNED FILTERS:

- REAL CAPACITOR and INDUCTORS are not pure capacitance and INDUCTANCE. ↝ MORE DISSIPATION, lower Q.
- GOOD CAPACITORS: pF – 1μF ; GOOD INDUCTORS: mH – few 100 mH limit FREQUENCY CHOICE.
- PARASITIC CAP: pF IND: mH (smaller in IC)
- RESONANCE is at $\boxed{f_0 = \frac{1}{\sqrt{2\pi LC}}}$ TO OBTAIN LOW VALUES, HIGH C and L
- for $f_0 > 100$ MHz $Q > 10$ is obtained
- for 1MHz $< f_0 <$ 100MHz, $Q \approx 10$ are obtained
- for $f < 1$MHz, difficult to obtain HIGH Q (≈ 5)
- for a given Q, NOISE BW is reduced as resonant freq is reduced $\Delta f_m = \frac{\pi}{2} f_0/Q$
- keep variations due to fabrication process/temp. low to avoid unwanted variations of the output.
- CASCADING simple FILTER STAGES is not practical for NARROW BW FILTERS (they should be very accurate)
- THE VALUE OF C INFLUENCES BOTH f_0 and Δf_s ADJUSTABLE f_0 is DIFFICULT TO IMPLEMENT.

ASYNCHRONOUS MEASUREMENT OF SINUSOIDAL SIGNALS

Measure a sinusoidal signal without an auxiliary reference that points the peaking time.

↳ avoid DC components (HPF or BPF)

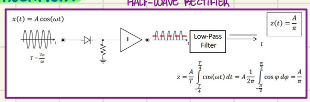


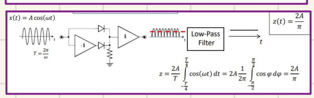
- IT'S A POWER METER, the output is a measure of the total INPUT MEAN POWER (sum of signal power plus noise power)
  ↳ LPF has no effect of NOISE REDUCTION; it does not average the input, but the SQUARE of the INPUT.
  TO IMPROVE (S/N) IT'S NECESSARY TO INSERT A FILTER BEFORE MEAN SQUARE DETECTOR.

$$\boxed{y(t) = A^2\cos^2(\omega t+\theta) = \frac{A^2}{2} + \cancel{\frac{A^2}{2}\cos(2\omega t+2\theta)}}$$ LPF — MEAN SQUARE DETECTOR

Other structures are HWR (HALF-WAVE RECTIFIER) and FWR (FULL-WAVE RECTIFIER):

HALF-WAVE RECTIFIER



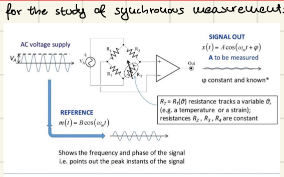


FULL-WAVE RECTIFIER

- not really ASYNCHRONOUS but SELF-SYNCHRONIZED
  ↓
  the sinusoidal signal itself decides when it has to be passed with positive polarity and when passed with negative polarity (or not passed, depending if it's FWR or HWR)
- a LPF can help improving the (S/N) but only when the signal is already higher than the noise.
- for LOW INPUTS, NOISE IS MORE INFLUENT ON THE SWITCHING TIME OF THE RECTIFIER, UNTIL THE SYNCHRONISM IS LOST (ZERO CROSSING OF THE NOISE)
  ↓
  WEAK S/N IMPROVEMENT due to LPF.
  NOT USEFUL with low S/N signals

SYNCHRONOUS (OR PHASE SENSITIVE) MEASUREMENTS OF SINUSOIDAL SIGNALS

for the study of synchronous measurements and narrow-band filtering.



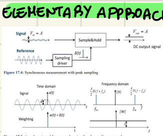
R_T: ex. STRAIN SENSOR (varies following mechanical strain θ)

a) in case of CONSTANT STRAIN θ:
constant A ⟶ x(t) is a pure SINUSOIDAL SIGNAL

b) in cases with SLOWLY VARIABLE θ = θ(t)
variable A = A(t) ⟶ x(t) is a MODULATED SINUSOIDAL SIGNAL

SLOW VARIATIONS: fourier components of $A(f) = \mathcal{F}[A(t)]$ have freqs $f \ll f_m$

- an ELEMENTARY APPROACH to SYNC-MEASUREMENT is the following:



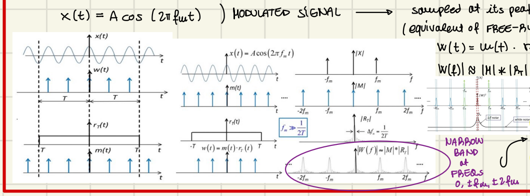
☹ NO FILTERING ACTION ⟶ All input noise power acquired.

- we can also exploit AVERAGING on MULTIPLE SAMPLES ($N \gg 1$): $N = f_m 2T \gg 1$

$x(t) = A\cos(2\pi f_m t)$ MODULATED SIGNAL ⟶ sampled at its peak N times (equivalent of FREE-RUNNING SAMPLER)

$w(t) = m(t)\cdot r_R(t)$ ⌐ rect of duration 2T

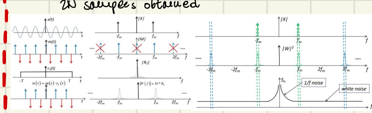
$W(f) \approx |M| * |R_T|$ centered at DC with $\Delta f = 1/2T$



NARROW BAND at FREQS 0, $\pm f_m$, $\pm 2f_m$

improved with respect to SINGLE SAMPLE
but
still
- DC BW acquire 1/f noise and NO SIGNAL
- $2f_m$, $3f_m$, ... only WHITE NOISE NO SIGNAL

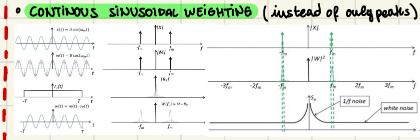
- DC SUPPRESSION + SUBTRACT NEGATIVE PEAK SAMPLES:
  2N samples obtained



☺ DC COMPONENT deleted

☹ still components at $3f_m$, $5f_m$ that acquire only white noise

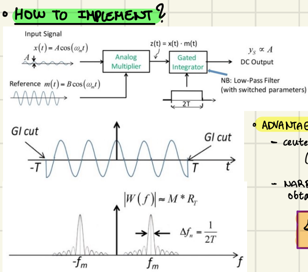
- CONTINOUS SINUSOIDAL WEIGHTING (instead of only peaks):



- choosing freq of cosinusoidal
  ↓
  choose f_m
- choose width of rect
  ↓
  choose BW

☺ SPLITTED

- HOW TO IMPLEMENT?



- ANALOG MULTIPLIER, multiplies the input signal x(t) by the reference signal m(t), shifting in freq. by $\pm f_m$
- GATED INTEGRATOR, which gates the output of the analog multiplier

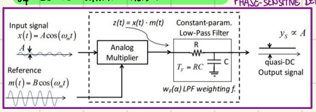
• ADVANTAGES with respect to TUNED FILTERS:
- center freq. f_m, width Δf_m independently set (from m(t) REF.) (from G.I.) $\Delta f_m = 1/2T$
- NARROW Δf_m and HIGH Q FACTOR easily obtainable at any f_m

$$\boxed{\Delta f_m \ll f_m \qquad Q = \frac{f_m}{\Delta f_m} \gg 1}$$

LOCK-IN AMPLIFIER

- with average performed by G.I., the amplitude A can be measured only at discrete time (spaced at least $2T$)
- by employing a CONSTANT PARAMETER LOW-PASS FILTER instead of GI, CONTINOUS MONITORING of SLOWLY VARYING $A(t)$ is obtained

PHASE-SENSITIVE DETECTOR



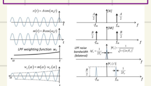
THE CONSTANT PARAMETER LPF performs a running average of the output $z(t)$ of the DEMODULATOR. OUTPUT is continuosly updated, and tracks the slowly varying amplitude $A(t)$ is obtained.

WEIGHTING FUNCTION OF LOCK-IN AMP.

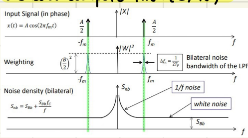
$$W_L(\alpha) = m(\alpha)\cdot w_F(\alpha)$$

$$|W_L(f)| \simeq |M| * |W_F|$$

$$\begin{cases} y(t) = \int_0^\infty z(\alpha)w_F(\alpha)d\alpha = \int_0^\infty x(\alpha)\,m(\alpha)w_F(\alpha)d\alpha \\ y(t) = \int_0^\infty x(\alpha)\,w_L(\alpha)\,d\alpha \end{cases}$$



- we can compute the (S/N):



OUTPUT SIGNAL:

- $y_s = 2\,\frac{A}{2}\cdot\frac{B}{2} = \frac{B}{2}A$
- OUTPUT NOISE: $\overline{n^2_{yL}} = 2\left(\frac{B}{2}\right)^2\cdot S_{Bb}\Delta f_m = \frac{B^2}{2}S_{Bb}\Delta f_m$
- $\left(\frac{S}{N}\right)_L = \frac{y_s}{\sqrt{\overline{n^2_{yL}}}} = \frac{A}{\sqrt{2S_{Bb}\Delta f_m}} = \frac{A}{\sqrt{2S_{Bu}\,f_{Fm}}}$

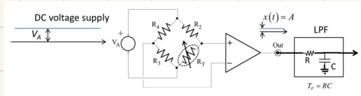
or in POWER TERMS $\left(\frac{S}{N}\right)^2_L = \frac{A^2/2}{S_{Bb}\Delta f_m}$

LOCK-IN AMPLIFIER POSSIBLE ISSUES

1) DC signal:

$$\begin{cases} y_C = A & \text{OUTPUT SIGNAL} \\ \overline{n^2_{yc}} = \hat{S}_{mu}\cdot f_{Fm} \end{cases}$$

MEAN DENSITY in LPF BAND

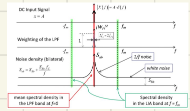


$$\left(\frac{S}{N}\right)_C = \frac{y_C}{\sqrt{\overline{n^2_{yc}}}} = \frac{A}{\sqrt{\hat{S}_{mu}\,f_{Fm}}}$$

seems an improvement of a factor $\sqrt{2}$

NO! $\hat{S}_{mu} \gg S_{Bu}$

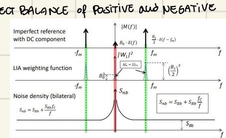
1/f noise is NOT CANCELED HERE



2) IMPERFECT MODULATION:

- DEVIATIONS FROM FREQUENCY f_m WITH AMPLITUDE B_1 CAN GENERATE SPURIOUS HARMONICS at kf_m with AMPLITUDE B_k ($\ll B_1$ for small deviations)
  - EACH HARMONIC COMPONENT OF $M(f)$ ADDS TO THE LIA WEIGHTING FUNCTION W_L A SPURIOS PASS-BAND AT FREQUENCY kf_m WITH AMPLITUDE B_k, and SHAPE given by LPF.
- a FAKE PASS-BAND at $f=0$ is DAMAGING even for $B_0 \ll B_1$ because of HIGH POWER SPECTRAL DENSITY of 1/f NOISE.

REMEMBER: unlucky deviations from PERFECT BALANCE of POSITIVE and NEGATIVE AREAS produces a DC component!



- the ratio $\hat{S}_{mu}/S_{Bu} > f_c/f_{Fm} \gg 1$ can match the AMPLITUDE RATIO $B_1^2/2B_0^2$

the noise in fake bandpass

$$\overline{n^2_{yL0}} = B_0^2\,\hat{S}_{mb}\Delta f_m = B_0^2\,\hat{S}_{mu}\cdot f_{Fm}$$

can equal or exceed that in correct band pass

$$\overline{n^2_{yL1}} = \frac{B_1^2}{2}S_{Bb}\Delta f_m = \frac{B_1^2}{2}S_{Bu}f_{Fm}$$

3) PHASE MISMATCH: between modulated signal and our reference

$$x(t) = A\cos(2\pi f_s t)$$
$$m(t) = B\cos(2\pi f_m t + \varphi_m)$$

$\Rightarrow y(t) = x(t)m(t) = \frac{AB}{2}\cos[\varphi_m] + \frac{AB}{2}\cos[2\pi(2f_m)t + \varphi_m]$

ATTENUATION OF SIGNAL AND (S/N) OF A FACTOR $\cos[\varphi_m]$

REAL LOCK-IN AMPLIFIERS

- in THEORY: a PHASE SENSITIVE DETECTOR is enough to achieve very NARROW BANDWIDTH and recover signals with much higher noise $S/N \ll 1$.
- in PRACTICE: non-idealities of real PSD pose more limits, but them can be overcome with MODIFICATIONS and STAGES to LIA STRUCTURE

1. THE SIGNAL provided at LIA OUTPUT must cover a 10V range of modern ADC. Therefore HIGH GAIN must be provided by LIA.
2. an HIGH GAIN amplifier AFTER the PSD is called POST-AMPLIFIER, which must be:
   1) DC-coupled amplifier with adequate BW limit
   2) receive a low noise signal
   3) has drift of baseline and low-freq. noise (they occur not filtered after the PSD)
3. if demodulated signal is very low (comparable with baseline drift) a PRE-AMPLIFIER is required, AC coupled, either wide-band or narrow band.

   POST-AMP NOISE ref. to INPUT

   It receive an HIGH NOISE signal (it operates before PSD)
   May have BASELINE DRIFT and LOW FREQ. NOISE, but their role is minimized thx to PSD
4. SIGNAL and NOISE must stay LINEAR to avoid damaging effects (self-modulation, spurious harmonics) in every stage!
5. with low signals ($S/N \ll 1$), PRE-AMP HIGH GAIN required for signals, acts on NOISE, making it out of LINEAR RANGE. PRE-FILTERING with WIDE-BAND is required before PRE-AMP. (TUNED PRE-AMP, SELECTIVE PRE-AMP)

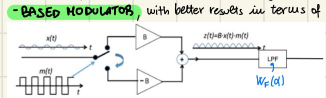
few more notes:

A) MULTIPLIER LINEAR RANGE independent of PRE-AMP GAIN.
B) PRE-AMP OUTPUT LINEAR RANGE independent of " " .
C) Maximum acceptable input signal limited by " " noise rms
D) both PRE and POST AMP GAINS determine INPUT FSR SIGNAL.

	SINUSOIDAL Reference	SQUAREWAVE Reference
SINUSOIDAL Signal amplitude A power $P = \frac{A^2}{2}$	$\frac{S}{N} = \frac{A}{\sqrt{2}\sqrt{S_{Bb}\Delta f_n}} = \frac{\sqrt{P}}{\sqrt{S_{Bb}\Delta f_n}}$	$\frac{S}{N} = \frac{A}{\frac{\pi}{2}\sqrt{S_{Bb}\Delta f_n}} = \frac{\sqrt{P}}{\frac{\pi}{2\sqrt{2}}\sqrt{S_{Bb}\Delta f_n}}$
A_{min} minimum measurable amplitude (at S/N=1)	$A_{min} = \sqrt{2}\sqrt{S_{Bb}\Delta f_n} = 1{,}41\sqrt{S_{Bb}\Delta f_n}$	$A_{min} = \frac{\pi}{2}\sqrt{S_{Bb}\Delta f_n} = 1{,}57\sqrt{S_{Bb}\Delta f_n}$
SQUAREWAVE Signal amplitude A power $P = A^2$	$\frac{S}{N} = \frac{A}{\frac{\pi}{2\sqrt{2}}\sqrt{S_{Bb}\Delta f_n}} = \frac{\sqrt{P}}{\frac{\pi}{2\sqrt{2}}\sqrt{S_{Bb}\Delta f_n}}$	$\frac{S}{N} = \frac{A}{\sqrt{S_{Bb}\Delta f_n}} = \frac{\sqrt{P}}{\sqrt{S_{Bb}\Delta f_n}}$
A_{min} minimum measurable amplitude (at S/N=1)	$A_{min} = \frac{\pi}{2\sqrt{2}}\sqrt{S_{Bb}\Delta f_n} = 1{,}11\sqrt{S_{Bb}\Delta f_n}$	$A_{min} = \sqrt{S_{Bb}\Delta f_n}$

LIA WITH SQUARE WAVE REF.

- ANALOG (COMPLEX) MULTIPLIERS are given up for SQUARE-WAVE, SWITCH-BASED MODULATOR, with better results in terms of semplicity, linearity, input-ref. noise.



$W_L(f)$ is a set of $W_F(f)$ REPLICAS centered on each line of $M(f)$, multiplied by LINE-WEIGHT $|W_L(f)| \simeq |M(f)| * |W_F(f)|$ and PHASE-SHIFTED.

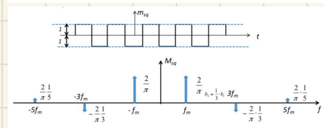
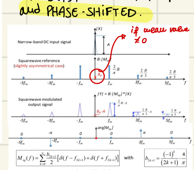
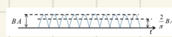


Figure 18.3: Square wave signal with zero mean value and module of the square wave spectrum.

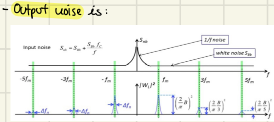


- Output signal is:

$$S_y = \int_{-\infty}^{+\infty} X(f)W_L(-f)df = \frac{1}{\pi}\int_0^\pi AB\sin(t)dt = \dots = \frac{2}{\pi}A\cdot B$$



- Output noise is:



$$\overline{n^2_{yL}} = \int_{-\infty}^{+\infty} S_{nb}(f)|W_L(f)|^2 df = 2S_{Bb}\Delta f_m\left(\frac{2}{\pi}B\right)^2\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right] = \dots = B^2 S_{Bb}\Delta f_m$$

and we obtain:

$$\left(\frac{S}{N}\right)_{sqw} = \frac{S_y}{\sqrt{\overline{n^2_{yL}}}} = \frac{A}{\frac{\pi}{2}\sqrt{S_{Bb}\Delta f_m}}$$

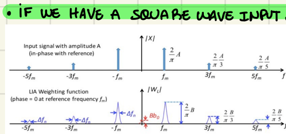
SQUARE-WAVE-MOD S/N

if we compare with ANALOG SINUSOIDAL REF.

$$\left(\frac{S}{N}\right)_{L,sin} = \frac{A}{\sqrt{2S_{Bb}\Delta f_m}} = 1{,}11\left(\frac{S}{N}\right)_{L,sqw}$$

we give up a bit of S/N (due to add freqs harmonics) to delete ANALOG!

- IF WE HAVE A SQUARE WAVE INPUT SIGNAL: let's first suppose perfectly IN-PHASE

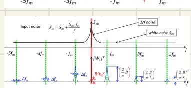


- comparing with sinusoidal input with same amplitude, we have double power to exploit!

$\Downarrow$

HIGHER S/N!!

$$S_y = \int_{-\infty}^{+\infty} X(f)W_L(-f)df = 2\left[\frac{2}{\pi}A\frac{2}{\pi}B\right]\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right] = 2\left[\frac{4}{\pi^2}AB\right]\frac{\pi^2}{8} = AB$$



- IF MOD. is not perfectly symmetrical additional DC noise

$$\overline{n^2_{yL0}} = \hat{S}_{nb}\Delta f_m B^2 b_0^2$$

$$\left(\frac{S}{N}\right)_{sqw} = \frac{S_y}{\sqrt{\overline{n^2_{yL}}}} = \frac{A}{\sqrt{S_{Bb}\Delta f_m}}$$

SQUARE-WAVE MOD and INPUT

with $\hat{S}_{nb} \gg S_{Bb} \Rightarrow$ even small spurious band $b_0 \ll 1$, added noise can be larger $\frac{\overline{n^2_{yL0}}}{\overline{n^2_{yL}}} = \frac{\hat{S}_{nb}}{S_{Bb}}b_0^2$

PHOTODETECTORS

def **LIGHT**: electromagnetic waves with frequency ν and wavelength λ.
PROPAGATION SPEED (in vacuum): $c = 2{,}998 \cdot 10^8$ m/s $\cong 30$ cm/ns

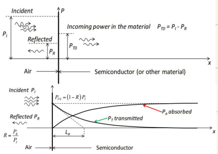
$\Longrightarrow$
- $\lambda < 400$ nm — UV - ultraviolet
- $400\,nm < \lambda < 750\,nm$ — VISIBLE
- $750\,nm < \lambda < 3\,\mu m$ — NEAR INFRA-RED (NIR)
- $3\,\mu m < \lambda < 30\,\mu m$ — MID - INFRARED (MIR)
- $30\,\mu m < \lambda$ — FAR - INFRARED (FIR)

def **PHOTON**: quantum of E-M energy

$$\boxed{E_{ph} = h\nu} \quad h = 7{,}6 \cdot 10^{-34}\ J \cdot s$$
$= qV_p$ in eV

$$\boxed{V_p = \frac{hc}{q} \cdot \frac{1}{\lambda} = \frac{1{,}24}{\lambda}}$$ λ in μm

REFLECTION and ABSORPTION of PHOTON



$$\boxed{n = \frac{c_0}{v} = \frac{1/\sqrt{\varepsilon_0\mu_0}}{1/\sqrt{\varepsilon_0\varepsilon_R\mu_0}} = \sqrt{\varepsilon_R}}$$
REFRACTIVE INDEX

$$\boxed{R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 = \frac{P_{REFL}}{P_{INC}}}$$
REFLECTION COEFF.

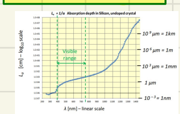
ANTI-REFLECTION COATING is often used to maximize ABSORBED POWER

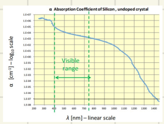
ABSORPTION OF PHOTONS described by: P_T is power transmitted

$$-dP_T = \alpha P_T\, dx = P_T\, dx/L_a$$

α: OPTICAL ABSORPTION COEFF
$L_a = 1/\alpha$. " " DEPTH
} strongly depends on WAVELENGTH

$$\boxed{P_a = P_{T0}\left(1 - e^{-x/L_a}\right)}$$ ABSORBED POWER

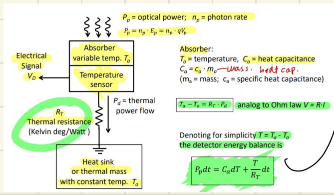




THERMAL PHOTODETECTORS

PRINCIPLE: employ their energy simply for HEATING A TARGET and MEASURE its ΔT↑ (TEMPERATURE)

☺ WIDE SPECTRAL RANGE
☹ POOR SENSITIVITY ~10^{15} blue photons to have a $\Delta T = 0.1K$
☹ SLOW DYNAMIC RESPONSE — THERMAL PROCESSES ARE SLOW.



$$\frac{dT}{dt} = \frac{P_p}{C_a} - \frac{T}{R_T C_a} \Rightarrow \boxed{T = P_p R_T \frac{1}{1 + sR_T C_a}}$$

OPTICAL POWER ⟶ MEASURED TEMP.

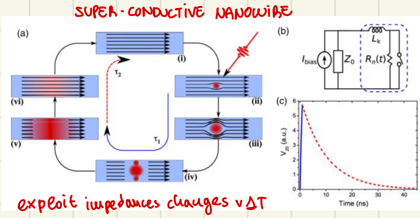
$C_a = c_a \cdot m_a$
STEADY STATE TEMP. for STEADY P_p

$$\boxed{S_D = \frac{\text{electrical output [voltage]}}{\text{optical power on detector's area [in W]}}}$$

RADIANT SENSITIVITY — independent of λ

Variations of REFLECTION/ABSORPTION vs λ are kept at moderate level with modern absorbers. Quite uniform S_D achieved.

SUPER-CONDUCTIVE NANOWIRE



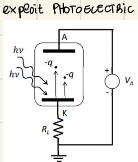
exploit impedances changes $\nu\Delta T$

QUANTUM PHOTODETECTORS

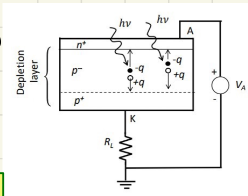
PRINCIPLE: exploit photo-electric effects for producing directly an electrical current in the detector.

energy absorbed generates FREE-CARRIERS, and then the CURRENT

VACUUM TUBE DETECTORS:
exploit PHOTOELECTRIC EFFECT



SEMICONDUCTOR DETECTOR:
based on PN Junctions (VALENCE BAND → CONDUCTIVE BAND)
electron generated in depletion region move
(e^- → n terminal; p^+ → p terminal)



QUANTUM PHOTODETECTORS TRANSDUCE OPTICAL SIGNALS in ELECTRIC CURRENT SIGNALS by COLLECTING FREE ELECTRONS generated by PHOTONS of RADIATION:

$$\boxed{\eta_D = \frac{N° \text{ of photogen. } e^- \text{ (or } e^-/p^+ \text{ pairs)}}{N° \text{ of photons reaching detector}} = \frac{N_e}{N_p}}$$
QUANTUM DETECTION EFFICIENCY

PHOTONS OF λ WITH STEADY RATE n_p gives
$$\boxed{P_L = n_p h\nu}$$ OPTICAL POWER
and ELECTRONS GENERATED with rate n_e
$$\boxed{I_D = n_e q}$$ CURRENT

↳ OPTICAL POWER ⟶ ELECTRIC CURRENT TRANSDUCTION FACTOR still used

$$\boxed{S_D = \frac{\text{eletr. output current [A]}}{\text{optical power in detector area [W]}} = \frac{I_D}{P_L} \left[\frac{A}{W}\right]}$$
RADIANT SENSITIVITY

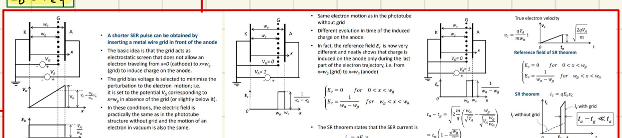
↳ we obtain:
$$\boxed{S_D = \frac{I_D}{P_L} = \frac{n_e}{n_p} \cdot \frac{q}{h\nu} = \frac{n_e}{n_p} \cdot \frac{\lambda}{\frac{hc}{q}} = \eta_D \frac{\lambda}{\frac{hc}{q}} = \eta_D \frac{\lambda\,[\mu m]}{1.24}}$$

PHOTON NOISE

OPTICAL RADIATION COMPOSED OF PHOTONS ARRIVING RANDOMLY IN TIME.
NUMBER OF PHOTONS N_p is a statistical variable:
mean $\overline{N_p}$; variance $\sigma_p^2 = \overline{N_p^2} - (\overline{N_p})^2$

RANDOM FLUCTUATIONS OF PHOTONS
POISSON STATISTICS

$$\boxed{\sigma_p^2 = \overline{N_p}} \qquad \boxed{S_p = 2h\nu P_p = \frac{2hc}{\lambda} P_p}$$
UNILATERAL SPECTRAL DENSITY

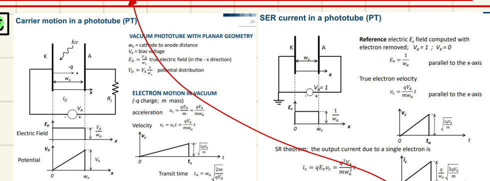


PHOTODETECTOR CURRENT

- RESPONSE to A MULTI-PHOTON OPTICAL SIGNAL is the linear superposition of SER, SINGLE ELECTRON RESPONSE (one photon generates one electron).
- SER is NOT δ-like current pulse, occurring when photogenerated charge carrier impacts on collector electrode.
  ⟹ CARRIER induce a CHARGE in COLLECTOR BEFORE REACHING IT. INDUCED charge varies with carrier position.
  ⟹ WAVEFORM OF CURRENT SIGNAL ⟶ **SHOCKLEY-RAMO THEOREM**

  1. COMPUTE MOTION OF ELECTRON (TRAJECTORY and VELOCITY v_c) at every point
  2. COMPUTE REFERENCE ELECTRIC FIELD, that would exist IF: E_v
     * NO ELECTRONS
     * OUTPUT AT 1V
     * ALL OTHER CONDUCTORS AT GND
  3. $$\boxed{i_c = q\, \vec{E_v} \cdot \vec{v_c} = q\, E_{v_c} \cdot v_c}$$ scalar product

PHOTOTUBE



In a phototube with planar geometry the **single electron response (SER)** is a pulse with triangular waveform

$$i_c = qE_v v_c = \frac{q^2 V_A}{m w_a^2} t \qquad (0 \le t \le t_a)$$

The frequency response is the Fourier transform of the SER pulse, which has a high frequency cutoff inversely proportional to the pulse width.
The pulse width is set by the transit time t_a of the electron from cathode to anode

$$t_a = \sqrt{2\frac{m}{q}} \frac{w_a}{\sqrt{V_A}} = 3{,}37 \cdot 10^{-6} \frac{w_a}{\sqrt{V_A}}$$

Typical values for phototubes are around $w = 1cm = 0{,}01m$ and $V_A = 100V$, which correspond to transit time around $t_a = 3{,}3$ ns

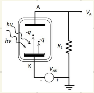
if we want a lower t_a (HIGHER FREQ.)
w_a↓ CAPACITANCE↑
V_A↑ POWER DISSIPATION↑
another idea: SCREENED ANODE

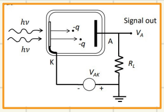
PHOTOTUBES

- Wide sensitive area (6 cm²)
- Fast response
- Noise limited by electronics

Two **structures**:

1) Side-window tube
2) End-window tube



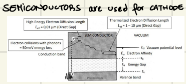


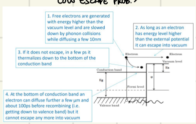
~ more efficient photons collection.

Photoemission

① Photon absorption. Free e^- generated. → ② Electron diffusion in photocathode layer → ③ Escape of e^- into vacuum

Semiconductors are used for cathode. Metal → high reflect. small diffusion, low escape prob.



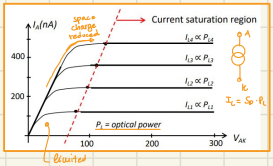


electrons generated in deep layers are not emitted. Escape prob. is high only for e^- generated in thin surface layer, about diffusion length

L_a optical penetration length must be not too high with respect to L_{eh} diffusion length of high energy e^-

$L_a \approx L_{eh}$ for good quantum detection efficiency η_D

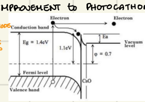
I-V characteristic:



limited by electron space charge effect

An improvement to photocathodes is:

Photo cathode with negative electron affinity (NEA)



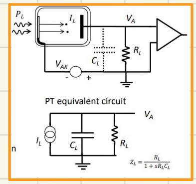
- deposition of few atomic layers of Cs-O activated thin layer of Cs⁺ ions. Electric field generated curves down energy band.
  → vacuum potential lower than the bottom of conduction band
- Higher η_D and broader spectral range
  electrons can escape in vacuum also when thermalized at bottom of conduction band.

Dynamic response:

limited by:

① Light-current trasduction $h_D(t)$ time-width T_D

② Load circuit (LPF) δ-response $h_L(t)$ with time width T_L



⇒ **Overall response:**

$$h_P(t) = h_D(t) * h_L(t)$$

$$T_P = \sqrt{T_D^2 + T_L^2} = \sqrt{T_D^2 + R_L^2 C_L^2}$$

⇒ **Exploit fast intrinsic response $h_D(t)$:** $T_L \leq T_D$ ($R_L C_L$)

⇒ **Parasitic capacitance C_L** is made of:

C_A: input capacitance of the amplifier
C_S: parasitic cap. of connections
C_D: electrode capacitance (depends on A_D area of photocathode)

$$C_D = \varepsilon_0 \frac{A_D}{w_a}$$

$\lesssim 20\,pF$

Dark current

described by j_B dark current density (per cathode area). Depends on cathode type and temperature.

$$\Rightarrow \boxed{I_B = j_B A_D} \text{ dark current}$$

we derive shot-noise.

$$\boxed{\sqrt{S_B} = \sqrt{2q \cdot I_B}} \text{ shot noise}$$

<< negl. to $\sqrt{S_i}$ of current noise of ampl.

NEP and detectivity

NEP = Noise equivalent power is a figure of merit that takes into account η_D and S_B (dark current noise).

↳ defined with reference to a situation where the limit to minimum measurable signal is set by internal noise of detector and not by circuit noise. That's not the case of phototubes but photomultipliers tubes.

NEP is the input optical power corresponding to minimum measurable signal.

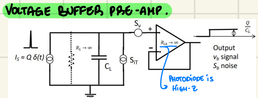
$$NEP = P_{P,min} = \frac{I_{P,min}}{S_D} = \frac{\sqrt{\overline{i_n^2}}}{S_D} = \frac{\sqrt{2qj_B}\sqrt{A_D}\sqrt{\Delta f}}{S_D}$$

it depends on area. If we normalize to area a bandwidth:

$$D^* = \frac{\sqrt{A_D}\sqrt{\Delta f}}{NEP}$$ detectivity

For electronics, we can have:

A) **Voltage buffer pre-amp.**



Photodiode is High-Z

C_L: load resistance $= C_D + C_{iA} + C_S$
R_L: total load resistance
S_V: ampl. voltage noise
S_{iT}: total current noise $= S_{iD} + S_{iA} + S_{iR}$ (detector, amplifier, load resistor)

Spectrum

$$S_b = S_V + S_{iT}\frac{1}{\omega^2 C_L^2}$$

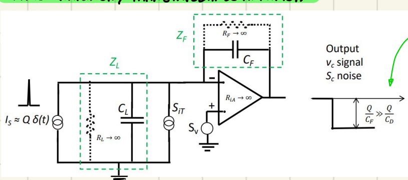
Output:

$$\boxed{v_b(t) = \frac{Q}{C_L}\mathbb{1}(t)}$$

☹ not well controllable

☹ if high input rate, decrease of photodetector bias voltage

B) **Charge amplifier / transimpedance amplifier:**



C_F: feedback capacitor chosen small ($\ll C_L$)
R_F: feedback resistor ($\to \infty$)
} ⇒ Z_F

C_L: total load resistance ($\to \infty$)
R_L: total load resistance
} ⇒ Z_L

S_V: amplifier voltage noise
S_{iT}: total current noise $= S_{iD} + S_{iA} + S_{iR}$ (detector, ampl., resistor load)

Output:

$$V_c = -QZ_F = -\frac{Q}{sC_F} \qquad v_c(t) = -\frac{Q}{C_F}\mathbb{1}(t)$$

$$\boxed{|v_c| = \frac{Q}{C_F} = \frac{C_L}{C_F}\cdot\frac{Q}{C_L} = \frac{C_L}{C_F}|v_b| = G_c\,|v_b|}$$

☺
- gain factor better than buffer: $G_c = C_L / C_F \gg 1$
  noise of next stages is less relevant
- much more stable
- detector terminal connected to virtual ground. (constant bias voltage)

Noise:

$$\boxed{S_c = S_V\left|1+\frac{Z_F}{Z_L}\right|^2 + S_{iT}|Z_F|^2 = \left(\frac{C_L}{C_F}\right)^2\left[S_V\left(1+\frac{C_F}{C_L}\right)^2 + S_{iT}\frac{1}{\omega^2 C_L^2}\right] \underset{\frac{C_F}{C_L}\ll 1}{\simeq} \left(\frac{C_L}{C_F}\right)^2\left[S_V + S_{iT}\frac{1}{\omega^2 C_L^2}\right] = \left(\frac{C_L}{C_F}\right)^2 S_b}$$

$Z_L \approx 1/sC_L \quad Z_F \approx 1/sC_F$

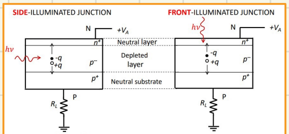
- same S/N of buffer

SEMICONDUCTORS PHOTODIODES

TWO TYPES OF DEVICES:

1. SIDE-ILLUMINATED JUNCTION
2. FRONT-ILLUMINATED JUNCTION

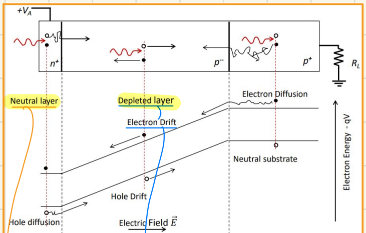
SIDE-ILLUMINATED JUNCTION — FRONT-ILLUMINATED JUNCTION



Employed for specific purposes, e.g. microsystems with integrated waveguides for on-chip optical connections

Most widely employed; the active area (illuminated area) can be designed with flexibility and can attain wide size

CARRIER MOTION



carrier generated in neutral region is surrounded by huge population of free carrier

- if moves other carriers automatically re-arrange to screen any circuit effect

NO CURRENT AS LONG AS IT DIFFUSES IN A NEUTRAL REGION.
IF per DIFFUSION REACHES EDGE OF DEPLETION LAYER (BEFORE RECOMBINE) THEN IT DRIFTS (YES CURRENT)

induce opposite charge in in the conductive electrodes (value depending on distance)

$\Rightarrow$ e^- MOVE. CHARGE MOVE. CURRENT INDUCED.

CIRCUIT MODEL

I-V of illuminated p-n junction

$$I_j = I_0 \exp\frac{qV_j}{kT} - I_0 - I_p$$

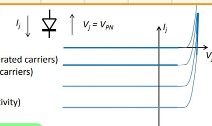
I_j $V_j = V_{PN}$

I_0 reverse current (thermally generated carriers)

I_p photocurrent (photogenerated carriers)

$$I_p = S_D \cdot P_L$$

(P_L optical power; S_d radiant sensitivity)



Photodetector (reverse-biased junction)

I_D detector current = - I_j

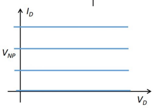
V_D detector voltage = - V_j

$$I_D = I_p + I_0 - I_0 \exp\left(-\frac{qV_D}{kT}\right)$$

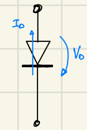
I_D $V_D = V_{NP}$

with $V_D \gg kT/q$ $\quad \boxed{I_D = I_p + I_0 \approx I_p}$

Detector photocurrent $\propto P_L$ — Detector dark current



PHOTODETECTOR

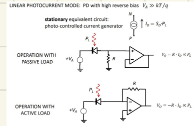


SOME CONFIGS:

Semiconductor photodiodes can be operated also without a bias voltage source. As outlined below, the short-circuit current is measured in the photoconductive mode and the open-circuit voltage in the photovoltaic mode. These configurations have modest sensitivity and slow response (see later), but their simplicity is attractive in some practical cases, e.g. for monitoring a steady light over a wide dynamic range.

LINEAR PHOTOCURRENT MODE: PD with high reverse bias $V_A \gg kT/q$

stationary equivalent circuit: photo-controlled current generator $I_D = S_D \cdot P_L$



PHOTOCONDUCTIVE MODE
PD in short-circuit $V_A = 0$
Linear output scale

$$I_D = I_p + I_0 \approx I_p$$

$$V_O = -R \cdot I_D$$

PHOTOVOLTAIC MODE
PD in open-circuit $I_D = 0$
Logarithmic output scale

$$V_j = \frac{kT}{q}\ln\left(1 + \frac{I_p}{I_0}\right) \approx \frac{kT}{q}\ln\left(\frac{I_p}{I_0}\right)$$

$$V_O = V_j$$

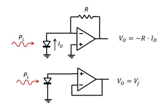
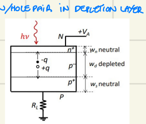


PHOTO GENERATION OF FREE CARRIERS and PHOTON DETECTION EFFICIENCY

We define PROBABILITY OF A PHOTON TO GENERATE A FREE ELECTRON/HOLE PAIR IN DEPLETION LAYER P_d.

$$\boxed{P_d = \underbrace{(1-R)}_{\text{NOT being REFLECTED at surface}} \cdot \underbrace{e^{-\alpha w_n}}_{\text{NOT BEING ABSORBED IN THE TOP NEUTRAL LAYER}} \cdot \underbrace{(1-e^{-\alpha w_d})}_{\text{BEING ABSORBED IN THE DEPLETION LAYER}}}$$



Often, we can neglect the contribution given from CARRIERS generated in neutral region, that reach depletion by diffusion:

$$\boxed{\eta_D = P_d}$$

QUANTUM DETECTION EFFICIENCY

SOURCES OF η_D LOSSES:

- SURFACE REFLECTION
- ABSORPTION IN NEUTRAL INPUT LAYER
- INCOMPLETE ABSORPTION BY DEPLETION REGION

L_a related to $\alpha(\lambda)$!

DARK CURRENT, DETECTOR NOISE, SENSITIVE AREA

Dark current I_B, due to THERMAL GENERATION OF FREE CARRIERS and TUNNELING IN HIGH-$\vec{F}$ STRUCTURES. Same shot noise of photodiode:

$$\sqrt{S_B} = \sqrt{2qI_B} \quad \text{(unilateral)} \quad \ll \quad \text{CIRCUIT NOISE (like vacuum tubes)}$$

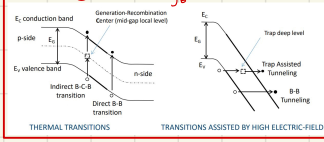
HOWEVER, ☹ MUCH HIGHER j_B $\longrightarrow$ AREA LIMITS for LOW NOISE

THERMAL GENERATION RATE has volume density:

$$\boxed{n_G = \frac{n_i}{2\tau}}$$

VOLUME DENSITY OF THERMALLY GENERATED CARRIERS

τ: MINORITY CARRIERS LIFETIME



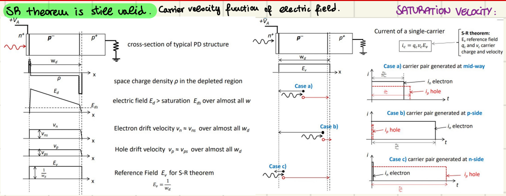
$\Rightarrow n_B = n_G \cdot A \cdot w$

a n_{BMAX} gives the A_{MAX} limit.

CARRIER MOTION and CURRENT

SR theorem is still valid. Carrier velocity function of electric field.

SATURATION VELOCITY: $v_{ns} \simeq 10^7$ cm/s, $v_{ps} \simeq 8 \cdot 10^6$ cm/s



- DURATION OF CURRENT PULSE = TRANSIT TIME T_t OF CARRIER in DEPLETED REGION (depends on POSITION of generation)

$$T_t = W_d / v_{sn} \longrightarrow \text{BAND LIMIT. } 1/2T_t$$

- FOR FAST MULTIPLE-PHOTON PULSE, CONVOLUTION of STANDARD SINGLE-CARRIER RESPONSE is a GOOD APPROX.
- TRADE-OFF: long w_d for $\eta_D \uparrow$ at long λ; short w_d for fast time-response

PHOTODIODE EQUIVALENT CIRCUIT

DYNAMIC RESPONSE limited by:

1. LIGHT-to-CURRENT TRANSDUCTION $h_D(t)$, width T_t, rectangular pulse
2. LOAD CIRCUIT with δ-response $h_L(t) = \frac{1}{C_L}\exp(-t/R_L C_L)$

$\Downarrow$

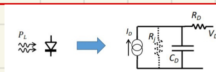
$$h_P(t) = h_D(t) * h_L(t)$$

$$T_P = \sqrt{T_t^2 + T_L^2}$$

to exploit fast response

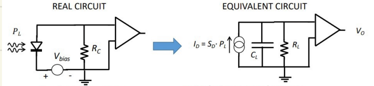
$$T_L \leq T_t$$

$$\begin{cases} C_L \simeq C_D = \varepsilon_{Si}\dfrac{A}{W_d} \\ T_t = \dfrac{W_d}{v_{sn}} \end{cases} \longrightarrow A \leq \frac{W_d^2}{v_{sn}} \cdot \frac{1}{R_L \varepsilon_{Si}}$$



- $I_D = S_D \cdot P_L$ photo-controlled generator (S_D radiant sensitivity or responsivity)
- C_D diode capacitance (p-n junction)
- R_D diode **series** resistance (of the input layer and substrate)
- R_j parallel resistance of the reverse biased junction is considered → ∞

REAL CIRCUIT — EQUIVALENT CIRCUIT

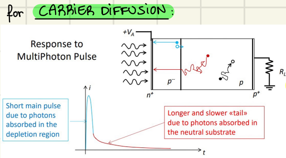


R_C circuit input resistance

R_L total Resistance Load = $R_C + R_D$
C_L total Capacitive Load = $C_D + C_S$ (stray) + C_A (amplifier)

the load circuit is a low-pass filter with time constant $R_L C_L$ in the transfer from detector current I_D to output voltage V_O

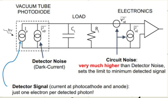
for CARRIER DIFFUSION:



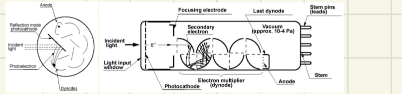
The shape and relative size of the «diffusion tail» are established by the photogeneration and by the diffusion dynamic of minority carriers in neutral regions. They strongly depend on the PD device geometry, on the material properties in the neutral regions (diffusion coefficient and minority carrier lifetime) and on the space distribution of the absorbed photons, hence on the photon wavelength.

The «diffusion tail»:

- increases the photon detection efficiency, by bringing to the output a contribution from photons absorbed in a neutral region
- downgrades the detector dynamic response, since the diffusion tail is definitely longer than the prompt pulse
- The time span of the tail increases with the thickness w_s of the neutral substrate and with the minority carrier lifetime, which is longer at lower doping level.

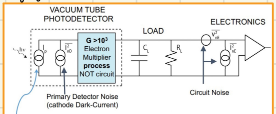


PHOTOMULTIPLIERS TUBES PHT



in previous configs. the (S/N) was limited by electronic noise, much higher than detector one.

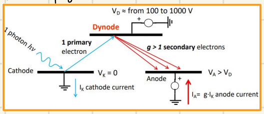
we want the opposite. We can add a photon multiplier BEFORE THE CIRCUIT, to make electronic noise negligibee.



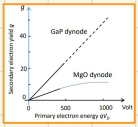
- Primary Signal (photocathode current): one electron per detected photon
- Output (anode) current: **G >10³** electrons per primary electron
- Dark-current noise and/or photocurrent noise at detector output are much higher than circuit noise, which has practically negligible effect

SECOND ELECTRON EMISSION and DYNODES:

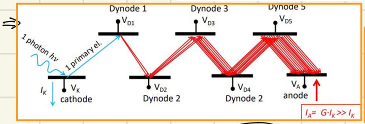
- it's a physical and statistical multiplication.



with HIGH $\vec{E}$ (δoov!!) between cathode and DYNODE, we accelerate the e^- so much that, when it impact, multiple e^- are emitted



- $g \approx 5$ to 25
- saturation (extra energy go to e^- in deep energy levels with low probability to escape)

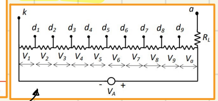


$V_K < V_{D1} < V_{D2} < \ldots < V_A$

$G = g_1 \cdot g_2 \cdot g_3 \ldots = g^k$

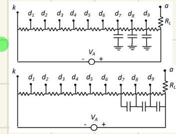
$I_A = G \, I_K \gg I_K$

VOLTAGE DIVIDER.



$$G = k_G \cdot V_A^n$$

$$k_G = k_S^n \, f_1 f_2 f_3 \ldots$$

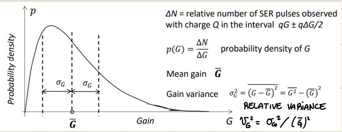


G is very sensitive to BIAS VOLTAGE V_A variations:

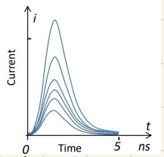
$$\frac{dG}{G} = n \frac{dV_A}{V_A} \Rightarrow \text{use CAPACITORS to have a LPF effect}$$

STATISTICAL MODEL OF PHT

- PHT output is superposition of SINGLE ELECTRON RESPONSE (SER)
- Pulses higher than noise, and each of them corresponds to a single photon. The amplitude is randomly varying
- STATISTICAL VARIATIONS OF G.



RELATIVE VARIANCE $v_G^2 = \sigma_G^2 / (\bar{G})^2$



$\sigma_P^2 = N_P$ N_P = primary e^- mean number

$v_P = \sigma_P^2 / N_P^2 = 1/N_P$

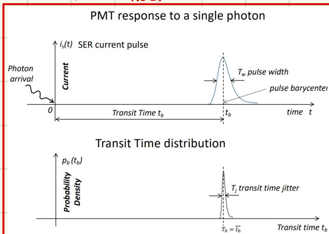
- we can derive an EXCESS NOISE due to GAIN FLUCTUATIONS:

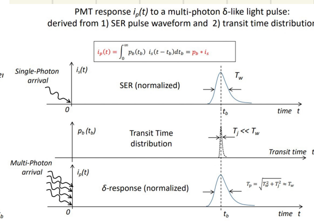
$$v_u^2 = \frac{\sigma_u^2}{N_u^2} = v_P^2 + \frac{v_G}{N_P} = \frac{1}{N_P}\left(1 + v_G^2\right) \quad \text{OUTPUT OF CASCADE}$$

$$\boxed{\sigma_u^2 = N_P^2 \cdot \bar{G}^2 \, v_u^2 = N_P \bar{G}^2 \left(1 + v_G^2\right) = \sigma_P^2 \, \bar{G}^2 \, \underbrace{\left(1 + v_G^2\right)}_{F}} \quad \text{INPUT VARIANCE}$$

$$\boxed{F = 1 + v_G^2 > 1} \quad \text{EXCESS FACTOR}$$

PHT DYNAMIC RESPONSE



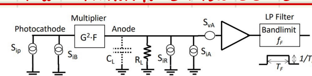


- Pulse width is:

$$T_p = \sqrt{T_w^2 + T_j^2} \underset{T_j/T_w \text{ small}}{\approx} T_w \left[1 + \frac{1}{2}\left(\frac{T_j}{T_w}\right)^2\right] \approx T_w \qquad f_p = \frac{1}{k_a \cdot T_w}$$

k_a: depends on SER pulse waveform (3 to 10)

SNR and MINIMUM MEASURABLE SIGNAL



Noise sources :

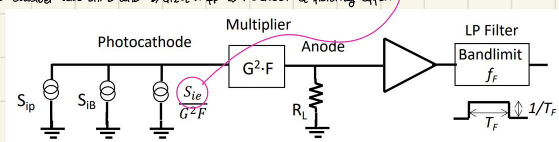
- at cathode: $S_{ip}=2qI_p=2q^2n_p$ photocurrent noise, **increases with the signal**
- at cathode: $S_{iB}=2qI_B=2q^2n_B$ background noise, **independent from the signal**
- at anode: resistor load noise S_{iR} and preamplifier noise S_{iA} and S_{iR}

CIRCUIT NOISE:

$$S_{ie} = S_{iA} + S_{iR} + \frac{S_{vA}}{R_L^2}$$

- n_p photoelectron rate → $I_p = n_p q$ photocurrent
- n_D dark electron rate → $I_D = n_D q$ cathode dark current
- n_b electron rate due to photon background → $I_b = n_b q$ photon background current
- $n_B = n_D + n_b$ total background electron rate → $I_B = n_B q$ total background current

if we consider WIDE-BAND LOAD $1/4R_LC_L \gg f_F \Rightarrow$ NEGLECT C_L filtering effect:



S_{ie} can be modeled as SHOT CURRENT at ANODE. $I_{ie} = S_{ie}/2q$ with electron rate $n_e^{(A)} = I_{ie}/q$

- We obtain a constant NOISE COMPONENT (NOT DEPENDENT on SIGNAL):

→ ref CATHODE $n_e^{(K)} = n_e^{(A)} / FG^2$

$$S_{iB} + \frac{S_{ie}}{G^2F} = 2qI_B + \frac{2qI_e}{G^2F} = 2q\left(n_B + \frac{n_e}{G^2F}\right) \quad \text{tipically negl.}$$

- By neglecting circuit noise:

$$\left(\frac{S}{N}\right) = \frac{I_P}{\sqrt{S_{ip} f_F + S_{iB} f_F}} \cdot \frac{1}{\sqrt{F}} = \frac{I_P}{\sqrt{2qI_P f_F + 2qI_B f_F}} \cdot \frac{1}{\sqrt{F}}$$

SIGNAL-TO-NOISE RATIO

$$\frac{S}{N} = \frac{I_p}{\sqrt{2qI_pf_F + 2qI_Bf_F}} = \frac{I_pT_F}{\sqrt{qI_pT_F + qI_BT_F}} = \frac{n_pT_F}{\sqrt{n_pT_F + n_BT_F}} = \frac{N_p}{\sqrt{N_p + N_B}}$$

if negl. BACKGROUND NOISE.

NOISE BW: $f_F = 1/2T_F \Rightarrow$

$$\left(S/N\right) = \frac{I_P}{\sqrt{2qI_P f_F}} = \ldots = \sqrt{N_P} \quad \text{NUMBER of PHOTOELECTRONS}$$

for $(S/N) = 1$ we obtain the minimum observabee number of photoelectrons:

$$\left(\frac{S}{N}\right)_{min} = 1 = \sqrt{\frac{I_{p,\min -p} T_F}{q}} = \sqrt{n_{p,\min -p} \; T_F} \sqrt{N_{p,\min -p}}$$

The minimum measurable photocurrent signal $I_{p,min-p}$ corresponds to just **one photoelectron in T_F**, the filter weighting time:

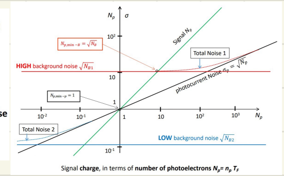
$$I_{p,\min -p} = \frac{q}{T_F} \qquad n_{p,\min -p} = \frac{1}{T_F} \qquad N_{p,\min -p} = 1$$

- The **opposite extreme case is with negligible photocurrent noise:** only background noise matters. More precisely, it's the case where the limit current $I_p = I_{p,min-p}$ computed with only the photocurrent noise is much lower than the background current I_B

$$\boxed{I_B \gg \frac{q}{T_F}} \qquad \boxed{n_B \gg \frac{1}{T_F}} \qquad \boxed{N_B \gg 1}$$

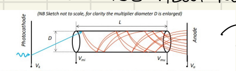
- There is now a different **minimum signal $I_{p,min-B}$ limited by the background noise**

$$\boxed{I_{p,\min -B} = \sqrt{\frac{qI_B}{T_F}}} \qquad \boxed{n_{p,\min -B} = \sqrt{\frac{n_B}{T_F}}} \qquad \boxed{N_{p,\min -B} = \sqrt{N_B}}$$



Signal **charge**, in terms of **number of photoelectrons** $N_p = n_p T_F$

SOME NOTES ABOUT NEW TECHNOLOGIES:



In order to get PMTs more simple, compact, robust and less sensitive to mechanical vibrations, minitubular electron multipliers were introduced (in the late years 60's)

- A special glass **capillary tube with D < 1mm**, called Continuous Channel Multiplier CCM or Channeltron, is at once **voltage divider and electron multiplier**; the inner surface is chemically treated and converted in a semiconductor layer with high resistivity and secondary electron emission yield g ≈ from 1,2 to 3.
- For a given applied voltage the **gain depends on the ratio *L/D***. As *L/D* increases the number of impacts increases, but the yield decreases because the impacting electron energy decreases. Maximum gain is attained with *L/D* ≈ 50
- Gain *G* from 10^5 to 10^6 is attained with applied voltage in the range 2 to 3kV
- **No need to focus electrons** within the multiplier, but the electron optics from cathode to multiplier input must be carefully designed to get good collection efficiency

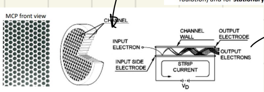
In order to exploit CCMs it is necessary to neutralize the effect of Ion Feedback.

- In the **last part of the channel the density of energetic electrons is high** and creation of free **heavy ions** (ionized atoms) by collision with residual gas molecules (or with the wall material) becomes probable.
- The free ions drift in the field and by impacting on the wall cause a strong emission of electrons. If the impact occurs near the channel input the emitted electrons undergo all the channel multiplication.
- This is a positive feedback effect, which enhances the current amplification in uncontrolled way and may even cause a self-sustaining breakdown current in the multiplier.
- The effect is avoided by bending the axis of the multiplier tube. Due to the large mass and small charge, a free ion has small acceleration in the electric field and its trajectory is almost straight; the ions thus impact in the last part of the channel, hence the emitted electrons undergo undergo a much lower amplification

- Dark current is lower in CCM-PMTs than in dynode-PMTs, which collect also electrons from auxiliary input electrodes contaminated in the photocathode fabrication
- The **excess noise factor F>2 is significantly higher than dynode-PMTs**, because the statistical dispersion in the electron multiplication is clearly greater
- The inner layer resistance is in *GΩ* range, the current in this voltage divider is low *<1μA*, hence for avoiding nonlinearity the **mean output current must not exceed a few nA**. This sets a strict limit to the product of mean photon rate and PMT gain.
- The amplification of a pulse signal leaves a charge on the multiplier surface near to the output. A high charge modifies the electric field, impairing the amplification of the following pulses during a long recovery transient (discharge through the inner layer resistance, with time constant of milliseconds or more). **To avoid this, the product of multiplier gain and input pulse charge and/or repetition rate must be limited**
- Strong nonlinearities due to space charge may occur for high pulses and high gain

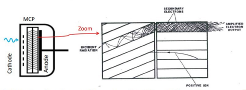
In conclusion, CCM-PMTs are

a) **well** suitable and provide very good performance for detecting **pulses with moderate repetition rate and small size** (down to single photons).
b) **NOT** well suitable for **many-photon-pulses** (e.g. for scintillation detectors of ionizing radiation) and for **stationary light intensity**.



- For overcoming the CCM limitations, the multiplier concept evolved (in early years 70's) to the MicroChannel Plate MCP, implemented with sophisticated glass technology
- An array of many thousands of multiplier microtubes is embedded in beehive structure into a plate. All channels are biased in parallel with the same high voltage V_D , applied via metal electrodes deposited on the two faces of the plate.
- The MCP has a **planar geometry**, well matched to a planar end-window photocathode; focusing of photoelectrons on the multiplier is simply provided by a high voltage from cathode to multiplier input (**proximity-focusing** geometry)

- MCPs are implemented with small diameter D from 50μm to 5μm and the useful area (sum of the channel input sections) is ≈50 to 60% of the total plate area
- Each channel operates as an individual miniaturized CCM: the gain is optimized still with L/D ≈ 50
- To avoid ion feedback by bending the channel axis is not convenient for MCPs; the same principle is exploited by two MCPs with inclined channel axis, mounted in series with channel axis of the first and second MCP forming an angle



Most of the limitations that plague CCMs are relaxed for MCPs with illumination distributed on the cathode because:

a) Electrons emitted from the same position of the cathode do not enter all in the same microchannel; they are distributed over a group of facing channels in the MCP.
b) The perturbation of the voltage distribution in a channel affects the multiplication and collection of electrons just in that channel and closest neighbors, not farther.

It follows that:

1) the limit to the output mean signal current is much higher; it is a small percentage of the **total** bias current of the MCP, not of a single microchannel
2) also many-photon optical pulses are correctly linearly processed, since the pulse photoelectrons are multiplied in parallel in different microchannels

- The statistical gain distribution of MCPs is similar to CCMs, significantly wider than for dynode-PMTs, with excess noise factor significantly higher F>2
- The dynamic response of MCPs is remarkably superior to that of dynode PMTs. The transit time T_b and its jitter T_j are remarkably shorter; in fast MCP types they are reduced down to T_b ≈1ns and T_j≈ a few 10ps. Also the SER pulse-width T_w is shorter, down to T_w≈ a few 100ps.

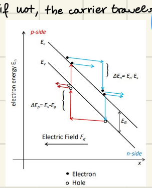
AVALANCHE PHOTODIODES (APD)

a FREE e^- in conduction band has a KINETIC ENERGY $\Delta E_m = E_m - E_c$ (in part transf. to lattice by vibrations of scattering).

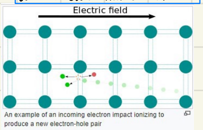
we want a IONIZING COLLISION, that occurs when

$$\Delta E > 1.5\, E_g$$

if not, the carrier travels without ionizing.



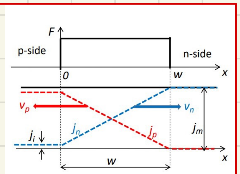
a IONIZED e^- can knock out some electrons from valence band to conduction band.



POSITIVE FEEDBACK: cascade of ionizing collisions produces AVALANCHE MULTIPLICATION OF CARRIERS.

CARRIER MULTIPLICATION

- current both from e^- and h^+
- total current $j_M = j_n + j_p$
- created a dipole-like space charge with field opposite to junction field.



example: for $\alpha = \beta$ (GaAs)

$$j_M = \frac{j_i}{1 - I_i}$$

$$I_i = \int_0^w \alpha(x)\,dx$$ IONIZATION INTEGRAL ~ strongly dependent on F_e

$$\Rightarrow \boxed{M = j_M / j_I = 1/(1 - I_i)}$$ MULTIPLICATION FACTOR

EFFECTIVE IONIZ. COEFF.

if $\alpha \neq \beta$, just: $I_i = \int_0^w \alpha_e(x)\,dx \Rightarrow \boxed{\alpha_e = \alpha \exp\left[-\int_0^x (\alpha - \beta)\,d\xi\right]}$

- M gets steeper if $w\uparrow$ (HIGH-FIELD ZONE gets wider $\Rightarrow$ higher number of collisions)

$\Rightarrow$ when $V_a = V_B$ BREAKDOWN VOLTAGE then $I_i \to 1$, $M \to \infty$, $j_M \to \infty$ ($V_B \uparrow$ if $T\uparrow$)

However, in reality BREAKDOWN CURRENT is not divergent and flows without requiring a primary injected current (it's self-sustaining); because of FEEDBACK EFFECT DUE TO MOBILE SPACE CHARGE which for $V_a > V_B$ reduces the FIELD that act on carriers. The MULTIPLICATION stabilize itself at self-sustaining level.

$\Rightarrow R_a = \Delta V_a / \Delta i_a$ AVALANCHE RESISTANCE

$\Delta V_a \uparrow \to \vec{F}\uparrow \to$ IMPACT IONIZATION $\uparrow \to I_a \uparrow \to$ SPACE CHARGE $\uparrow \to \Delta V_a \downarrow \longrightarrow \Delta i_a \downarrow$

- close to V_B we have a LINEAR AMPLIFICATION of current with APD (Aval. PhotoDiodes)

$$M = \frac{1}{1 - (V_a/V_B)^n}$$

MULTIPLICATION FACTOR empirical

n depends on FIELD PROFILE 3 to 6

stabilize T!

- EXCESS FACTOR:

after impact, holes re-inject, further enhancing the multiplication

EXCESS FACTOR

$$F = 1 + v_M^2 = 1 + \frac{1}{1 + \mu} \cong 2$$

Si $k \ll 1$, $\mu \ll 1$

if $k = 0$ NEGL. POSITIVE FEEDBACK

mean number of secondary electrons

with k constant, field constant:

$$F \approx M\left[1 - (1-k)\left(1 - \frac{1}{M}\right)^2\right]$$

- there is a F_{Max}, which corresponds to M_{Max}, which is the limit where GAIN FLUCTUATIONS noise become higher than circuit noise

---

- STATISTICAL PROCESS
- GOOD MODEL APPROX if WIDTH OF MULTIPLICATION REGION (high field) is larger than the MEAN PATH BETWEEN IONIZING COLLISIONS.
- Model considers the PROBABILITY OF IONIZING IMPACT OF A CARRIER AS CONTINOUSLY DISTRIBUTED in SPACE

IONIZING COEFF.

$\alpha\, dx$ prob % of IMPACT IONIZATION for (e^-)

$\beta\, dx$ " " for (h^+)

MEAN PATH between IONIZING COLLISIONS

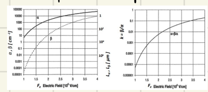
$L_\alpha = 1/\alpha$ for e^-	$L_\beta = 1/\beta$ for h^+

- MULTIPLICATION PROCESS strongly depends on RELATIVE INTENSITY OF POSITIVE FEEDBACK

$$\boxed{k = \beta / \alpha}$$

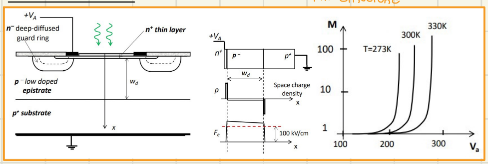
$$\alpha = \alpha_0 \exp(-F_{n0}/F_e)$$
$$\beta = \beta_0 \exp(-F_{p0}/F_e)$$

$\downarrow$ if $T\uparrow$ LOSS OF ENERGY in MORE VIBRATIONS

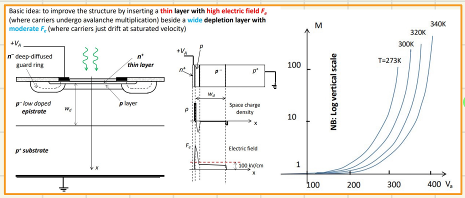


SOME STRUCTURES:

PIN STRUCTURE



M increase too STEEPLY $\Downarrow$ UNSTABLE M



REACH-THROUGH APD (RAPD)

$\Rightarrow$ still F: excess noise factor due to STATISTICAL FLUCTUATIONS OF M

Input:	primary carriers with mean number N_p variance $\sigma_p^2 = N_p$ (Poisson statistics)
Output:	multiplied carriers with mean number $N_u = M N_p$ variance $\sigma_u^2 = F M^2 \sigma_p^2 = F M^2 N_p$

a COMPARISON with PMT:

The physical processes exploited for multiplying electrons in PMTs and in APDs are remarkably different and the detector gain has remarkably different features.

- In **PMTs**, the accelerated electron that hits a dynode is lost and the number of emitted secondary electrons fluctuates in a set of values that includes zero. The resulting mean number of carriers coming from the dynode is just the mean number of emitted secondary electrons and is definitely **higher than unity**.
- In **APDs**, the accelerated electron that undergoes a ionizing impact is not lost, it remains available for further impacts; the generation of a further electron (plus a hole) is statistical and the mean number of generated electrons is definitely **lower than unity**. The resulting mean number of electrons after the impact is **one plus the mean number of generated** electrons.

- In **PMTs** the gain is produced by an unidirectional sequence of events, the cascade of statistical multiplications at the various dynodes. Cascaded statistical processes can be well analyzed by known mathematical approaches (as the Laplace probability generating function)
- In **APDs** the **statistical process is much more complicated** than a simple cascade because of the intrinsic **positive feedback** in the impact-ionization. Rather than a cascade, it is a complex of interwoven feedback loops, each one originating from the other type of carrier (the hole in our case) generated in the impact.

SINGLE-PHOTON AVALANCHE DIODES

With respect to PMT, the SENSITIVITY is much lower due to:

*) lower mean value $\overline{M}$

*) much stronger GAIN FLUCTUATIONS

CAN WE EMPLOY LINEAR AMPLIFYING APDs (instead of PMTs) in SINGLE PHOTON COUNTING and TIMING TECHNIQUES?

almost NO for Si APD

absolutely NO for ALL OTHER MATERIALS.

SPC : Single Primary Carrier
+
Noise of circuit

PULSE COMPARATOR with THRESHOLD
- higher : pulse accepted
- lower : pulse discarded

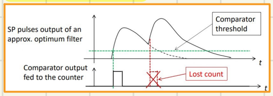
IF NOISE VARIANCE is σ_n, THRESHOLD N_{nr} must be set at least at $\geq 2.5\sigma_n$

$$\sigma_n = \frac{\sqrt{2C_i\sqrt{S_v}\sqrt{S_i}}}{q}$$

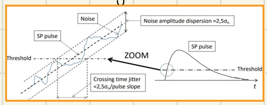
$N_{nr} \sim 100/300\ e^-$

$\Rightarrow M \gg N_{nr}$ is required!

- FINITE WIDTH OF SP PULSES CAN CAUSE PROBLEMS:



- pulses occurs randomly in time. Therefore if WIDTH ↑, PULSE PILE-UP% ↑ LOSS PULSES ↑
- ARRIVAL TIME of PULSE is STATISTICAL and can have a JITTER (if long T_{RC} of pulse, less steep pulse, wider time jitter)

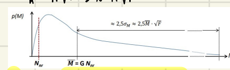


it's NOT SUFFICIENT to have $\overline{M} > N_{nr}$ also most of $M < \overline{M}$ must be $> N_{nr}$ for single photon detection.

M has strong fluctuations, and therefore, $F \gg 1$:

$$F = 1 + v_M^2 = \frac{1+\sigma_M^2}{(\overline{M})^2}$$

$$\sigma_M = \overline{M}\sqrt{F-1} \approx \overline{M}\sqrt{F}$$



M should be $\propto G N_{nr}$ with $G \gg 1$

for REDUCING COUNT-LOSSES and TIME JITTER we must process the ADP pulses with FILTER BW WIDER than OPTIMUM FILTER

but

this implies also higher-noise and higher threshold and higher gain required for ADP.

AVALANCHE DIODES ABOVE V_B (SPAD)

it's called GEIGER-MODE OPERATION:

① SINGLE PHOTON SWITCH-ON AVALANCHE (MACROSCOPIC CURRENT) → TRIGGERED MODE AVALANCHE → BISTABLE DETECTOR

② AVALANCHE is TURNED-OFF BY PULLING-DOWN DIODE VOLTAGE $V_D \simeq V_B$ (or BELOW)

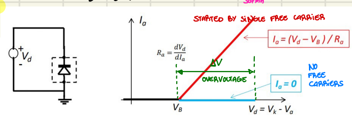
③ DIODE VOLTAGE is then RESET ABOVE THE BREAKDOWN

(1) Quiescent state: Bias voltage V_d above breakdown V_B (with excess bias V_{exc}) is applied and no current flows

(2) Avalanche current flowing: it is triggered by a photon or noise

(3) Quenching: bias voltage V_d is lowered below the breakdown to stop the avalanche current flowing

(4) Reset: voltage across the junction is restored to the initial value



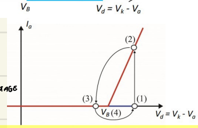
NOTE:

A) avoid DEFECTS (local fields → lower V_B)

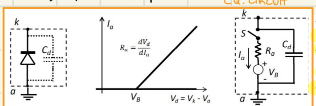
B) thermal generated carriers → generate pulses → DARK COUNT RATE (DCR)

C) AVAL. triggering, η_D, pulse amplit., DCR, delay, jitter → depends on DIODE VOLTAGE

D) V_B depends on temperature T.



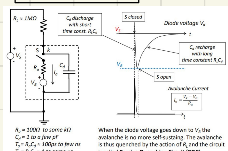
EQ. CIRCUIT

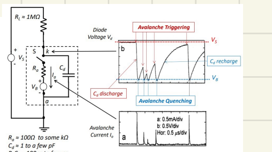


- at $V_d > V_B$ the switch S can be closed or open; when it is closed, the avalanche current flows. At $V_d \leq V_B$ it is always open.
- **Closing the switch** is the equivalent of **triggering the avalanche** in the diode. Therefore, S is closed when a carrier injected or generated in the high field region succeeds in triggering the avalanche
- S then is open when the avalanche current is quenched (i.e. terminated) by the decrease of the diode voltage down to $V_d \approx V_B$

some CONFIGURATIONS:

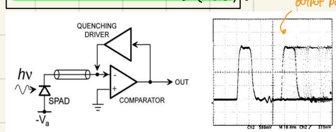
- PASSIVE QUENCHING:





- In a passive-quenching circuit, after each quenching the diode voltage **slowly recovers from the breakdown voltage V_B** to the supply level V_S.
- In photon counting with a PQC, count losses are caused by the gradual recovery of the detection efficiency from nil to the correct level after each quenching.
- In photon timing with an avalanche diode in PQC, for photons arriving during a voltage recovery the arrival time measured on the electrical output pulse suffers **increased delay and time-jitter with respect to the operation** at the correct diode voltage. This effect progressively degrades the time resolution as the pulse counting rate is increased
- In conclusion, the application to photon counting and timing of avalanche diodes in Geiger mode with a PQC has very limited interest. It is restricted to favorable cases, that is **cases with low dark-count rate, low count-rate of background photons and low count-rate of the signal photons**

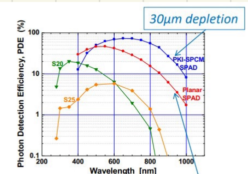
- ACTIVE QUENCHING CIRCUITS (AQC):

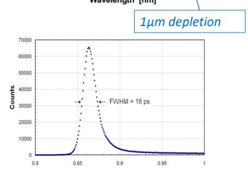


by providing

- short, well-defined deadtime
- high counting rate > 1 Mc/s
- good photon timing
- standard output

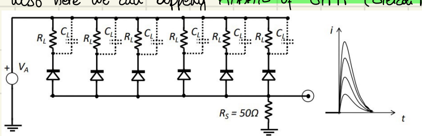
- A COMPARISON with PMTs:





- **microelectronic advantages:**
  miniaturized, low voltage, etc.
- **improved performance:**
  higher Photon Detection Efficiency
  better photon timing
  comparable or lower noise

- also here we can apply ARRAYS of SiPM (Silicon Photon Multipliers)



This detector is a SPAD array where

- each pixel has an individual integrated quenching resistance $R_L \approx 100k\Omega$.
- each pixel has a very small individual load capacitance $C_L \approx 100\ fF$
- All pixels have a common ground terminal, connected to a low resistance external load, typically $R_S = 50\Omega$. The pixel currents all flow in this terminal, they are added

The detector pixels are thus

a) individually triggered by incident photons,

b) individually quenched by the discharge of the pixel capacitance

c) individually reset by the recharge of C_L with short time constant $R_L C_L \approx 10ns$

- The signal charge at the common output is proportional to the number of incident photons (at least as long as the light intensity on the detector is low enough to have negligible probability of more than one photon arriving on a pixel at the same time)
- Each pixel is a digital SPAD detector, but the pixel ensemble provides an analog information about the number of incident photons. The operation is indeed fairly similar to that of PMTs with microchannel plate multiplier. The detector was indeed conceived and is currently denoted as «Silicon PhotoMultiplier» SiPM.

With respect to PMTs, SiPMs offer various advantages

a) The typical properties of microelectronic devices (miniaturization; low voltage and low power; ruggedness; etc.)

b) remarkably higher detection efficiency, particularly in the red spectral range

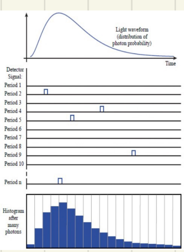
c) operation insensitive to magnetic fields, which are detrimental for PMTs

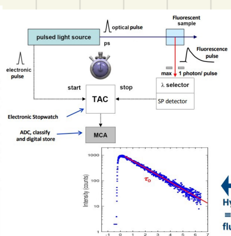
However, SiPMs have also drawbacks with respect to PMTs

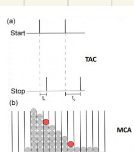
1. active area not as wide as PMTs
2. lower filling factor, with corresponding reduction of the photon detection efficiency
3. Fairly high dark current, that is, much higher dark current density over the active area

- another technique is TIME-CORRELATED SINGLE PHOTON COUNTING (TCSPC):

- Use a periodical illumination with a pulsed laser
- Measure the time of arrival of each photon re-emitted by the sample
- Build a histogram of the photons time of arrivals
- Upon the collection of a statistically significant amount of events, the histogram corresponds to the waveform you would have obtained with a single "analog" measurement.
- The equivalent bandwidth is not limited by the Single Electron Response (SER)







Hystogram of many trials = fluorescence decay curve

TCSPC speed has some intrinsic limitations

- In a TCSPC measurement, the laser power has to be tuned in order to ensure that the system works in single-photon regime
- Otherwise, detector dead time affects photon recording probability and it can lead to a distortion of the reconstructed curve because of pile-up
- In order to limit pile-up distortion, the average count rate of a single TCSPC acquisition chain is kept below 10% of the excitation rate (typically 1% or 5%)
- In this way, the probability of having two photons in a single excitation period is negligible and pile-up distortion is avoided
- Example: with a laser operating at 80MHz, the count rate is typically up to 4Mcps
- In order to have 10000counts, Tmeas=10000/4Mcps = 2.5ms for a single spot

TEMPERATURE SENSORS

METAL RTD PRINCIPLE :

- RESISTANCE R_S OF METAL CONDUCTORS INCREASES WITH TEMP. T

$$\boxed{R_S = R_0(1+\alpha\Delta T) \quad T_0 = \text{reference temperature} \quad R_0 = R_S(T_0)}$$

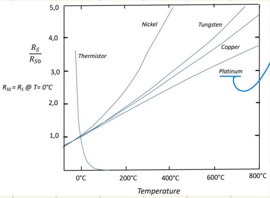
α : TEMP. COEFF. OF RESISTANCE

Metal	α
Platinum Pt	$3{,}9 \cdot 10^{-3}$
Copper Cu	$4{,}3 \cdot 10^{-3}$
Tungsten W	$4{,}6 \cdot 10^{-3}$
Nickel Ni	$6{,}8 \cdot 10^{-3}$

Platinum has useful qualities:

- **Chemically inert and resistant to contamination**, hence stable properties
- **R_S(T) linear with very good approximation** from -200°C to about 500°C and with small deviation from linearity up to 800°C
- **small quantity of Pt necessary** in a RTD, cost is not high

Pt is the material of choice in many cases and is used in official metrology to define the International Practical Temperature Scale (from 13,81 K to 903,89 K).

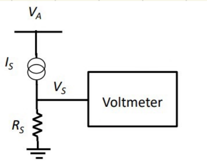


Because of requirements for correct operation, the **RTD fabrication technology is not so simple** :

- The package must be compact and ensure good **thermal contact** of the resistor to the object measured and good **electrical isolation** from it
- Small size is required with **R_0 > some 10 Ω , typically R_0 =100 Ω**, in order to have to measure not very small ΔR_S. Thin wire wrapped in spiral on a support is used
- The mechanical structure must **avoid strain** of the metal wire due to thermal expansion or contraction: the **piezoresistive effect** would cause unwanted resistance variations and consequent errors in ΔT

- POWER SUPPLY is required to generate an electrical signal.
- SELF-HEATING causes a $T_S > T_a$ to be measured

$\Delta T_S = T_S - T_a \uparrow$ for $P_S \uparrow$ POWER DISSIPATION $P_S = V_S^2 / R_S < P_{MAX} \implies V_S \leq \sqrt{R_S \cdot P_S}$



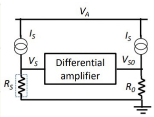
$$\Delta R_S = f(\Delta T) \approx \alpha R_0 \cdot \Delta T$$

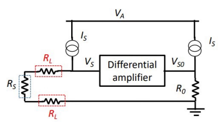
$$V_{S0} = I_S R_0$$

$$\Delta V_S = V_S - V_{S0} = I_S \cdot \Delta R_S = V_{S0}\frac{\Delta R_S}{R_0} \approx V_{S0}\alpha\Delta T$$

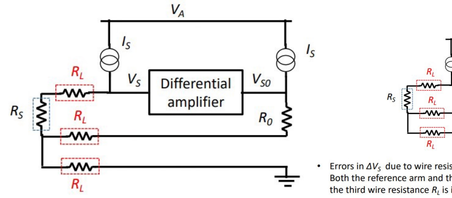
In modern electronics a simple approach is possible and practical thanks to the routine availability of current generators :

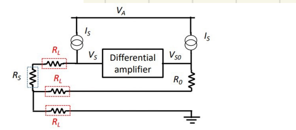
- R_S is biased with a **constant current** generator I_S ,
- voltage V_S on R_S is measured
- at any T, **V_S is exactly proportional to R_S** : the difference ΔV_S from measured V_S to reference voltage V_{S0} gives an accurate measure of ΔR_S
- ΔR_S is an accurately known function of ΔT= T-T_0 , in many cases approximately linear





- Since ΔV_S is much smaller than V_S , it is advisable to include in the circuit a reference V_{S0} and take **directly differential measurements of ΔV_S** , instead of measuring V_S and then subtracting V_{S0}
- However, in various cases the RTD is placed on a measured object not near to the circuit, the **long connecting wires** have resistance R_L not negligible with respect to R_S and their **effect is significant** and must be taken into account
- In the simplest configuration, called «Two-wire-connection», the two wire resistances are in series with R_S and their voltage drop $2I_S R_L$ is added to V_S , thus causing a significant error in the measured ΔV_S





- Errors in ΔV_S due to wire resistances R_L are avoided by a **«Three-wire-connection».** Both the reference arm and the RTD arm include in series a wire resistance R_L ; the third wire resistance R_L is inserted in the common return to the circuit ground

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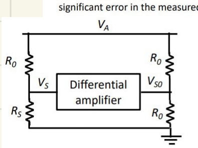
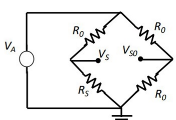


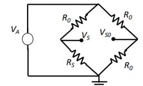


- An alternative configuration, devised when current generators were not available, requires only resistors and due to its simplicity is still widely exploited
- A **voltage divider** is implemented by the **R_S of the RTD** in series with a **reference resistor R_0** and the variations of the divider output voltage corresponding to the variations of R_S are measured
- This is the principle of the **Wheatstone bridge**, invented in 1833 by Samuel Hunter Christie and popularized by Charles Wheatstone and usually drawn as sketched above at right

$$R_S = R_0 + \Delta R_S$$

$$V_{S0} = V_A\frac{R_0}{R_0+R_0} = \frac{V_A}{2}$$

$$V_S = V_A\frac{R_S}{R_0+R_S}$$



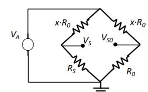
For **small resistance variation** $\Delta R_S < 0{,}05\ R_0$ the voltage variation ΔV_S is **approximately linear** with ΔR_S and can be computed by first-order development

$$\Delta V_S = \Delta R_S\left(\frac{dV_S}{dR_S}\right)_{R_S=R_0} = \frac{V_A}{4}\frac{\Delta R_S}{R_0} = \frac{V_A}{4}\alpha\Delta T$$

$$R_S = R_0 + \Delta R_S$$

$$V_{S0} = V_A\frac{R_0}{R_0+xR_0} = \frac{V_A}{1+x}$$

$$V_S = V_A\frac{R_S}{xR_0+R_S}$$



The Wheatstone bridge can be employed with **any ratio x** of the voltage divider, i.e. R_S can be in series with a resistor $x{\cdot}R_0$ with any value of the factor x. However, it is intuitive and readily verified that **with x=1 the highest output ΔV_S** is obtained

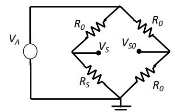
$$\Delta V_S = \left(\frac{dV_S}{dR_S}\right)_{R_S=R_0} \Rightarrow \Delta R_S = V_A\frac{x}{(1+x)^2}\frac{\Delta R_S}{R_0}$$

$$\max\left[\frac{x}{(1+x)^2}\right] = \frac{1}{4} \quad for \quad x=1$$

$$R_S = R_0 + \Delta R_S$$

$$V_{S0} = V_A\frac{R_0}{R_0+R_0} = \frac{V_A}{2}$$

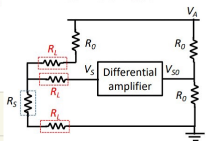
$$V_S = V_A\frac{R_S}{R_0+R_S}$$



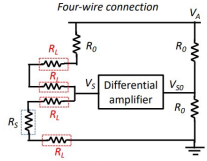
The cheap availability of integrated electronics for digital data processing and storage makes practical to extend the application of the Wheatstone bridge also to cases with **greater variations ΔR_S** , that have a **non-linear but known dependance** of ΔV_S on ΔR_S

$$\Delta V_S = V_S - V_{S0} = V_A\frac{R_0+\Delta R_S}{2R_0+\Delta R_S} - \frac{V_A}{2} = \frac{V_A}{2}\cdot\frac{\frac{\Delta R_S}{2R_0}}{1+\frac{\Delta R_S}{2R_0}}$$

Three-wire connection



Four-wire connection



- «**Two-wire-connection**» causes error also in this case by adding $2R_L$ to R_S
- «**Three-wire-connection**» adds one R_L to the RTD and one to the balancing resistance R_0 . The R_L of the connection to the differential amplifier is not compensated, but its effect is negligible because the current in it is negligible
- «**Four-wire-connection**» achieves complete simmetry between RTD arm and balancing arm, with complete cancellation of the errors due to wire resistances (and also cancellation of other minor thermoelectric effects caused by electrical current flowing in conductors with a temperature gradient)

THERMISTORS :

- Commonly used temperature transducers called Thermistors are made of semiconductor ceramic materials, oxides of Cr, Mn, Fe, Co, Ni
- The dependence of thermistor resistance R on temperature is strikingly different from RTDs (see the plot in slide 29): strongly **nonlinear**, **decreases with increasing temperature** and the R values are **much larger** (some 100 kΩ at room temperature) and have much **greater relative variation**
- The resistance-temperature relationship can be described by the equation

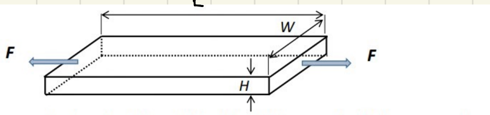
$$R = \exp\left(\frac{B}{T}\right)$$

where **T is the absolute temperature** in Kelvin degrees, B is constant. B is called characteristic temperature of the termistor and usually ranges from 2000 K to 4000 K.
Making reference to the resistance value R_0 at a known reference temperature T_0 we get

$$R = R_0\exp\left[B\left(\frac{1}{T}-\frac{1}{T_0}\right)\right]$$

- Thermistors can be made **much smaller** than RTDs.
- The smaller mass enables them to respond **more quickly** to temperature variations
- The **smaller size**, however, makes **less efficient the dispersion of the self-heating** power, which must be limited to low level
- The basic advantage of thermistors with respect to RTDs is **higher sensitivity**, i.e. larger relative variation ΔR/R for a given ΔT, which eases measurements of very small ΔT
- The main disadvantages are **lower accuracy and lower reproducibility** and strongly nonlinear characteristics, which limit the application of thermistors in automatic control systems

STRAIN GAUGES



L = length; W = width; H = thickness; A= W·H cross section
F = pull force applied to the ends

STRESS $N = F/A$

EXTENSION of L DUE TO F ΔL

STRAIN $\varepsilon = \Delta L / L$

in μstrain $\Delta L / L = 10^{-6}$

$\Rightarrow \boxed{N = \overset{\frown}{E}\,\varepsilon}$ YOUNG'S MODULUS $\frac{N}{m^2}$

valid up to ε_L ELASTIC LIMIT OF MATERIAL

CONTRACTION OF W and H and AREA

$$-\frac{\Delta W}{W} = -\frac{\Delta H}{H} = \nu \cdot \varepsilon$$

$$\frac{\Delta A}{A} \approx \frac{\Delta W}{W} + \frac{\Delta H}{H} = 2\nu\varepsilon$$

RESISTANCE change:

$$R = \rho \frac{L}{A} = \frac{L}{\sigma A}$$

ρ RESISTIVITY

$$\rho = \rho_0 (1 + \beta N)$$

$$\frac{\Delta \rho}{\rho_0} = \beta N = \beta E \varepsilon$$

$\Rightarrow$ a STRAIN GAUGE (SG) is a long and thin metal slab (small cross-section $H \ll L$, $W \ll L$) employed to measure strain ε along L, in elastic range

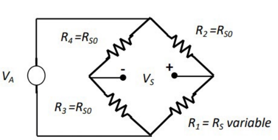
$$\frac{\Delta R}{R_0} = \frac{\Delta L}{L_0} - \frac{\Delta A}{A} + \frac{\Delta \rho}{\rho_0} = \varepsilon (1 + 2\nu + \beta E)$$

$$\Rightarrow \boxed{G = 1 + 2\nu + \beta E}$$ GAUGE FACTOR

WHEATSTONE BRIDGE:

For a W-bridge with
- one SG of variable R_S
- three constant R_{S0}

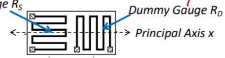
$$V_S = \frac{V_A}{4}\frac{\Delta R_S}{R_{S0}} = \frac{V_A}{4} G \cdot \varepsilon$$



we have also to consider variations due to temperature

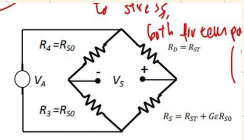
$\Rightarrow$ SOL: insert DUMMY GAUGE to compensate variation of temperature.
It's orthogonal to force to be measured (NO VARIATIONS due to stress)

Compensated dual SG outline



$R_S = R_{S0} + \alpha\Delta T R_{S0} + G\varepsilon R_{S0} = R_{ST} + G\varepsilon R_{S0}$

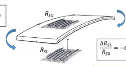
$R_D = R_{S0} + \alpha\Delta T R_{S0} = R_{ST}$



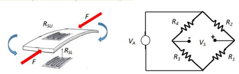
BENDING and COMPOSITE STRAIN:

- Let's consider bending a long board with rectangular section (see the figure). The upper surface experiences a tensile strain ε_B, the lower surface a symmetrical compressive strain $-\varepsilon_B$. In fact, the strain linearly varies in the board section from ε_B to $-\varepsilon_B$ and is zero in the mid, which is called «neutral plane»
- Let's consider to apply on the two surfaces of the board two matched SG (with equal resistance R_{S0} and Gauge factor G), denoted as R_{SU} on the upper surface and R_{SL} on the lower surface. Due to bending we get

$$\boxed{\frac{\Delta R_{SU}}{R_{S0}} = G\varepsilon_B}$$



$$\boxed{\frac{\Delta R_{SL}}{R_{S0}} = -G\varepsilon_B}$$



- With R_{SU} inserted in the bridge as R_1 and R_{SL} as R_3, we measure the bending strain ε_B

$$V_{SB} = \frac{V_A}{4}\frac{\Delta R_{SU}}{R_{S0}} - \frac{V_A}{4}\frac{\Delta R_{SL}}{R_{S0}} = \frac{V_A}{2} G \cdot \varepsilon_B$$

- Let's consider now that a compressive force is added at the board ends: equal strain ε_F is added at the upper and lower surface, but the two SG have equal variation and the added contribution to the bridge output voltage is zero

$$\frac{\Delta R_{SU}}{R_{S0}} = \frac{\Delta R_{SL}}{R_{S0}} = G\varepsilon_F \Rightarrow V_{SF} = \frac{V_A}{4}\frac{\Delta R_{SU}}{R_{S0}} - \frac{V_A}{4}\frac{\Delta R_{SL}}{R_{S0}} = 0 \Rightarrow \boxed{V_S = V_{SB} + V_{SF} = \frac{V_A}{2} G \cdot \varepsilon_B}$$

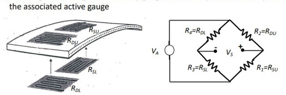
- In conclusion, by suitably employing two SG we can separately measure the net bending strain ε_B also in presence of an axial strain ε_F
- On the other hand, with the same two SG we can also measure separately the net axial strain ε_F in presence of the bending strain ε_B
- It is sufficient to change the configuration of the bridge. In fact, with R_{SU} inserted as R_1 and R_{SL} as R_4 we get

$$V_{SB} = \frac{V_A}{4}\frac{\Delta R_{SU}}{R_{S0}} + \frac{V_A}{4}\frac{\Delta R_{SL}}{R_{S0}} = 0 \qquad V_{SF} = \frac{V_A}{4}\frac{\Delta R_{SU}}{R_{S0}} + \frac{V_A}{4}\frac{\Delta R_{SL}}{R_{S0}} = \frac{V_A}{2} G \cdot \varepsilon_F$$

Therefore

$$\boxed{V_S = V_{SB} + V_{SF} = \frac{V_A}{2} G \cdot \varepsilon_F}$$

- The measurements of ε_B and ε_F obtained with two matched SG as illustrated are correct only if the two SG are at the same temperature, but in many cases this is not achieved because the two SG are not in close proximity
- The drawback is avoided and the approach extended to all cases simply by
  a) employing dual compensated SGs instead of simple SGs and
  b) inserting in the bridge each dummy gauge in suitable position to compensate the associated active gauge



SEMICONDUCTOR GAUGES:

- **Semiconductors such as Germanium and Silicon have very strong piezoresistive effect.** Strain Gauges in such materials thus provide large Gauge Factor G in the range from 100 to 300
- Magnitude and sign of the piezoresistive effect are governed by the type and level of doping. In p-type Silicon the effect is positive (tensile strain increases the resistivity) and in n-type silicon it is negative (tensile strain decreases the resistivity)
- **The effect is markedly dependent on the temperature**, with G decreasing significanly as the temperature is increased. A typical example is a reduction from G=120 at 10°C to G=105 at 65°C.
- **The Gauge Factor G is not constant as the strain is increased**, i.e. the gauge is not linear, with G decreasing significantly at moderately high strain. A typical example is a decrease from G=125 at 2000 microstrain down to G=100 at 4000 microstrain
- **The elastic range of these semiconductor materials is quite narrower** than that of metals, the elastic limit is typically at ≈4000 microstrain