POLIMI GRADUATE SCHOOL OF MANAGEMENT

Master in Finanza Quantitativa

Valutazione dei derivati

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1 Si calcolino i prezzi forward del tasso di cambio $\boldsymbol{X_0(T)}$ alle date di scadenza T delle opzioni plain vanilla quotate dal mercato

In order to calculate forward prices of exchange rate, the following relation has been considerate:

$$X_t(T) = \frac{P_t^f(T;b)}{P_t(T;c)}\chi_t$$

where $P_t^f(T;b)$ is the zero coupon foreign bond price summarized in the FX market, $P_t(T;c)$ is the price of zero coupon domestic bond e χ_t spot exchange rate at time t.

In order to obtain prices at expiry date of interest, interest rates curve of JPN and EUR have been interpolated through linear interpolation and, after, prices of the zero coupon bonds have been re-calculated, respectively in domestic currency and foreign currency, for expiry dates of interest:

$$P_t^f(T;b) = \exp\left(-r_t^f(T-t)\right)$$

$$P_t(T;c) = \exp\left(-r_t^d(T-t)\right)$$

Through the showed method, following forward prices $X_t(T)$ have been obtained at the time $t = 0$:

Expiry	Forward price
06-Feb-18	134,7781321
13-Feb-18	134,7865981
28-Feb-18	134,8034984
30-Mar-18	134,8337631
30-Apr-18	134,8667186
30-Jul-18	134,9637316
30-Jan-19	135,1646577
30-Jan-20	135,3512135
30-Jan-23	132,6150164

2 Si calcolino gli strike K delle opzioni plain-vanilla quotate dal mercato in termini di delta

In order to obtain strike values K in terms of δ we have used definition of delta pips/percentage (Clark 3. Deltas and Market Conventions) respectively in the following two different cases:

1. caso $\delta = 50\%, K^{atm}$
2. caso $\delta \neq 50\%$

1. caso $\delta = 50\%, K^{atm}$

Considering the result which strike at ATM is the level where straddles (call+put) are delta neutral. In particular, from definition of delta percentage we obtain the following results, respectively for the short term case ($T \leq 1\ year$) and for the long term case ($T > 1\ year$) (equation 3.8 Clark):

<u>SPOT CASE - percentage:</u>

$$\Delta_t^{\%} = \chi_t \frac{\partial}{\partial \chi_t}\frac{PV_t}{\chi_t} = \Delta_t^{pips} - \frac{\mathrm{PV_t}}{\chi_t}$$

$$\Delta_t^{\%}\big(T, K^{atm}, 1, \sigma(K^{atm})\big) + \Delta_t^{\%}\big(T, K^{atm}, -1, \sigma(K^{atm})\big) = 0$$

$$P_t^f(T;b)\phi(d_+) - \frac{Call_t}{\chi_t} - P_t^f(T;b)\phi(-d_+) - \frac{Put_t}{\chi_t} =$$

$$= P_t^f(T;b)\phi(d_+) - \frac{1}{\chi_t}\left[\chi_t P_t^f(T;b)\phi(d_+) - KP_t(T;c)\phi(d_-)\right] - P_t^f(T;b)\phi(-d_+)$$
$$- \frac{1}{\chi_t}\left[-\chi_t P_t^f(T;b)\phi(-d_+) + KP_t(T;c)\phi(-d_-)\right] =$$

$$= P_t^f(T;b)\phi(d_+) - \left[P_t^f(T;b)\phi(d_+) - \frac{K}{\chi_t}P_t(T;c)\phi(d_-)\right] - P_t^f(T;b)\phi(-d_+)$$
$$- \left[-P_t^f(T;b)\phi(-d_+) + \frac{K}{\chi_t}P_t(T;c)\phi(-d_-)\right] =$$

$$= \frac{K}{\chi_t}P_t(T;c)\phi(d_-) - \frac{K}{\chi_t}P_t(T;c)\phi(-d_-)$$

$$\frac{K}{\chi_t}\; P_t(T;c)\big(\phi(d_-) - \phi(-d_-)\big) = 0$$

From where we get the following result:

$$\big(\phi(d_-) - \phi(-d_-)\big) = 0$$

From monotonicity of the function ϕ we have $d_- = -d_-$, from which:

$$2\frac{1}{\sigma\sqrt{T-t}}\log\frac{X_t(T)}{K} - \sigma\sqrt{T-t} \;\; = 0$$

$$K \;\; = X_t(T)\exp\left(-\frac{1}{2}\sigma^2(T-t)\right)$$

$$K^{atm,spot} \; = X_t(T)\exp[-\frac{1}{2}(\sigma^2(K^{atm})(T-t)]$$

FORWARD CASE - Percentage:

$$\Delta_t^{\%,fwd} = \frac{X_t}{P_t(T;c)}\frac{\partial}{\partial X_t}\frac{PV_t}{\chi_t} = \Delta_t^{pips,fwd} - \frac{1}{P_t^f(T;b)}\frac{\mathrm{PV_t}}{\chi_t}$$

Similar to spot case we obtain:

$$\Delta_t^{\%,fwd}\big(T, K^{atm}, 1, \sigma(K^{atm})\big) + \Delta_t^{\%,fwd}\big(T, K^{atm}, -1, \sigma(K^{atm})\big) = 0$$

$$\frac{K}{X_t(T)}\phi(d_-) - \frac{K}{X_t(T)}\phi(-d_-) = 0$$

$$\big(\phi(d_-) - \phi(-d_-)\big) = 0$$

From which we get, with a reasoning similar to the one developed for the spot case, that $d_- = -d_-$ and then:

$$K^{atm,fwd} \; = X_t(T)\exp[-\frac{1}{2}(\sigma^2(K^{atm})(T-t)]$$

1. case $\delta \neq 50\%$

Considering market quotations in terms of $\Delta_t^{\%}$ and $\Delta_t^{\%,fwd}$, listed on the market, respectively for short term maturity $T \leq 1\ year$ (equation 3.3 Clark) and long-term maturity $T > 1\ year$:

$$\Delta_t^{\%} = \chi_t \frac{\partial}{\partial \chi_t} \frac{PV_t}{\chi_t} = \Delta_t^{pips} - \frac{\mathrm{PV_t}}{\chi_t}$$

$$\Delta_t^{\%,fwd} = \frac{X_t}{P_t(T;c)} \frac{\partial}{\partial X_t} \frac{PV_t}{\chi_t} = \Delta_t^{pips,fwd} - \frac{1}{P_t^f(T;b)} \frac{\mathrm{PV_t}}{\chi_t}$$

Considering definitions of Δ_t^{pip} and $\Delta_t^{\%}$, we obtain, respectively for the spot case and forward case, following results:

<u>SPOT CASE - Percentage:</u>

$$\Delta_t^{\%}(T,K,\omega,\sigma) = \omega P_t^f(T;b)\phi(\omega d_+) - \frac{\mathrm{PV_t}}{\chi_t} = \omega P_t^f(T;b)\phi(\omega d_+) - \frac{1}{\chi_t}\left[Call_t\, 1_{\{\omega=1\}} + Put_t 1_{\{\omega=-1\}}\right] =$$

$$\begin{aligned} = \omega P_t^f(T;b)\phi(\omega d_+) & \\ & - \frac{1}{\chi_t}\left[\chi_t P_t^f(T;b)\phi(\omega d_+)1_{\{\omega=1\}} - K P_t(T;c)\phi(\omega d_-)1_{\{\omega=1\}}\right. \\ & \left. - \chi_t P_t^f(T;b)\phi(\omega d_+)1_{\{\omega=-1\}} + K P_t(T;c)\phi(\omega d_-)1_{\{\omega=-1\}}\right] \end{aligned}$$

$$= \omega P_t^f(T;b)\phi(\omega d_+) - \frac{1}{\chi_t}\left[\omega \chi_t P_t^f(T;b)\phi(\omega d_+) - \omega K P_t(T;c)\phi(\omega d_-)\right] =$$

$$= \omega P_t^f(T;b)\phi(\omega d_+) - \left[\omega P_t^f(T;b)\phi(\omega d_+) - \frac{\omega K}{\chi_t} P_t(T;c)\phi(\omega d_-)\right] =$$

$$= \frac{\omega K}{\chi_t} P_t(T;c)\phi(\omega d_-)$$

$$d_\pm = \frac{1}{\sigma\sqrt{T-t}} \log\frac{X_t(T)}{K} \pm \frac{1}{2}\sigma\sqrt{T-t}$$

We obtain the following relationship:

$$\Delta_t^{\%}(T,K,\omega,\sigma) = \frac{\omega K}{\chi_t} P_t(T;c)\phi(\omega d_-)$$

<u>FORWARD CASE - Percentage:</u>

$$\Delta_t^{\%,fwd}(T,K,\omega,\sigma) = \omega\phi(\omega d_+) - \frac{1}{P_t^f(T;b)} \frac{\mathrm{PV_t}}{\chi_t} = \omega\phi(\omega d_+) - \frac{1}{P_t^f(T;b)\chi_t}\left[Call_t\, 1_{\{\omega=1\}} + Put_t 1_{\{\omega=-1\}}\right]$$
$$=$$

$$\begin{aligned} = \omega\phi(\omega d_+) - \frac{1}{P_t^f(T;b)\chi_t} & \left[\chi_t P_t^f(T;b)\phi(\omega d_+)1_{\{\omega=1\}} - K P_t(T;c)\phi(\omega d_-)1_{\{\omega=1\}}\right. \\ & \left. - \chi_t P_t^f(T;b)\phi(\omega d_+)1_{\{\omega=-1\}} + K P_t(T;c)\phi(\omega d_-)1_{\{\omega=-1\}}\right] \end{aligned}$$

$$= \omega\phi(\omega d_+) - \frac{1}{P_t^f(T;b)\chi_t}\left[\omega\chi_t P_t^f(T;b)\phi(\omega d_+) - \omega K P_t(T;c)\phi(\omega d_-)\right] =$$

$$= \omega\phi(\omega d_+) - \left[\omega\phi(\omega d_+) - \frac{\omega K}{P_t^f(T;b)\chi_t} P_t(T;c)\phi(\omega d_-)\right] =$$

$$= \frac{\omega K}{P_t^f(T;b)\chi_t} P_t(T;c)\phi(\omega d_-) = \frac{\omega K}{X_t(T)}\phi(\omega d_-)$$

Where $P_t^f(T;b) = \frac{P_t(T;c)X_t(T)}{\chi_t}$

We obtain the following relation (equation 3.5 Clark):

$$\Delta_t^{\%,fwd}(T,K,\omega,\sigma) = \frac{\omega K}{X_t(T)}\phi(\omega d_-)$$

In order to obtain the strikes from the equations previously solved, we have used a numerical optimization method *(function python: optimize.fsolve).*

	Δ				
Expiry	-10%	-25%	50%	25%	10%
06-Feb-18	132,6415	133,7207	134,7700	135,7504	136,6295
13-Feb-18	131,7679	133,2928	134,7704	136,1595	137,4054
28-Feb-18	130,3745	132,6371	134,7700	136,7908	138,6350
30-Mar-18	127,5274	131,3025	134,7478	137,9766	140,8791
30-Apr-18	125,7326	130,5106	134,7377	138,7292	142,4319
30-Jul-18	122,1387	128,8739	134,7086	140,5032	146,2643
30-Jan-19	116,3757	126,3277	134,6304	143,3385	152,5585
30-Jan-20	108,0009	122,0293	134,1856	147,0442	162,2025
30-Jan-23	85,00075	112,9018	129,1440	157,6627	207,3618

In order to prove the accuracy of previously method showed, strikes have been obtained from the forward moneyness provided, reversing the following relationship:

$$FM_t(T) = \frac{K}{X_t(T)}$$

$$K = FM_t(T)X_t(T)$$

From results obtained, come to light an insignificant difference that seems to confirm the coherence of the numerical process followed.

3 **Si implementi un codice per valutare numericamente le equazioni di Dupire con il metodo di Eulero Implicito e si studi la convergenza delle volatilità implicite predette del modello verso quelle di mercato all'infittirsi delle griglie.**

In order to study the convergence of implicit predicted volatility of Local Volatility Model towards those of the market, the following method has been implemented:

a. Interpolation of the local volatility surface $\{\eta(T_i,k_j); i \in [1,N], j \in [1,M]\}$;

b. Definition of an expiry grid and strike, and valuation of the local volatility through interpolating function in the nodes as defined;
c. Resolution of the Dupire Equation through discretization with Euler Implicit Method, in order to obtaining the call options prices for considerate grid;
d. Calculation of implicit volatilities originated from the prices calculated with the Local Volatility Model, through the Black&Scholes formula reverserd;
e. Analysis of convergence of implicit volatility predicted by the model with those of the market.

Firstly, the interpolating function of Local Volatility surface has been calculated using a cubic spline for the interpolation between the nodes; we have used a constant value for extrapolation, and a constant at times along the direction of expiry dates. In this way, the following interpolating surface has been obtained.

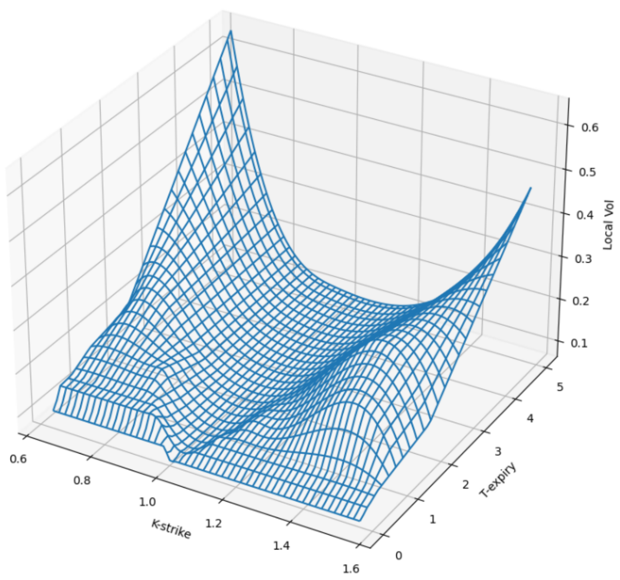

Figure 1 - Local Volatility

Secondly, a strike and expiry grid has been defined in order to valuating the Local Volatility and calculating the call options prices through the resolution of the Dupire Equation, with the Euler Implicit Method:

$$\partial_T c(T,k) - \frac{1}{2} k^2 \eta^2(T,k) \partial_k^2 c(T,k) = 0$$

$$c(0,k) = (1-k)^+$$

In particular, the following discretization has been considered along times and strikes, and the following approximations of first and second derivatives:

$$\{t = T_0, \dots., T_i = T\}\ i = 0, \dots, N$$

$$\{L = k_0, \dots., k_j = U\}\ j = 0, \dots, M+1$$

$$\partial_T c(T,k) \simeq \frac{c(T,k) - c(T-\delta T,k)}{\delta T}$$

$$\partial_k^2 c(T,k) \simeq \frac{c(T,k+\delta k) - 2c(T,k) + c(T,k-\delta k)}{\delta k^2}$$

$$\xi_{i+1,j} = \frac{\delta T}{2\delta k^2} k^2 \eta^2(T_{i+1}, k_j)$$

Also, we have developed a suitable choice of approximation's values of strike's range starting from analysis of underlying distribution:

$$dx_t = \eta(t, x_t) x_t dW_t, \qquad x_0 = 1$$

$$x_t = \frac{\chi_t}{X_0(t)}$$

In particular, from Ito's Formula we can infer that:

$$\mathrm{dlog}(x_t) = \frac{1}{x_t} dx_t - \frac{1}{2x_t^2} x_t^2 \eta^2(t, x_t) dt$$

$$\mathrm{dlog}(x_t) = \frac{1}{x_t} \eta(t, x_t) x_t dW_t - \frac{1}{2x_t^2} x_t^2 \eta^2(t, x_t) dt$$

$$\mathrm{dlog}(x_t) = \eta(t, x_t) dW_t - \frac{1}{2} \eta^2(t, x_t) dt$$

$$\log(x_T) - \log(x_0) = \int_0^T \eta(t, x_t) dW_t - \frac{1}{2} \int_0^T \eta^2(t, x_t) dt$$

$$\log(x_T) = \log(x_0) + \int_0^T \eta(t, x_t) dW_t - \frac{1}{2} \int_0^T \eta^2(t, x_t) dt$$

$$x_T = \exp\left(\int_0^T \eta(t, x_t) dW_t - \frac{1}{2} \int_0^T \eta^2(t, x_t) dt\right)$$

From which, starting from log normal distribution of underlying:

$logx_T \backsim N(-\frac{1}{2}\int_0^T \eta^2(t, x_t)dt, \int_0^T \eta^2(t, x_t)dt)$ and, in order to approximate consistently the domain, avoiding to leave out relevant tails, the following extreme values have been chosen:

$$L = \exp\left(-\frac{1}{2} {\eta_T^*}^2 - 6\eta_T^* \sqrt{T}\right)$$

$$U = \exp\left(-\frac{1}{2} {\eta_T^{**}}^2 + 6\eta_T^{**} \sqrt{T}\right)$$

Where η_T^* is the maximum value of local volatility at expiry date T and η_T^{**} is medium value of local volatility at expiry date T.

Thanks to estimations previously described, we have obtained the following discretized equation:

$$\frac{c(T_{i+1}, k_j) - c(T_i, k_j)}{\delta T} - \frac{1}{2} k_j^2 \eta^2(T_{i+1}, k) \frac{c(T_{i+1}, k_{j+1}) - 2c(T_{i+1}, k_j) + c(T_{i+1}, k_{j-1})}{\delta k^2} = 0$$

$$\frac{c(T_{i+1}, k_j) - c(T_i, k_j)}{\delta T} - \frac{\delta k_j^2 \xi_{i+1,j}}{\delta T} \frac{c(T_{i+1}, k_{j+1}) - 2c(T_{i+1}, k_j) + c(T_{i+1}, k_{j-1})}{\delta k^2} = 0$$

$$c(T_{i+1}, k_j) - c(T_i, k_j) - \xi_{i+1,j}\left(c(T_{i+1}, k_{j+1}) - 2c(T_{i+1}, k_j) + c(T_{i+1}, k_{j-1})\right) = 0$$

$$c(T_{i+1}, k_j) - \xi_{i+1,j}\left(c(T_{i+1}, k_{j+1}) - 2c(T_{i+1}, k_j) + c(T_{i+1}, k_{j-1})\right) = c(T_i, k_j)$$

$$-\xi_{i+1,j}\, c(T_{i+1}, k_{j-1}) + (1 + 2\xi_{i+1,j})c(T_{i+1}, k_j) - \xi_{i+1,j}c(T_{i+1}, k_{j+1}) = c(T_i, k_j)$$

$$c(T_i, k_j) = (1 + 2\xi_{i+1})c(T_{i+1}, k_j) - \xi_{i+1,j}c(T_{i+1}, k_{j+1}) - \xi_{i+1,j}c(T_{i+1}, k_{j-1}) \;\; j = 1, \dots, M$$

$$c(T_i, L) = 1 \quad c(T_i, U) = 0 \quad i = 0, \dots, N$$

$$c(0, k_j) = (1 - k_j)^+ \; j = 1, \dots, M$$

To interpolate prices at expiry dates we have used a cubic spline, because we have noticed a better fitting of dates.

Figure 2 – Log - Error N=500

4 Si calcoli il prezzo delle opzioni digitali caratterizzate dal payoff $\mathbf{1}_{\{\chi_T > K\}}$ considerando le date di scadenza T e gli strike K delle opzioni plain vanilla quotate dal mercato

In order to price a digital option on exchange rate, we have used Breeden's result from Model Indipendent Pricing that permits to value the price of FX derivatives without introducing a developmental model of underlying. In particular, from Breeden's result we can infer that:

$$\partial_K^2 Call_t(T, K) = P_t(T) p_{\chi_t|t}(K)$$

For any European option with payoff $f(\chi_t)$, assuming determinists rates, we have:

$$V_t = \mathbb{E}_t[P_t(T) f(\chi_t)] = \int_0^{+\infty} P_t(T) f(s) p_{\chi_t|t}(s) ds = \int_0^{+\infty} f(s) \partial_s^2 Call_t(T, s) ds$$

In our specific case, we have:

$$f(\chi_t) = 1_{\{\chi_t > K\}}$$

From which we get, assuming *K* as strike value:

$$V_t(K) = \int_0^{+\infty} 1_{\{s > K\}} \partial_s^2 Call_t(T, s) dx = \int_K^{+\infty} \partial_s^2 Call_t(T, s) ds$$

To use this result, we need an instrument that permits to interpolate all the value at market along strikes.

Is also possible use the instrument built on previous step to price a call at every strike and at every expiry date of interest, discretizing Dupire Equation with Euler Implicit method. The choice was this one because doesn't need none modeling hypothesis about the evolution of underlying, this permits us to obtain coherent prices in relation to market quotation.