# Operative Research

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First Semester 2022/2023

# Linear Programming

Teorema: Cnsider a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\} = \{x : A^{=}x = b^{=}, A^{<}x \leq b^{<}\}$ It can be shown that $dim(P) = n - rank(A^{=})$. This implies that:

- a full-dimensional polytope cannot have any implicit equations
- a single point cannot have true inequalities and there are n linearly independent equations

$\alpha^T x \leq \beta$ is a valid inequality for P if satisfied by all $v \in P$. The set $F : H(\alpha, \beta) \top P(! = 0)$ is called a face of P, defined or induced by the inequality $\alpha^T x \leq \beta$ The most important faces of a polyhedron are those whose dimension is $= d-1$ where $d$ is the dimensions of the whole polyhedron, as described before In the definition of a polyhedron some inequalities may be redundant and could be removed. However the inequalities that result, that create facets are never implied by any other inequality and are thus essential. If one is removed the polyhedron changes.
Let ext(P) be tbhe set of vertices of P. A polyhedron P is pointed if $ext(P)! = \emptyset$. Some simple condition for being pointed are:

- P does not contain any line infinite in both directions
- P contains the inequalities $x_i \geq 0 f or i = 1, \ldots, n$
- P is a polytope

The minkowski-weil theorem states: A set $P \subseteq \mathbb{R}^n$ is a polytope if and only if $P = conv(S)$ for some finite set S of vectors in $\mathbb{R}^n$ In view of this theorem there are two possible representations of a polytope P:

- external representation: a set of hyperplanes such that P is the intersection of their half-spaces
- internal representation: a set of vectros such that P is their convex hull

# Modelling for Linear Programming

Per convenzione in problemi di minimo mettiamo tutti i vincoli $\geq$, viceversa per i massimi mettiamo $\leq$.

The goal function is a linear function. Each variable $x_i$ has a cost that depends, linearly, on the variable's value. By doubling the value, the cost also doubles and so on. Let $c_i$ be the

For many problems the objective function is non-linear. Start-up costs can be modeled using suitable binary variables. The decision is then shifted to how much more than the threshold $\tau$ do i want to produce? The meaning of the binary variable z is: "is y¿0?". The same reasoning can be applied multiple times to represent economies of scale setting up steps at which each individual unit lowers its cost.

### Linear Constraints

A disequality is loose for $(x_i, x_2, x_k) \leftarrow (\gamma_1, \gamma_2, \gamma_k)$

If a non-negative variable must take only values within a range [L,U], we enforce this condition by two constraints, $x \geq L$, $x \leq U$. There might be cases in which we want a variable to be either 0 or within the range [L,U]. The variable must then take values in the set (which is not convex)

$$X := \{0\} \cup [L, U]$$

we can enforce this by adding a new variable y:

$$x \geq Ly$$

$$x \leq Uy$$

Big-M Method: suppose we have a series of OR constraints, these create a union of half-spaces which is not convex.
Let M be "large enough" (a virtual infinity) such that

$$-M \leq a_i^T x \leq M$$

is always true when x satisfies all the remaining constraints of the model. If y is a binary variable, constraints as such:

$$a^T x \leq b + My \quad a^T x \geq b - My$$

Coming back to the union of half spaces we can introduce k binary variables $y_1, y_2, \dots y_k$ and k+1 constraints:

$$a_i^T x \leq b_i + My_i, \forall i = 1 \dots k$$

and

$$\sum_{i=1}^{k} y_i \leq k - 1$$

This forces at least one of these binary variables to be binding, creating a non-trivial constraints. Modelling the absolute value of a variable: let $x \in \mathbb{R}$ To make z be equal to $|x|$

$$z \geq -x \quad z \geq x$$

$$z \leq -x + My \quad z \geq x + M(1 - y)$$

## Linear Programming

A linear programming problem is the maximization or minimization of a linear functino over a set of real-valued vectros constrained by a finite number of linear inequalities and/or equations. An instance of a linear programming problem is a linear program.

We can define a matrix A whose rows are the vectors $a_1^T, \ldots, a_m^T$. Assume $|I| = m_1, |E| = m_2, |C| = n_1 and |U| = n_2$. We can partition A:

$$A^1 1, A^1 2$$

Canonical Form: When E= U $= \emptyset$ we say the LP is in canonical form and can be written in short: $\max\{c^T x : Ax \leq b, x \geq 0\}$

Standard Form: When I=U=$\emptyset$ we say that the LP is in standard form and can be written in short: $\max\{c^T x : Ax = b, x \geq 0\}$ It's always possible to transform a LP in general form into standard from. All inequalities can be turned into equalities by simply adding a slack variable to turn the disequality into an equality. Inversely a LP in standard form can be turned into a canonical form LP by splitting the equalities in two symmetric disequalities

## Simplex

The simplex algorithm is the algorithm that solves linear programming problems. The worst theoretical complexity of the simplex algorithm is exponential but the cases in which it behaves as such must be crafted on purpose. Otherwise, in a general scenario the algorithm is polynomial. The simplex solves an LP assuming it's in standard form, let the LP be:

$$max\{c^T x : Ax = b, x \geq 0\}$$

Let's denote P as the feasable set, P has vertices, possibly more than one and

$$dim(P) = d - rk(A) = d - m =: n$$

Once we are inside $P^=$ (an affine space of dimension n=d-m), we can think of (and visualize) P as a full-dimensional polyhedron of dimension n.
The LP can either be infeasable, unbounded or has optimum.

**Theorem 1** *The LP is unbounded if and only if there existts a vertex* $\bar{x} \in P$ *and a vector* $r$ *with* $c^T r > 0$ *such that the semi-line* $R := \bar{x} + \lambda r, \lambda > 0$ *is an edge of P (a face of dimension 1)*

The direction r is called an unbounded improving extreme ray of P. Any algorithm which at some point detects an unbounded improving extreme ray can stop and report: problem unbounded.
The other important possibility is when there exists an optimum. In this case at least one of the vertices of P must be optimal.

**Theorem 2** *Assume* $x*$ *is an optimal solution of LP. Then there exists at least one verex of P which also is an optimal solution. Proof: Let* $c_0 := c^T x*$ *be the optimal value and cosider the inequality* $c^t x \leq c_0$*. This inequality is valid. Let* $F := P \cap x : c^T x ? c_0$ *be the set of all optimal solutions. We have* $x^* \in F$ *and so F is a face of P. Since P has vertices also F has vertices and the vertices of F are vertices of P.*

From the above discussion an algorithm could move from one vertex to another looking for the best one.

1. if none of the edges meeting at $v^i$ is imporoving, stop $v^i$ is optimal (in convenx programming local optimality implies global iptimality)

2. if one of the improving edges is an unbouded extreme ray, stop: problema unbounded

3. otherwise follow an improving edge up to the next verteex $v^{i+1}$ and repreat from there

**Vertices and basic solutions**

P has dimension n. Its faces are associated to the constraints defining $P^{>}$. A vertex is identified by the facets intersercting at v (at least n). We can indeed say something stronger: the columns of $A^j$ such that $v_j > 0$ are linearly independent (this implies that at each vertex there are at least n components = 0)

Insert theorem

Let us call the tyupe of solutions like those happeneing at the verices of P basic solutions. Namely, a solution is basic if the columns corresponding to its non-zero comonents are linearly independetnt.

- a basic solution in which the linearly independet columns are in fact a basis of $\mathbb{R}^m$ so they are m, is also called non-degenerate
- an independent solution in which the linearly independet columns are less than m is also called degenerate

We have seen that all vertifes are basic solutions. We can easily show the converse, i.e. that each basic solution is a vertex of P.

Insert Theorem

Let us call a set $\mathcal{B}$ of columns a feasible basic set in A if the columns are linearly independent and $b = \sum_{i \in \mathcal{B}} \lambda_i A^i$ for some $\lambda > 0$

Therefore:

- the cost of a vertex becomes costo fo a feasible basic set,
- the search for the best vertex becomes the search for the best feasible basic set
- the adjacency of the vertices in P becomes adhacency of two feasible basic sets and so on.

The geometrical problem has become an analytical problem, consisting in determining a sequence of baisc sets of aA until the best one is found.
Given a feasible basic set $\mathcal{B}$, it can always be extended to a basis B. Let $A_B$ be the corresponding submatrix of A. Then $\lambda$, which are the coordinates of b w.r.t., are the non-zero coordinates of b w.r.t. B. These last can be obtained as:

$$x_B := A_B^{-1} b$$

So the basic solution corresponding to $\mathcal{B}$ is

$$x(\mathcal{B}) = (X_b, x_{[d]\mathcal{B}}) = (A_B^{-1} b, 0)$$

The cost of the feasible basic set $\mathcal{B}$ is then

$$\sum_{i=1}^{k} c_{j_i} x_i$$

The simplex algorithm works with bases rather than baisc sets. If there is no degeneracy it's the same, but in presence of degernarcy a basi set can be extende dto a basis in many ways, and therefore to a verx of P there can correspond more than one basis.

Given a basis B, let N =[d] -B. Let A$_B bethem \times msubmatrixofAconsisteingofthecomulnsinB, andA_n them \times nsubmatrixofthecolumnsinN.Wepartitionxasx?(x_B, x_N) wherex_B arethebasicvariablesandX_N arethenon-basicvariables.Simil\grave{o}arly, wepartiioncasc = (x_B, C_N) Let\bar{A} = A^-1_B and\bar{b} = A^-1_b \bar{A}_N arethecoefficientsofthenon-basicvariables, and\bar{b}istherhs, whenthesystemAx = bisputinechelonformwrtthebasicvariablesA_B x_B + A_N x_N = bbecomesIx_B +_N$
$x_N = \bar{b}$