Operations Research Exam. 28/06/2017 - Solution sheet

Problem 1

A businessman is going to operate abroad for one month with his firm, and he needs to acquire a few SIMs to use for phone calls and internet. In the country he will move to there are two mobile providers, company 1 and company 2. Each of them offers a 1-month plan, which can be sumarized as follows:

	Company 1	Company 2
Monthly cost	20 Euros	15 Euros
Free calls	1000	800
Cost of each extra call	10 cent	7 cent
Free internet	4GB	3 GB
Extra data traffic	0.5 Euro / 500MB	0.2 Euro / 100MB

Notice that

1. Free calls are **only vs numbers of the same provider**. Each call vs another company, or a non-mobile number, or vs the same provider but in excess of the maximum limit, are considered extra calls.

2. Extra-data traffic is billed at the maximum even if it is used only partially. I.e., if we use 300MB extra in company 1, we have still to pay 0.5Euro and not 0.3. Similarly, if we use 50MB extra in company 2, we must still pay 0.2Euro

The firm is going to make phone calls directed to clients and partners living in the foreign country. As far as these phone calls are concerned, it is expected that there will be:

Number of calls versus Company-1 clients	4100
Number of calls versus Company-2 clients	2600
Number of calls versus non-mobile numbers	1200

Furthermore, the firm will need to use the internet, for a total expected traffic of **30 GB**.

Write an Integer Linear Programming model to find out how many SIMs of company 1 and how many of company 2 the businessman needs to buy in order to fulfill his needs at minimum cost. Compute the cost of the solution in which the businessman acquires 4 SIMs of company 1 and 3 of company 2.

Solution

Let us introduce an integer variable x_i, $i = 1, 2$ to denote how many SIMs of company i he buys.

We denote by y_{ij} (with $i = 1, 2$ and $j = 1, 2, 3$) the number of extra calls made from SIMs of company i to numbers of company j (where $j = 3$ stands for "non-mobile"). We also denote by w_i ($i = 1, 2$) the number of extra-data packages he buys (where a package has 500MB in company 1 and 100MB in company 2).

A possible ILP model is

$$\min 20x_1 + 15x_2 + 0.1(y_{11} + y_{12} + y_{13}) + 0.07(y_{21} + y_{22} + y_{23}) + 0.5w_1 + 0.2w_2$$

subject to

$$1000x_1 + y_{11} + y_{21} \geq 4100 \tag{1}$$

$$800x_2 + y_{21} + y_{22} \geq 2600 \tag{2}$$

$$y_{13} + y_{23} \geq 1200 \tag{3}$$

$$40x_1 + 5w_1 + 30x_2 + w_2 \geq 300 \tag{4}$$

Some comments: inequalities (1) and (2) are $\geq$ rather than $=$. Putting $=$ is a minor error. If we put $=$ we might get absurd situations, such as if we need to make 13 calls and the plan has 1000 free calls, the 13 should have to be all extra or there is no solution. To use $=$ the constraint should be, e.g.,

$$y_{11} + y_{21} = \max\{4100 - 1000x_1, 0\}$$

which is not a linear equation.

(Another possibility would be to introduce a variable z_i ($i = 1, 2$) to represent the number of free calls made from SIM of company i. Then we would have

$$z_1 + y_{11} + y_{21} = 4100 \qquad z_2 + y_{21} + y_{22} = 2600$$

$$z_1 \leq 1000x_1 \qquad z_2 \leq 800x_2$$

)

Constraint (4) is more interesting. We find the maximum common unit of data traffic, i.e., the block of 100MB, and start thinking in terms of that block size. Then, plan 1 gives 40 free blocks and then each extra package gives 5 blocks at a cost of 0.5 Euro. Plan 2, gives 30 free blocks and then each extra package gives 1 block at a cost of 0.2 Euro. The total need is 300 blocks.

We can also notice that the variables y_{12} and y_{13} are useless. Indeed, each extra call of company 1 should be made from company-2 SIMs since they are either free or cheaper.

(As a final consideration, notice that there should be big-M constraints, such as

$$\sum_{j=1}^{3} y_{ij} \leq Mx_i, \qquad i = 1, 2$$

to make sure that extra calls can be made only if we have at least purchased a plan. These constraints are not needed with reasonable numbers (such as ours in the example), since, e.g., it is better to get the 1000 free calls at cost 20 euro than paying for 1000 extra calls. However, if the numbers can be any values there should be these constraints to enforce the meaning of the extra calls. Think of an absurd situation such as the plan costs 100 Euro and gives 10 free calls, and then the extra calls are 1cent each. Then the solution would use only extra calls without purchasing the plan, which is impossible in reality. Not making this kind of considerations is a minor error, given that in reality -and in our instance- this situation does not arise)

Let us compute the value of the solution in which we use four SIMs of company 1 and three of company 2.

First, there is the cost of the plan:

$$20 \times 4 + 15 \times 3 = 125$$

Then, we make the calls

1. to company 1: 4000 from company 1 (free) and 100 from company 2 at a cost of 7 euro
2. to company 2: 2400 from company 2 (free) and 200 from company 2 at a cost of 14 Euro
3. to non-mobile: 1200 from company 2 at a cost of $1200 \times 0.07 = 84$ euro

Finally, the free traffic is $4 \times 4 + 3 \times 3 = 25$GB. The extra 5GB of traffic should be all done from SIM-1 since it is cheaper, and it costs $0.5 \times 10 = 5$ Euro.

In conclusion, the cost of this solution is

$$125 + 7 + 14 + 84 + 5 = 235$$

euro.

Problem 2

Write the dual of the following LP model:

$$\min 3x_2 - x_3 + 2x_5$$

subject to

$$x_1 - 4x_2 + 4x_4 \geq 2$$

$$-x_1 + 2x_2 + x_3 \leq -1$$

$$x_2 + x_3 - x_4 - x_5 = 6$$

$$x_1 + x_4 - 2x_5 \leq 2$$

$$x_2 - x_3 = 4$$

$$x_1 \geq 0, \quad x_4 \geq 0, \quad x_5 \geq 0$$

Solution

We first rewrite the model so as all inequalities are $\geq$, since the objective is a minimization:

$$\min 3x_2 - x_3 + 2x_5$$

subject to

$$x_1 - 4x_2 + 4x_4 \geq 2$$

$$x_1 - 2x_2 - x_3 \geq 1$$

$$x_2 + x_3 - x_4 - x_5 = 6$$

$$-x_1 - x_4 + 2x_5 \geq -2$$

$$x_2 - x_3 = 4$$

$$x_1 \geq 0, \quad x_4 \geq 0, \quad x_5 \geq 0$$

There will be 5 dual variables, $u_1, \ldots, u_5$. Since the 3rd and 5th constraints are equations, then u_3 and u_5 are unconstrained while $u_1, u_2, u_4 \geq 0$. Similarly, the 2nd and 3rd constraints of the dual will be equations.

The dual is

$$\max 2u_1 + u_2 + 6u_3 - 2u_4 + 4u_5$$

subject to

$$u_1 + u_2 - u_4 \leq 0$$

$$-4u_1 - 2u_2 + u_3 + u_5 = 3$$

$$-u_2 + u_3 - u_5 = -1$$

$$4u_1 - u_3 - u_4 \leq 0$$

$$-u_3 + 2u_4 \leq 2$$

$$u_1 \geq 0, \quad u_2 \geq 0, \quad u_4 \geq 0$$

Multiple-choice questions

1. (T/F) If we use branch and bound to solve a polynomial problem (such as minimum spanning tree, maximum flow, shortest path...), the running time will always be polynomial

   SOL: false, the B-and-B may take exponential time, expecially if the bound is poor

2. (T/F) Let $G = (\{1, \ldots, n\}, E)$ be a connected graph, weighted on the edges. The weight of each edge is 3. Let T be a tree of shortest paths from node 1 to all other nodes. Then T is a minimum spanning tree of G.

   SOL: true: each spanning tree is a MST, since each tree has the same cost, i.e., $3(n-1)$ (since every spanning tree has exactly $n-1$ edges)

3. (T/F) Same question as before, only that now T is a tree which happens to be made by longest paths from node 1 to all other nodes.

   SOL: true, for the same reason as before

4. (T/F) Each matrix with 0,1 entries which has exactly two '1' in each column is totally unimodular

   SOL: false. See, e.g., the matrix
$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$
   for which $det(A) = 2 \notin \{0, 1, -\}$

5. Let $G = (V, E)$ be a connected graph weighted on the edges. Let the edges be $e_1, \ldots, e_m$ and assume that
$$c(e_1) \leq c(e_2) \leq \cdots \leq c(e_m) \tag{5}$$
   Which the following statements are always true?

   (a) e_1 is in each minimum spanning tree of G

   (b) if all the inequalities in (5) are strict then e_1 is in each minimum spanning tree of G

   (c) if $c(e_j) \neq c(e_k)$ for each pair of edges $1 < j < k$ then e_1 is in each minimum spanning tree of G

   (d) if $c(e_1) = c(e_2) = c(e_3)$ then e_1 is not in each minimum spanning tree of G

   (e) if $m > n - 1$ then there is at least one minimum spanning tree that does not contain e_m

   SOL:

   (a) is false (think of a complete graph and each edge has cost 1. Then, every spanning tree is a MST, including those without e_1).

(b) true: in this case e_1 is the edge of minimum cost and it is unique, so it belongs to each MST (e.g., Kruskal will select it as the first edge)

(c) true: in this case, it is not guaranteed that e_1 is the unique minimum (it might be $c(e_2) = c(c_1)$) but, if it is not, it will be selected as first or second edge by Kruskal's algorithm (it cannot yield a cycle when it is selected as first or second, so it will always be added to the MST)

(d) false. It depends on the graph G. For instance, if e_1 is the only edge incident on a certain node, it will be selected in each MST

(e) false. The edge e_m might be needed in every spanning tree (e.g. if it leads to a node of degree 1)

6. (T/F) We recall that, given a graph $G = (V, E)$, its complement $\bar{G} = (V, \bar{E})$ is defined by $ij \in \bar{E}$ iff $ij \notin E$.

   For a set $S \subset V$, with $0 < |S| < |V|$, let us denote by

   $$\delta(S) = \{ij \in E : i \in S, j \in V - S\}$$

   and

   $$\gamma(S) = \{ij \in \bar{E} : i \in S, j \in V - S\}$$

   Notice that $\delta(S)$ is a cut in G and $\gamma(S)$ is a cut in $\bar{G}$.

   Then $\delta(S)$ is a minimum-size cut of G if and only if $\gamma(S)$ is a maximum-size cut of $\bar{G}$.

   SOL:

   The statement is false. In fact, for a given partition $(S, V - S)$ of the nodes, it is always

   $$\delta(S) + \gamma(S) = |S| \times |V - S|$$

   If the right-hand side of this equation were a constant, then it would be true that by minimizing $\delta(S)$ we are indeed maximizing $\gamma(S)$, but the rhs is not a constant. Consider, for instance, the case that G is a path $(1, \ldots, n)$. Then, the partition $S = \{1\}$, $V - S = \{2, \ldots, n\}$ is certainly a min-cut, since $|\delta(S)| = 1$. In this case, $|\gamma(S)| = (n-1) - 1$. However, the partition $S' = \{1, 2\}$, $V - S' = \{3, \ldots, n\}$ has the same value $|\delta(S')| = 1$, but $|\gamma(S')| = 2(n-2) - 1 > |\gamma(S)|$. The largest difference is when $S = \{1, \ldots, \lfloor n/2 \rfloor\}$, in which case $|\delta(S)|$ is still 1 and $|\gamma(S)|$ is about $n^2/4$

7. Given a graph $G = (V, E)$ weighted on the edges by d_e, $e \in E$, and two constants $\lambda \geq 0$ and $K \geq 0$ let us change the cost of each edge as

   $$d'_e := \lambda\, d_e + K$$

   Let H be an optimal solution of TSP on d (a tour of G minimizing $\sum d_e$) and H' an optimal solution on d'(a tour of G minimizing $\sum d'_e$).

   Which of the following statements are true?

(a) H' is also optimal for d (irrespective of λ, K)

(b) If $\lambda > 0$ then H' is optimal for d, irrespective of K

(c) If $K > 0$ then H' is optimal for d, irrespective of λ

(d) H is optimal for d' (irrespective of λ, K)

SOL:

(a) is false. In fact, if, e.g. $\lambda = K = 0$ then each solution on d' would be optimal and so H' has no relation to H

(b) is true. In fact, in this case it is $d'(H) = \lambda d(H) + nK$, where $n = |V|$ so that, for two tours A and B, it is $d'(A) \leq d'(B)$ if and only if $\lambda d(A) \leq \lambda d(B)$, i.e., if and only if $d(A) \leq d(B)$.

(c) is false. In fact, if $\lambda = 0$ then each solution on d' would be optimal (of cost nK) and so H' has no relation to H

(d) is true. Assume $d(H) \leq d(X)$ for each tour X. Then for each $\lambda \geq 0$ it is also $\lambda d(H) \leq \lambda d(X)$, and, adding nK to both sides, it is $\lambda d(H) + nK \leq \lambda d(X) + nK$, i.e.,

$$d'(H) \leq d'(X)$$

Hence H is an optimum for d'.