Operations Research Exam. 14/07/2017

General Rules No books, lecture notes, computers, cell phones, calculators or other electronic devices are allowed.

Each answer must be motivated, including answers to true/false and multiple choice questions (e.g., for a `true` answer, truth must be proved. For a `false` answer a counterexample must be given). The full solutions, with all details, must be reported on separate sheets. **Put your name and student number on each sheet that you hand in.** It is important that you are mathematically strict and rigourous in all your passages. No "common sense" reasoning and/or vague statement unsupported by a mathematical proof of its validity will be accepted.

NB: Before leaving the room, sign the exam text and return it to the professor.

Name:

Student number:

Course:

() Ingegneria Gestionale () Informatica () Informatica Internazionale

Signature:

Problem 1

A zoo must transfer its n animals to a new location. In order to do so, the animals will be put in cages and shipped to their destination by train, all in the same trip.

The cages are all identical, and they have a capacity C (say, expressed in cubed meters). Each animal i takes up some space in a cage, which we denote by s_i. Therefore, if a set S of animals are put in the same cage, it must be

$$\sum_{i \in S} s_i \leq C$$

Not all animals are compatible with each other. E.g., if any lion and any sheep are put in the same cage, the lion will eat the sheep. The compatibilities of the animals are expressed by a directed graph $G = (V, A)$ where

- V is the set of all animals
- there is an arc (i, j) for each pair of animals i and j such that i would eat j if put in the same cage

Write an Integer Linear Programming model which, given as input G, C, and $s_1, \dots, s_n$, determines how to put the animals in the cages so as to use the minimum possible number of cages.

Solution

No more than n cages will ever be needed. Assume we label n cages as $1, \dots, n$ and assign a 0-1 variable y_i for $i = 1, \dots, n$ to denote if cage i is used or not.

The objective is

$$\min \sum_{i=1}^{n} y_i \tag{1}$$

We also use 0-1 variables x_{ij} for $i = 1, \dots, n$ and $j = 1, \dots, n$ to specify if animal i is put in cage j. The constraints are

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i = 1, \dots, n \tag{2}$$

that says that each animal must be put in one cage. Moreover we have

$$\sum_{i=1}^{n} s_i x_{ij} \leq C y_j \qquad \forall j = 1, \dots, n \tag{3}$$

These are capacity constraints that say that the animals put in the same cage must not overflow the cage capacity (which is 0 if the cage is not used and C if it is used).

Finally

$$x_{uj} + x_{vj} \leq 1 \qquad \forall (u, v) \in A, \quad \forall j = 1, \dots, n$$

which says that no pair of incompatible animals can be put in the same cage

Problem 2

Consider the complete graph $G = (V, E)$, with nodes $V = \{1, \ldots, 6\}$ and edges $E = \{\{i, j\} : 1 \leq i < j \leq 6\}$. Let the cost of each edge $\{a, b\}$ be defined as

$$c(a, b) := |\,\mathrm{lcm}(a, b) - 15\,| + |\,\min(a, b) - 4\,|$$

(we remind that $\mathrm{lcm}(n, m)$ is the least common multiple of n and m. For instance, $\mathrm{lcm}(6, 15) = 30$, $\mathrm{lcm}(2, 6) = 6$ etc.) Determine the Minimum Spanning Tree of G by using Kruskal's and Prim's algorithms.

- Kruskal: write down the sequence of edges as they are considered by the algorithm, specifying if the edges are inserted in the tree or discared.
- Prim: Start growing the tree from node 1. Write down the sequence of edges inserted in the tree by the algorithm.

Solution

First of all we must compute the cost of the edges $\{i, j\}$. The following table reports all costs

	j=2	j=3	j=4	j=5	j=6
i=1	16	15	14	13	12
i=2		11	13	7	11
i=3			4	1	10
i=4				5	3
i=5					16

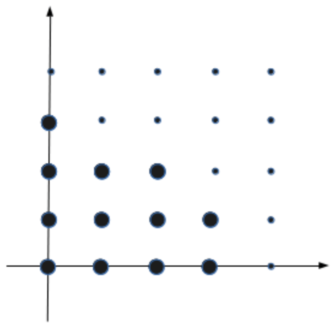
Kruskal's procedure considers the following edges (A=Accepted, D=Discarded):
A $(3, 5)$, A $(4, 6)$, A $(3, 4)$, D $(4, 5)$, A $(2, 5)$, D $(3, 6)$, D $(2, 3)$, D $(2, 6)$, A $(1, 6)$.
Prim's algorithm, starting from node 1, inserts the following edges:
$(1, 6), (4, 6), (3, 4), (3, 5), (2, 5)$.

Problem 3

Write a set of linear inequalities (constraints of a Linear Programming problem) in $\mathbf{R}^2$ such that the points of integer coordinates contained in the polytope of the feasible solutions are exactly the following ones

$$\{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1)\}$$

(which are drawn as larger circles in the following figure)



Solution

There are many possible solutions. The tightest would be a set of inequalities that precisely define the convex hull of the integer points, but this is not necessarily required by the problem.

Let $O = (0,0)$, $A = (0,3)$, $B = (2,2)$, $C = (3,1)$, $D = (3,0)$ Consider the following valid constraints:

1. $x_1 \geq 0$ (line through O, A)
2. $x_2 \geq 0$ (line through O, D)
3. $x_1 \leq 3$ (line through C, D)
4. $x_2 \leq 3$
5. $x_1 + 2x_2 \leq 6$ (line through A, B)
6. $x_1 + x_2 \leq 4$ (line through B, C)

Then, for instance, inequalities 1., 2., 3., and 5. are a possible solution.

Adding any other of the inequalities to these four is, of course, another possible solution. The inequalities 1., 2., 3., 5. and 6. define the convex hull of the integer points.

True-False questions

1. Let $G = (V, E)$ be a directed graph with positive weights $w(i)$ defined for each vertex $i \in V$. For each pair of nodes i, J, let us define the lenght of the arc (i, j) as $l(i, j) := 5w(i) - 2w(j)$. Then the Traveling Salesman Problem (TSP) on the complete graph with these lengths is difficult (i.e., NP-hard).

SOL: The statement is false. In fact, each TSP solution (i.e., a permutation of the n cities, corresponding to a hamilton cycle in G) has the same cost under these lengths, and so each solution is optimal. To see, this, consider any cycle

$$(\pi_1, \pi_2, \ldots, \pi_n, \pi_{n+1} = \pi_1)$$

Then, the length of this cycle is

$$(5w(\pi_1) - 2w(\pi_2)) + (5w(\pi_2) - 2w(\pi_3)) + (5w(\pi_3) - 2w(\pi_4)) + \cdots + (5w(\pi_n) - 2w(\pi_1))$$

which is equal to

$$3 \sum_{i=1}^{n} w(i)$$

2. Let G be a graph with nodes $1, \ldots, n$ and positive weights on the edges. Let T be a tree of shortest paths from node 1 to all other nodes and let T' be a minimum spanning tree. Then T and T' have at least one edge in common.

   SOL: true. Consider Dijkstra's algorithm for the shortest path and Prim's algorithm for the MST. Wlog, assume $(1, 2)$ is a cheapest edge out of 1. If the cheapest edge is unique, then $(1, 2)$ is added to the tree of shortest paths and to the MST and so it is in common. Otherwise, assume, for instance, that $(1, 3)$ has the same cost as $(1, 2)$ so that it is possible that at the first iteration, Dijkstra selects $(1, 2)$ but Prim selects $(1, 3)$. However, since the costs are positive, the arc $(1, 3)$ is a shortest path from 1 to 3, and so, eventually, it will be added by Dijkstra to the tree of shortest paths.

3. For every $n \geq 1$, the $n \times n$ matrix $A = (a_{ij})$ such that $a_{ij} = 1$ for each $i, j \in \{1, \ldots n\}$ is totally unimodular.

   SOL: this is true. In fact, each square $k \times k$ submatrix B of A has the same property (each entry is $= 1$) and therefore $\det(B) = 0$ if $k > 1$ and $\det(B) = 1$ if $k = 1$. At any rate, $\det(B) \in \{0, 1, -1\}$ always.