Complexity and Information Theory

Pietro Marcatti

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1 Introduction

The information comunicated with a sentence, or any other way, is always dependant on the context. Same goes for the quality for the information. If an event has a low probability of occurring in a given context the information it provides is high, inversely if the probability is high the information is little.
We could try to describe information with a simple inverse model:

$$Information = \frac{1}{p(E)}, \quad \text{E=event}$$

$$p(E) \rightsquigarrow 0 \quad Information \rightsquigarrow \infty$$

$$p(E) \rightsquigarrow 1 \quad Information \rightsquigarrow 1$$

Instead if we were to adopt a logaritmic model:

$$Information = \log \frac{1}{p(E)} = -\log p(E), \quad \text{E=event}$$

$$p(E) \rightsquigarrow 0 \quad Information \rightsquigarrow \infty$$

$$p(E) \rightsquigarrow 1 \quad Information \rightsquigarrow 0$$

Let's consider an alphabet as a group of possible events for which we have a probability distribution:

$$A = a_1, a_2, a_3, \ldots, a_k$$

$$P = p_1, p_2, p_3, \ldots, p_k$$

meaning that p_i is the probability to observe the event a_1. We can then calculate the average quantity of information as follows:

$$\sum_{i=1}^{k} p_i \cdot (-\log p_i) = \underset{ShannonEntropy}{\mathbb{H}(P)} \tag{1}$$

1.1 Coin Flip Example

Fair Coin

$$A = \{H, T\}$$

$$P(H) = \frac{1}{2}, \quad P(T) = \frac{1}{2}$$

$$\mathbb{H}(P) = \frac{1}{2} \cdot (-\log \frac{1}{2}) + \frac{1}{2} \cdot (-\log \frac{1}{2}) = 1$$

Every time we filp the fair coin we expect to gain a bit of information. Unfari Coin

$$A = \{H, T\}$$

$$P(H) = \frac{9}{10}, \quad P(T) = \frac{1}{10}$$

$$\mathbb{H}(P) = \frac{9}{10} \cdot (-\log \frac{9}{10}) + \frac{1}{10} \cdot (-\log \frac{1}{10}) \rightsquigarrow 0.32$$

1.2 Properties of Entropy

- $\mathbb{H}(P) \geq 0$
- $\mathbb{H}(P) = 0$ if there is an event with probability 1
- $\mathbb{H}$ is continuous with respect to P
- If an event gets split the entropy should be additive
- $\mathbb{H}(p_1, p_2, \ldots, p_k) \leq \log k = \mathbb{H}(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k})$

To prove the last property we can first show that for a equi-distribution we get $\mathbb{H} = \log k$. To prove that this is also the maximum we use Jensen disequality:

$$\forall f, f''(x) < 0 \quad in\ [a, b], (a, b) \in \mathbb{R}^2$$

$$f\Big(\sum_{i=1}^{k} \lambda_i \cdot x_i\Big) \geq \sum_{i=1}^{k} \lambda_i \cdot f(x_i), \quad \sum \lambda_i = 1$$

2 Data Compression

Before we can talk about data compression it is necessary to explore the meaning and the functiong of encodings. Given two alphabets $A = a_1, a_2, \ldots, a_k$ and $B = b_1, b_2, \ldots, b_k$ an encoding is function:

$$\varphi : A^* \longrightarrow B^* \tag{2}$$

For a function to be a suitable encoding it must be at least an injective function (one-way function). In information theory and in data encoding in particular we refer to injective encoding as **uniquely decodable** encodings. Prefix codes are uniquely decodable codes that can be decoded without delay and are written in the form:

$$A = \{a_1, a_2, \ldots, a_k\} \xrightarrow{\varphi} B = \{b_1, b_2, \ldots, b_D\} \tag{3}$$

where $\varphi_i = |\varphi(a_i)|$

2.1 Kraft-McMillen 1958 Inverse Theorem

If φ is a uniquely decodable code, then:

$$\sum_{i=1}^{k} D^{1-l_i} <= 1 \tag{4}$$

D is the cardinality of the output alphabet, l_i is the lenght of the encoding for a_i. We can quickly see that, there can't be an uniquely decodable code that uses three encodings of length 1 to map over the binary alphabet:

$$A = \{a, b, c\}, \quad B = \{0, 1\}, \quad l_a = l_b = l_c = 1 \longrightarrow \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1 \tag{5}$$

Instead, if we were to insist on using a length 10 encoding for everyone of the input characters:

$$A = \{a, b, c\}, \quad B = \{0, 1\}, \quad l_a = l_b = l_c = 10 \longrightarrow \frac{1}{2^{10}} + \frac{1}{2^{10}} + \frac{1}{2^{10}} << 1 \tag{6}$$

2.1.1 Proof 1

Assume the code is uniquely decodable and it is a prefix code, we can sort the input alphabet so that the length of the encoding satisfies the following:

$$a_1 \leq a_2 \leq \ldots \leq a_k$$

A character with an enconding of length i generates a sub-tree rooted in a_i with height $l - l_i$. This sub-tree contains $D^(l - l_i)$ nodes that cannot be used in the encoding because they would share an encoded prefix.

Insert diagram
In the original three the following number of leaves

$$D^l - \sum_{i=1}^{k-1} D^{l-l_1}$$

The $k-1$ term that bounds the sum is needed because we need to make sure that one last leaf is available to assign it to the last character a_k.

$$
\begin{aligned}
D^l - \textstyle\sum_{i=1}^{k-1} D^{l-l_i} &\geq 1 \\
&\Big\downarrow \textit{Group by } D^l \\
D^l(1 - \textstyle\sum_{i=1}^{k-1} D^{-l_i}) &\geq 1 \\
&\Big\downarrow \textit{Divide by } D^l \\
1 - \textstyle\sum_{i=1}^{k-1} D^{-l_i} &\geq D^{-l} \\
&\Big\downarrow D^{-l} \textit{ is the last term of the sum} \\
1 &\geq \textstyle\sum_{i=1}^{k} D^{-l_i}
\end{aligned}
$$

2.1.2 Proof 2 - General case

Let's define $N(n,h)$ the number of strings of alphabet A^n having encoding length h. It must be true that $N(n,h) \leq D^h$ because φ is uniquely decodable.

$$D^{-l_1} + D^{-l_2} + \ldots + D^{-l_k} \leq 1 \tag{7}$$

$\forall n, n \in \mathbb{N}$ let's consider the object $(D^{-l_1} + D^{-l_2} + \ldots + D^{-l_k})^n$. If written as a product it takes the following form:

$$D^{-l_1 \cdot n} + D^{-l_1 \cdot (n-1) - l_2} + \ldots + D^{-l_k \cdot n}$$

As n tends towards infinity the exponent will behave differently depending on the value of the base. If the base is < 1 it will be bound between 0 and 1, and in any case < 1. If the base is > 1 it will grow towards infinity faster than any polynomial function. Lastly, if the base is $= 1$ it will remain constant. To prove the theorem we will demonstrate that the object is dominated by a linear function, meaning that it cannot be in the second case. All of the members' exponents follow this chain of disequalities: $l_1 \cdot n \leq exp \leq l_k \cdot n$. That is, because we sorted the character by encoding length, 1 being the shortest while k the longest.

$$
\begin{aligned}
&\underset{prob.0}{N(n,1)}D^{-1} + \ldots + N(n, l_1 \cdot n)D^{l_1 \cdot n} + \ldots + N(n, l_k \cdot n)D^{-l_k \cdot n} \\
&\qquad\qquad\qquad\qquad \leq D^1 D^{-1} + \ldots + D^l_k D^{-l_k} \leq l_k \cdot n
\end{aligned}
$$

Since the object is dominated by $l_k \cdot n$ which is linear, it must mean that it also grows at most linearly, meaning that the base of the exponent must be ≤ 1.

2.2 Kraft-McMillen Direct Theorem

If given $l_1, l_2, \ldots, l_k$ and D such that $\sum_{i=1}^{k} D^{-l_i} \leq 1$ there must exist a prefix code φ having $l_1, l_2, \ldots, l_k$ as lengths of the encoding.

2.2.1 Proof

Proof can be given by construction by producing the complete D-ary tree

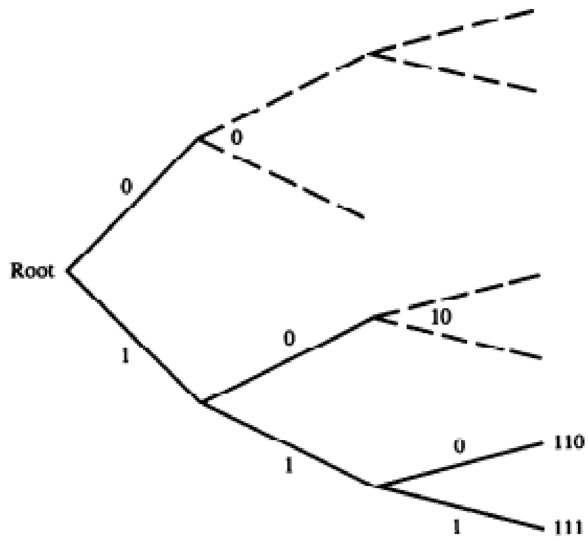

Figure 5.2. Code tree for the Kraft inequality.

Observation: Prefix codes compress as good as any other uniquely decodable code.

2.2.2 Average expected code length - Definition

Given the input alphabet A, P the probability and B the output we define the Expected Length as:

$$EL(\varphi) = \sum_{i=1}^{k} p_i \cdot l_i = \sum_{i=1}^{k} p_i |\varphi(a_i)| \tag{8}$$

Our goal is to find prefix codes that minimize EL.

2.3 1st Shannon Theorem 1948

If φ is uniquely decodable code then $EL(\varphi) \geq \mathbb{H}_D(P) = \sum_{i=1}^{k} p_i \cdot log_D p_i$ The proof relies on Kraft-McMillen theorem and this simple logarithmic property:

$$log_e x \leq x - 1, \quad -log_e x \geq 1 - x$$

2.3.1 Proof

$$
\begin{aligned}
&EL(\varphi) - \mathbb{H}_D(P) = \textstyle\sum_{i=1}^{k} p_i \cdot l_i + \sum_{i=1}^{k} p_i \cdot log_D p_i && \textit{Group by } p_i \textit{ and force } l_i \textit{ as } log_D \\
&= \textstyle\sum_{i=1}^{k} p_i \cdot log_D(D^{l_i} p_i) && \textit{Change to base e, take out the common term} \\
&= \tfrac{1}{log_e D} \cdot \textstyle\sum_{i=1}^{k} p_i \cdot log_e(D^{l_i} p_i) && \textit{Group by } -1 \\
&= \tfrac{-1}{log_e D} \cdot \textstyle\sum_{i=1}^{k} p_i \cdot log_e \tfrac{1}{D^{l_i} p_i} && \textit{Using the above property} \\
&\geq \tfrac{-1}{log_e D} \cdot \textstyle\sum_{i=1}^{k} p_i \cdot (\tfrac{1}{D^{l_i} p_i} - 1) \\
&= \tfrac{-1}{log_e D} \cdot \left(\textstyle\sum_{i=1}^{k} \tfrac{1}{D^{l_i}} - \underbrace{\cancel{\textstyle\sum_{i=1}^{k} p_i}}_{=1} \right) && \textit{Group by } -1 \\
&= \tfrac{1}{log_e D} \left(\underbrace{1 - \textstyle\sum_{i=1}^{k} \tfrac{1}{D^{l_i}}}_{KMM \longrightarrow \geq 0} \right) \geq 0
\end{aligned}
$$