On the Conservativity of First-order Minimalist Foundation over Arithmetic

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Conservativity

A central notion in the foundations of mathematics is the following.

Definition

Let T be a theory. We say that an extension T^+ of T is *conservative over* T if every judgment expressible in the language of T and provable in T^+, is already provable in T.

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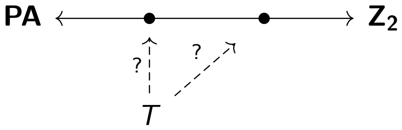
Example (Beeson, 1985)

The first-order fragment of Martin-Löf's type theory is conservative over Heyting Arithmetic.

Two uses for Conservativity

1. To determine the proof-theoretic strength of a theory T in terms of subsystems of second-order arithmetic.



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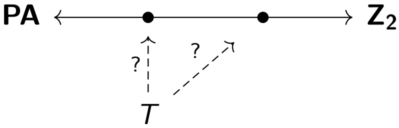

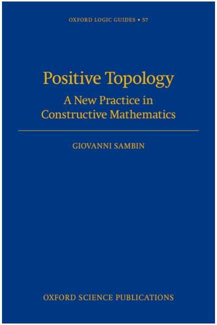

2. To reduce *ideal mathematics* to *real mathematics*, as advocated in G. Sambin. "Positive Topology. A New Practice in Constructive Mathematic". Oxford Logic Guides. Cary, NC: Oxford University Press, 2025.

The Minimalist Foundation

The *Minimalist Foundation* is a *common-core* foundation, compatible with the most relevant foundations of mathematics. Once you prove a theorem in it, you should be able to mechanically translate the proof into any other foundation.

M. E. Maietti, G. Sambin. "Toward a minimalist foundation for constructive mathematics". 2005

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Example

The Minimalist Foundation is compatible with the following foundational systems:

- Martin-Löf's type theory
- Constructive Zermelo-Fraenkel set theory
- Calculus of Inductive Constructions
- Internal language of toposes

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Three levels of abstraction

The Minimalist Foundation consists of two intuitionistic dependent type theories:

- an intensional one, called *minimal Type Theory*
- an extensional one, called *extensional minimal Type Theory*

H. Ishihara, M. E. Maietti, S. Maschio, T. Streicher. "Consistency of the intensional level of the Minimalist Foundation with Church's thesis and axiom of choice". 2018

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$$
\mathbf{MF}\left\{
\begin{array}{ll}
\mathbf{emTT} & \text{for mathematician}\\
\text{forget}\;\uparrow\downarrow\;\text{restore} & \\
\mathbf{mTT} & \text{for computer scientists}\\
\downarrow\;\text{implement} & \\
\widehat{\mathbf{ID}}_1 & \text{for machines}
\end{array}\right.
$$

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Proof-theoretic analysis of **MF** (I)

Open Problem

Determine the exact proof-theoretic strength of **MF**.

M. E. Maietti, P. Sabelli. "Equiconsistency of the Minimalist Foundation with its classical version". 2024

P. Sabelli. "Around the Minimalist Foundation: (Co)Induction and Equiconsistency". 2024

Proof-theoretic analysis of **MF** (I)

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Determine the exact proof-theoretic strength of **MF**.

Firstly, the following theorems get the additional complexity posed by the presence of two levels out of the way.

Theorem (Maietti, S.)

The theories **emTT** *and* **mTT** *are equiconsistent.*

Theorem

The theories **emTT** *and* **mTT** *prove the same second-order arithmetic statements.*

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Proof-theoretic analysis of **MF** (II)

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The Minimalist Foundation has the following proof-theoretic bounds.

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In this work, we focus on the *real* or *first-order* fragment $\mathbf{MF}_{set}$, for which we readily have a definite result.

Corollary (to Beeson's theorem)

$\mathbf{MF}_{set}$ *is conservative over* $\mathbf{HA}$.

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Warning

The whole Minimalist Foundation is *not* conservative over its real fragment because $\mathbf{MF}_{set} \cong \mathbf{HA} < \mathbf{ACA} \leq \mathbf{MF}$.
We conjecture that such a result can be achieved by weakening the induction of natural numbers to act only towards sets.

Enter Classical Logic

We want to determine the proof-theoretic strength of $\mathbf{MF}^c_{set}$, the *classical version* of $\mathbf{MF}_{set}$.

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... that $\mathbf{MF}^c_{set}$ is conservative over **PA**.

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... that $\mathbf{MF}^c_{set}$ is conservative over **PA**.

Warning

This is not obvious at all. For example, even if first-order Martin-Löf's type theory is conservative over **HA**, its classical version is way stronger than **PA** (it is, in fact, fully impredicative!).

The Double-Negation Translation

If φ is a first-order arithmetic formula, let φ^N be the formula obtained by prefixing a double-negation to each subformula of φ.

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Theorem (Gödel, 1933)

$\mathbf{PA} \vdash \varphi$ *if and only if* $\mathbf{HA} \vdash \varphi^N$.

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Theorem (Gödel, 1933)

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Idea

Lift Gödel's double-negation translation to **MF**.

$$
\begin{tikzcd}
\mathbf{PA} \arrow[r, hook, "?"] \arrow[d, "N"'] & \mathbf{MF}^c_{set} \arrow[r, hook] \arrow[d, dashed, "N"] & \mathbf{MF}^c \arrow[d, dashed, "N"] \\
\mathbf{HA} \arrow[r, hook, "cons."'] & \mathbf{MF}_{set} \arrow[r, hook] & \mathbf{MF}
\end{tikzcd}
$$

The Challenge

In dependent type theories, and especially in **MF**, logical and set-theoretical constructors are highly intertwined:

- terms appear in formulas through equality $a = b$ (as in predicate logic)
- types appear in formulas as domains of quantification $(\exists x \in A)\varphi(x)$
- formulas appear in types as in the quotient set constructor A/R
- formulas appear in terms as in the subset term constructor $\{x \in A \mid \varphi(x)\} \in \mathcal{P}(A)$.

...we need to extend Gödel's $(-)^N$ translation to every entity!

$(-)^N$ translation for the Minimalist Foundation

The definition of the translation turns out to be surprisingly simple. The relevant cases are the following.

$$((\exists x \in A)\varphi)^N :\equiv \neg\neg(\exists x \in A^N)\varphi^N$$

$$\mathcal{P}(A)^N :\equiv \{U \in \mathcal{P}(A^N) \mid (U^{\complement})^{\complement} = U\}$$

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We say that a proposition φ is *stable* if $\neg\neg\varphi \Rightarrow \varphi$.
We say that a type A has *stable equality* if $x =_A y$ is stable.

Theorem (Maietti, S.)

- *if A is a type, then A^N has stable equality;*
- *if φ is a proposition, then φ^N is stable;*
- *a judgment $\mathcal{J}$ is derivable in the classical version if and only if $\mathcal{J}^N$ is derivable in the intuitionistic version.*

Conservativity over Peano Arithemtic

We get to our main result.

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Proof.

Let φ be an arithmetical proposition, and assume $\mathbf{MF}^c_{set} \vdash \varphi$ *true*.

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1. By the previous theorem, we know $\mathbf{MF}_{set} \vdash \varphi^{N}$ *true*.
2. Since φ^{N} is still an arithmetical proposition, by conservativity of $\mathbf{MF}_{set}$ over $\mathbf{HA}$ we obtain $\mathbf{HA} \vdash \varphi^{N}$.

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2. Since φ^{N} is still an arithmetical proposition, by conservativity of $\mathbf{MF}_{set}$ over $\mathbf{HA}$ we obtain $\mathbf{HA} \vdash \varphi^{N}$.
3. Finally, by Gödel's theorem we conclude $\mathbf{PA} \vdash \varphi$.

□

Calculus of Constructions

The Calculus of (Inductive) Constructions[1] can be thought as the impredicative version of **MF**.

Recently, its conservativity over *Higher Order Heyting Arithmetic* has been established.[2]

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[2] Daniël Otten and Benno van den Berg. "Conservativity of Type Theory over Higher-Order Arithmetic". In: 32nd EACSL Annual Conference on Computer Science Logic (CSL 2024)

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Recently, its conservativity over *Higher Order Heyting Arithmetic* has been established.[2]

We can adapt the technique previously described to obtain the classical counterpart:

Theorem (Contente, S.)

The classical version *of the Calculus of Inductive Constructions is conservative over Higher Order* Peano *Arithmetic.*

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[2] Daniël Otten and Benno van den Berg. "Conservativity of Type Theory over Higher-Order Arithmetic". In: 32nd EACSL Annual Conference on Computer Science Logic (CSL 2024)

Conclusions

Often, classical logic does not interact well with predicative foundations from a proof-theoretic perspective.
In most cases, such as:

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- Homotopy Type Theory
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the addition of classical logic explodes the strength of the system.

On the other hand, we have shown that $\mathbf{MF}^{c}_{set}$, with all its rich type theoretical constructs, requires no more ontological commitment than good old Peano's axioms.

Thanks for your attention!