Double-negation for the Foundation of Constructive Mathematics

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Outline

Part I. Constructive Mathematics

Part II. Foundations of Constructive Mathematics

Part III. Double-Negation Interpretation

Part I. Constructive Mathematics

Figure: Ca' Dario, an (apparently haunted) Venetian palace.

Constructive Mathematics

The metaphors:

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Non-constructive proofs (i)

Theorem

There exist two irrational numbers a and b, such that a^b is rational.

Proof.

We know that $\sqrt{2}$ is irrational. Consider the number $\sqrt{2}^{\sqrt{2}}$. If it is rational, then we are done. Otherwise we have
$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2 \in \mathbb{Q}$. $\square$

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We did not construct an explicit choice for a and b. We know the pair is out there but cannot see it – just like a ghost.

Non-constructive proofs (ii)

Theorem

There exists a digit which occurs infinitely often in the decimal expansion of π.

Proof.

By contradiction, suppose that each digit occurs finitely many times, then the decimal expansion would be finite, but it is not since π is irrational. $\square$

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From a proof by contradiction we did not learn *which* digit(s) occurs infinitely often.

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Proof (First Theorem).

Take $a = \sqrt{2}$ and $b = \log_2(9)$. Then $a^b = 3$. $\square$

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Remark (Second Theorem)

Finding a constructive proof for the second theorem is still an open problem.

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To be constructive means giving proofs a **computational interpretation**:

- the proof of a disjunctive statement $\varphi \vee \psi$ consists of either a proof of φ or one of ψ.
- the proof of an existential statement $\exists x \in A.P(x)$ consists of **a witness** $a \in A$ and a proof of $P(a)$;
- the proof of a universal statement $\forall x \in A.P(x)$ consists of **an algorithm** that, given $a \in A$ as input, returns a proof of $P(a)$ as output.

Examples of computational interpretation (i)

Consider the following formulation of Euclid's theorem on the infinity of prime numbers.

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$$\forall n \in \mathbb{N}\, \exists p \in \mathbb{N}\, (\, p \ \mathit{prime} \wedge \ p > n\,)$$

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The original Euclid's proof is thus constructively valid.

Proof.

Given n, take as p a prime factor of $(n! + 1)$. $\square$

Examples of computational interpretation (ii)

Given a property P on the natural numbers, consider the statement *for each natural number, either P holds or it does not*.

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It is **constructively unprovable** in some others, e.g. when $P(n) =$ "the nth Turing machine never enters an endless loop." (Turing, 1936).

Examples of computational interpretation (iii)

Another example of a constructively unprovable statement.

$$\forall x \in \mathbb{R}\,(x = 0 \vee x \neq 0)$$

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$$\forall x \in \mathbb{R}\,(x = 0 \vee x \neq 0)$$

You cannot write a computer program which checks infinitely many digits.

However, the following weaker statement is constructively provable.

$$\forall x \in \mathbb{R}\,\forall n \in \mathbb{N}^+\,(|x| < \frac{1}{n} \vee |x| \geq \frac{1}{n})$$

Examples of computational interpretation (iv)

Quote

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Let $f : [0,1] \to \mathbb{R}$ *a continuous function. If* $f(0) < 0$ *and* $f(1) > 0$*, then there exists* $x_0 \in [0,1]$ *such that* $f(x_0) = 0$*.*

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Let $f : [0,1] \to \mathbb{R}$ *a continuous function such that, for each* $x \in [0,1]$*, either* $f(x) < 0$ *or* $f(x) > 0$*. Then* f *is either always positive or always negative.*

Part II. Foundations of Constructive Mathematics

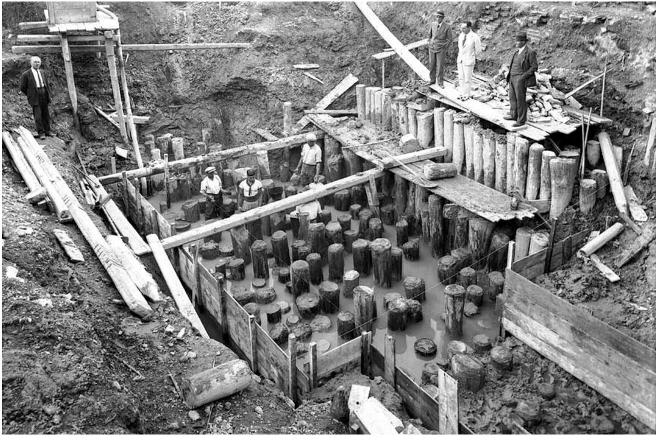

Figure: Wooden foundations of a Venetian building.

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2. Constructive mathematics is the kind of mathematics that constructs buildings just for human beings.
3. Logicians are people ~~delving in the mud~~ who ensure that the building does not collapse.

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It allows bifurcating a proof in two paths **without necessarily knowing** which is the true one.

Example

We used the fact that either $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ or $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$.

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The point is that it must be proved each time and can never be given for granted.

Negation in constructive mathematics

The excluded middle deals with **negation** in an essential way. Recall that the meaning of a negated statement $\neg\varphi$ is just

$$\varphi \Rightarrow \perp$$

where $\perp$ can be any contradiction you like, e.g. $0 = 1$ in arithmetic, $\exists x \forall y . y \in x$ in set-theory, $e = 2$ in analysis, etc.

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Let us also review **its computational meaning**:

- the proof of a negative statement $\neg\varphi$ consists of **an algorithm** that, given as input a proof of φ, returns a proof of a contradiction.

Negation in constructive mathematics

As an example, consider the ordinary proof of irrationality of $\sqrt{2}$.

Theorem

$$\neg\, \exists a, b \in \mathbb{N} \,.\, \sqrt{2} = \frac{a}{b}$$

Proof.
Suppose there exist natural numbers a and b such that $\sqrt{2} = \frac{a}{b}$. Then, $a^2 = 2b^2$. The prime factor 2 appears an even number of times on the left and an odd number of times on the right. Then we can conclude $0 \equiv 1 \pmod 2$, a contradiction. $\square$

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It is a constructively valid **proof of a negation**.

Double Negation Elimination

The law of **double negation elimination** asserts: "If the negation of a proposition is contradictory, then the proposition is true".

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Proof of a negation $\neg\varphi$ *(constructive)*
Suppose φ holds. Derive a contradiction to conclude $\neg\varphi$.

Proof of φ **by contradiction** *(non-constructive)*
Suppose $\neg\varphi$ holds. Derive a contradiction to prove $\neg\neg\varphi$.
Conclude φ using double negation elimination.

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The situation in constructive mathematics: we have a zoo of proposals with different scopes, philosophies, and techniques. Most of the time, they are mutually incompatible.

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Luckily, there is **MF**!

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- Extrinsically, since it is a common core compatible with all the most relevant existing foundations.

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- Intrinsically, since it does not assume strong principles (e.g. power-set, axioms of choice...)

Part III. Double-Negation Translation

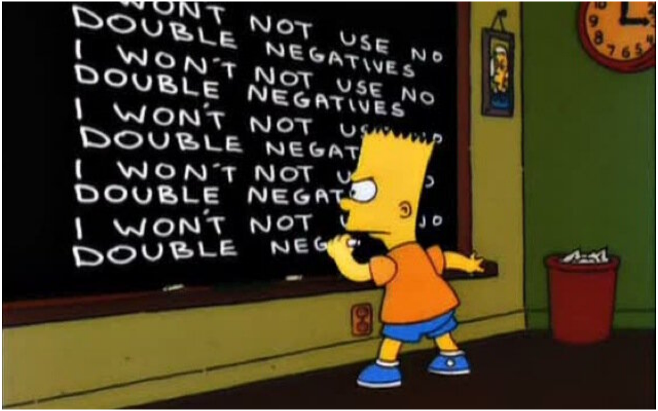

Figure: Simpson chalkboard gag S.11 E.6

The Goal

Recall

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Let $f : [0,1] \to \mathbb{R}$ *a continuous function. If* $f(0) < 0$ *and* $f(1) > 0$*, then there exists* $x_0 \in [0,1]$ *such that* $f(x_0) = 0$*.*

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Goal

Is there a mechanical way to reformulate every classical theorem so that it is also constructively provable?

The Result

Theorem

We can translate each proposition φ into another proposition $\varphi^{\mathcal{N}}$ such that:

- φ *and* $\varphi^{\mathcal{N}}$ *are classically equivalent;*
- φ *is classically provable if and only if* $\varphi^{\mathcal{N}}$ *is intuitionistically provable.*

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Gödel (1933)

Proved in the context of Peano Arithmetic.

Maietti, S.

Proved in the context of the Minimalist Foundation.

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Example

The propostion	$\forall x \in \mathbb{N}\,(x = n \vee \exists y \in \mathbb{N}\,.\,x = y + 1)$
is translated to	$\forall x \in \mathbb{N}\,.\,\neg\neg(x = n \vee \neg\neg\exists y \in \mathbb{N}\,.\,x = y + 1)$

Double-negation translation

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The following set defined by comprehension

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The non-trivial part of the proof consists in checking that the translation satisfies the theorem desiderata.

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Thank you for the attention...

...it is now *not* the case
that I am *not* going to wish you...

Happy Holidays!