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Velocity Control of a Wheeled Inverted Pendulum by Partial Feedback Linearization

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***Abstract*— In this paper, the dynamic model of a wheeled inverted pendulum (e.g. Segway [9], Quasimoro [8], Joe [6]) is analyzed from a controllability and feedback linearizability point of view. First, a dynamic model of this underactuated system is derived with respect to the wheel motor torques as inputs while taking the nonholonomic no-slip constraints into considerations. This model is compared with the previous models derived for similar systems. The strong accessibility condition is checked and the maximum relative degree of the system is found. Based on this result, a partial feedback linearization of the system is obtained and the internal dynamics equations are isolated. The resulting equations are then used to design a two-level controller for tracking vehicle orientation and heading speed set-points, while controlling the vehicle pitch within a specified range. Simulation results are provided to show the efficacy of the controller.**

I. Introduction

Mobile wheeled inverted pendulum models have evoked a lot of interest recently ([6], [7], [8]) and at least one commercial product (Segway) is available [9]. Such vehicles are of interest because they have a small foot-print and can turn on a dime. The kinematic model of the system has been proved to be uncontrollable [8] and therefore balancing of the pendulum is only achieved by considering dynamic effects. Ha et al [5] developed an early prototype with a trajectory controller based on linearization of dynamic equations. Using a similar approach, Grasser et al [6] derived a dynamic model using a Newtonian approach and the equations were linearized around an operating point to design a controller. In Salerno et al [8], the dynamic equations were studied, with the pendulum pitch and the rotation angles of the two wheels as the variables of interest. Various controllability properties of the system in terms of the state variables were analyzed using a differential-geometric approach.

In contrast, in this paper, the dynamic modeling is done directly in terms of variables which are of interest with respect to the planning and control of vehicle's position and orientation. A Lagrangian approach is used to derive the equations and the nonoholonomic constraint forces are eliminated using the approach in [4]. The equations are then simplified, reduced in order and then checked for the strong-accessibility condition. It has been proved, using a result from [1], that the maximum relative-degree possible for the system is 4. A set of outputs is then chosen with the correct total relative degree and the system is put in the normal form [3] with the internal dynamics singled- out. In Sec. V, a two level controller is designed which makes use of the partial feedback linearization results. The controller keeps the vehicle pitch within specified limits and tracks the given orientation and heading-speed set-points which are assumed to be coming from a higher-level controller for motion planning.

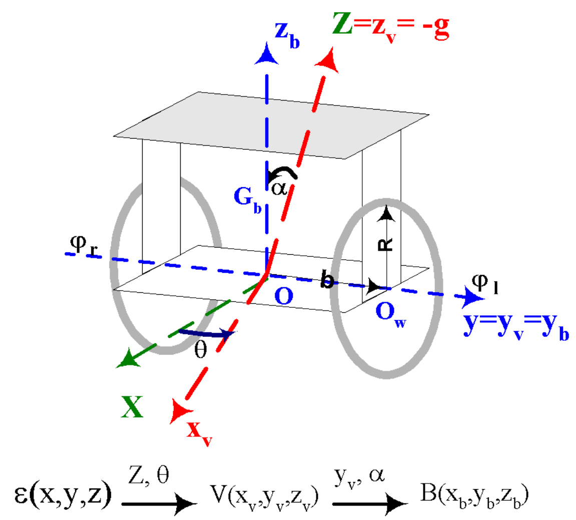

Fig. 1. Geometric parameters and coordinate systems for the system

II. Dynamic Model

Referring to Fig. 1, b is distance OO_w, where O is the point midway between the two wheel centers. R is the radius of both wheels. The pendulum body parameters have the subscript b, the wheel parameters have subscript w. $\mathcal{E}(\mathbf{X},\mathbf{Y},\mathbf{Z})$ is an inertial frame. $\mathcal{B}(\mathbf{x_b},\mathbf{y_b}=\mathbf{y},\mathbf{z_b})$ is a frame attached to the pendulum body. $\mathcal{V}(\mathbf{x_v},\mathbf{y_v}=\mathbf{y},\mathbf{z_v}=\mathbf{Z})$ are the vehicle-fixed coordinates. The pitch angle α is $\angle(\mathbf{Z},\mathbf{z_b})$, and the vehicle orientation angle θ equals $\angle(\mathbf{X},\mathbf{x_v})$. The center of mass of the pendulum body G_b is at coordinates $OG_b=(c_x,0,c_z)$ in $\mathcal{B}$. c_x would later be taken as 0 to simplify equations. M_b is the mass of the pendulum body.

$$I_{b/\mathcal{B}} = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{pmatrix},\ I_{xy}=0, I_{yz}=0,$$

is the inertia matrix of the pendulum body about its center of mass G_b, in the basis $(\mathbf{x_b},\mathbf{y_b},\mathbf{z_b})$. $I_{xz}=0$ if we take $c_x=0$ and also assume that the body is symmetric about the $\mathbf{x_b}$ axis. $[I_{wa}, I_{wd}]$ are the moment of inertia of a wheel about its axis and about a diameter respectively and M_w is its mass. ϕ_r and ϕ_l are the angles of rotation of the right and the left wheel respectively. The position-vector of point O in $\mathcal{E}$ is (x_o, y_o, R).

The *rotational* kinetic energy of the pendulum body T_b^R, and its *translational* kinetic energy are computed as

$$2T_b^R = \mathbf{\Omega}_{\mathcal{B}/\mathcal{E}}^T I_{b/\mathcal{B}} \mathbf{\Omega}_{\mathcal{B}/\mathcal{E}},\ 2T_B^T = M_b \mathbf{v}_{\mathbf{G_b}/\mathcal{E}} \cdot \mathbf{v}_{\mathbf{G_b}/\mathcal{E}}. \tag{1}$$

The gravitational potential energy of the system is

$$U_B = M_b g(R + c_z \cos(\alpha)). \tag{2}$$

The rotational and translational kinetic energy due to the two wheels can be given by

$$\begin{aligned} 2T_w^R &= I_{wa}\dot{\phi}_r^2 + I_{wa}\dot{\phi}_l^2 + 2I_{wd}\dot{\theta}^2, \\ 2T_w^T &= M_w R^2(\dot{\phi}_r^2 + \dot{\phi}_l^2). \end{aligned} \tag{3}$$

The configuration variables of the system are initially taken as

$$\mathbf{q}_{6\times 1} = [x_o, y_o, \theta, \alpha, \phi_r, \phi_l]^T. \tag{4}$$

The motor-rotor inertial quantities are considered negligible compared to those of the wheels. The Lagrangian is therefore $L(\mathbf{q}) = T_B^T + T_B^R + T_w^R + T_w^T - U_B$. Using the Euler-Lagrange equations, the equations of motion can then be derived as

$$M(\mathbf{q})\ddot{\mathbf{q}} + V(\mathbf{q},\dot{\mathbf{q}}) = E(\mathbf{q})\tau + A^T(\mathbf{q})\lambda, \tag{5}$$

where $A(\mathbf{q})$ is the nonholonomic constraints matrix derived in the next section. τ is the input motor-torque vector given by

$$\tau = \begin{pmatrix} \tau_r \\ \tau_l \end{pmatrix},\ E(\mathbf{q}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

Note the structure of the fourth row of $E(\mathbf{q})$ which arises because the motors are mounted on the pendulum-body. λ is the constraint-force vector.

The three nonholonomic constraints due to no-slip can be written as $A(\mathbf{q})_{3\times 6}\dot{\mathbf{q}} = 0$, where

$$A(\mathbf{q})_{3\times 6} = \begin{pmatrix} -\sin(\theta) & \cos(\theta) & 0 & 0 & 0 & 0 \\ \cos(\theta) & \sin(\theta) & b & 0 & -R & 0 \\ \cos(\theta) & \sin(\theta) & -b & 0 & 0 & -R \end{pmatrix}. \tag{7}$$

The null-space of $A(\mathbf{q})$ is given by the matrix $S(\mathbf{q})$ as

$$S(\mathbf{q})_{6\times 3} = \begin{pmatrix} 0 & \cos(\theta) & 0 \\ 0 & \sin(\theta) & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1/R & b/R \\ 0 & 1/R & -b/R \end{pmatrix}. \tag{8}$$

The vector $\dot{\mathbf{q}}$ has to lie in this null-space, therefore

$$\dot{\mathbf{q}} = S(\mathbf{q})\nu_{3\times 1}, \tag{9}$$

$$\nu = [\dot{\alpha}, v, \dot{\theta}]^T. \tag{10}$$

where v is the forward heading speed of the vehicle in the direction $\mathbf{x_v}$.

Next, we follow the standard procedure for the elimination of Lagrange multipliers λ by premultiplication with S^T to obtain

$$(S^T M S)\dot{\nu} + S^T(M\dot{S}\nu + V(\mathbf{q},\dot{\mathbf{q}})) = S^T E(\mathbf{q})\tau. \tag{11}$$

We note that ϕ_r, ϕ_l are decoupled entirely from the rest of the state-variables and $\dot{\phi}_r, \dot{\phi}_l$ can be found uniquely given v, $\dot{\theta}$ from Eq. (9). From a control point of view we are not interested in these variables, and can thus reduce the order of the system by 2.

We now define the configuration vector ($\mathbf{q_r}$) and the state vector ($\mathbf{x}$) as follows

$$\mathbf{q_r} = [x_o, y_o, \theta, \alpha]^T,\ \mathbf{x} = \begin{pmatrix} \mathbf{q_r} \\ \nu \end{pmatrix}_{7\times 1}. \tag{12}$$

This results in the last two rows of the $S(\mathbf{q})$ being removed and we get a truncated matrix $S_r(\mathbf{q_r})_{4\times 2}$. Another simplification can be obtained by assuming that the body is symmetric about the $\mathbf{x_b}$ axis. This implies that $I_{xz} = 0$ and $c_x = 0$. The input-affine form is then given by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \\ \mathbf{g}(\mathbf{x}) &= \begin{pmatrix} \mathbf{0}_{4\times 2} \\ (S^T M S)^{-1} S^T E \end{pmatrix}_{7\times 2},\ \mathbf{u} = \begin{pmatrix} \tau_r \\ \tau_l \end{pmatrix}, \\ \mathbf{f}(\mathbf{x}) &= \begin{pmatrix} S_r\nu \\ -(S^T M S)^{-1} S^T \left(M\dot{S}\nu + V(\mathbf{q_r},\dot{\mathbf{q}}_\mathbf{r})\right) \end{pmatrix}. \end{aligned} \tag{13}$$

We partition $\mathbf{g}(\mathbf{x})$ and $\mathbf{f}(\mathbf{x})$ in the following way for later derivations:

$$\mathbf{g}(\mathbf{x})_{7\times 2} = [\mathbf{g_1}(\mathbf{x}),\ \mathbf{g_2}(\mathbf{x})],\ \mathbf{f}(\mathbf{x}) = \begin{pmatrix} \mathbf{f_1}(\mathbf{x}) \\ \mathbf{f_2}(\mathbf{x}) \end{pmatrix}, \tag{14}$$

$$\mathbf{f_1}(\mathbf{x}) = (\cos(\theta)\,v \quad \sin(\theta)\,v \quad \dot{\theta} \quad \dot{\alpha})^T. \tag{15}$$

The detailed expressions for $\mathbf{g}(\mathbf{x})$ and $\mathbf{f_2}(\mathbf{x})$ have been provided in Appendix .

III. Strong Accessibility Condition and Maximum Relative Degree

This section is devoted to study of the strong accessibility condition and the feedback linearization of the system, either full or partial. The strong accessibility condition [2] is defined as follows: Defining:

$$\begin{aligned} \mathcal{C} = \langle \{ [X_k, [X_{k-1}, [\ldots, [X_1, g_j]\ldots]]], \\ X_i \in \{f, g_1, \ldots, g_m\},\ j = 1, \ldots, m,\ k \geq 1\} \rangle. \end{aligned} \tag{16}$$

where the notation $\langle \cdots \rangle$ denotes a distribution. If the dimension of $\mathcal{C}$ is n, then the system

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x)u_j \quad x \in \Re^n \tag{17}$$

is *strongly accessible*.

Regarding the *largest feedback linearizable subsystem*, we refer to the construction in [1]. Consider the following sets and distributions:

$$G_0 = \langle g_1, \ldots, g_m \rangle,\ G_f = \{f + g,\ g \in G_0\}, \tag{18}$$

$$G_i = \langle G_{i-1}, \{[G_f, G_{i-1}]\} \rangle,\ Q_i = \langle \{ad_f^i G_0, \overline{G}_{i-1}\} \rangle, \tag{19}$$

where $[G_f, G_{i-1}] = \{[X,Y],\ \forall X \in G_f,\ Y \in G_{i-1}\}$, $\overline{G}_{i-1}$ means the involutive closure of G_{i-1}, $ad_f^i g = [f, ad_f^{i-1} g]$, while $ad_f^1 g = [f, g]$. Based on these definitions, compute the following integers,

$$r_0 = \dim G_0 \tag{20}$$

$$r_i = \dim Q_i - \dim \overline{G}_{i-1} \quad \forall i \geq 1 \tag{21}$$

$$K_j = \#\{r_i \geq j,\ \forall i \geq 0\} \quad j \geq 1. \tag{22}$$

Then, it follows that $r_1 \geq r_2 \geq \ldots$ and the maximum relative degree that one can achieve for the system (17) is $r_1 + r_2 + \ldots \leq n$.

Before computing these distributions and integers for f, g_1, g_2 given in the last section and Appendix , we will apply a feedback law in order to simplify as much as possible the input vector fields. Since the fifth and sixth coordinate of g_1 and g_2 are equal, while the seventh is the same but with different sign, we suggest the feedback law

$$v_1 = \frac{(u_1+u_2)}{D_\alpha}, \quad v_2 = \frac{(u_1-u_2)Rb}{G_\alpha}. \tag{23}$$

Here G_α and D_α are as defined in Appendix . Therefore, the new input vector fields are (we use the same notation for the new input vector fields)

$$g_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & g_1[5] & g_1[6] & 0 \end{pmatrix}^T, \tag{24}$$
$$g_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T, \tag{25}$$

where,

$$g_1[5] = M_b R^2 + 2M_w R^2 + 2I_{wa} + M_b \cos(x_4)\, c_z R, \tag{26}$$
$$g_1[6] = -R\left(M_b \cos(x_4)\, c_z R + I_{yy} + M_b c_z^2\right). \tag{27}$$

Clearly, $G_0 = \langle g_1, g_2\rangle$ has dimension 2. It is a straightforward computation to see $[g_1, g_2] = 0$, from where we infer that G_0 is involutive or, equivalently, $G_0 = \overline{G}_0$. This involutivity property is useful to see the equality

$$G_1 = \langle G_0, \{[G_f, G_0]\}\rangle = \langle G_0, [f, G_0]\rangle.$$

After Lie bracket computations,

$$G_1 = \langle g_1, g_2, [f, g_1], [f, g_2]\rangle, \tag{28}$$

where,

$$[f, g_1] = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \end{pmatrix}^T,$$

$$[f, g_2] = \begin{pmatrix} 0 & 0 & \beta_3 & 0 & \beta_5 & \beta_6 & \beta_7 \end{pmatrix}^T,$$

where α_i and β_j are certain nonzero functions that depend on the state variables. This distribution is not involutive since

$$[g_1, [f, g_1]] = \begin{pmatrix} 0 & 0 & 0 & 0 & \gamma_5 & \gamma_6 & 0 \end{pmatrix}^T,$$

where, γ_5, γ_6 are functions depending on the state variables. Let us remark that dimension of $\langle g_1, [g_1, [f, g_1]]\rangle = 2$. This is the reason why $[g_1, [f, g_1]] \notin G_1$. Hence,

$$\begin{aligned} \overline{G}_1 \supset \widetilde{G}_1 &= \langle g_1, g_2, [f, g_1], [f, g_2], [g_1, [f, g_1]]\rangle \\ &= \langle [f, g_1], \eta_3, \eta_5, \eta_6, \eta_7\rangle. \end{aligned} \tag{29}$$

where,

$$\begin{aligned} \eta_3 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \eta_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \\ \eta_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\ \eta_7 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{30}$$

$\widetilde{G}_1$ is still not involutive since $[\eta_3, [f, g_1]] \notin \widetilde{G}_1$. Therefore,

$$\overline{G}_1 \supset \langle \eta_3, \eta_5, \eta_6, \eta_7, [f, g_1], [\eta_3, [f, g_1]]\rangle.$$

It can be seen that the distribution on the right hand side is not involutive because $[\eta_3, [\eta_3, [f, g_1]]]$ is not in that distribution. Therefore,

$$\overline{G}_1 \supset \langle \eta_3, \eta_5, \eta_6, \eta_7, [f, g_1], [\eta_3, [f, g_1]], [\eta_3, [\eta_3, [f, g_1]]]\rangle = \Re^7.$$

This implies that $\overline{G}_1 = \Re^7$. Since $\overline{G}_1 \subset \mathcal{C} \subset \Re^7$, it follows that $\mathcal{C} = \Re^7$ and, hence, the system is strongly accessible.

On the other hand,

$$Q_1 = \langle [f, G_0], G_0\rangle = G_1.$$

has dimension 4. Therefore, the integers $r_i,\ i \geq 0$ in Eqs. (20) and (21) are

$$\begin{aligned} r_0 &= \mathrm{dim} G_0 = 2, \\ r_1 &= \mathrm{dim} Q_1 - \mathrm{dim}\overline{G}_0 = 4 - 2 = 2, \\ r_2 &= \mathrm{dim} Q_2 - \mathrm{dim}\overline{G}_1 = 0. \end{aligned} \tag{31}$$

There is no need to compute more of these integers since they are positive and non-increasing. Finally, the controllability indices $K_j,\ j \geq 1$ in Eq. (22) are $K_1 = \#\{r_i \geq 1,\ \forall i \geq 0\} = 2$, $K_2 = \#\{r_i \geq 2,\ \forall i \geq 0\} = 2$, $K_3 = \#\{r_i \geq 3,\ \forall i \geq 0\} = 0$. As before, there is no need to compute more of these indices because successive ones vanish. Summarizing, the largest feedback linearizable subsystem for the system has dimension 4.

IV. Partial Feedback Linearization

In this section, we partially feedback linearize the system. Using a feedback law and a change of variables, we write the system as two chains of integrators, one for each input, plus three nonlinear equations which represents the internal dynamics of the system. By definition [3], the inputs do not appear explicitly in the internal dynamics equations.

We shall find two variables whose relative degrees together match the maximum relative degree found in the previous section. Later on, three new variables ϕ_1, ϕ_2, ϕ_3 will be obtained such that they are independent of the variables, whose relative degree is two, and their derivatives.

It is not difficult to see that both $x_3 = \theta$ and $x_4 = \alpha$ have relative degree 2. Their equations are:

$$\begin{aligned} \dot{x}_3 = x_7,\ \dot{x}_7 = f_2[3] + v_2, \\ \dot{x}_4 = x_5,\ \dot{x}_5 = f_2[1] + g_1[5]v_1. \end{aligned} \tag{32}$$

In order to convert (32) into two chains of integrators, a new feedback law is applied:

$$w_1 = f_2[3] + v_2, \tag{33}$$
$$w_2 = f_2[1] + g_1[5]v_1. \tag{34}$$

After this feedback law, the input-vector fields become:

$$\begin{aligned} G_m &= \begin{pmatrix} \bar{\mathbf{g}}_1 & \bar{\mathbf{g}}_2 \end{pmatrix}, \\ \bar{\mathbf{g}}_1(\mathbf{x}) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T, \\ \bar{\mathbf{g}}_2(\mathbf{x}) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & g_1[6]/g_1[5] & 0 \end{pmatrix}^T, \end{aligned} \tag{35}$$

and the drift vector field becomes

$$\bar{\mathbf{f}}(\mathbf{x}) = \begin{pmatrix} x_6 \cos(x_3) \\ x_6 \sin(x_3) \\ x_7 \\ x_5 \\ 0 \\ f_2[2] - (g_1[6]/g_1[5]) f_2[1] \\ 0 \end{pmatrix}, \tag{36}$$

where the terms $f_2[1], f_2[2], f_2[3]$ are as defined in Appendix . For the operational range of $\alpha \in (-\pi/2, \pi/2)$, $g_1[5] > 0$ unconditionally.

In order to obtain the three remaining coordinates of the change of variables, the condition

$$\nabla\phi_i G_m = \mathbf{0}, \quad i = 1, 2, 3 \tag{37}$$

must be fulfilled. Moreover, the gradients of ϕ_1, ϕ_2, ϕ_3 must be linearly independent of the gradients of the coordinates already chosen (x_3, x_7, x_4, x_5). Let us write $\nabla\phi_i =$

$(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$. The orthogonality condition (37) implies:

$$a_7 = 0, \; a_5 g_1[5] + a_6 g_1[6] = 0. \tag{38}$$

Eq. (38) is satisfied if one takes:

$$a_5 = -\lambda g_1[6], \; a_6 = \lambda g_1[5], \tag{39}$$

for some nonzero λ which could be in general a function of $\mathbf{x}$.

Note that the terms $g_1[5], g_1[6]$ are functions of x_4 only. For integrability, the following conditions must be also satisfied:

$$\frac{\partial a_4}{\partial x_5} = \frac{\partial a_5}{\partial x_4} = -\lambda \frac{\partial g_1[6]}{\partial x_4}, \; \frac{\partial a_4}{\partial x_6} = \frac{\partial a_6}{\partial x_4} = \lambda \frac{\partial g_1[5]}{\partial x_4}. \tag{40}$$

while a_1, a_2, a_3 can be chosen as desired. Therefore, one feasible choice for the remaining coordinates is

$$\phi_1 = x_1, \; \phi_2 = x_2, \; \phi_3 = -x_5 g_1[6] + x_6 g_1[5]. \tag{41}$$

All of the these can be verified to satisfy Eqs. (38) and (40). Summarizing, the change of coordinates is given by:

$$\mathbf{z} = T(x) = \begin{pmatrix} x_3 \\ x_7 \\ x_4 \\ x_5 \\ x_1 \\ x_2 \\ -x_5 g_1[6] + x_6 g_1[5] \end{pmatrix}. \tag{42}$$

Note that $g_1[5], g_1[6]$ are now functions of $\alpha = z_3$ only. Using the change of variables (42) and the feedback law (33)-(34), the equations of the system become

$$\begin{aligned}
\dot{z}_1 &= z_2, \; \dot{z}_2 = w_1 && (43)\\
\dot{z}_3 &= z_4, \; \dot{z}_4 = w_2 && (44)\\
\dot{z}_5 &= \left(\frac{z_7 + g_1[6] z_4}{g_1[5]}\right) \cos(z_1) \\
\dot{z}_6 &= \left(\frac{z_7 + g_1[6] z_4}{g_1[5]}\right) \sin(z_1) \\
\dot{z}_7 &= z_4 \left(-z_4 \frac{\partial g_1[6]}{\partial z_3} + \frac{\partial g_1[5]}{\partial z_3} \frac{z_7 + g_1[6] z_4}{g_1[5]}\right) \\
&\quad + g_1[5] \left(f_2[2] - f_2[1] \frac{g_1[6]}{g_1[5]}\right). && (45)
\end{aligned}$$

where $f_2[1], f_2[2], f_2[3], g_1[5], g_1[6]$ are understood to be written in terms of $\mathbf{z}$. Note that inputs are explicitly absent from the last three equations and they therefore represent the internal dynamics.

V. Velocity Controller Design Using Partial Linearization

The results from the sections above can be used to design a nonlinear controller which can track specified reference heading speed v and vehicle orientation θ, these are denoted as v_d and θ_d respectively. The controller makes sure that $\alpha \in A_s = \{|\alpha| < \alpha_m < \pi/2\}$ for a given $\alpha_m > 0$. Note that this is a physically meaningful problem because using these reference commands one can safely follow a motion plan. The essential idea is to use the pitch angle α as a 'gas pedal' for the vehicle and use it to accelerate and decelerate till the specified speed is attained.

The following should be noted

1) $\alpha(t_0) \in A_s$ should be assured before starting the controller at time t_0.
2) Referring to Eqs. (26) and (27), we see that $g_1[5] > 0, g_1[6] < 0$ if $\alpha \in (-\pi/2, \pi/2)$ which is its operating range. The limiting case is when the pendulum body pitches forward or backward such that it is horizontal. Thus, making sure that $\alpha(t) \in A_s, t > t_0$, ensures safe operation.

First, the dependence of the acceleration $\dot{v}$ on α is derived.

$$v = x_6 = \left(\frac{z_7 + g_1[6] z_4}{g_1[5]}\right) \tag{46}$$
$$z_3 = \alpha, \; z_4 = \dot{\alpha}, \; z_1 = \theta.$$

Plugging these in the last of Eqs. (45), we get

$$\begin{aligned}
\dot{z}_7 + \dot{\alpha}^2 \frac{\partial g_1[6]}{\partial \alpha} \equiv \Gamma &= (v\dot{\alpha}) \frac{\partial g_1[5]}{\partial \alpha} \\
&\quad + g_1[5] f_2[2] - g_1[6] f_2[1]. && (47)
\end{aligned}$$

Differentiating Eq. (46) and substituting expressions from Eq. (47), one gets

$$\begin{aligned}
\dot{v} &= \frac{1}{g_1[5]} (\Gamma + g_1[6] w_2) - \frac{(v\dot{\alpha})}{g_1[5]} \frac{\partial g_1[5]}{\partial \alpha} \\
&= \frac{1}{g_1[5]} (g_1[5] f_2[2] - f_2[1] g_1[6] + g_1[6] w_2) && (48)
\end{aligned}$$

Now the basic idea is to have $\theta = \theta_d, \dot{\theta} = 0$ and $\alpha = \alpha_r, \dot{\alpha} = 0$, where α_r is yet unspecified. This control is actually simple to achieve due to the linear structure in the θ-system given by Eq. (43), and in the α-system of Eq. (44). Having achieved this, in the steady state $\dot{\alpha} = 0, \dot{\theta} = 0$ and the acceleration expression from Eq. (48) can be written as:

$$\dot{v}_{ss} = f_{ss}(\alpha_r) = \left[\frac{1}{g_1[5]} (g_1[5] f_2[2]_{ss} - f_2[1]_{ss} g_1[6])\right]_{\alpha = \alpha_r} \tag{49}$$

Note that in steady-state, $w_2 = 0$ because it is used to control only the α-system and $\alpha = \alpha_r$. Expressions for $f_2[1]_{ss}$ and $f_2[2]_{ss}$ are given in Appendix and are marked by underbraces. The function $f_{ss}(\alpha)$ is plotted in Fig. 2.

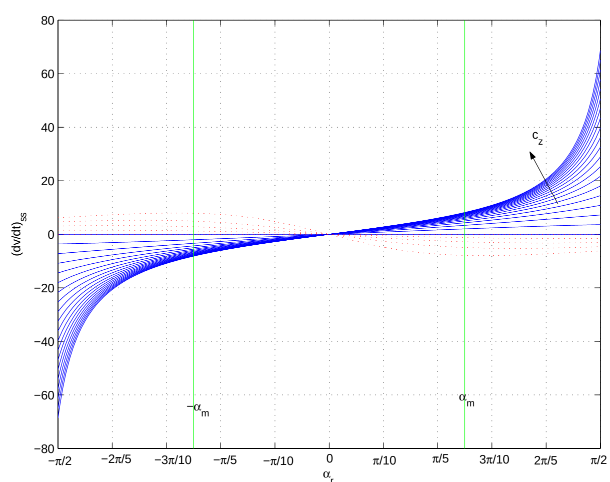

Fig. 2. $f_{ss}(\alpha_r)$ plotted for $\alpha_r \in (-\pi/2, \pi/2)$, for different values of c_z. Negative c_z curves are shown in dotted(red) lines. The arrow denotes the direction of increase of c_z. c_z values vary from $-0.75R$ to $10R$, where R is the radius of the wheels. For $c_z = 0$, $f_{ss}(\alpha_r) \equiv 0$ and α_r has no effect on the steady-state forward acceleration.

One observes that $f_{ss}(\alpha)$ is monotonic and odd in the range $\alpha \in (-\pi/2, \pi/2)$. Also for a considerable range of α's near zero, the behaviour is linear. We can now construct a two-layer controller: A higher-level controller with slower dynamics for setting α_r and a lower level controller with much faster dynamics for tracking α_r and θ_d using controls w_2 and w_1 respectively. The lower level controller is easy to design due to the linear structure of Eqs. (43) and (44).

A. Lower-level Controller C_l

One can choose

$$w_1 = -k_{qv}\dot{\theta} - k_q(\theta - \theta_d) \quad (50)$$
$$w_2 = -k_{av}\dot{\alpha} - k_a(\alpha - \alpha_r) \quad (51)$$

The gain constants $(k_{qv}, k_q, k_{av}, k_a)$ should be high enough to ensure sub-second convergence. To get the actual motor torques one has to apply the reverse feedback laws $\mathbf{w} \to \mathbf{v} \to \tau/\mathbf{u}$. This controller tracks θ_d, α_r while taking $\dot{\theta}, \dot{\alpha}$ to 0 in steady-state.

B. Higher-level Controller C_h

This controller C_h has slower dynamics and assumes that the lower-level controller C_l is able to track its reference α_r 'fast enough'. C_h also has to make sure that $\alpha_r \in A_s$ for a given $\alpha_m \in (0, \pi/2)$. It has to then vary α_r to track the specified desired heading speed v_d. However, for $c_z = 0$, this approach cannot be used as α_r has no effect on the steady-state forward acceleration.

Consider the Lyapunov function

$$V_\Sigma = \frac{1}{2({\alpha_m}^2 - {\alpha_r}^2)} + \frac{k_v(v_{\text{ss}} - v_d)^2}{2} \quad (52)$$
$$\dot{V}_\Sigma = \frac{\alpha_r\dot{\alpha}_r}{({\alpha_m}^2 - {\alpha_r}^2)^2} + k_v(v_{\text{ss}} - v_d)\dot{v}_{\text{ss}} \quad (53)$$

Substituting Eq. (49) in Eq. (53), one gets

$$\dot{V}_\Sigma = \frac{\alpha_r\dot{\alpha}_r}{({\alpha_m}^2 - {\alpha_r}^2)^2} + k_v(v_{\text{ss}} - v_d)f_{\text{ss}} \quad (54)$$

This suggests the control law

$$\dot{\alpha}_r = -k_r f_{\text{ss}} - k_v({\alpha_m}^2 - {\alpha_r}^2)^2(v_{\text{ss}} - v_d)\frac{f_{\text{ss}}(\alpha_r)}{\alpha_r} \quad (55)$$

Note the following properties of f_{ss}, the first of which is due to the fact that the function is odd.

$$\alpha_r f_{\text{ss}}(\alpha_r) > 0, \alpha_r \neq 0 \quad (56)$$
$$\lim_{\alpha_r \to 0} \frac{f_{\text{ss}}(\alpha_r)}{\alpha_r} = \frac{M_b R c_z g}{M_b R^2 + 2M_w R^2 + c_z R M_b + 2I_{wa}} \quad (57)$$

Substituting expression (55) in Eq. (54) one gets

$$\dot{V}_\Sigma = -\frac{k_r \alpha_r f_{\text{ss}}}{({\alpha_m}^2 - {\alpha_r}^2)^2} \leq 0 \quad (58)$$

On using LaSalle's Invariant Set Theorem and properties (56) and (57), one deduces that the system converges to

$$\lim_{t\to\infty} \alpha_r = 0,\ \dot{\alpha}_r = 0 \Rightarrow v_{\text{ss}} = v_d \quad (59)$$

Also, as $\alpha_r(t_0) \in A_s$, and the Lyapunov function is non-increasing under the control law, the reference pitch angle α_r cannot go out of A_s due to the barrier offered by the first term in V_Σ.

The dynamics of this controller can be slowed down by adjusting the gains k_r, k_v. Another emperical but effective way of making the dynamics of α_r slower than that of the actual pitch angle α is to multiply the right hand side of Eq. (55) by $e^{-\hat{k}|\dot{\alpha}|}, \hat{k} \gg 1$. The rationale of this is that as long as the real pitch rate is high, $\dot{\alpha}_r$ remains small, i.e. α_r remains almost constant till the lower level controller catches up. This approach is found to work well in simulations.

TABLE I
SIMULATION PARAMETERS

M_b	35.00	Kg
M_w	5	Kg
R	0.25	m
c_z	R, $3R$	m
b	0.20	m
I_{xx}	2.1073	Kg m^2
I_{yy}	1.8229	Kg m^2
I_{zz}	0.6490	Kg m^2
I_{wa}	0.1563	Kg m^2
I_{wd}	0.0781	Kg m^2
α_m	$\pi/9, \pi/4$	rad
k_q	100	N/rad
k_{qv}	5	N/(rad/s)
k_a	1000	N/rad
k_{av}	50	N/(rad/s)
k_r	20	N/rad
k_v	$10/\alpha_m^4$	N/(rad/s)
$\hat{k}$	100	s/rad

C. Simulation Results

For the simulations, the values taken are listed in Table I.

Fig. 3 shows the response of a stop command $v_d = 0$ for non-zero high intial speed and pitch. In addition, the vehicle is commanded to stop at orientation $\theta_d = \pi/6$ rads. The responses for two values of c_z are compared. A higher c_z shows higher oscillation amplitudes in v before settling down. The controller was able to stop the vehicle in about 15 seconds. The simulation is for illustration only and in a real system, the controller parameters should be adjusted to halt the system sooner for safety reasons.

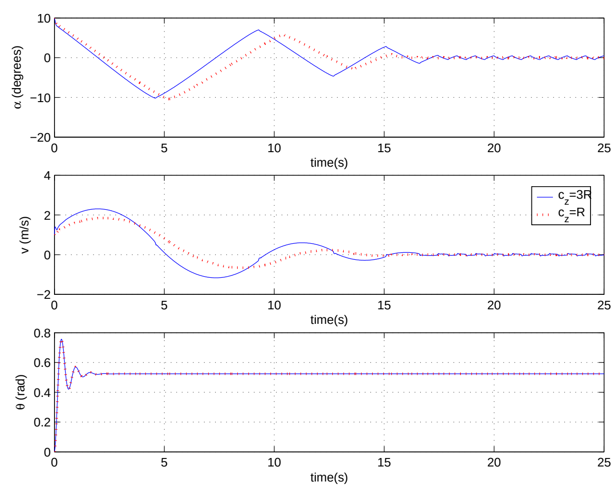

Fig. 3. Comparison of responses to a stop command $v_d = 0, \theta_d = \pi/6 = 0.52$ for $c_z = R$ and $c_z = 3R$ with $\alpha_m = \pi/9 \equiv 20^o$. The initial speed $v(t_0) = 1$ m/s and initial angles were $\theta(t_0) = 0, \alpha(t_0) = \pi/18 \equiv 10^o$.

Fig. 4 shows the result of specifying a step-command for a high speed $v_d = 7$ m/s and $\theta_d = \pi/6$ rads from zero initial speed, pitch and orientation for two different pitch-limits α_m. Note that the controller follows the limits rather conservatively. A higher α_m leads to quicker acceleration but also more oscillation in v and hence the settling time is increased.

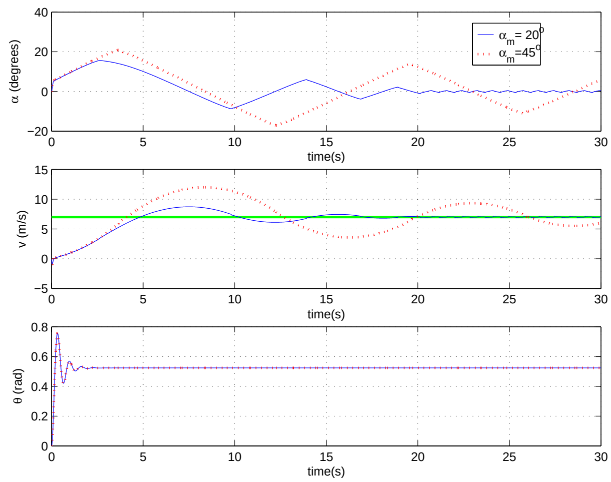

Fig. 4. Comparison of responses to a step command $v_d = 7$m/s, $\theta_d = \pi/6 = 0.52$ for $\alpha_m = \pi/9 \equiv 20^o$ and $\alpha_m = \pi/4 \equiv 45^o$. The initial speed $v(t_0) = 0$ m/s and initial angles were $\theta(t_0) = 0, \alpha(t_0) = 0$.

VI. Conclusions

The main contributions of this paper are as follows: (i) Complete derivation of the dynamic equations of a wheeled inverted pendulum in terms of physical variables useful for motion planning and control; (ii) Study of the system's controllability properties and maximum relative degree and derivation of a partial feedback linearized form; (iii) Derivation of a two level velocity controller which makes use of this partial feedback linearized form.

Simulations with realistic data show that this approach is effective. Further work needs to be done to make the controller design robust with respect to parameter-uncertainties.

Appendix

Detailed expressions from Sec. II.

$$
\begin{aligned}
D_\alpha \equiv\ & {M_b}^2 \cos^2(\alpha) {c_z}^2 R^2 \\
& + \left(\left(-{M_b}^2 - 2 M_w M_b \right) {c_z}^2 - 2 I_{yy} M_w - I_{yy} M_b \right) R^2 \\
& - 2 M_b {c_z}^2 I_{wa} - 2 I_{yy} I_{wa} \qquad (60) \\
G_\alpha \equiv\ & \left(-M_b {c_z}^2 + I_{zz} - I_{xx} \right) R^2 \cos^2(\alpha) + 2 b^2 I_{wa} \\
& + \left(M_b {c_z}^2 + I_{xx} + 2 I_{wd} + 2 b^2 M_w \right) R^2 \qquad (61)
\end{aligned}
$$

$$\mathbf{g}(\mathbf{x}) = [\mathbf{g_1}(\mathbf{x}), \mathbf{g_2}(\mathbf{x})] \qquad (62)$$

$$\mathbf{g_1}(\mathbf{x}) = \begin{pmatrix} \mathbf{0}_{4\times 1} \\ \frac{M_b\, R^2 + 2\, M_w\, R^2 + 2\, I_{wa} + M_b\, \cos(\alpha) c_z\, R}{D_\alpha} \\ -\frac{R\left(M_b\, \cos(\alpha) c_z\, R + I_{yy} + M_b\, {c_z}^2\right)}{D_\alpha} \\ \frac{Rb}{G_\alpha} \end{pmatrix} \qquad (63)$$

$$\mathbf{g_2}(\mathbf{x}) = \begin{pmatrix} \mathbf{0}_{4\times 1} \\ \frac{M_b\, R^2 + 2\, M_w\, R^2 + 2\, I_{wa} + M_b\, \cos(\alpha) c_z\, R}{D_\alpha} \\ -\frac{R\left(M_b\, \cos(\alpha) c_z\, R + I_{yy} + M_b\, {c_z}^2\right)}{D_\alpha} \\ -\frac{Rb}{G_\alpha} \end{pmatrix} \qquad (64)$$

$$
\begin{aligned}
\bar{H} \equiv\ & 1/2 M_b R^2 I_{zz} + I_{wa} I_{zz} - M_w R^2 I_{xx} \\
& - I_{wa} I_{xx} - M_b {c_z}^2 M_w R^2 - M_b {c_z}^2 I_{wa} \\
& - 1/2 M_b R^2 I_{xx} + M_w R^2 I_{zz} \qquad (65) \\
K_\alpha\,(4 D_\alpha) \equiv\ & (-4 I_{yy} M_b R^2 c_z - 3 R^2 {M_b}^2 {c_z}^3 \\
& + M_b R^2 c_z (I_{xx} - I_{zz})) \sin(\alpha) \\
& + (M_b R^2 c_z (I_{xx} - I_{zz}) \\
& + R^2 {M_b}^2 {c_z}^3) \sin(3\alpha) \qquad (66)
\end{aligned}
$$

$$\mathbf{f_2}[\mathbf{1}](\mathbf{x}) = \underbrace{\frac{\sin(2\alpha)\dot{\theta}^2 \bar{H}}{D_\alpha}}_{f_{21}^{\dot{\theta}}} + 1/2 \frac{{M_b}^2 {c_z}^2 R^2 \sin(2\alpha)(\dot{\alpha})^2}{D_\alpha} + \underbrace{1/2 \frac{\left(-2{M_b}^2 R^2 c_z - 4 I_{wa} M_b c_z - 4 M_w R^2 M_b c_z\right) g \sin(\alpha)}{D_\alpha}}_{f_{21}^{\alpha}} \qquad (67)$$

$$\mathbf{f_2}[\mathbf{2}](\mathbf{x}) = \underbrace{K_\alpha \dot{\theta}^2}_{f_{22}^{\dot{\theta}}} + 1/2\, \underbrace{\frac{{M_b}^2 {c_z}^2 R^2 g \sin(2\,\alpha)}{D_\alpha}}_{f_{22}^{\alpha}} + 1/4\, \frac{\left(-4\, I_{yy}\, M_b\, R^2 c_z - 4\, R^2 {M_b}^2 {c_z}^3\right) \sin(\alpha)\, (\dot{\alpha})^2}{D_\alpha} \qquad (68)$$

$$
\begin{aligned}
\mathbf{f_2}[\mathbf{3}](\mathbf{x}) =\ & \frac{\left(-(I_{xx} - I_{zz}) R^2 - M_b {c_z}^2 R^2\right) \sin(2\,\alpha)\, \dot{\alpha}\dot{\theta}}{G_\alpha} \\
& - \frac{\sin(\alpha) R^2 M_b c_z v \dot{\theta}}{G_\alpha} \qquad (69)
\end{aligned}
$$

References

[1] Marino, R., "On the largest feedback linearizable subsystem". *Systems and Control Letters* vol. 6, pp. 345-351, 1986.

[2] Nijmeijer, van der Schaft, H., A., "Nonlinear Dynamical Control Systems". Ed. *Springer-Verlag*, 1990.

[3] Slotine, J-J., and Li, W., "Applied nonlinear control", *Prentice Hall*, 1991.

[4] Sarkar, N., Yun, X., and Kumar V. "Control of Mechanical Systems with Rolling Constraints: Application to dynamic control of mobile robots", *Int. Journal of Robotics Research*, Vol. 13, No. 1, Feb. 1994.

[5] Ha, Y-S. and Yuta, S., "Trajectory tracking control for navigation of the inverse pendulum type self-contained mobile robot". *Robotics and Autonomous Systems*, 17, pp. 65-80, 1996.

[6] Grasser, F., D'Arrigo, A., Colombi, S., and Rufer, A., "Joe: A mobile, inverted pendulum", *IEEE Trans. Industrial Electronics*, vol. 49, no. 1, pp. 107, no. 114, 2002.

[7] Baloh, M. and Parent, M., "Modeling and Model Verification of an Intelligent Self-Balancing Two-wheeled Vehicle for an Autonomous Urban Transportation System". *Conf. Comp. Intelligence, Robotics and Autonomous Systems, Singapore*, Dec, 15, 2003.

[8] Salerno, A. and Angeles, J., "On the Nonlinear Controllability of a Quasiholonomic Mobile Robot". *Proc. IEEE I.C.R.A, Taiwan* 2003.

[9] "Segway Human Transporter". *http://www.segway.com* 2003.

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