Section 1

Basic Properties of Real Numbers

We will begin this resource guide with a discussion of the real numbers—the numbers with which you are most familiar. The set of the real numbers is denoted by $\mathbb{R}$. This set is equipped with two operations: ***addition***, $+$, and ***multiplication***, $\cdot$. When two real numbers are added, the result is a real number. We often express this fact by saying that $\mathbb{R}$ is closed under addition. Likewise, when two real numbers are multiplied, the result is a real number. Thus, $\mathbb{R}$ is also closed under multiplication.

Let us now turn to the basic properties of addition and multiplication. These properties are very important for understanding the structure of the real numbers. In high school mathematics, however, they are mainly used to simplify computations, especially computations that involve algebraic expressions.

The numbers 0 and 1 are special—they have the following properties:

1. For any real number x, $x+0=x$
2. For any real number x, $x\cdot 1=x$

No real numbers other than 0 and 1 have these properties. The rest of the properties are as follows:

3. **Additive Inverse**. For any real number x there is a unique number, denoted by $-x$, such that $x+(-x)=0$.
4. **Multiplicative Inverse**. For any real number x different from 0, there is a unique number, denoted by x^{-1}, such that $x\cdot x^{-1}=1$.
5. **Associative Law of Addition**. For all real numbers x, y, and z, $x+(y+z)=(x+y)+z$
6. **Commutative Law of Addition**. For all real numbers x, y, $x+y=y+x$
7. **Associative Law of Multiplication**. For all real numbers x, y, and z, $x\cdot(y\cdot z)=(x\cdot y)\cdot z$
8. **Commutative Law of Multiplication**. For all real numbers x, y, $x\cdot y=y\cdot x$
9. **Distributive Law**. For all real numbers x, y, and z, $x\cdot(y+z)=x\cdot y+x\cdot z$
10. **Zero Factor Property**. For all real numbers x and y, $x\cdot y=0$ if and only if $x=0$ or $y=0$ or both x and y are equal to 0.

Note that the last property consists of two properties. The first property is: If $x \cdot y = 0$, then $x = 0$ or $y = 0$ or both x and y are equal to 0. The second property is: If $x = 0$ and/or $y = 0$, then $x \cdot y = 0$.

You might be wondering, what about the operations ***subtraction*** and ***division***? These two operations are defined in terms of ***addition*** and ***multiplication***, respectively, as follows:

DEFINITION

If x and y are real numbers, their difference $x - y$ is the real number $x + (-y)$.

And if $y \neq 0$, $\frac{x}{y}$ is the real number $x \cdot y^{-1}$.

$\mathbb{R}$, the set of all real numbers, includes the ***integers***: $0, \pm 1, \pm 2, \pm 3, \ldots$. The set of all integers is denoted by $\mathbb{Z}$. The set of all positive integers is denoted by $\mathbb{Z}^+$, and the set of all negative integers is denoted by $\mathbb{Z}^-$. The real numbers also include the ***rational numbers***. A *rational number* is a ratio, or division, of two integers. For example, $\frac{2}{3}, \frac{-4}{17}, \frac{-23}{42}, \frac{3}{-18}, 2.13, -0.98123$ are all rational numbers. Any integer n is rational because it can be written as $\frac{n}{1}$, which is a ratio of two integers. The set of all rational numbers is denoted by $\mathbb{Q}$.

You can see from this description that:

a. $\mathbb{Z}^+$ and $\mathbb{Z}^-$ are subsets of $\mathbb{Z}$,

b. $\mathbb{Z}$ is a subset of $\mathbb{Q}$, and

c. $\mathbb{Q}$ is a subset of $\mathbb{R}$.

FIGURE 1–1 illustrates these relations:

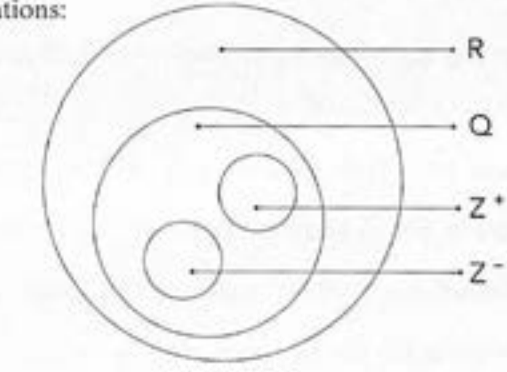


FIGURE 1–1

Another important subset of the real numbers is the set of ***irrational numbers***. A real number that is not rational is called irrational. For example, $\sqrt{2}$ is irrational. Do you know why $\sqrt{2}$ is irrational?

THEOREM

$\sqrt{2}$ is an irrational number.

PROOF

1. Assume that $\sqrt{2}$ is rational. Then by definition $\sqrt{2}=\frac{m}{n}$, where m and n are integers and $n \neq 0$.
2. If indeed $\sqrt{2}=\frac{m}{n}$, then we should be able to reduce this fraction to the lowest term.
3. Let m and n be so that $\frac{m}{n}$ is a reduced fraction.
4. Since $\sqrt{2}=\frac{m}{n}$, $2n^2=m^2$. This implies that m^2 is even, and, hence, m is also even. (Why?)
5. Since m is even, we can write $m=2k$ for some integer k.
6. Substituting $m=2k$ in $2n^2=m^2$, we get $2n^2=(2k)^2$, and so $n^2=2k^2$. This implies that n^2 is even, and, hence, n is also even.
7. Since m and n are both even, $\frac{m}{n}$ is not a reduced fraction—a contradiction. This shows that our assumption that $\sqrt{2}$ is rational is not true. Hence $\sqrt{2}$ is irrational.

We will conclude this section with another set of properties that concern inequalities between real numbers:

11. For any two real numbers x and y, one and only one of the following is true: $x<y$, $x=y$, or $x>y$.
12. For any three real numbers x, y, z, if $x<y$, $y<z$, then $x<z$.
13. For any three real numbers x, y, z, if $x<y$ then $x+z<y+z$.
14. For any three real numbers x, y, z, if $x<y$ and $z>0$, then $zx<zy$.
15. For any three real numbers x, y, z, if $x<y$ and $z<0$, then $zx>zy$.

Section 2
Linear and Quadratic Equations

2.1 LINEAR EQUATIONS

The simplest equations are those in the form $ax+b=0$. These equations are called ***linear equations***, because, as we will see in SECTION 8, they are associated with straight *lines*. The following equations are all linear:

(1) $2x+6=0$

(2) $-3x+4=0$

(3) $-\frac{2}{5}x+3.1=0$

Equations (1), (2), and (3) have the form $ax+b=0$. In equation (1), $a=2$ and $b=6$; in equation (2), $a=-3$ and $b=4$; and in equation (3) $a=-\frac{2}{5}$ and $b=3.1$. Hence, equations (1), (2), and (3) are linear.

Now, let's consider the following equations:

(4) $4x=17$

(5) $2x-9=34$

(6) $-6+2x=4x+9$

These equations do not have the form $ax+b=0$, but they are ***equivalent*** to equations of this form. They, too, are considered linear equations. When are two equations equivalent? Two equations are *equivalent* if they have the same solution set, that is to say, if each solution of one equation is also a solution of the second equation, and vice versa.

EXAMPLE 2.1a: The equations $2x-9=34$ and $2x-43=0$ are equivalent.

• •

REASON: The solution set of each equation consists of the same number: $x=21.5$.

Now, let's solve a few linear equations.

EXAMPLE 2.1b: Solve the equation $8x - 34 = 15$.

SOLUTION: We can add 34 to both sides of the equation without changing the solution set of the equation. And, in doing so, we get a simpler equation, an equation that is easier to solve. Thus:

$8x - 34 + 34 = 15 + 34$

$8x + 0 = 49$

$8x = 49$

The last equation is equivalent to the original equation and is easier to solve. To get x, we simply divide both sides of the equation by 8.

$\frac{8x}{8} = \frac{49}{8}$

$x = 6.125$

The solution set of the given equation $8x - 34 = 15$ consists of one number: $x = 6.125$.

Let's check that $x = 6.125$ is indeed a solution to this equation:

$8(6.125) - 34 = 49 - 34 = 15$

EXAMPLE 2.1c: Solve the equation $-5x + \frac{4}{7} = 3x - \frac{1}{7}$.

SOLUTION:

$-5x + \frac{4}{7} - \frac{4}{7} = 3x - \frac{1}{7} - \frac{4}{7}$ Subtract $\frac{4}{7}$ from both sides of the equation.

$-5x + 0 = 3x - \frac{5}{7}$

$-5x = 3x - \frac{5}{7}$

$-5x - 3x = 3x - \frac{5}{7} - 3x$ Subtract $3x$ from both sides of the equation.

$-8x = -\frac{5}{7}$

$$8x = \frac{5}{7}$$ Multiply both sides of the equation by -1.

$$\frac{8x}{8} = \frac{\frac{5}{7}}{8}$$ Divide both sides of the equation by 8.

$$x = \frac{5}{56}$$

In the next series of examples, we will present the solution process without a detailed explanation as was given in the previous examples.

EXAMPLE 2.1d: Solve the equation $2(x-4)+3=2x-5$.

SOLUTION:

$2(x-4)+3=2x-5$

$2x-8+3=2x-5$

$2x-5=2x-5$

Any number x is a solution to the last equation because with any number that we substitute for x, we get a true statement. Hence, our original equation, $2(x-4)+3=2x-5$, has infinitely many solutions.

EXAMPLE 2.1e: Solve the equation $x+7=2(x+3)-x$.

SOLUTION:

$x+7=2(x+3)-x$

$x+7=2x+6-x$

$x+7=x+6$

There is no number x that satisfies the last equation because if there were such a number, then we would get $7=6$, which is absurdity. Hence, the original equation $x+7=2(x+3)-x$ has no solution.

Section 2.1
EXERCISES

1. Solve the following equations:

   a. $\frac{4}{57}(x+8)=\frac{1}{3}$

   b. $2\left(x-\frac{3}{4}\right)+3\left(x+\frac{2}{3}\right)+4\left(x-\frac{1}{5}\right)=9x+17$

   c. $\frac{8x-3}{11}-\frac{5x+2}{2}=1$

   d. $\frac{x+3}{6}+\frac{x-7}{3}=\frac{x+11}{2}$

   e. $x+2x+3x+\ldots+33x=51$

2. Sam took a number, multiplied it by 12, added 60 to the result, and got 372. What was Sam's initial number?

3. Tom took a number and subtracted $2\frac{4}{15}$ from it. Then, he divided $3\frac{2}{3}$ by the result and got $3\frac{5}{13}$. What was Tom's initial number?

4. John left town A at the same time that Mary left town B. They walked toward each other and met 40 minutes after their departure. John arrived at town B 32 minutes after he met Mary. How long did it take Mary to arrive at town A?

5. Cities A and B are 70 miles apart. A biker leaves town A at the same time that a bus leaves town B. They travel toward each other and meet 84 minutes after their departure at a point between A and B. The bus arrives at City A, stays there for 20 minutes, and then heads back to City B. The bus meets the biker again 2 hours and 41 minutes after their first meeting. What are the speeds of the bus and the biker?

2.2 QUADRATIC EQUATIONS

We have learned how to solve linear equations. In this section, we will learn how to solve a new kind of equation, ***quadratic equations***. These are equations of the form $ax^2 + bx + c = 0$, where $a \neq 0$. We add the condition $a \neq 0$ because if $a = 0$, then we would just have a linear equation, $bx + c = 0$, which we have already discussed. When we say "quadratic equation," we always assume that the coefficient of x^2 is nonzero. We will begin with two special cases of quadratic equations, and then we will deal with the general case.

2.2.1 Equations of the Form $x^2 - p = 0$

In this case, $a = 1$, $b = 0$, $c = -p$. Equations of this form are easy to solve:

$x^2 - p = 0$

$x^2 = p$

$x = \sqrt{p}$ or $x = -\sqrt{p}$

Or, we just write $x = \pm\sqrt{p}$. Note, however, that p must be non-negative. If $p < 0$, there is no real solution to the given equation.

EXAMPLE 2.2a: Solve the equation $x^2 - 16 = 0$.

SOLUTION:

$x^2 - 16 = 0$

$x^2 = 16$

$x = \pm\sqrt{16}$

$x = \pm 4$

2.2.2 Equations of the form $k(x + r)^2 - p = 0$, Where $k \neq 0$

This type of equation can be reduced to the format we just discussed. To do this, we need to divide both sides of the equation by k. Then, we get $(x+r)^2 = \frac{p}{k}$ (remember that $k \neq 0$). If $\frac{p}{k} \geq 0$, then $x + r = \pm\sqrt{\frac{p}{k}}$, and so $x = -r \pm \sqrt{\frac{p}{k}}$. If $\frac{p}{k} < 0$, the equation has no real solution.

EXAMPLE 2.2b: Solve the equation $8(x+3)^2 - 72 = 0$.

SOLUTION:

$8(x+3)^2 - 72 = 0$

$8(x+3)^2 = 72$

$(x+3)^2 = \frac{72}{8}$

$(x+3)^2 = 9$

$x + 3 = \pm\sqrt{9}$

$x + 3 = \pm 3$

$x = -3 \pm 3$

$x = 0$ or $x = -6$

2.2.3 Equations of the form $ax^2 + bx + c = 0$, Where $a \neq 0$

To solve this equation, we first must change it into the form we just discussed, $k(x+r)^2 - p = 0$, which we already know how to solve. Let's begin with a specific example:

EXAMPLE 2.2c: Solve $2x^2 - 3x + 1 = 0$.

SOLUTION: We solve this equation by making it an equivalent equation of the form $k(x+r)^2 - p = 0$.

When opening the parentheses in the last equation, we get $kx^2 + 2krx + kr^2 - p = 0$. Comparing the coefficients of this equation with the coefficients of the given equation (see the figure below), we get:

$$\begin{cases} k = 2 \\ 2kr = -3 \\ kr^2 - p = 1 \end{cases}$$

$$\begin{array}{ccccccc} 2\,x^2 & + & -3\,x & + & 1 & = & 0 \\ \updownarrow & & \updownarrow & & \updownarrow & & \\ k\,x^2 & + & 2kr\,x & + & kr^2 - p & = & 0 \end{array}$$

By substituting $k = 2$ in $2kr = -3$, we get $r = \frac{-3}{4}$.

And, by substituting $k=2$ and $r=\frac{-3}{4}$ in $kr^2-p=1$, we get $p=2\cdot\left(\frac{-3}{4}\right)^2-1=\frac{1}{8}$.

Thus, our equation can be written as: $2\left(x+\left(\frac{-3}{4}\right)\right)^2-\frac{1}{8}=0$.

Now, we have arrived at an equation of the form $k(x+r)^2-p=0$, which we know how to solve.

$$2\left(x+\left(\frac{-3}{4}\right)\right)^2=\frac{1}{8}$$

$$\left(x+\left(\frac{-3}{4}\right)\right)^2=\frac{1}{16}$$

$$x+\frac{-3}{4}=\pm\frac{1}{4}$$

$$x=\frac{3}{4}\pm\frac{1}{4}$$

$$x=1 \text{ or } x=\frac{1}{2}$$

Thus, the solution set of our equation consists of two numbers: $x=1$ and $x=\frac{1}{2}$.

EXAMPLE 2.2d:

SOLUTION: This process is applicable to the general case $ax^2+bx+c=0$, where $a\neq 0$. As before, our aim is to bring this equation into the form $k(x+r)^2-p=0$. This form, as has just been shown, is $kx^2+2krx+kr^2-p=0$.

Comparing the coefficients of the two equations, we get $\begin{cases} k=a \\ 2kr=b \\ kr^2-p=c \end{cases}$

Substituting $k=a$ in $2kr=b$, we get $r=\frac{b}{2a}$.

Substituting $k=a$ and $r=\frac{b}{2a}$ in $kr^2-p=c$, we get $p=a\cdot\left(\frac{b}{2a}\right)^2-c$.

Thus, our equation can be written as:

$$a\left(x+\left(\frac{b}{2a}\right)\right)^2-\left(a\left(\frac{b}{2a}\right)^2-c\right)=0.$$

Now we can solve for x:

$$a\left(x+\left(\frac{b}{2a}\right)\right)^2=a\left(\frac{b}{2a}\right)^2-c$$

$$\left(x+\left(\frac{b}{2a}\right)\right)^2=\frac{a\cdot\left(\frac{b}{2a}\right)^2-c}{a}$$

$$\left(x+\left(\frac{b}{2a}\right)\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

$$\left(x+\left(\frac{b}{2a}\right)\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$\left(x+\left(\frac{b}{2a}\right)\right)^2=\frac{b^2-4ac}{4a^2}$$

$$x+\left(\frac{b}{2a}\right)=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x=-\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

We have just developed a formula for solving quadratic equations $ax^2+bx+c=0$. This formula is called the ***quadratic formula***: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

2.2.4 The Discriminant

The expression b^2-4ac tells us when a quadratic equation has a real solution, and if it does, whether it has one or two real solutions. This expression is called the ***discriminant***. It is denoted by the Greek letter Δ (delta).

From the quadratic formula we developed previously, we get:

1. When $\Delta = b^2 - 4ac > 0$, the equation has two real solutions: $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ or $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$.
2. When $\Delta = b^2 - 4ac = 0$, the equation has only one real solution: $x = \frac{-b}{2a}$.
3. When $\Delta = b^2 - 4ac < 0$, the equation has no real solution.

It is not always necessary to use the quadratic formula to solve a quadratic equation. Quadratic equations of the form $ax^2 + bx = 0$ are easy to solve by factoring, as is shown below:

$ax^2 + bx = 0$

$x(ax + b) = 0$

$x = 0$ or $ax + b = 0$

$x = 0$ or $x = -\frac{b}{a}$

Here we utilized the fact that the product of two numbers is zero if and only if at least one of the numbers is zero.

When using the quadratic formula, we naturally get the same solution:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-b \pm \sqrt{b^2-4a0}}{2a} = \frac{-b \pm \sqrt{b^2-0}}{2a} = \frac{-b \pm \sqrt{b^2}}{2a} = \frac{-b \pm b}{2a}$$

Thus, $x = 0$ and $x = \frac{-2b}{2a} = -\frac{b}{a}$ is the solution set of the equation $ax^2 + bx = 0$.

Section 2.2
EXERCISES

1. Solve the equations:

   a. $x^2 - x - 90 = 0$

   b. $25x^2 + 90x + 81 = 0$

   c. $(3x-2)(x-3) = 20$

   d. $3(5x+3)(4x^2-1) = 8(4x^2-1)^2$

e. $x^2 - 13x + 4 = 0$

f. $\frac{3x^2 - 14x + 11}{14} = \frac{x+9}{2} - \frac{x^2 + x + 1}{5}$

g. $x^2 + 2(1+\sqrt{8})x + 8\sqrt{2} = 0$ (Do not round off the coefficients.)

2. One of the digits of a two-digit number is 3 less than the other digit. When the digits are interchanged, we get a new number. The sum of the squares of the new number and the initial number is 1877. Find the two numbers.

3. Make up a word problem whose solution can be found by solving the equation $x(x+36) = 180$.

4. Two planes start flying from the same airport at the same time. One plane flies directly north, and the other flies directly east. After 2 hours from the time they started, the distance between the two planes is 2000 miles. Find the speeds of each plane if the speed of one is 75% of the speed of the other.

5. A company has decided to double its production within two years. By how many percents should the company increase its production each year if it wants the percentage increase in the first year to be equal to the percentage increase in the second year?

Section 3

Polynomial Equations

So far we have learned how to solve linear and quadratic equations. Now, you may be wondering, what about equations of similar form but with higher degrees? For example, how do we solve equations like $2x^3+4x^2-x+9=0$ and $-3x^4+5x^3-2x+12=0$? Equations of this kind are called ***polynomial equations***. A polynomial is an expression of the form $a_nx^n+a_{n-1}x^{n-1}+\ldots+a_3x^3+a_2x^2+a_1x^1+a_0$, where n is a non-negative integer, and $a_0,a_1,a_2,\ldots,a_n$ are real numbers with $a_n\neq 0$. $a_0,a_1,a_2,\ldots,a_n$ are called the ***coefficients*** of the polynomial. n, the highest power in the polynomial, is called the ***degree*** of the polynomial. The expressions, $a_0, a_1x^1, a_2x^2, a_3x^3, \ldots, a_{n-1}x^{n-1}, a_nx^n$, are called the ***terms*** of the polynomial. Expressions of the form x^k, where k is a non-negative integer, are called ***monomials***.

EXAMPLE 3a:

The coefficients of the polynomial $4x^5-3x^4+\frac{2}{5}x^3+\sqrt{3}x^2-6x+1$ are:
$a_5=4$, $a_4=-3$, $a_3=\frac{2}{5}$, $a_2=\sqrt{3}$, $a_1=-6$, $a_0=1$ and the degree of this polynomial is $n=5$.

The terms of this polynomial are: 1, $-6x$, $\sqrt{3}x^2$, $\frac{2}{5}x^3$, $-3x^4$, and $4x^5$.

EXAMPLE 3b:

The coefficients of the polynomial $17x^4+0\cdot x^3-12x^2+0\cdot x-\frac{3}{4}$ are:
$a_4=17$, $a_3=0$, $a_2=-12$, $a_1=0$, $a_0=-\frac{3}{4}$
This polynomial is of degree 4.

The terms of this polynomial are: $-\frac{3}{4}$, $0\cdot x$, $-12x^2$, $0\cdot x^3$, and $17x^4$.

When a coefficient is zero, there is no need to write the corresponding term. For example, instead of writing $-4x^5+0x^4+\sqrt{5}x^3+0x^2-4x+7$, we just write $-4x^5+\sqrt{5}x^3-4x+7$. When all the coefficients of a polynomial are zeros, the polynomial is called the ***zero polynomial***. For convenience, we denote polynomials by $Q(x)$, $R(x)$, etc.

3.1 WHEN ARE TWO POLYNOMIALS EQUAL?

Two polynomials $Q(x)$ and $R(x)$ are ***equal*** if they have the same terms.

EXAMPLE 3.1a:

The polynomials $P(x)=x^3+x^2+x$ and $Q(x)=0\cdot x^4+x^3+x^2+x+0$ are equal. The polynomials $R(x)=x^3+x^2+x$ and $S(x)=x^4+x^3+x^2+x$, on the other hand, are not equal because the term x^4 is in $S(x)$ but not in $R(x)$.

3.2 HOW DO WE ADD AND SUBTRACT POLYNOMIALS?

Naturally, the sum and difference of two polynomials are defined. We simply add or subtract the coefficients of the like terms.

EXAMPLE 3.2a:

$Q(x)=-4x^5+3x^3+2x^2-6x+9$

$P(x)=2x^5+7x^3-2x^2+6x+19$

$$Q(x)+P(x)=(-4+2)x^5+(3+7)x^3+(2+(-2))x^2+(-6+6)x+(9+19)=$$
$$=-2x^5+10x^3+0x^2+0x+28=-2x^5+10x^3+28$$

$$Q(x)-P(x)=(-4-2)x^5+(3-7)x^3+(2-(-2))x^2+(-6-6)x+(9-19)=$$
$$=-6x^5-4x^3+4x^2-12x-10$$

EXAMPLE 3.2b:

$Q(x) = 65x^7 + 13x^3 + \sqrt{2}x^2 - 8$

$P(x) = 32x^5 + 4x^4 - 2x^3 + 6x + 19$

$$Q(x) + P(x) = (65+0)x^7 + (0+32)x^5 + (0+4)x^4 + (13-2)x^3 + (\sqrt{2}+0)x^2 + (0+6)x + (-8+19) =$$
$$= 65x^7 + 32x^5 + 4x^4 + 11x^3 + \sqrt{2}x^2 + 6x + 11$$

EXAMPLE 3.2c:

$Q(x) = 6x^4 + 23x^3 + \sqrt{2}x^2 - 3x - 8$

$P(x) = 32x^5 - 2x^3 - 18x + 19$

$$Q(x) - P(x) = (0-32)x^5 + (6-0)x^4 + (23-(-2))x^3 + (\sqrt{2}-0)x^2 + (-3-(-18))x + (-8-19) =$$
$$= -32x^5 + 6x^4 + 25x^3 + \sqrt{2}x^2 + 15x - 27.$$

3.3 HOW DO WE MULTIPLY POLYNOMIALS?

To multiply two polynomials, we multiply each term of one polynomial by each term of the second polynomial and add like terms.

EXAMPLE 3.3a:

$Q(x) = 5x^3 - 3x^2 + 2x$

$P(x) = -4x^4 - 12x^3 + 15$

$$Q(x)\cdot P(x) = \left(5x^3 - 3x^2 + 2x\right)\cdot\left(-4x^4 - 12x^3 + 15\right) =$$
$$= \left(5x^3\right)\cdot\left(-4x^4 - 12x^3 + 15\right) + \left(-3x^2\right)\cdot\left(-4x^4 - 12x^3 + 15\right) + \left(2x\right)\cdot\left(-4x^4 - 12x^3 + 15\right) =$$
$$= \left(5x^3\right)\cdot\left(-4x^4\right) + \left(5x^3\right)\cdot\left(-12x^3\right) + \left(5x^3\right)\cdot\left(15\right) + \left(-3x^2\right)\cdot\left(-4x^4\right) + \left(-3x^2\right)\cdot\left(-12x^3\right) + \left(-3x^2\right)\cdot\left(15\right) +$$
$$+ \left(2x\right)\cdot\left(-4x^4\right) + \left(2x\right)\cdot\left(-12x^3\right) + \left(2x\right)\cdot\left(15\right) =$$
$$= -20x^7 - 60x^6 + 75x^3 + 12x^6 + 36x^5 - 45x^2 - 8x^5 - 24x^4 + 30x =$$
$$= -20x^7 - 48x^6 + 28x^5 - 24x^4 + 75x^3 - 45x^2 + 30x$$

$$Q(x)\cdot P(x) = \left(5x^3 - 3x^2 + 2x\right)\cdot\left(-4x^4 - 12x^3 + 15\right) = -20x^7 - 48x^6 + 28x^5 - 24x^4 + 75x^3 - 45x^2 + 30x$$

3.4 HOW DO WE DIVIDE POLYNOMIALS?

Division of polynomials is similar to the long division algorithm you learned in elementary school. Before we show how to divide one polynomial by another, it is worth discussing a few properties of polynomials that are analogous to those of whole numbers.

Any integer can be factored into the product of other, smaller integers, unless the integer is a prime number. For example, $12 = 2^2 \cdot 3$, $60 = 2^2 \cdot 3 \cdot 5$, $525 = 3 \cdot 5^2 \cdot 7$, etc. 3 divides 12 because $12 \div 3 = 4$, or $12 = 3 \cdot 4$. On the other hand, 5 does not divide 12, because when we divide 12 by 5, we get a remainder of 2. We can say the same thing using different words: 5 does not divide 12 because there is no integer whose product with 5 equals 12. In general, we say an integer $a \neq 0$ divides an integer b, if there exists an integer c such that $b = ac$.

A similar situation exists with polynomials. Polynomials can also be factored into the product of other polynomials with smaller degrees. For example, $3x^2 + 5x - 2 = (3x-1)(x+2)$. Therefore, we say that the polynomial $3x-1$ divides the polynomial $3x^2 + 5x - 2$. Also, the polynomial $x+2$ divides the polynomial $3x^2 + 5x - 2$. We note that $2(x+2)$ also divides the polynomial $3x^2 + 5x - 2$ because $3x^2 + 5x - 2 = \left(\frac{3}{2}x - \frac{1}{2}\right)(2x+4)$. The coefficients of the polynomials can be real numbers and even complex numbers in general. But, for computational simplicity, most of our polynomials will have integer coefficients.

Similarly, each of the polynomials $x-2$ and $x+2$ divides the polynomial $x^2 - 4$ because $x^2 - 4 = (x-2)(x+2)$. On the other hand, the polynomial x does not divide $x-2$ because there is no polynomial whose product with x results in $x-2$.

DEFINITION

A polynomial $K(x)$ divides a polynomial $P(x)$ if there exists a polynomial $Q(x)$ such that $P(x) = K(x)Q(x)$.

We will now learn how to carry out the division of polynomials.

EXAMPLE 3.4a: Let $P(x) = 2x^4 + 5x^3 + 3x^2 - 2x - 8$ and $K(x) = x - 1$. Divide $P(x)$ by $K(x)$.

SOLUTION: As we said before, the division of polynomials is similar to the division of multi-digit numbers. Each step in the following division process is accompanied by an explanation on the right.

$$\begin{array}{r|l} \begin{array}{r} \overline{2x^3+7x^2+10x+8} \\ x-1\overline{)\;2x^4+5x^3+3x^2-2x-8} \\ \underline{2x^4-2x^3} \\ 7x^3+3x^2-2x-8 \\ \underline{7x^3-7x^2} \\ 10x^2-2x-8 \\ \underline{10x^2-10x} \\ 8x-8 \\ \underline{8x-8} \\ 0 \end{array} & \end{array}$$

1. Divide $2x^4$ by x. You get $2x^3$.
2. Multiply $2x^3$ by the divisor $x-1$ and subtract the result from the polynomial. You get $7x^3+3x^2-2x-8$.
3. Divide $7x^3$ by x. You get $7x^2$.
4. Multiply $7x^2$ by the divisor $x-1$ and subtract the result from the polynomial. You get $10x^2-2x-8$.
5. Divide $10x^2$ by x. You get $10x$.
6. Multiply $10x$ by the divisor $x-1$ and subtract the result from the polynomial. You get $8x-8$.
7. Divide $8x$ by x. You get 8.
8. Multiply 8 by the divisor $x-1$ and subtract the result from the polynomial. You get 0.

Thus, $P(x)$ divided by $K(x)$ is the polynomial $Q(x) = 2x^3 + 7x^2 + 10x + 8$.

We write: $\dfrac{2x^4+5x^3+3x^2-2x-8}{x-1} = 2x^3+7x^2+10x+8$

or

$2x^4+5x^3+3x^2-2x-8 = (2x^3+7x^2+10x+8)(x-1)$

EXAMPLE 3.4b: Let $P(x) = 2x^3 + 7x^2 + 10x + 8$ and $K(x) = x + 2$. Divide $P(x)$ by $K(x)$.

SOLUTION:

$$\begin{array}{r} \overline{2x^2+3x+4} \\ x+2\overline{)\;2x^3+7x^2+10x+8} \\ \underline{-(2x^3+4x^2)} \\ 3x^2+10x+8 \end{array}$$

1. Divide $2x^3$ by x. You get $2x^2$.
2. Multiply $2x^2$ by the divisor $x+2$ and subtract the result from the polynomial. You get $3x^2+10x+8$.
3. Divide $3x^2$ by x. You get $3x$.

$$-(3x^2+6x)$$

← 4. Multiply $3x$ by the divisor $x+2$ and subtract the result from the polynomial. You get $4x+8$.

$$4x+8$$

← 5. Divide $4x$ by x. You get 4.

$$-(4x+8)$$

← 6. Multiply 4 by the divisor $x+2$ and subtract the result from the polynomial. You get 0.

$$0$$

Therefore, we get: $2x^3+7x^2+10x+8=(x+2)(2x^2+3x+4)$.

In both of these examples, the remainder of dividing $P(x)$ by $K(x)$ is zero. This is not always the case, as is shown in the following example.

EXAMPLE 3.4c: Let $P(x)=-4x^4+9x^3-12x+5$ and $K(x)=x^2+2$. Divide $P(x)$ by $K(x)$.

SOLUTION: Note, the coefficient of x^3 in polynomial $K(x)$ is zero.

$$\begin{array}{r l}
 & -4x^2+9x+8 \\
x^2+2 \,\big) & -4x^4+9x^3+0\cdot x^2-12x+5 \\
 & \underline{-(-4x^4 \qquad -8x^2)} \\
 & 9x^3+8x^2 \\
 & \underline{-(9x^3 \qquad +18x)} \\
 & 8x^2-30x \\
 & \underline{-(8x^2 \qquad +16)} \\
 & -30x-11
\end{array}$$

← 1. Divide $-4x^4$ by x^2. You get $4x^2$.

← 2. Multiply $4x^2$ by the divisor x^2+2 and subtract the result from the polynomial. You get $9x^3+8x^2$.

← 3. Divide $9x^3$ by x^2. You get $9x$.

← 4. Multiply $9x$ by the divisor x^2+2 and subtract the result from the polynomial. You get $8x^2-30x$.

← 5. Divide $8x^2$ by x^2. You get 8.

← 6. Multiply 8 by the divisor x^2+2 and subtract the result from the polynomial. You get $-30x-11$.

Since $-30x$ cannot be divided by x^2, we have the remainder $-30x-11$. Therefore, we get:

$$\frac{-4x^4+9x^3+0\cdot x^2-12x+5}{x^2+2}=-4x^2+9x+8+\frac{-30x-11}{x^2+2}$$

or

$$-4x^4+9x^3-12x+5=(x^2+2)(-4x^2+9x+8)+(-30x-11)$$

The division of polynomials just shown is based on the following theorem, which we accept without proof.

DIVISION ALGORITHM

Let $P(x)$ and $K(x)$ be two polynomials, where $K(x)$ is not the zero polynomial. Then, there exist two unique polynomials $Q(x)$ and $R(x)$ such that $P(x) = K(x)Q(x) + R(x)$; the polynomial $Q(x)$ is the quotient, and the remainder $R(x)$ is either the zero polynomial or of a degree that is less than the degree of $K(x)$.

EXAMPLE 3.4d:

In the last example, we saw that $-4x^4 + 9x^3 - 12x + 5 = (x^2 + 2)(-4x^2 + 9x + 8) + (-30x - 11)$. Here $P(x) = -4x^4 + 9x^3 - 12x + 5, K(x) = x^2 + 2$, $Q(x) = -4x^2 + 9x + 8$, and $R(x) = -30x - 11$. The degree of the remainder $R(x)$ is less than the degree of the divisor $K(x)$.

EXAMPLE 3.4e:

We have seen in previous examples that $2x^3 + 7x^2 + 10x + 8 = (x + 2)(2x^2 + 3x + 4)$
Here $Q(x) = 2x^2 + 3x + 4$, and the remainder $R(x)$ is the zero-polynomial.

The following theorem relates evaluating a polynomial to the division of polynomials.

REMAINDER THEOREM

If R is the remainder when a polynomial $P(x)$ is divided by $x - a$, then $R = P(a)$.

PROOF

We use the Division Algorithm. Let $Q(x)$ be the quotient and R be the remainder when $P(x)$ is divided by $x - a$. Thus, $P(x) = Q(x)(x - a) + R$.

Setting $x = a$ in the above equation, we get $P(a) = Q(a)(a - a) + R$. That is, $R = P(a)$.

EXAMPLE 3.4f:

When dividing the polynomial $P(x)=3x^3-4x^2+5$ by $x+2$ using the Division Algorithm, we get the remainder -35. Using the Remainder Theorem, we obtain the same result:

$P(-2)=3\cdot(-2)^3-4\cdot(-2)^2+5=-35$

3.5 HOW DOES THE DIVISION OF POLYNOMIALS HELP US WITH POLYNOMIAL EQUATIONS?

Division of polynomials can be very helpful in solving polynomial equations. We have no general method of solving polynomial equations of any degree. However, sometimes we can solve a polynomial equation of a high degree by factoring the polynomial into the product of lower-degree polynomials.

EXAMPLE 3.5a: Solve the equation $x^3+3x^2-18x-40=0$.

SOLUTION: We first factor the polynomial $x^3+3x^2-18x-40$ (we will learn about factoring polynomials later): $x^3+3x^2-18x-40=(x+2)(x^2+x-20)$. With this factorization, we reduce the task of solving an unfamiliar equation $x^3+3x^2-18x-40=0$ into solving two familiar equations: $x+2=0$ or $x^2+x-20=0$.

The solution of $x+2=0$ is $x=-2$.
The solutions of $x^2+x-20=0$ are $x=-5$, $x=4$.
Hence, the given equation has three solutions: $x=-2$, $x=-5$, and $x=4$.

The question now is: How would one know how to factor the polynomial $x^3+3x^2-18x-40$ or any other high-degree polynomial? The following two theorems, together with the Division Algorithm, can be very helpful in factoring certain polynomials.

THE RATIONAL ROOT THEOREM

Let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ be a polynomial with integer coefficients. If a reduced fraction $\frac{m}{k}$ is a solution to the equation $P(x)=0$, then m divides a_0 and k divides a_n.

FACTOR THEOREM

The polynomial $x-a$ is a factor of $P(x)$ if and only if $P(a)=0$. The Factor Theorem follows from the Remainder Theorem, and a direct proof of it by using the Division Algorithm is presented in Section 3.7. But, before we present the proofs of these theorems, we will first consider how to apply them to solve polynomial equations.

EXAMPLE 3.5b: Solve $x^3-2x-1=0$.

SOLUTION: Let $P(x)=x^3-2x-1$. In this polynomial, $a_0=-1$ and $a_n=1$. By the Rational Root Theorem, if a reduced fraction $\frac{m}{k}$ is a solution to $P(x)=0$, then m divides -1 and k divides 1.

The integers that divide -1 are $m=\pm 1$.

And, the integers that divide 1 are $k=\pm 1$.

Therefore, any fraction $\frac{m}{k}$, with these values of m and k, is a candidate to solve $P(x)=0$.

The following are all the possible fractions $\frac{m}{k}$: $\frac{m}{k}=\frac{+1}{+1},\frac{-1}{-1},\frac{+1}{-1},\frac{-1}{+1}$

After removing repetitions, we get two candidates for rational solutions to $P(x)=0$. These are: $\frac{+1}{+1}=1$ and $\frac{+1}{-1}=-1$

Now we only need to find out which of these two numbers is a solution to $P(x)=0$.

We substitute $x=1$ in $P(x)=0$, and get $1\cdot 1^3-2\cdot 1-1=-2\neq 0$ Thus, $x=1$ is not a solution.

We substitute $x=-1$ in the same equation, and get $1\cdot(-1)^3-2\cdot(-1)-1=0$ Thus, $x=-1$ is a solution.

By the Factor Theorem, $x-(-1)=x+1$ divides x^3-2x-1. Let's do the division:

$$
\begin{array}{r}
x^2 - x - 1 \\
x+1 \overline{\smash{\big)}\, x^3 + 0\cdot x^2 - 2x - 1} \\
\underline{-(x^3 + x^2)} \\
-x^2 - 2x - 1 \\
\underline{-(-x^2 - x)} \\
-x - 1 \\
\underline{-(-x - 1)} \\
0
\end{array}
$$

This shows that $x^3 - 2x - 1 = (x^2 - x - 1)(x+1)$

Solving $(x^2 - x - 1)(x+1) = 0$, we get

$x + 1 = 0 \qquad x^2 - x - 1 = 0$

$x = -1 \qquad x = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

Thus, our polynomial equation has three solutions: $x = -1$, $x = \dfrac{1+\sqrt{5}}{2}$, $x = \dfrac{1-\sqrt{5}}{2}$.

Thus, we can see how the Division Algorithm, the Rational Root Theorem, and the Factor Theorem help us to solve a polynomial equation of degree three.

EXAMPLE 3.5c: Solve $2x^3 - 5x + 3 = 0$.

• •

SOLUTION: Let $P(x) = 2x^3 - 5x + 3$. In this polynomial, $a_0 = 3$ and $a_3 = 2$. By the Rational Root Theorem, if a reduced fraction $\dfrac{m}{k}$ is a solution to $P(x) = 0$, then m divides 3 and k divides 2.

The integers that divide 3 are $m = \pm 3, \pm 1$.

And the integers that divide 2 are $k = \pm 2, \pm 1$.

Therefore, any fraction $\dfrac{m}{k}$ with these values of m and k is a candidate to solve the equation $P(x) = 0$.

The following are the possible fractions $\dfrac{m}{k}$:

$\frac{m}{k}=\frac{+3}{+2},\frac{+3}{-2},\frac{-3}{+2},\frac{-3}{-2}$ $\frac{m}{k}=\frac{+3}{+1},\frac{+3}{-1},\frac{-3}{+1},\frac{-3}{-1}$

$\frac{m}{k}=\frac{+1}{+2},\frac{+1}{-2},\frac{-1}{+2},\frac{-1}{-2}$ $\frac{m}{k}=\frac{+1}{+1},\frac{+1}{-1},\frac{-1}{+1},\frac{-1}{-1}$

After removing all the repetitions, we get the following eight candidates for solutions to $P(x)=0$: $\pm\frac{3}{2}$, ±3, $\pm\frac{1}{2}$, ±1

Now we only need to examine which of these 8 numbers are solutions to $P(x)=0$. We start with the easy ones: ±1 and ±3.

$2\cdot1^3-5\cdot1+3\cdot1=0$ $x=1$ is a solution

$2\cdot(-1)^3-5\cdot(-1)+3=6$ $x=-1$ is not a solution

$2\cdot3^3-5\cdot3+3=42$ $x=3$ is not a solution

$2\cdot(-3)^3-5\cdot(-3)+3=-36$ $x=-3$ is not a solution

You can check on your own that none of the other candidates is a solution to $P(x)=0$.

Now we can apply the Factor Theorem. Since, $P(1)=0$, $x-1$ must divide $P(x)$. After dividing $P(x)$ by $x-1$, we get $2x^3-5x+3=(x-1)(2x^2+2x-3)$. Therefore, the problem of solving the equation $2x^3-5x+3=0$ is reduced to the problem of solving the equations $x-1=0$ and $2x^2+2x-3=0$.

Solving, we get: $x=1$, $x=\frac{-1-\sqrt{7}}{2}$, $x=\frac{-1+\sqrt{7}}{2}$

Hence, these three values are solutions of the equation $2x^3-5x+3=0$.

EXAMPLE 3.5d: Solve $2x^4+5x^3+3x^2-2x-8=0$.

SOLUTION: Let $P(x)=2x^4+5x^3+3x^2-2x-8$. In this polynomial, $a_0=-8$ and $a_n=2$. By the Rational Root Theorem, if a reduced fraction $\frac{m}{k}$ is a solution to $P(x)=0$, then m divides -8 and k divides 2.

The integers that divide $a_0=-8$ are $m=\pm8,\pm4,\pm2,\pm1$. And, the integers that divide $a_n=2$ are $k=\pm2,\pm1$.

Therefore, any fraction $\frac{m}{k}$, with these values of m and k, is a solution candidate for $P(x)=0$. The following are all the possible fractions $\frac{m}{k}$:

$$\frac{m}{k}=\frac{+8}{+2},\frac{+8}{-2},\frac{-8}{+2},\frac{-8}{-2} \qquad \frac{m}{k}=\frac{+8}{+1},\frac{+8}{-1},\frac{-8}{+1},\frac{-8}{-1}$$
$$\frac{m}{k}=\frac{+4}{+2},\frac{+4}{-2},\frac{-4}{+2},\frac{-4}{-2} \qquad \frac{m}{k}=\frac{+4}{+1},\frac{+4}{-1},\frac{-4}{+1},\frac{-4}{-1}$$
$$\frac{m}{k}=\frac{+1}{+2},\frac{+1}{-2},\frac{-1}{+2},\frac{-1}{-2} \qquad \frac{m}{k}=\frac{+1}{+1},\frac{+1}{-1},\frac{-1}{+1},\frac{-1}{-1}$$

After removing all the repetitions, we get the following candidates as possible solutions for $P(x)=0$: $\pm 8, \pm 4, \pm 2, \pm 1, \pm\frac{1}{2}$.

Now we only need to examine which of these 10 numbers are solutions to $P(x)=0$.

We start with the easy ones: ± 1 and ± 2.

$2\cdot 1^4+5\cdot 1^3+3\cdot 1^2-2\cdot 1-8=0$ $\qquad x=1$ is a solution

$2\cdot(-1)^4+5\cdot(-1)^3+3\cdot(-1)^2-2\cdot(-1)-8=-6$ $\qquad x=-1$ is not a solution

$2\cdot 2^4+5\cdot 2^3+3\cdot 2^2-2\cdot 2-8=72$ $\qquad x=2$ is not a solution

$2\cdot(-2)^4+5\cdot(-2)^3+3\cdot(-2)^2-2\cdot(-2)-8=0$ $\qquad x=-2$ is a solution

You can check on your own that none of the other candidates is a solution to $P(x)=0$. After dividing $2x^4+5x^3+3x^2-2x-8$ by $x-1$, we get $2x^3+7x^2+10x+8$.

Therefore, we have $2x^4+5x^3+3x^2-2x-8=\left(2x^3+7x^2+10x+8\right)\left(x-1\right)$.

Now we solve two equations:

1. $K(x)=x-1=0$
2. $Q(x)=2x^3+7x^2+10x+8=0$

The first will give us $x=1$. To solve the second equation, we need to factor $Q(x)$. We have: $P(x)=Q(x)K(x)$.

Recall $x=-2$ solves the equation $P(x)=0$. Hence, $P(2)=Q(-2)K(-2)=0$.

Since $K(-2)\neq 0$, $Q(-2)=0$.

Now, we use the Factor Theorem again and divide $Q(x)$ by $x+2$ to get:

$2x^3+7x^2+10x+8=(2x^2+3x+4)(x+2)$

The solution of equation $2x^3+7x^2+10x+8=0$ is reduced to the solution of two equations:

1. $x+2=0$

2. $2x^2 + 3x + 4 = 0$

Solving the two equations, we get: $x = -2$ in the first one and no real solutions in the second one. Thus, the given equation has two real solutions: $x = -2$ and $x = 1$.

3.6 PROOF OF THE RATIONAL ROOT THEOREM

THE RATIONAL ROOT THEOREM

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$, where all the coefficients are integers, and let $\frac{m}{k}$ be a reduced fraction. If $\frac{m}{k}$ is a solution to the equation $P(x) = 0$, then m divides a_0 and k divides a_n.

PROOF

Let a reduced fraction $\frac{m}{k}$ be a solution to the equation $P(x) = 0$.

Then, $a_n\left(\frac{m}{k}\right)^n + a_{n-1}\left(\frac{m}{k}\right)^{n-1} + \cdots + a_1\left(\frac{m}{k}\right) + a_0 = 0$.

Multiplying both sides of this equation by k^{n-1} we get

$$a_n\frac{m^n}{k} + a_{n-1}m^{n-1} + a_{n-2}m^{n-2}k + \cdots + a_1 m^1 k^{n-2} + a_0 k^{n-1} = 0.$$

Since $a_0, a_1, a_2, \ldots, a_{n-1}$, k and m are all integers,

$a_{n-1}m^{n-1} + a_{n-2}m^{n-2}k + \cdots + a_1 m^1 k^{n-2} + a_0 k^{n-1}$ is an integer.

We have: $a_n\frac{m^n}{k} + \left[\mathit{Integer}\right] = 0$.

Hence, $a_n\frac{m^n}{k}$ must be an integer as well.

This implies that k divides $a_n m^n$.

Since $\frac{m}{k}$ is a reduced fraction, k and m have no common factor (other than 1), therefore k and m^n have no common factor too.

Hence, k divides a_n.

In a similar manner, we can prove that m divides a_0. To do this, we multiply both sides

of the equation $a_n\left(\frac{m}{k}\right)^n + a_{n-1}\left(\frac{m}{k}\right)^{n-1} + \cdots + a_1\left(\frac{m}{k}\right) + a_0 = 0$ by $\frac{k^n}{m}$.

We get $a_n m^{n-1} + a_{n-1} m^{n-2} k + \cdots + a_1 k^{n-1} + a_0 \frac{k^n}{m} = 0$.

As before, $a_n m^{n-1} + a_{n-1} m^{n-2} k + \cdots + a_1 k^{n-1}$ is an integer.

Again, since $\left[\mathit{Integer}\right] + a_0 \frac{k^n}{m} = 0$, $a_0 \frac{k^n}{m}$ must be an integer as well.

Since m and k have no common factor, m and k^n have no common factor either.

Hence m divides a_0. This completes the proof.

3.7 PROOF OF THE FACTOR THEOREM

FACTOR THEOREM

The polynomial $x-a$ divides a polynomial $P(x)$ if and only if $P(a)=0$.

PROOF

We have two assertions in this theorem:

1. If $x-a$ divides $P(x)$, then $P(a)=0$.
2. If $P(a)=0$, then $x-a$ divides $P(x)$.

PROOF OF 1

Since $x-a$ divides $P(x)$, there exists a polynomial $Q(x)$ such that $P(x)=Q(x)(x-a)$.

Substituting a for x, we get: $P(a)=Q(a)(a-a)=Q(a)\cdot 0=0$.

PROOF OF 2

By the Division Algorithm for polynomials, there exist two unique polynomials $Q(x)$ and $R(x)$ such that: $P(x)=Q(x)(x-a)+R(x)$, where either $R(x)$ is the zero polynomial or the degree of $R(x)$ is less than the degree of the $x-a$.

We will show that if $P(a)=0$, then $R(x)$ is the zero polynomial, and so $x-a$ divides $P(x)$. Since the degree of the $x-a$ is 1, $R(x)$ must be either the zero polynomial or a number, say r.

Let's examine these two cases:

1. Let $R(x)$ be the zero polynomial. Then $P(x)=Q(x)(x-a)$, and so $x-a$ divides $P(x)$.

2. Let $R(x) = r$. Substituting $x = a$ in $P(x) = Q(x)(x-a) + r$, we get $P(a) = (a-a)Q(a) + r$.

Since $P(a) = 0$, we get $0 = (a-a)Q(a) + r$. This implies that $r = 0$ and so $x - a$ divides $P(x)$. This completes the proof.

Before finishing our discussion of polynomial equations, we should note that polynomials of degree three or higher usually have rational roots when they are studied in high school.

Sections 3.1–3.7
EXERCISES

1. Solve the following equations
   a. $x^3 + x^2 - 4x - 4 = 0$
   b. $24x^3 + 16x^2 - 3x - 2 = 0$
   c. $x^3 + 1991x + 1992 = 0$

2. Find the quotient and the remainder of the division of $P(x)$ by $K(x)$ for
   $P(x) = 30x^4 - 31x^3 - 180x^2 + 7x + 6$, $K(x) = x + 1$.

3. The remainder of the division of $2x^5 - 3x^3 + 11x^2 - x + a$ by $x + 2$ is 3. Find the value of a.

4. Solve the equation $ax^3 - 2x^2 - 5x + 6 = 0$, knowing 2 is one of its solutions.

5. Find all rational solutions to the following equations:
   a. $6x^4 + 7x^3 - 22x^2 - 28x - 8 = 0$
   b. $x^3 + 3x^2 - 18x - 40 = 0$
   c. $x^6 - x^4 - x^2 + 1 = 0$

6. For what values of b and c are $x = -3$ and $x = -4$ solutions to the equation $x^6 + bx^5 + cx^4 = 0$?

7. Solve the following equations:
   a. $(x^2 + x + 1)(x^2 + x + 2) = 12$

b. $(x+1)(x+3)(x+5)(x+7)+15=0$

8. Prove that $K(x)=x$ does not divide $P(x)=x-2$ (Hint: Assume that there exists a polynomial $Q(x)$ such that $P(x)=Q(x)K(x)$, and conclude that $Q(x)$ must be a number. Proceed from there to conclude that such $Q(x)$ does not exist.)

3.8 COMPLEX NUMBERS

Earlier we concluded that there are no real solutions to a quadratic equation when its discriminant is less than zero. What is meant by this is that there are no ***real*** solutions, numbers of the set of real numbers $\mathbb{R}$. For example, if the discriminant is -1, there are no real solutions because there is no real number x for which $x^2=-1$. That is, the expression $\sqrt{-1}$ does not represent a ***real number***.

Mathematicians encountered certain problems in science and mathematics that necessitated extending the real numbers into a new set of numbers called ***complex numbers***. This was done by defining the expression $\sqrt{-1}$ as a number that solves the equation $x^2=-1$. They denoted this number by the letter *i*. This new number, in combination with the real numbers, generates the complex numbers.

Let's consider the equation $x^2-2x+5=0$.

Solving, we get: $x=\dfrac{2\pm\sqrt{4-4\cdot5}}{2}=\dfrac{2\pm\sqrt{-16}}{2}=\dfrac{2\pm4\sqrt{-1}}{2}=\dfrac{2\pm4i}{2}=1\pm2i$

The expressions $1+2i$ and $1-2i$ are complex numbers; they are solutions to the equation $x^2-2x+5=0$.

A complex number is an expression of the form $a+bi$, where a and b are real numbers. The set of all complex numbers is denoted by $\mathbb{C}$. Clearly, any real number r is a complex number since it can be written as $r+0i$. Hence, $\mathbb{R}$ is a subset of $\mathbb{C}$. With this relation, we can extend Figure 1–1 from Section 1 of the guide, as shown in Figure 3–1:

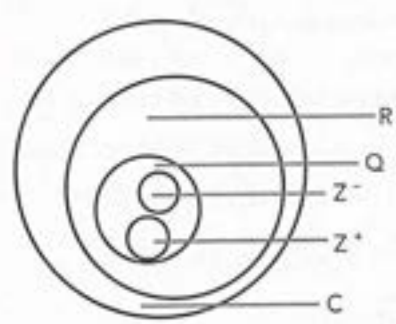


FIGURE 3–1

Properties 1–8 from SECTION 1 hold for the complex numbers as well.

Every complex number $z = a + ib$ corresponds to the point (a, b) in the coordinate plane, and, conversely, every point (a, b) in the coordinate plane corresponds to the complex number $z = a + ib$. (See FIGURE 3-2 below.)

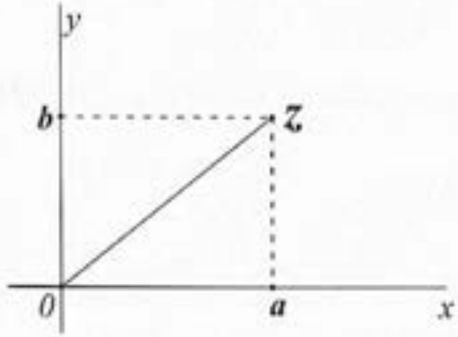


FIGURE 3-2

3.8.1 How Do We Add Complex Numbers?

Addition and subtraction of complex numbers are defined as follows:

Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ be two complex numbers; then $z_1 \pm z_2 = (a_1 \pm a_2) + (b_1 \pm b_2)i$.

EXAMPLE 3.8a:

1. $(5+3i)+(4-5i)=(5+4)+(3-5)i=9-2i$

2. $(3+15i)-(4+5i)=(3-4)+(15-5)i=-1+10i$

3. $\left(\frac{1}{2}+\frac{3}{4}i\right)-\left(\frac{2}{3}-2i\right)+\frac{1}{4}i=\left(\frac{1}{2}-\frac{2}{3}\right)+\left(\frac{3}{4}-(-2)+\frac{1}{4}\right)i=-\frac{1}{6}+3i$

3.8.2 How Do We Multiply Complex Numbers?

Multiplication of complex numbers is defined as follows:

Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ be two complex numbers. Then

$$z_1 \cdot z_2 = a_1 \cdot (a_2 + b_2 i) + b_1 i \cdot (a_2 + b_2 i) = a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 i^2 =$$
$$= a_1 a_2 + b_1 b_2 \left(\sqrt{-1}\right)^2 + (a_1 b_2 + a_2 b_1)i = a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1)i.$$

Notice that $i^2 = \left(\sqrt{-1}\right)^2 = \sqrt{-1} \cdot \sqrt{-1} = -1$.

EXAMPLE 3.8b:

1. $(4+15i)\cdot(17-2i)=4\cdot17+4\cdot(-2)i+15i\cdot17+15\cdot(-2)i^2=68-8i+255i-30\cdot(-1)=$
$=68+30+(255-8)i=98+247i$

2. $\left(1+\frac{1}{3}i\right)\cdot\left(4-\frac{1}{5}i\right)=1\cdot4+1\cdot\left(-\frac{1}{5}i\right)+\frac{1}{3}i\cdot4+\frac{1}{3}\left(-\frac{1}{5}\right)i^2=4-\frac{1}{5}i+\frac{4}{3}i-\frac{1}{15}\cdot(-1)=$
$=4+\frac{1}{15}+\left(\frac{4}{3}-\frac{1}{5}\right)i=4\frac{1}{15}+1\frac{2}{15}i$

3. $(3-5i)^2=3^2-2\cdot3\cdot5i+(5i)^2=9-30i+25\cdot(-1)=9-25-30i=-16-30i$

4. $(12+3i)\cdot(12-3i)=12^2-(3i)^2=144+9=153$

In the last example, we multiplied two complex numbers that differ by the sign in front of $3i$. One is referred to as the ***conjugate*** of the other. In general, we say that $a-bi$ is the conjugate of $a+bi$, and vice versa: $a+bi$ is the conjugate of $a-bi$.

The product of a complex number and its conjugate is a real number:
$(a+bi)\cdot(a-bi)=a^2-(bi)^2=a^2-b^2i^2=a^2+b^2$.

The conjugate of a complex number z is denoted by $\overline{z}$. Therefore, we have: $z\cdot\overline{z}=a^2+b^2$.

And, $z\cdot\frac{\overline{z}}{a^2+b^2}=\frac{z\overline{z}}{a^2+b^2}=\frac{a^2+b^2}{a^2+b^2}=1$

From the last equality, we see that $\frac{\overline{z}}{a^2+b^2}$ is the reciprocal of z because their product is 1. As with real numbers, the reciprocal of z is denoted by z^{-1}.

If $z=a+bi$, then $z^{-1}=\frac{\overline{z}}{a^2+b^2}=\frac{a-bi}{a^2+b^2}=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i$.

As with real numbers, the absolute value of a complex number z, also denoted by $|z|$, is defined as the distance of z from the origin. Therefore: $|z|=\sqrt{a^2+b^2}$.

3.8.3 How Do We Divide Complex Numbers?

Let $z_1=a_1+b_1i$ and $z_2=a_2+b_2i$ be two complex numbers, and $z_2\neq0$. And, let z be a complex number such that: $z=\frac{z_1}{z_2}$.

We rewrite the last expression as $z = z_1 \cdot z_2^{-1}$.

Since $z_2^{-1} = \frac{\overline{z}_2}{a_2^2 + b_2^2} = \frac{a_2}{a_2^2 + b_2^2} - \frac{b_2}{a_2^2 + b_2^2} i$, we have:

$$z = z_1 \cdot z_2^{-1} = (a_1 + b_1 i) \cdot \frac{\overline{z}_2}{a_2^2 + b_2^2} = \frac{(a_1 + b_1 i)\cdot(a_2 - b_2 i)}{a_2^2 + b_2^2} = \frac{(a_1 + b_1 i)\cdot(a_2 - b_2 i)}{(a_2 + b_2 i)\cdot(a_2 - b_2 i)} = \frac{z_1 \cdot \overline{z}_2}{z_2 \cdot \overline{z}_2}.$$

or

$$z = \frac{z_1}{z_2} = \frac{z_1 \cdot \overline{z}_2}{z_2 \cdot \overline{z}_2}$$

Substituting in the last equation $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ and multiplying, we get:

$$z = \frac{(a_1 + b_1 i)\cdot(a_2 - b_2 i)}{(a_2 + b_2 i)\cdot(a_2 - b_2 i)} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \frac{a_2 b_1 + a_1 b_2}{a_2^2 + b_2^2} i.$$

This shows that the result of dividing two complex numbers is a complex number.

EXAMPLE 3.8c:

1. $\frac{2+5i}{3-5i} = \frac{(2+5i)\cdot(3+5i)}{(3-5i)\cdot(3+5i)} = \frac{6+10i+15i+5i^2}{3^2+5^2} = \frac{6-25+(10+15)i}{34} = \frac{-19+25i}{34} = -\frac{19}{34} + \frac{25}{34} i$

2. $\frac{\sqrt{2}-i}{\sqrt{3}-4i} = \frac{\left(\sqrt{2}-i\right)\cdot\left(\sqrt{3}+4i\right)}{\left(\sqrt{3}-4i\right)\cdot\left(\sqrt{3}+4i\right)} = \frac{\sqrt{2}\cdot\sqrt{3}+\sqrt{2}\cdot 4i - i\sqrt{3} - 4i^2}{\left(\sqrt{3}\right)^2 - (4i)^2} = \frac{\sqrt{6}+4+\left(4\sqrt{2}-\sqrt{3}\right)i}{3+4} =$

   $= \frac{\sqrt{6}+4}{7} + \frac{4\sqrt{2}-\sqrt{3}}{7} i$

EXAMPLE 3.8d: Solve the equation $5x^2 - 6x + 5 = 0$.

SOLUTION: $5x^2 - 6x + 5 = 0$

$\Delta = 36 - 5\cdot 5\cdot 4 = 36 - 100 = -64$ The equation does not have real solutions.

$$x = \frac{6 \pm \sqrt{-64}}{10} = \frac{6 \pm \sqrt{64}\cdot\sqrt{-1}}{10} = \frac{6 \pm 8\cdot i}{10} = \frac{3}{5} \pm \frac{4}{5} i$$

Thus, $x = \frac{3}{5} \pm \frac{4}{5} i$.

EXAMPLE 3.8e: Solve the equation $x^3+3x^2+7x+10=0$.

SOLUTION: $x^3+3x^2+7x+10=0$

By the Rational Root Theorem, a rational solution to this equation must be one of the following:

$\frac{m}{k}=\frac{+10}{+1},\frac{+10}{-1},\frac{-10}{+1},\frac{-10}{-1}$, $\quad \frac{m}{k}=\frac{+5}{+1},\frac{+5}{-1},\frac{-5}{+1},\frac{-5}{-1}$,

$\frac{m}{k}=\frac{+2}{+1},\frac{+2}{-1},\frac{-2}{+1},\frac{-2}{-1}$, $\quad \frac{m}{k}=\frac{+1}{+1},\frac{+1}{-1},\frac{-1}{+1},\frac{-1}{-1}$.

After removing all the repetitions, we get the following candidates for rational solutions:

$\pm 1, \pm 2, \pm 5, \pm 10$.

Of all these, only $x=-2$ is a rational solution of the equation.

By the Factor Theorem, the polynomial $x^3+3x^2+7x+10$ is divisible by $x+2$. We get:

$x^3+3x^2+7x+10=(x+2)(x^2+x+5)$

$(x+2)(x^2+x+5)=0$
$x+2=0$

or

$x^2+x+5=0$.

Solving the first equation, we get $x=-2$.

Solving the second equation, we get $x=\frac{-1\pm\sqrt{-19}}{2}=\frac{-1\pm\sqrt{-1}\cdot\sqrt{19}}{2}=\frac{-1\pm i\sqrt{19}}{2}$.

Thus, the solutions to the given equation are: $x=-2$, $x=\frac{-1\pm i\sqrt{19}}{2}$.

EXAMPLE 3.8f: Solve the equation $(\sqrt{2}+i)x^2+2x-(\sqrt{2}-i)=0$.

SOLUTION: $(\sqrt{2}+i)x^2+2x-(\sqrt{2}-i)=0$

$\Delta=4+4(\sqrt{2}-i)(\sqrt{2}+i)=4+4\cdot(2+1)=16$

$$x = \frac{-2 \pm 4}{2(\sqrt{2}+i)} = \frac{-1 \pm 2}{\sqrt{2}+i}$$

$$x = \frac{1}{\sqrt{2}+i}$$

or

$$x = \frac{-3}{\sqrt{2}+i}.$$

Each of these solutions can be expressed in the form of $a+bi$. For this, we utilize the division of complex numbers as we learned earlier:

$$\frac{1}{\sqrt{2}+i} = \frac{1 \cdot \left(\sqrt{2}-i\right)}{\left(\sqrt{2}+i\right) \cdot \left(\sqrt{2}-i\right)} = \frac{\sqrt{2}-i}{2+1} = \frac{\sqrt{2}}{3} - \frac{1}{3}i$$

$$\frac{-3}{\sqrt{2}+i} = \frac{-3 \cdot \left(\sqrt{2}-i\right)}{\left(\sqrt{2}+i\right) \cdot \left(\sqrt{2}-i\right)} = \frac{-3\sqrt{2}+3i}{2+1} = -\sqrt{2}+i$$

Thus, the given equation has the solutions: $x = \frac{\sqrt{2}}{3} - \frac{1}{3}i$ and $x = -\sqrt{2}+i$.

Section 3.8
EXERCISES

1. Find the conjugate $\overline{z}$ and the absolute value $|z|$ of the given complex numbers z:

   a. $4-13i$

   b. $(1-i)^2$

   c. $\frac{(3+i)^2}{2-5i}$

2. Given $z_1 = -i+3$, $z_2 = 3i-1$, find:

   a. $z_1 + z_2$

   b. $z_1 \cdot z_2$

c. $\dfrac{|z_1 + z_2| i}{z_2}$

3. Present the following complex number in the standard form $a + bi$: $\dfrac{(2+i)^3 - (2-i)^3}{(2+i)^2 - (2-i)^2}$

4. Solve the following equation in the set $\mathbb{C}$: $x^5 + 2x^4 + 2x^3 + 2x^2 + x = 0$

Section 4
Functions

4.1 PRELIMINARIES

The term ***function*** is often used in everyday language to indicate dependencies between quantities. For example, we say "the commuting time from work to home is a function of traffic density"; "the price of gasoline is a function of supply from oil-producing countries"; "population growth is a function of time"; etc. In order to understand the properties of such dependencies, it is necessary to express them mathematically. For example, to study the growth rate of a population, it is necessary to model the size of the population at a given time. The equation $P(t) = 67.38(1.026)^t$ is an example of such a model; it models the population growth in a particular country at any given time. The equation says that the size of the population (in millions) at a given time t is the number $P(t) = 67.38(1.026)^t$. For example, the initial population (when the measurement started) is 67.38 million ($P(0) = 67.38(1.026)^0 = 67.38$), and after 5 years, it is 76.607 ($P(5) = 67.38(1.026)^5 = 76.607$).

Earlier we dealt with linear equations—equations of the form $ax + b = 0$ —and quadratic equations—equations of the form $ax^2 + bx + c = 0$. The expressions $ax + b$ and $ax^2 + bx + c$ are functions. We can think of the expression $2x - 7$ in terms of a dependency between quantities: the value of $2x - 7$ depends on the value of x, or we can say $2x - 7$ is a function of x. For $x = 3$, $2x - 7 = -1$; for $x = 0$, $2x - 7 = -7$; for $x = \sqrt{2}$, $2x - 7 = 2\sqrt{2} - 7 \sim -4.1716$. Functions of the form $y = ax + b$ are called ***linear functions***, and functions of the form $y = ax^2 + bx + c$, where $a \neq 0$, are called ***quadratic functions***.

When thinking of an expression as a function, we use symbols such as $f(x)$, $g(x)$, $k(x)$, etc. For example, the function $f(x) = 2x - 7$ indicates that for any value of x, there corresponds a value $f(x)$ (read *f of x*). For example, $x = 3$ corresponds to $f(3) = -1$; $x = 0$ corresponds to $f(0) = -7$; $x = \sqrt{2}$ corresponds to $f(\sqrt{2}) = 2\sqrt{2} - 7$, etc. Often, instead of $f(x) = 2x - 7$ we write $y = 2x - 7$, meaning the value of x corresponds to the value $y = 2x - 7$.

4.2 DEFINITION OF A FUNCTION

The function $f(x)=2x-7$ admits any value of x. That is to say, for any real number x, there is a corresponding value $f(x)=2x-7$. Likewise, the function $u(x)=3x^2-2x+4$ admits any value of x. Not all functions admit all real values. For example, the function $g(x)=\frac{1}{x}$ does not admit $x=0$, because $f(0)=\frac{1}{0}$ is not a number. Likewise, the function $h(x)=\sqrt{x}$ admits only non-negative numbers.

A number that can be admitted into a function is called an ***input***, and its corresponding number under the function is called an ***output***. Thus, for example, for the function $h(x)=x^2$, any real number can be an input. Can any real number be an output for this function? The answer is obviously no. For example, -1 cannot be an output, because there is no real number x for which $x^2=-1$.

The set of all possible inputs of a function is called the ***domain*** of the function; and the set of all the outputs of a function is called the ***range*** of the function. Thus, the domain of $h(x)=x^2$ is the set of all real numbers. The range of this function is the set of all non-negative numbers.

EXAMPLE 4.2a: What are the domain and range of the function $h(x)=\sqrt{x}$?

ANSWER: Both the domain and range of this function are the set of all non-negative numbers.

Sometimes determining the domain and range of a function requires a little more work.

EXAMPLE 4.2b: What are the domain and range of the function $k(x)=\frac{2}{x-3}$?

ANSWER: Clearly, any real number different from 3 is in the domain of this function. So, the domain consists of real numbers $x \neq 3$.

What about the range of this function?

By definition, a real number a is in the range if there exists x such that $\frac{2}{x-3}=a$.

Multiplying both sides of this equation by $x-3$, we get $2=a(x-3)$.

Clearly $a = 0$ is not in the range of the function because otherwise we get $2 = 0(x-3) = 0$, which is not possible.

On the other hand, any number $a \neq 0$ is in the range. To show this, we need to show that for any $a \neq 0$, the equation $\frac{2}{x-3} = a$ is solvable for x. Indeed:

$$\frac{2}{x-3} = a$$

$$2 = a(x-3)$$

$$\frac{2}{a} = x-3$$

$$x = \frac{2}{a} + 3$$

To summarize, the domain of the function $k(x) = \frac{2}{x-3}$ consists of all the real numbers different from 3, and its range consists of all the real numbers different from 0.

With this background, we can now introduce the formal definition of a ***function***.

DEFINITION

A function f is a correspondence between two sets, A and B, that assigns to each element x in A one and only one element $f(x)$ in B.

This state of affairs is indicated by the following notation: $f : A \to B$

A is called the ***domain*** of the function, and B is called the ***range*** of the function. You might think of a function f as a machine with an input and output. The domain of f is all the numbers that can be fed into the machine, and the range is all the numbers that come out of the machine. (See FIGURE 4–1.) The machine never produces two outputs for the same input.

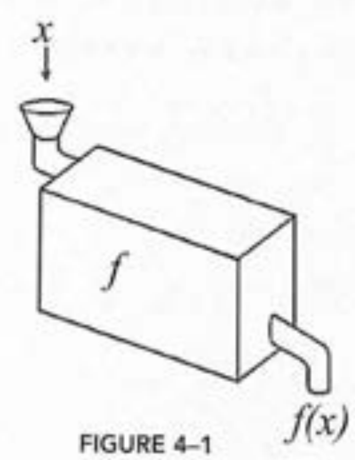


FIGURE 4–1

Thinking of a function as a machine helps us see how functions are composed of more elementary functions. For example, the function $f(x)=(x+2)^2$ is composed of two functions: $g(x)=x+2$ and $h(x)=x^2$. Any given input x corresponds to the output $g(x)=x+2$. This output, in turn, is an input for the function h. We get: $h(g(x))=h(x+2)=(x+2)^2=f(x)$. Thus, f is a composition of h and g: $f(x)=h(g(x))$. FIGURE 4–2 below illustrates this composition.

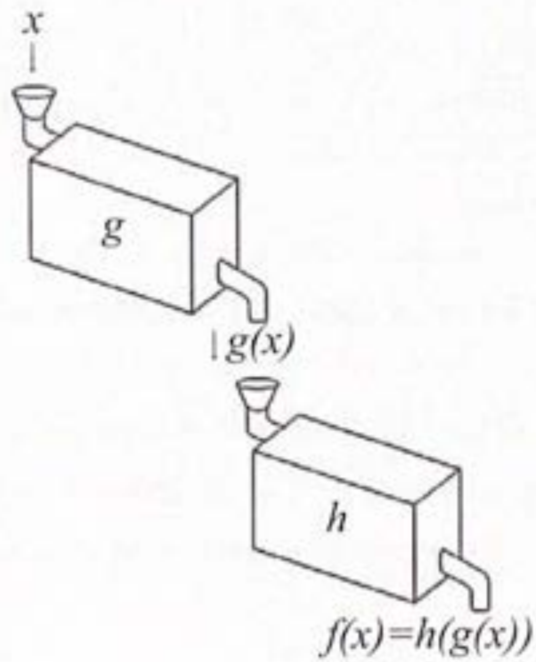


FIGURE 4–2

EXAMPLE 4.2c: Consider the functions $g(x)=2x-3$ and $h(x)=\sqrt{x}$. These functions can be composed in two ways: $h(g(x))$ and $g(h(x))$.

1. For a given input x, we get the output $g(x)=2x-3$, which, in turn, is an input for the function h. Thus, we have $h(g(x))=\sqrt{g(x)}=\sqrt{2x-3}$.
2. For a given input x, we get the output $h(x)=\sqrt{x}$, which, in turn, is an input for the function g. We have $g\left(h(x)\right)=2h(x)-3=2\sqrt{x}-3$.

EXAMPLE 4.2d: The function $f(x)=\left(\sqrt{x^2-9}+3\right)^2$ can be looked at as a composition of three functions: $g(x)=x^2-9$, $h(x)=\sqrt{x}$ and $k(x)=(x+3)^2$.

1. Every non-negative output of the function g is an input of the function h: $h(g(x))=\sqrt{g(x)}=\sqrt{x^2-9}$;
2. Every output of the function h is an input of the function k: $k\left(h\right)=\left(h+3\right)^2=\left(\sqrt{x^2-9}+3\right)^2$;
3. Finally, every output of the function k determines the function f: $f(x)=k\left(h\left(g(x)\right)\right)=\left(\sqrt{x^2-9}+3\right)^2$.

4.3 MANY-TO-ONE FUNCTIONS VERSUS ONE-TO-ONE FUNCTIONS

Is it possible that different inputs of a function correspond to one output of the same function? The answer is yes. For example, for the function $h(x)=x^2$, the inputs $x=2$ and $x=-2$ correspond to the same output 4. Such functions are often called ***many-to-one*** functions, meaning that more than one input corresponds to one output.

We have already learned that any quadratic equation $ax^2+bx+c=0$ for which the discriminant $\Delta=b^2-4ac>0$ has two distinct solutions: $x_1=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$ and $x_2=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$. This shows that quadratic functions (for which the discriminant is positive) are many-to-one functions.

Is it possible that functions are ***one-to-many***: that is, one input of a function corresponds to more than one output of the same function? The answer is no. This possibility is excluded in order to avoid confusion. If we allowed, for example, $f(x)=\pm\sqrt{x}$ to be a function, upon evaluating an expression such as $f(x)+x+1$ at $x=4$, we wouldn't know whether the outcome is $2+4+1=7$ or $-2+4+1=3$.

Finally, there are functions that are ***one-to-one***, meaning that two distinct inputs must correspond to two distinct outputs.

EXAMPLE 4.3a: Any linear function $y=ax+b$, for which $a\neq 0$, is one-to-one.

PROOF: Take any two distinct inputs x_1 and x_2; their corresponding outputs $y_1=ax_1+b$ and

$y_2 = ax_2 + b$ must be different. Assume this is not so, namely that $y_1 = y_2$. Then:

$ax_1 + b = ax_2 + b$

$ax_1 = ax_2$

$ax_1 - ax_2 = 0$

$a(x_1 - x_2) = 0$

Since $a \neq 0$, $x_1 = x_2$.

This, of course, is not possible since we started with two distinct inputs x_1 and x_2. Hence, $y_1 \neq y_2$.

EXAMPLE 4.3b: $y = \sqrt{x}$ is *one-to-one*

PROOF: As before, take any two distinct inputs x_1 and x_2. We are to show that their corresponding outputs $y_1 = \sqrt{x_1}$ and $y_2 = \sqrt{x_2}$ are different. Assume this is not so, namely that $y_1 = y_2$. Then:

$\sqrt{x_1} = \sqrt{x_2}$

$\left(\sqrt{x_1}\right)^2 = \left(\sqrt{x_2}\right)^2$

$x_1 = x_2$

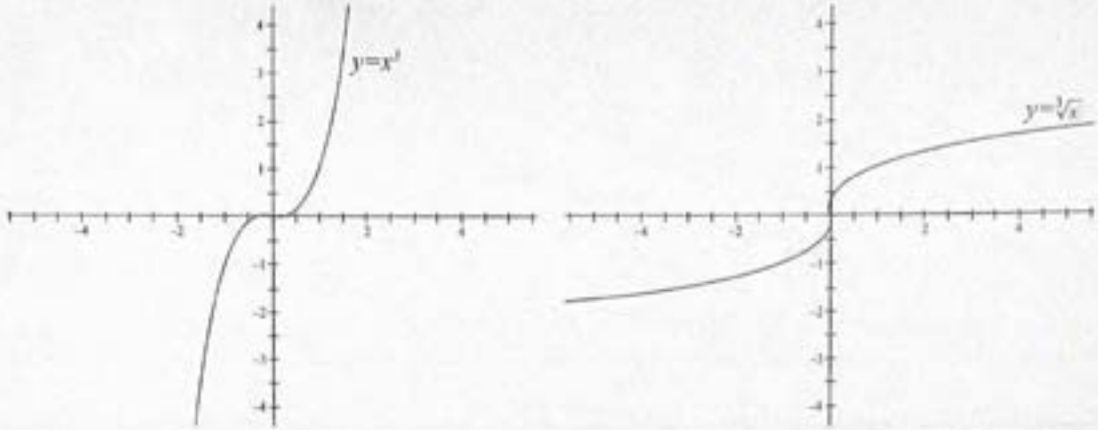
This of course is not possible since we started with two distinct inputs x_1 and x_2. Hence, it is not true that $y_1 = y_2$.

4.4 INVERSE FUNCTIONS

Let f be a function. If there exists a function g such that $y = f(x)$ if and only if $x = g(y)$, then we say that f is ***invertible*** and g is the ***inverse*** of f. The function g is unique (see the next page for a proof) and is denoted by f^{-1}.

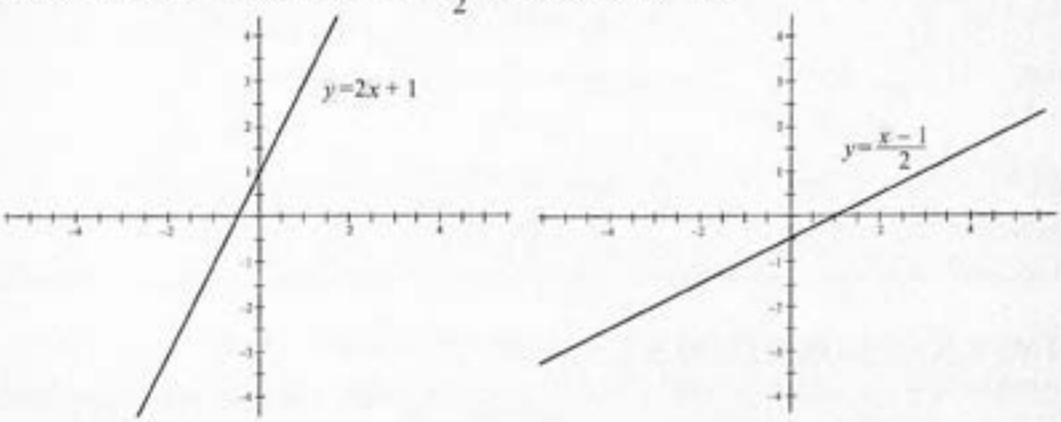
EXAMPLE 4.4a: The inverse of the function $f(x) = x^3$ is $g(x) = \sqrt[3]{x}$.

PROOF: If $y = x^3$ then $\sqrt[3]{y} = x$, and conversely, if $\sqrt[3]{y} = x$, then $y = x^3$. Thus, $g(x) = f^{-1}(x)$.



EXAMPLE 4.4b: The inverse of $f(x)=2x+1$ is $g(x)=\dfrac{x-1}{2}$.

PROOF: $y=2x+1$ if and only if $x=\dfrac{y-1}{2}$. Thus $g(x)=f^{-1}(x)$



EXAMPLE 4.4c: Find the inverse of $f(x)=\dfrac{2x+5}{-3x+7}$.

SOLUTION: $y=\dfrac{2x+5}{-3x+7}$ if and only if $x=\dfrac{7y-5}{3y+2}$ (check). Hence the function $g(x)=\dfrac{7x-5}{3x+2}$ is the inverse of the function $f(x)=\dfrac{2x+5}{-3x+7}$.

The domain of f is the set all real numbers different from $\dfrac{7}{3}$, and the domain of g is the set of all real numbers different from $-\dfrac{2}{3}$.

THEOREM

An invertible function f can have only one inverse function.

PROOF

Let f_1 and f_2 be two different inverse functions of f.

Then $y = f(x)$ if and only if $x = f_1(y)$

and

$y = f(x)$ if and only if $x = f_2(y)$

This implies that $x = f_1(y)$ if and only if $x = f_2(y)$. We see then that if f_1 and f_2 are both inverse functions of f, then $f_1 = f_2$.

THEOREM

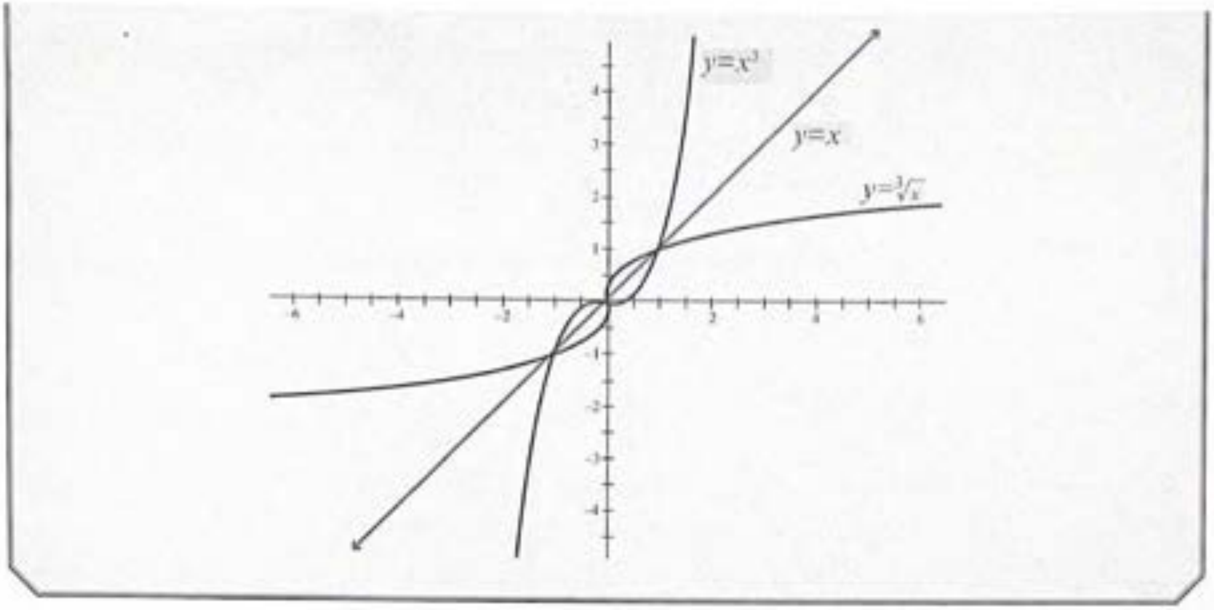
The graph of f^{-1} is the reflection of f in the line $y = x$.

PROOF

Implied from the definition of an inverse function is that $A(x_0, y_0)$ is on the graph of f if and only if $A'(y_0, x_0)$ is on the graph of f^{-1}. Notice that A and A' are symmetric with respect to the line $y = x$. This is so, because the midpoint between A and A' is $M\left(\frac{x_0 + y_0}{2}, \frac{y_0 + x_0}{2}\right)$, and M is on the line $y = x$.

EXAMPLE 4.4d:

The property we have just proven can clearly be seen in the graphs of $f(x) = x^3$ and $f^{-1}(x) = \sqrt[3]{x}$ which we discussed earlier. This property is useful because if we know the graph of an invertible function f, then we can graph its inverse f^{-1}. The two graphs are symmetric with respect to $y = x$.



Not all functions are invertible of course. For example, $y = x^2$ is not invertible. To see why, we graph $y = x^2$, and then graph its symmetric image with respect to the line $y = x$. (See FIGURE 4–3 below.) We see that the resulting graph is not a function because for one input, x, there are two outputs, y_1 and y_2.

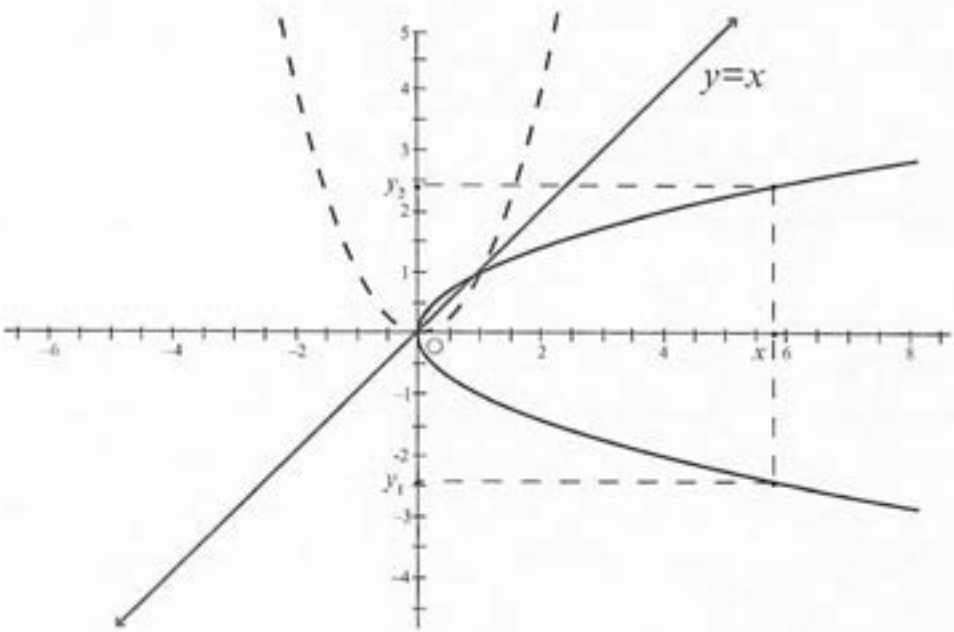


FIGURE 4–3

Section 4

EXERCISES

1. Find the domain and the range of the following functions:

   a. $f(x)=\dfrac{3}{2-x}$

   b. $f(x)=\sqrt{x-5}$

   c. $f(x)=3+(x-2)^2$

2. Determine which of the functions have an inverse. If the function is invertible, find its inverse.

   a. $f(x)=x-3$

   b. $f(x)=x^3-3$

3. a. $h(x)=x^2$, $g(x)=2-x$. Find $h(g(x))$ and $g(h(x))$.

   b. $h(x)=\sqrt{6x-1}$, $g(x)=x+1$, $k(x)=\dfrac{1}{x}$. Find $h(g(x))$, $h(g(k(x)))$ and $k(g(h(x)))$.

4. Find the range and domain of $f(x)=\dfrac{\frac{2}{7}-5.7x}{\sqrt{23}+\frac{1}{6}x}$.

Section 5
Graphing

The graph of a function f is the set of all ordered pairs $(x, f(x))$, where x belongs to the domain of f. In this section, we will learn what the graphs of different functions look like.

5.1 WHAT DOES THE GRAPH OF A LINEAR FUNCTION $y = ax + b$ LOOK LIKE?

In SECTION 8 we will prove that the graph of a linear function $y = ax + b$ is a line. The coefficients a and b determine the form of the graph. For $x = 0$, $y = b$. Therefore, $(0, b)$ is the point where the line intersects the **y-axis**. This number b is called the **y-intercept**. The graph is shown in FIGURE 5–1.

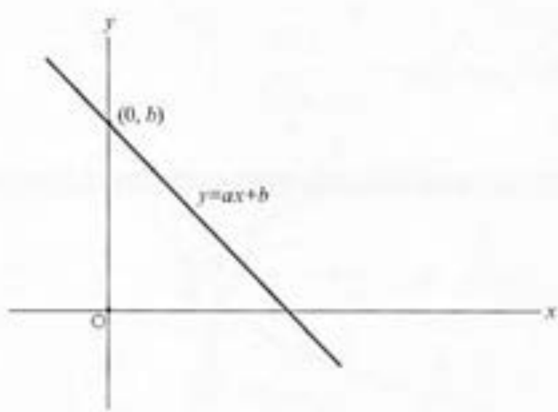


FIGURE 5–1

When $a = 0$, we have $y = 0 \cdot x + b = b$. We see that the output value for any x is b. Hence, the graph is simply a line parallel to the **x-axis**. Its y-intercept is b. (See FIGURE 5–2.)

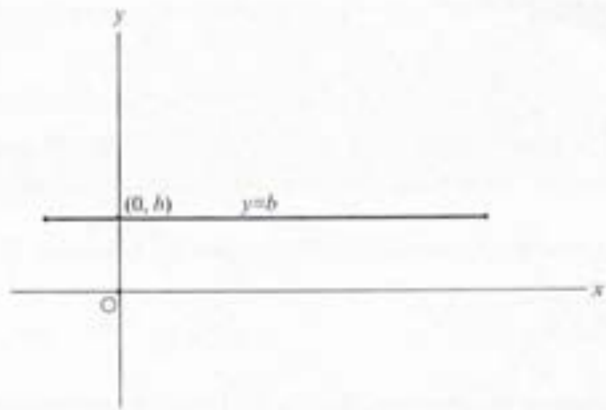


FIGURE 5–2

When $a \neq 0$, the graph intersects the x-axis at $x = -\frac{b}{a}$ (why?). This point is called the **x-intercept**. (See FIGURE 5-3 below.)

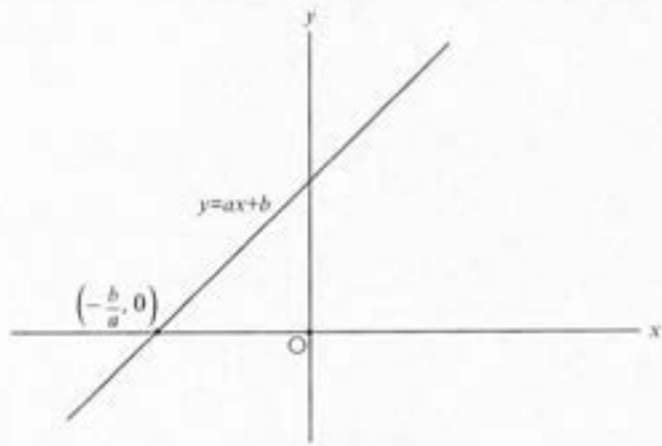


FIGURE 5–3

When $a > 0$, the function $y = ax + b$ is increasing. This means the following: For any numbers x_1 and x_2, if $x_1 \geq x_2$, then $ax_1 + b \geq ax_2 + b$ (See FIGURE 5–4.).

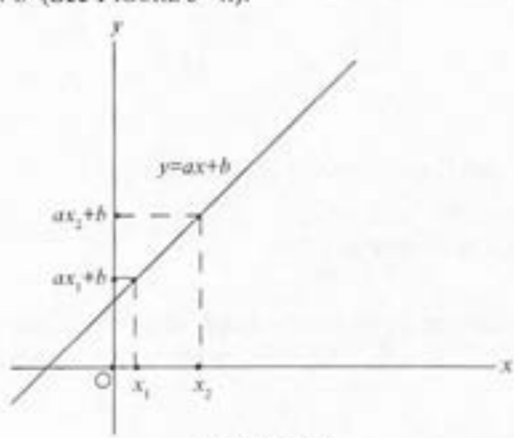


FIGURE 5–4

PROOF

Since $a > 0$ and $x_1 \geq x_2$, $a(x_1 - x_2) \geq 0$. Therefore, $ax_1 - ax_2 \geq 0$. Therefore, $ax_1 \geq ax_2$. Adding b to both sides of this inequality, we get: $ax_1 + b \geq ax_2 + b$.

When $a < 0$, the function $y = ax + b$ is decreasing. This means the following: For any numbers x_1 and x_2, if $x_1 \geq x_2$, then $ax_1 + b \leq ax_2 + b$.

PROOF

Since $a < 0$ and $x_1 \geq x_2$, $a(x_1 - x_2) \leq 0$. Therefore, $ax_1 - ax_2 \leq 0$. Therefore, $ax_1 \leq ax_2$. Adding b to both sides of the inequality, we get: $ax_1 + b \leq ax_2 + b$.

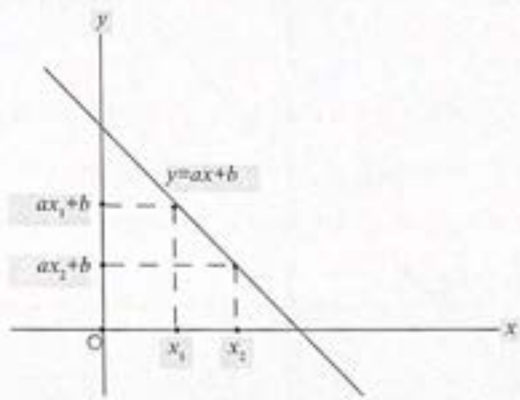


FIGURE 5–5

5.2 WHAT DOES THE GRAPH OF A QUADRATIC FUNCTION $y = ax^2 + bx + c$ LOOK LIKE?

5.2.1 The Case $y = x^2$

We can observe several properties of the graph of the function $y = x^2$.

PROPERTY 1: The graph has a minimum at $x = 0$.

It easy to see that $(0,0)$ is the lowest point in the graph of this function. It is so because for any real number x, $x^2 \geq 0$.

PROPERTY 2: The graph is symmetric.

The graph of $y=x^2$ is symmetric with respect to the y-axis. This means that any line parallel to the x-axis that intersects the graph will intersect it in two points that are equidistant from the y-axis.

Let $y=d$ be such a line, and let P and Q be the intersection points of the line with the graph. Also, let Y be the intersection of the line with the y-axis. We are to show that $PY=QY$. Clearly the coordinates of Y are $(0,d)$. To find the coordinates of P and Q, we solve the system

$$\begin{cases} y=x^2 \\ y=d \end{cases}$$

Solving the system, we get

$x^2=d$

$x=\pm\sqrt{d}$

$y=d$

Thus, $P=(\sqrt{d},d)$ and $Q=(-\sqrt{d},d)$. From this, we get that $PY=QY=\sqrt{d}$. Hence, the graph $y=x^2$ is symmetric.

PROPERTY 3: The function $y=x^2$ is increasing on the positive side of the x-axis and decreasing on the negative side of the x-axis. This means:

A. If $x_1>x_2\geq 0$, then $x_1^2>x_2^2$ (the function is increasing)

And

B. If $0\geq x_1>x_2$, then $x_1^2<x_2^2$ (the function is decreasing)

PROOF OF A: $x_1^2>x_2^2$ if and only if $x_1^2-x_2^2>0$ if and only if $(x_1-x_2)(x_1+x_2)>0$.

Since x_1 and x_2 are non-negative and different,

(1) $x_1+x_2>0$.

Since $x_1>x_2\geq 0$, (2) $x_1-x_2>0$

From (1) and (2), $(x_1-x_2)(x_1+x_2)>0$.

Therefore, $x_1^2-x_2^2>0$. Hence, $x_1^2>x_2^2$.

PROOF OF B: The proof for B is similar to the Proof of A, and you can complete the proof for B on your own as an exercise.

The three properties just discussed determine the graph of the function $y=x^2$. The graph is shown in FIGURE 5-6:

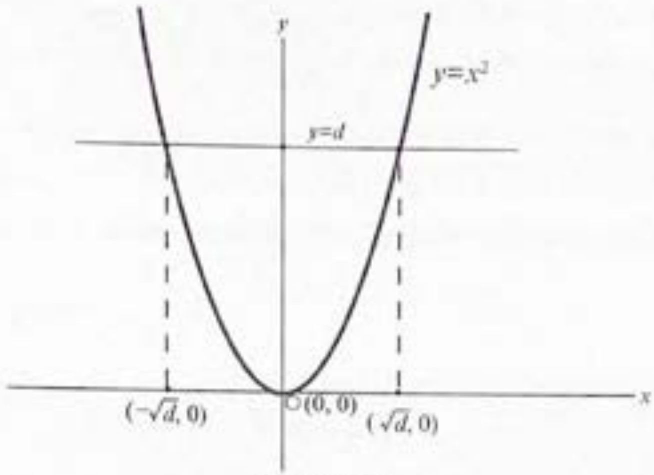


FIGURE 5–6

5.2.2 The General Case $y = ax^2 + bx + c$

In SECTION 2, we showed that $y = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$

Let $k = -\frac{b^2 - 4ac}{4a}$. Then, $y = a\left(x + \frac{b}{2a}\right)^2 + k$.

As before, we can observe several properties of the graph of $y = a\left(x + \frac{b}{2a}\right)^2 + k$.

PROPERTY 1: When $a > 0$, the graph has a minimum at $x = -\frac{b}{2a}$.

PROOF: We are to show that for any x, $y\left(-\frac{b}{2a}\right) \leq y(x)$.

(1) $y\left(-\frac{b}{2a}\right) = a\left(-\frac{b}{2a} + \frac{b}{2a}\right)^2 + k = a \cdot 0 + k = k$

(2) $y(x) = a\left(x + \frac{b}{2a}\right)^2 + k$

Since $a > 0$ and $\left(x + \frac{b}{2a}\right)^2 \geq 0$, $a\left(x + \frac{b}{2a}\right)^2 \geq 0$.

Therefore:

(3) $k \leq a\left(x + \frac{b}{2a}\right)^2 + k$

From (1), (2), and (3), we get $y\left(-\frac{b}{2a}\right) \leq y(x)$.

PROPERTY 2: When $a < 0$, the graph has a maximum at $x = -\frac{b}{2a}$.

PROOF: The proof is very similar to the proof of Property 1, and you can complete it on your own as an exercise.

PROPERTY 3: The graph is symmetric with respect to the line $x=-\frac{b}{2a}$.

PROOF: We are to show that any line parallel to the x-axis that intersects the graph will intersect it at two points that are equidistant from the line $x=-\frac{b}{2a}$.

Let $y=d$ be such a line, and let P and Q be the intersection points of the line with the graph. Also, let Y be the intersection of the line $y=d$ with the line $x=-\frac{b}{2a}$. We will show that $PY=QY$.

Clearly, the coordinates of Y are $\left(-\frac{b}{2a},d\right)$.

To find the coordinates of P and Q, we solve the system

$$\begin{cases} y=a\left(x+\frac{b}{2a}\right)^2+k \\ y=d \end{cases}$$

Solving the system, we get

$$a\left(x+\frac{b}{2a}\right)^2+k=d$$

$$\begin{cases} x=-\frac{b}{2a}\pm\sqrt{\frac{d-k}{a}} \\ y=d \end{cases}$$

Thus, $P=\left(-\frac{b}{2a}+\sqrt{\frac{d-k}{a}},d\right)$ and $Q=\left(-\frac{b}{2a}-\sqrt{\frac{d-k}{a}},d\right)$.

Now we calculate PY and QY, using the distance formula (which we will prove in SECTION 8):

$$PY=\sqrt{\left(\left(-\frac{b}{2a}+\sqrt{\frac{d-k}{a}}\right)+\frac{b}{2a}\right)^2+(d-d)^2}=\sqrt{\frac{d-k}{a}}$$

$$QY=\sqrt{\left(\left(-\frac{b}{2a}-\sqrt{\frac{d-k}{a}}\right)+\frac{b}{2a}\right)^2-(d-d)^2}=\sqrt{\frac{d-k}{a}}$$

Hence: $PY=QY$. This completes the proof that the graph of the function $y=ax^2+bx+c$ is symmetric with respect to the line $x=-\frac{b}{2a}$.

Note that Property 3 implies that the quadratic function has no inverse since it is not a one-to-one function.

PROPERTY 4: When $a>0$, the function $y=ax^2+bx+c$ is increasing on the right of $x=-\frac{b}{2a}$ and decreasing on the left of $x=-\frac{b}{2a}$.

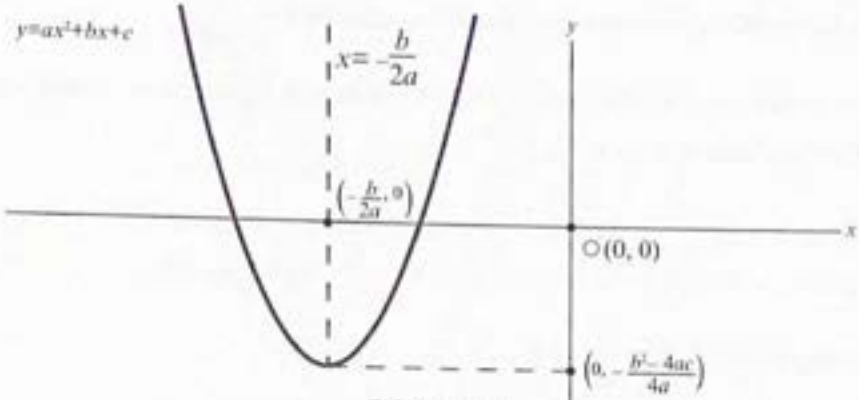


FIGURE 5–7

PROOF: We are to prove:

A. For any $x_1 \geq x_2 \geq -\frac{b}{2a}$, $ax_1^2 + bx_1 + c \geq ax_2^2 + bx_2 + c$

B. For any $x_1 \geq x_2 \geq -\frac{b}{2a}$, $ax_1^2 + bx_1 + c \leq ax_2^2 + bx_2 + c$

We will prove A here. The proof for B is similar, and you can complete it on your own as an exercise. As before, we write:

(1) $ax_1^2 + bx_1 + c = a\left(x_1 + \frac{b}{2a}\right)^2 + k$

(2) $ax_2^2 + bx_2 + c = a\left(x_2 + \frac{b}{2a}\right)^2 + k$

Since $x_1 > x_2 > -\frac{b}{2a}$,

(3) $\left(x_1 + \frac{b}{2a}\right)^2 > \left(x_2 + \frac{b}{2a}\right)^2$

Since $a > 0$

(4) $a\left(x_1 + \frac{b}{2a}\right)^2 > a\left(x_2 + \frac{b}{2a}\right)^2$

Adding k to both sides of this inequality, we get

(5) $a\left(x_1 + \frac{b}{2a}\right)^2 + k > a\left(x_2 + \frac{b}{2a}\right)^2 + k$

From (1), (2), and (5), we get $ax_1^2 + bx_1 + c \geq ax_2^2 + bx_2 + c$, as desired.

The graph of a quadratic function is called a ***parabola***, and the minimum or maximum point is called the ***vertex*** of the parabola.

PROPERTY 5: The domain of the quadratic function $y = ax^2 + bx + c$ is the set of all real numbers. If $a > 0$, the range of the quadratic function is the set of all numbers y for which $y \ge a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c$. If $a < 0$, the range of the quadratic function is the set of all numbers y for which $y \le a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c$.

You can complete the proof of this property on your own as an exercise.

5.3 WHAT DO THE GRAPHS OF SOME POLYNOMIALS LOOK LIKE?

EXAMPLE 5.3a: Graph the polynomial $y = x^3 - 2x^2 - 5x + 6$.

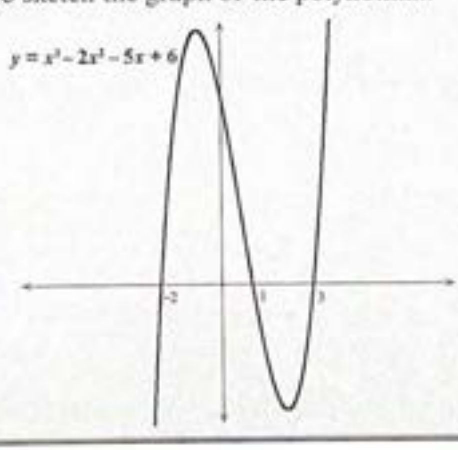
SOLUTION: We first find where the polynomial intersects the x-axis. To do this, we need to solve the equation $x^3 - 2x^2 - 5x + 6 = 0$.

We use the Rational Root Theorem to find that this equation has three solutions: $x = 1$, $x = -2$, and $x = 3$.

Also, substituting $x = 0$ in the polynomial, we find that the polynomial intersects the y-axis at $y = 6$.

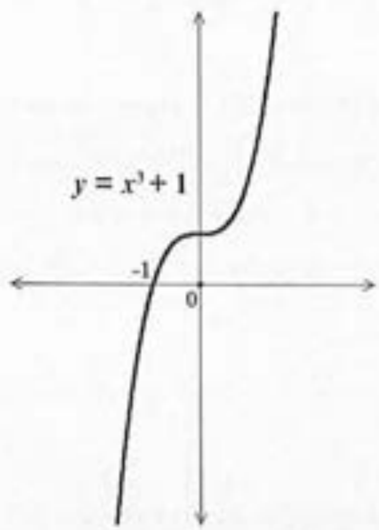
We also note that as x gets larger moving toward infinity, y gets larger moving toward infinity; and as x gets smaller moving toward negative infinity, y gets smaller moving toward negative infinity.

We can use this information to sketch the graph of the polynomial:



EXAMPLE 5.3b: Graph the polynomial $y = x^3 + 1$.

SOLUTION: We first find where the polynomial intersects the x-axis. To do this, we need to solve the equation $x^3 + 1 = 0$. This equation has one solution: $x = -1$. The polynomial intersects the y-axis at $y = 1$. We also note that as x gets larger toward infinity, y gets larger toward infinity; and as x gets smaller toward negative infinity, y gets smaller toward negative infinity. We can use this information to sketch the graph of the polynomial:



5.4 WHAT DOES THE GRAPH OF THE EXPONENTIAL FUNCTION $y = a^x$ LOOK LIKE?

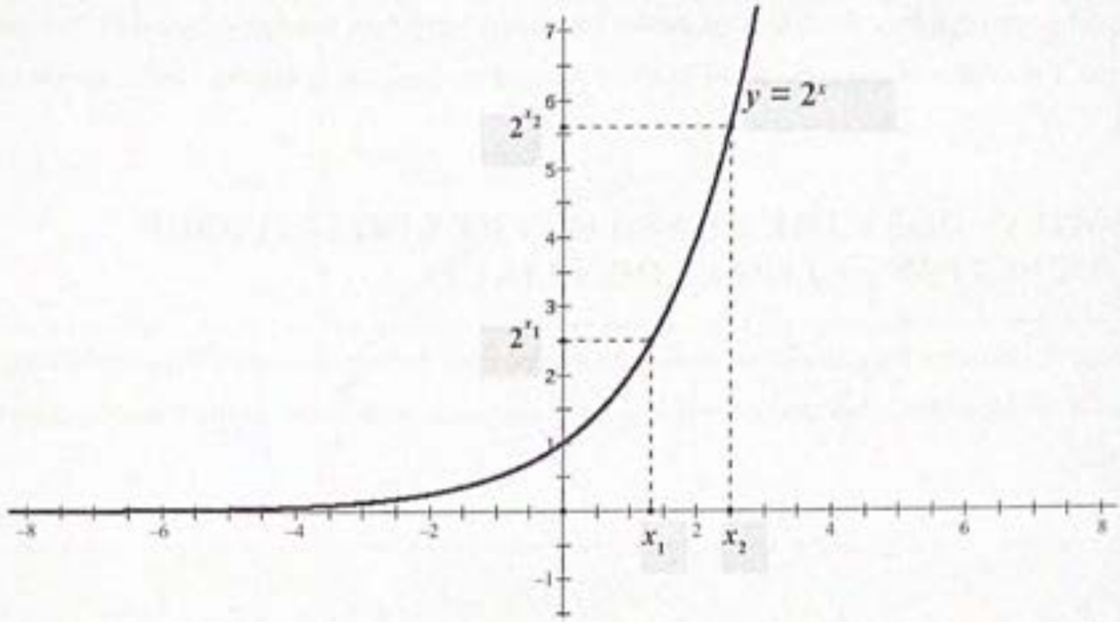
The function $y = a^x$ is called the ***exponential function***. The exponential function is only defined for $a > 0$ since if $a > 0$, $y = a^x$ is positive for any real number x, and so its graph is above the x-axis.

Another feature of this function is that when $a > 1$, the function is increasing, and when $0 < a < 1$, the function is decreasing.

EXAMPLE 5.4a:

Consider the functions $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$:

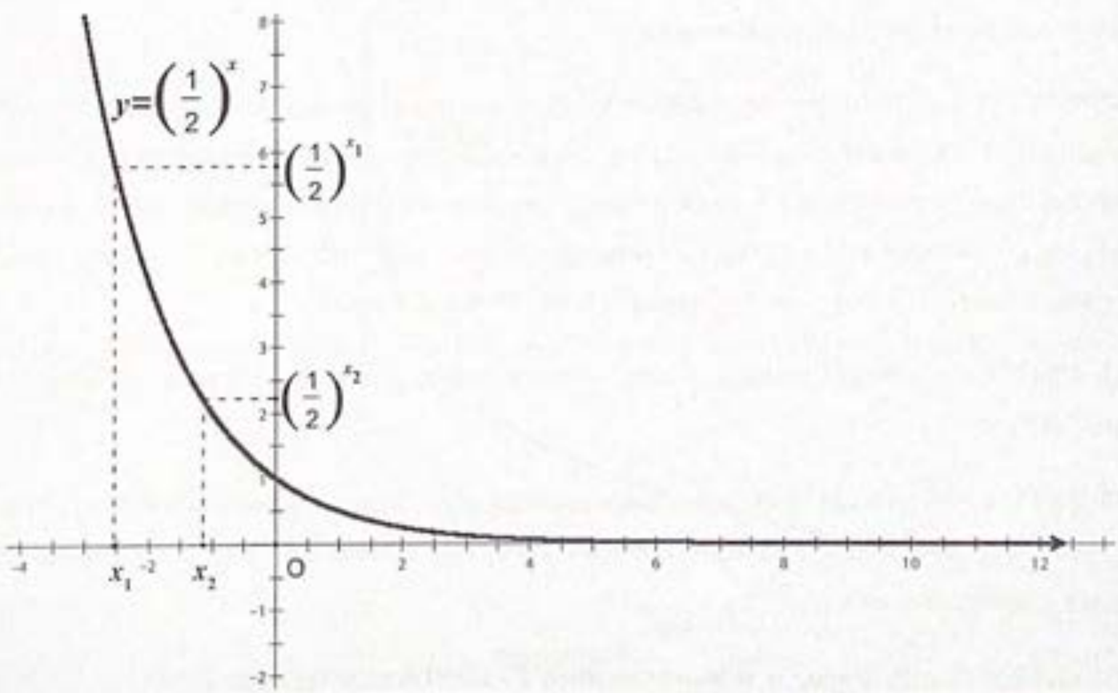
In $y = 2^x$, as x gets larger, 2^x gets larger.



That is: if $x_1 > x_2$, then $2^{x_1} > 2^{x_2}$. For example, $2^4 > 2^3$.

In $y = \left(\frac{1}{2}\right)^x = \frac{1}{2^x}$, on the other hand, as x gets larger, $\left(\frac{1}{2}\right)^x$ gets smaller.

That is: if $x_1 > x_2$, then $\left(\frac{1}{2}\right)^{x_1} < \left(\frac{1}{2}\right)^{x_2}$. For example, $\left(\frac{1}{2}\right)^3 < \left(\frac{1}{2}\right)^2$.



The domain of the exponential function is the set of all real numbers, whereas its range is the set of all positive real numbers. Since the exponential function is increasing, it is one-to-one, and thus has an inverse. The inverse of the exponential function is called the **logarithmic function**, which we will study next.

5.5 WHAT DOES THE GRAPH OF THE LOGARITHMIC FUNCTION $y = \text{LOG}_a x$ LOOK LIKE?

The properties of the logarithmic function can be derived from the properties of the exponential function because the logarithmic function is the inverse of the exponential function, as the following definition indicates.

> **DEFINITION**
>
> $c = \log_a b$ if and only if $a^c = b$.

The function $y = \log_a x$ is called the ***logarithmic function***. a is called the ***base*** of the logarithm.

PROPERTY 1: The expression $y = \log_a x$ is equivalent to $a^y = x$. Since a^y is defined only for $a > 0$ and a^y is always positive (as we showed earlier), x must be positive. Thus, the domain of the logarithmic function consists of the positive real numbers.

PROPERTY 2: Another important condition to observe is that a, the base of the logarithm, must be different from 1. This is so because for $a = 1$, $\log_a x$ is not a function. Recall, a function cannot be *one-to-many*. If $a = 1$, then the input $x = 1$ would have infinitely many outputs. For example, $\log_1 1 = 4$ because $1^4 = 1$; $\log_1 1 = -3$ because $1^{-3} = 1$; $\log_1 1 = 6.9$ because $1^{6.9} = 1$; $\log_1 1 = \sqrt{5}$ because $1^{\sqrt{5}} = 1$, etc. Thus, the logarithmic function $y = \log_a x$ is meaningful only for $x > 0$ and $1 \neq a > 0$.

PROPERTY 3: Since the function admits only positive inputs, its graph is located in the half plane to the right of the y-axis.

PROPERTY 4: Next, we can observe that the x-intercept of the logarithmic function is $x = 1$, because $\log_a 1 = 0$. Recall that $y = \log_a x$ is defined only for $x > 0$. Hence, there is no y-intercept—the function does not intersect the y-axis.

PROPERTY 5: Finally, $y = \log_a x$ is increasing when $a > 1$ and decreasing when $0 < a < 1$.

Putting all these properties together, we can see that the graph of $y = \log_a x$ when $a > 1$ is of the following form:

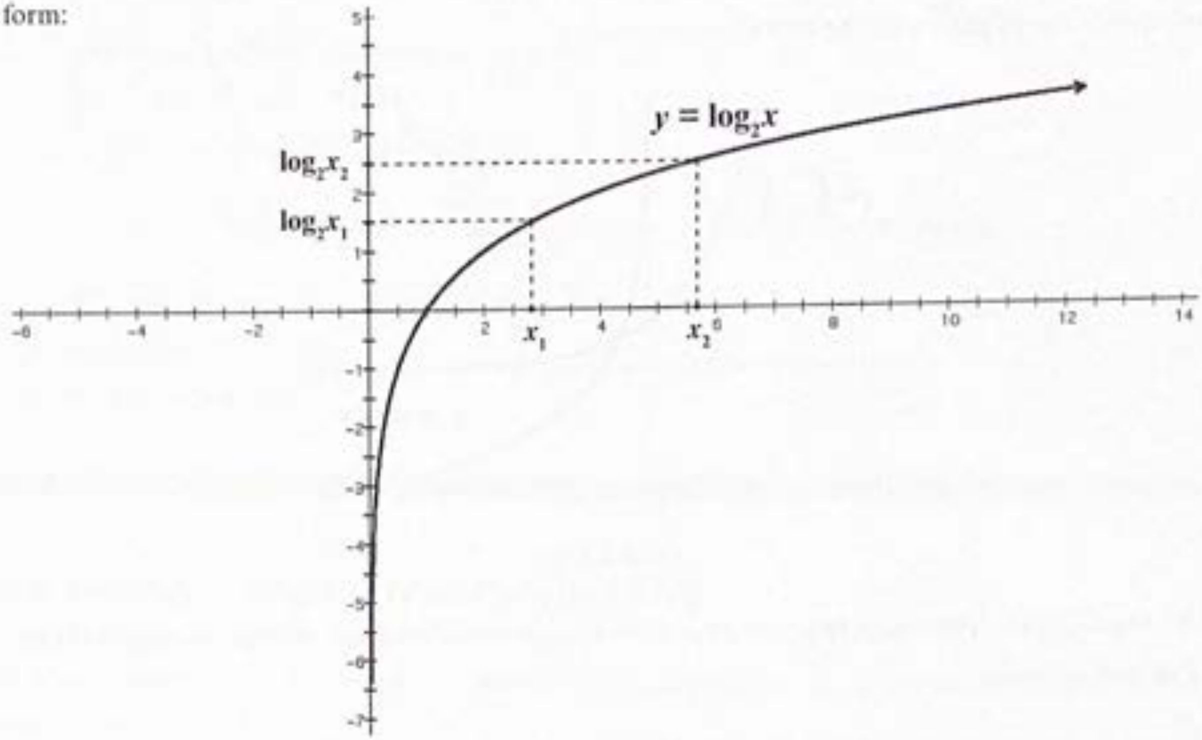


FIGURE 5–8

And, the graph of $y = \log_a x$ when $0 < a < 1$ is of the following form:

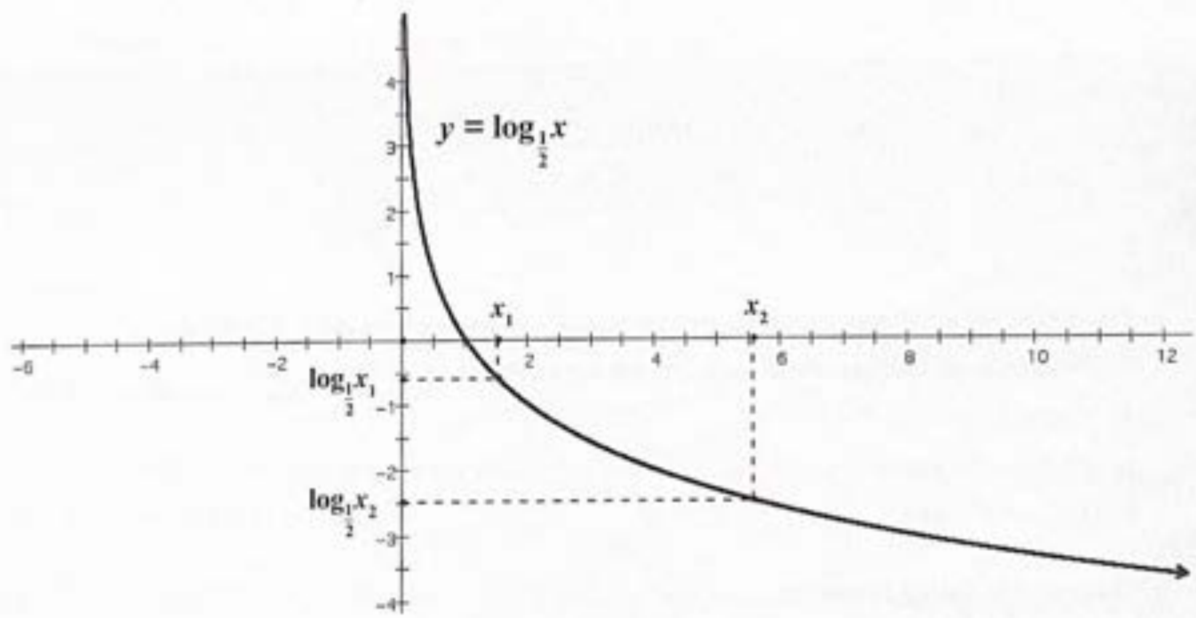


FIGURE 5–9

The graph of $y=\log_{\frac{1}{2}} x$ is symmetric to the graph of $y=\left(\frac{1}{2}\right)^x$ as is shown in Figure 5–10. This is because the two functions are the inverse of one another.

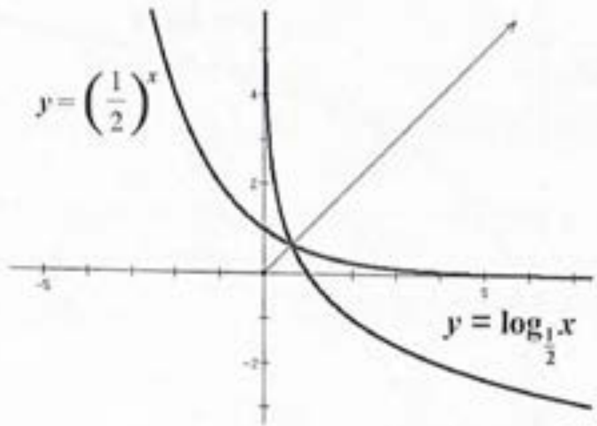


FIGURE 5–10

The domain of the logarithmic function is the set of all positive real numbers, whereas its range is the set of all real numbers.

In science, the most frequently used base is the number $e = 2.71828182845904523536.....$ Logarithms with base e are called ***natural logarithms***. The number e is irrational since it cannot be expressed as a repeating decimal. The natural logarithm is written **ln *x*** and is defined to be the inverse function of e^x.

Sections 5.1-5.5
EXERCISES

1. Determine the maximum or minimum and axis of symmetry of the following quadratic functions. Graph each parabola including the vertex and the axis of symmetry.
   a. $f(x)=x^2-6x+8$
   b. $f(x)=-x^2+6x-9$
   c. $f(x)=2x^2-3x-3$

2. Graph the following functions:
   a. $f(x)=\left(\frac{1}{3}\right)^x$

b. $f(x)=\left(\frac{3}{2}\right)^{x}$

3. Sketch the graphs of the following functions:

   a. $f(x)=4x^3-8x^2-15x+9$

   b. $f(x)=x^4-6x^3-4x^2+54x-45$

4. Let $f(x)=x^2+px+q$. Find the values of p and q if the minimum of this function is $(1,-2)$.

5. Find the inverse of the following functions:

   a. $f(x)=3-2^{2-x}$

   b. $f(x)=2-\log_3\left(\frac{3+x}{x}\right)$

5.6 TRANSFORMATIONS OF GRAPHS

It is recommended that you review the section on the composition of functions (Section 4) before studying this section.

If we know the graph of a function $y=f(x)$, we are able to construct the graphs of the functions $y=f(x+c)$, $y=f(x)+C$, $y=f(ax)$, and $y=Af(x)$, where a, c, A, and C are constants.

5.6.1 Graphing $y=f(x+c)$ from the Graph of $y=f(x)$

The function $y=f(x+c)$ is the composition of two functions: $y=x+c$ and $y=f(x)$. For any input x, we first get the output $x+c$. This output, in turn, becomes an input for the function $y=f(x)$, giving the output $f(x+c)$.

EXAMPLES:

(1) $y=(x+2)^2$ is the composition $y=x+2$ and $y=x^2$: to any given x, we first add the 2, and then we square the resulting sum.

(2) $y=2^{x-4}$ is the composition of $y=x-4$ and $y=2^x$: to any given x, we first subtract 4, and then we raise 2 to the power of $x-4$.

(3) $y=\log_2(x-5)$ is the composition of $y=x-5$ and $y=\log x$: to any given x, we first subtract 5, and then we take the log of $x-5$.

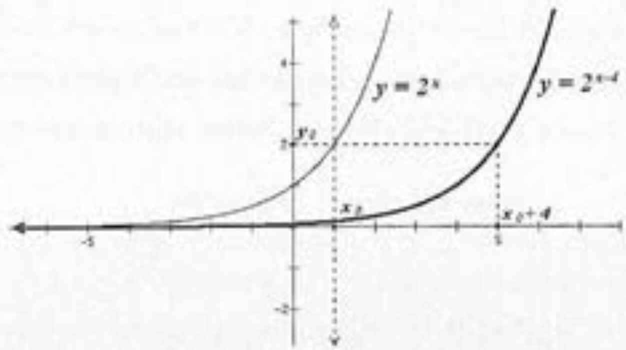
Notice that the two functions $y = f(x)$ and $y = f(x+c)$ have the same output for any two inputs that differ by c. For the inputs x and $x-c$, we have $f(x) = f((x-c)+c)$. On the basis of this fact, we can now construct the graph of $y = f(x+c)$ from the graph of $y = f(x)$.

For any point (x_0, y_0) on the graph of $y = f(x)$, we have $y_0 = f(x_0)$. What input x of the function $y = f(x+c)$ gives the output y_0? The answer is $x = x_0 - c$ because $f(x_0 + c) = f\big((x_0 - c)+c\big) = f(x_0) = y_0$. Conversely, if $(x_0 - c, y_0)$ is on the graph of $y = f(x+c)$, then (x_0, y_0) is on the graph of $y = f(x)$.

The conclusion that we can draw from this is that to construct the graph of the function $y = f(x+c)$ from the graph of $y = f(x)$, we simply shift each point of the graph $y = f(x)$ by c units along the x-axis. We move the graph by c units to the left if $c > 0$ and by c units to the right if $c < 0$.

EXAMPLE 5.6a: Construct $y = 2^{x-4}$.

The function $y = 2^{x-4}$ is of the form $y = f(x+c)$ where $f(x) = 2^x$ and $c = -4 < 0$. Therefore, the graph of $y = 2^{x-4}$ is the result of shifting the graph of $y = 2^x$ along the x-axis to the right by 4 units.



5.6.2 Graphing $y = f(x) + C$ from the Graph of $y = f(x)$

The function $y = f(x) + C$ is the composition of two functions: $y = f(x)$ and $y = x + C$. For any input x, we get the output $f(x)$. This output, in turn, becomes an input for the function $y = x + C$, giving the output $f(x) + C$.

EXAMPLES:

(1) $y = x^2 - 3$ is the composition of $y = x^2$ and $y = x - 3$: We first square an input x, and then add -3 to the result.

(2) $y=\left(\frac{1}{3}\right)^x+2$ is the composition of $y=\left(\frac{1}{3}\right)^x$ and $y=x+2$: We first raise $\frac{1}{3}$ to the power x and then add 2 to the result.

(3) $y=\log_2 x-\frac{1}{3}$ is the composition of $y=\log_2 x$ and $y=x-\frac{1}{3}$: We first compute the log of an input x and then add $-\frac{1}{3}$ to the result.

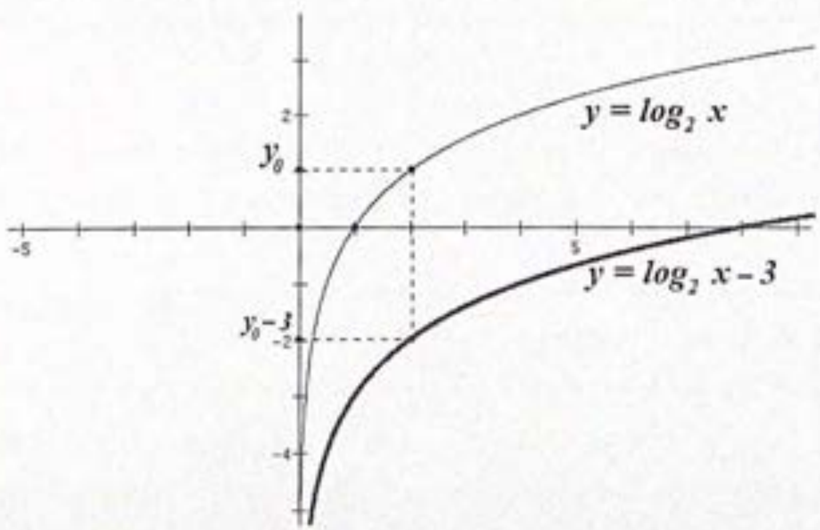
Notice that for any input x, the outputs $y=f(x)+C$ and $y=f(x)$ differ by C units. This fact helps us construct the graph of $y=f(x)+C$ from the graph of $y=f(x)$. For any point (x_0, y_0) on the graph of $y=f(x)$, we have $y_0=f(x_0)$. Clearly, the point $(x_0,\ y_0+C)$ belongs to the graph of $y=f(x)+C$ (because $y_0+C=f(x_0)+C$).

Conversely, if the point $(x_0,\ y_0+C)$ belongs to the graph of $y=f(x)+C$, then the point (x_0, y_0) is on the graph of $y=f(x)$.

The conclusion that we can draw from this is that to construct the graph of the function $y=f(x)+C$, we simply shift the graph $y=f(x)$ by C units along the y-axis. The graph moves up by C units if $C>0$, and it moves down by C units if $C<0$.

EXAMPLE 5.6b: Construct the graph of $y=\log_2 x-3$.

The function $y=\log_2 x-3$ is of the form $y=f(x)+C$ where $f(x)=\log_2 x$ and $C=-3<0$. Therefore, moving $f(x)=\log_2 x$ down the y-axis by 3 units results in the graph of $y=\log_2 x-3$.



5.6.3 Graphing y = f(ax) from the Graph of y = f(x)

The function $y = f(ax)$, $a \neq 0$, is the composition of $y = ax$ and $y = f(x)$. For any input x, we first get the output ax. This output, in turn, becomes an input for the function $y = f(x)$, giving the output $y = f(ax)$.

EXAMPLES:

(1) $y = \left(\frac{x}{2}\right)^2$ is the composition of $y = \frac{x}{2}$ and $y = x^2$ (How?)

(2) $y = 2^{3x}$ is the composition of $y = 3x$ and $y = 2^x$ (How?)

(3) $y = \log_2 \frac{1}{7}x$ is the composition of $y = \frac{1}{7}x$ and $y = \log_2 x$ (How?)

Let (x_0, y_0) be a point on the graph of $y = f(x)$, then $y_0 = f(x_0)$. What input x for the function $y = f(ax)$ gives the output y_0? Clearly, the answer is $x = \frac{x_0}{a}$. Indeed: $f(ax_0) = f\left(a\frac{x_0}{a}\right) = f(x_0) = y_0$.

This shows that if the point (x_0, y_0) belongs to the graph of $y = f(x)$, then point $\left(\frac{x_0}{a},\ y_0\right)$ belongs to the graph of $y = f(ax)$. And, conversely, if the point $\left(\frac{x_0}{a},\ y_0\right)$ belongs to the graph of $y = f(ax)$, then the point (x_0, y_0) belongs to the graph of $y = f(x)$.

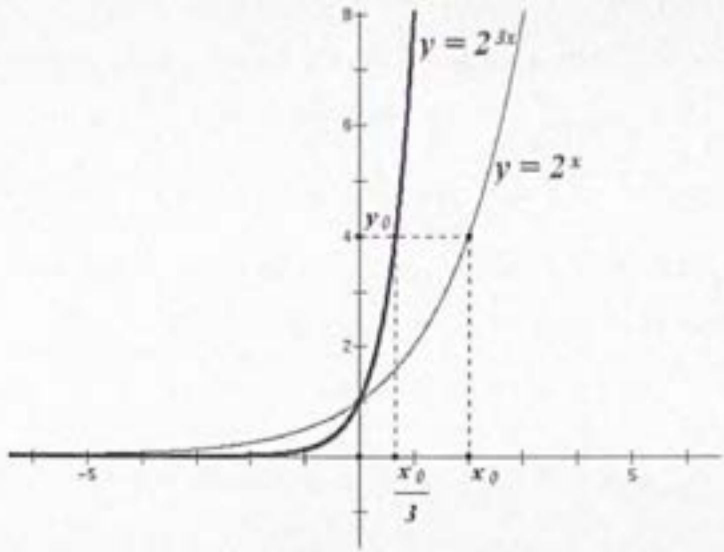
The relation between the graph of $y = f(ax)$ and the graph of $y = f(x)$ depends on the value of a:

- If $a > 1$, then we get the graph of $y = f(ax)$ by "shrinking" the graph of $y = f(x)$ along the x-axis by a factor of $\frac{1}{a}$.
- If $0 < a < 1$, then we get the graph of $y = f(ax)$ by "expanding" the graph of $y = f(x)$ along the x-axis by a factor of $\frac{1}{a}$.

You can investigate the cases $a < -1$ and $-1 < a < 0$ on your own as an exercise. (Hint: note that for any function $y = g(x)$ the graphs of $y = g(x)$ and $y = g(-x)$ are symmetric with respect to the y-axis.)

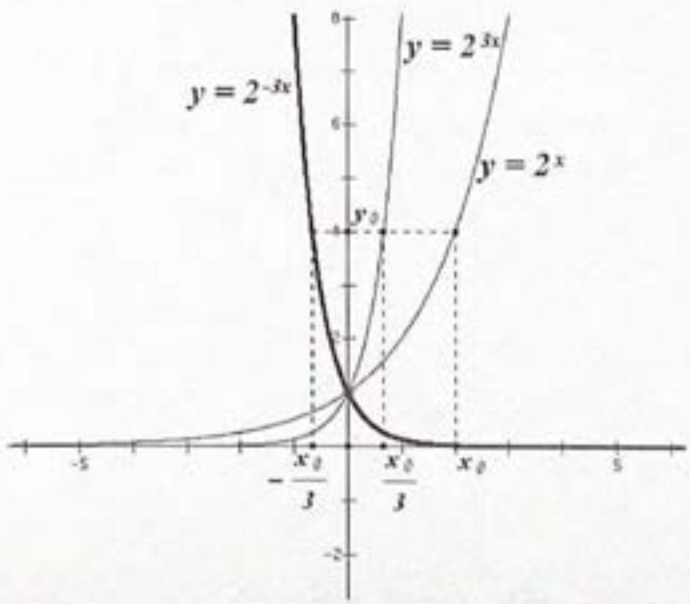
EXAMPLE 5.6c: Construct the graph of $y = 2^{3x}$.

The function $y = 2^{3x}$ is of the form $y = f(ax)$, $a \neq 0$, where $f(x) = 2^x$ and $a = 3 > 1$. To get the graph of $y = 2^{3x}$, we "shrink" the graph of $f(x) = 2^x$ by a factor of $\frac{1}{3}$ along the x-axis toward the y-axis.



EXAMPLE 5.6d: Construct the graph of $y = 2^{-3x}$.

The function $y = 2^{-3x}$ is of the form $y = f(ax)$, $a \neq 0$, where $f(x) = 2^x$ and $a = -3 < -1$. We first obtain the graph of $y = 2^{3x}$ by expanding the graph $f(x) = 2^x$ by a factor of $\frac{1}{3}$ along the x-axis toward the y-axis. Following this, we reflect the graph of $y = 2^{3x}$ with respect to the y-axis to obtain the graph of $y = 2^{-3x}$.



5.6.4 *Graphing y = Af(x) from the Graph of y = f(x)*

To obtain the graph of the function $y = Af(x)$, $A \neq 0$, from the graph of the function $y = f(x)$, we reason as before:

1. (x_0, y_0) is on the graph of $y = f(x)$ if and only if (x_0, Ay_0) is on the graph of $y = Af(x)$ (Why?)

2. Therefore, to plot the graph of $y = Af(x)$, we simply transform each point (x_0, y_0) on the graph $y = f(x)$ to the point $\left(x_0,\ Ay_0\right)$.

From there:

3. If $A > 1$, then we get the graph of $y = Af(x)$ by "expanding" the graph of $y = f(x)$ along the y-axis by a factor of A.

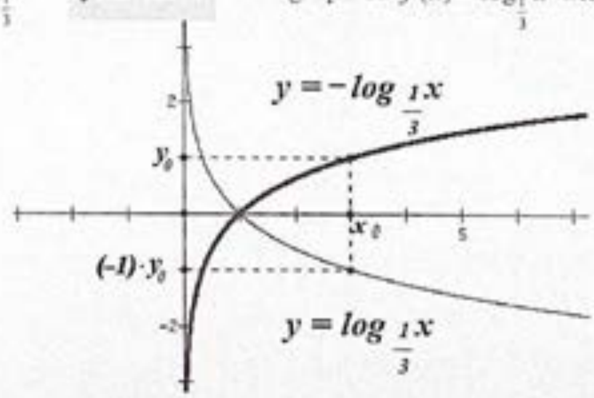
4. If $0 < A < 1$, then we get the graph of $y = Af(x)$ by "shrinking" the graph of $y = f(x)$ along the y-axis by a factor of A.

5. If $A = -1$, we get the graph of $y = Af(x)$ by simply reflecting the graph of $y = f(x)$ with respect to the x-axis.

You can work out the cases $-1 < A < 0$ and $A < -1$ as an exercise on your own. (Hint: note that for any function $y = g(x)$, the graphs of $y = g(x)$ and $y = -g(x)$ are symmetric with respect to the x-axis.

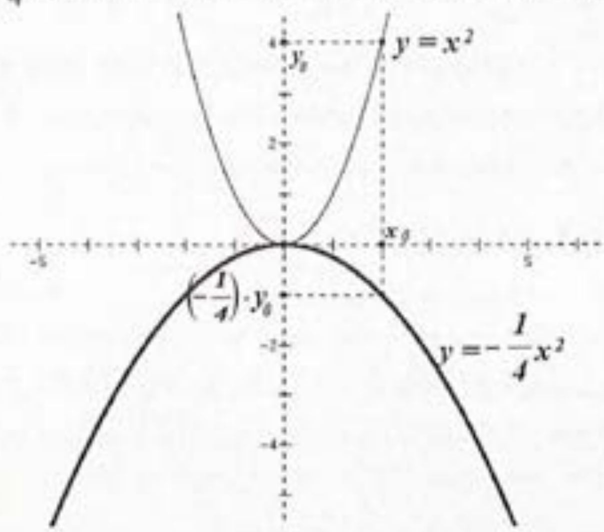
EXAMPLE 5.6e: Construct the graph of $y = -\log_{\frac{1}{3}} x$.

The function $y = -\log_{\frac{1}{3}} x$ is of the form $y = Af(x)$ where $f(x) = \log_{\frac{1}{3}} x$ and $A = -1$. Therefore, the graph of $y = -\log_{\frac{1}{3}} x$ is symmetrical to the graph of $f(x) = \log_{\frac{1}{3}} x$ with respect to the x-axis.



EXAMPLE 5.6f: Construct the graph of $y=-\frac{1}{4}x^2$.

The function $y=-\frac{1}{4}x^2$ is of the form $y=Af(x)$ where $f(x)=x^2$ and $A=-\frac{1}{4}$. Therefore, the graph of $y=-\frac{1}{4}x^2$ is obtained in two steps: first we shrink the graph of $f(x)=x^2$ along the y-axis by a factor of $\frac{1}{4}$, and then we take the reflection of the latter graph with respect to the x-axis.



Section 5.6
EXERCISES

1. Construct the graphs of the following functions:
   a. $y=-(x+3)^2-6$
   b. $y=\log_2(x+\frac{1}{2})$
   c. $y=-\log_3 x+4$

2. Find the inverse of the following functions and sketch both the function and its inverse in the same graph:
   a. $f(x)=\frac{1}{3}\ln 2x$
   b. $f(x)=e^{-2x}+2$

Section 6

Non-Polynomial Equations

In the previous sections, we learned how to solve linear equations, quadratic equations, and some techniques for solving higher-degree polynomial equations. In this section, we will learn how to solve a new type of equation that is not a polynomial equation.

6.1 RATIONAL EQUATIONS

The first non-polynomial equations we will learn about in this section are called ***rational equations***. A rational equation is an equation that involves only polynomials or *rational expressions*. By "*rational expression*," we mean a fraction whose numerator and denominator are polynomials.

EXAMPLES:

1. $\frac{1}{x} - x = 0$ is a rational equation. It involves the polynomials x and 0 and the rational expression $\frac{1}{x}$.
2. $\sqrt{x} + x^3 - 2 = x$ is not a rational equation since $\sqrt{x}$ is not a rational expression.
3. $\frac{x-3}{(x-4)^2} + \sqrt{7}x^2 + 10 = 5x$ is a rational equation since it involves only polynomials ($\sqrt{7}x^2 + 10$ and $5x$) and a rational expression $\frac{x-3}{(x-4)^2}$.
4. $2x(x-4)^2 = 0$ is a polynomial equation, but we can view it as a rational equation since we can write it as $\frac{2x(x-4)^2}{1} = 0$.
5. $(x-5)(x-4) - (x-2)(2x+9) = \frac{17}{\sqrt{2}x + 1.9}$ is a rational equation for the same reason given in 1 and 3.
6. $\frac{(x+3)(x+5)}{3} - \frac{\sqrt{11x+13}}{10} = 4 + \frac{x-2}{7}$ is not a rational equation (why?)

A polynomial admits any real number x; that is, its domain is the set of all real numbers. A rational expression, on the other hand, may not admit all real numbers. For example, the rational expression $\frac{17}{x^2-8} + 1$ does not admit the values $x = \pm 2\sqrt{2}$ because the denominator $x^2 - 8 = 0$ is zero for these values. We say that the rational expression is not defined for $x = \pm 2\sqrt{2}$ or the domain of the rational function $f(x) = \frac{17}{x^2 - 8} + 1$ is the set of all real numbers $x \neq \pm 2\sqrt{2}$. On the other hand, the expression $\frac{1}{5x^2 + 1} - 4$ is defined for all real numbers because there is no real number x for which the denominator $5x^2 + 1$ is zero. So, the domain of the rational function $f(x) = \frac{1}{5x^2 + 1} - 4$ is the set of all real numbers.

6.1.1 How Do We Solve Rational Equations?

When solving rational equations often we need to simplify rational expressions. Rational expressions are dealt with like fractions. However, since they involve variables, we must pay attention to their domains. Consider, for example, the rational expression $A(x) = \frac{x^2 - 1}{(x-1)(x+3)}$.

One might hastily simplify this expression into $B(x) = \frac{x+1}{x+3}$ by factoring the numerator into $(x-1)(x+1)$ and canceling the expression $x - 1$, which appears in both the numerator and denominator.

Is $A(x) = B(x)$ for all x? The answer is no because while for $x = 1$, $B(x) = \frac{1}{2}$, $A(x)$ is not defined for $x = 1$. The lesson we learn from this is that before we simplify an expression, we must pay attention to the domain of the expression. Thus, for example, we can say $A(x) = B(x)$ for all $x \neq 1$.

Likewise, $\frac{x^2 - 4x + 4}{x - 2} = \frac{(x-2)^2}{x-2} = x - 2$ for all $x \neq 2$.

Let us now turn to solving rational equations.

EXAMPLE 6.1a: Solve the equation $\frac{5}{x-2} - 4x = 0$.

SOLUTION:

1. We first exclude the numbers that cannot be solutions. Since $x - 2$ appears in the denominator, $x = 2$ cannot be a solution.

2. Next, as we do with fractions, we write all expressions in equivalent forms with the same denominator. In this case, $x - 2$ is a common denominator. We write: $\frac{5}{x-2} - \frac{4x(x-2)}{x-2} = 0$.

3. Now we add the two rational expressions as we do with fractions: $\frac{5 - 4x(x-2)}{x-2} = 0$.

4. Since a fraction is zero if and only if its numerator is zero, we get $5 - 4x(x-2) = 0$.

5. Now we proceed to solve this familiar equation:

$$5-4x^2+8x=0$$

$$4x^2-8x-5=0$$

$$x=\frac{8\pm\sqrt{64+80}}{8}$$

$$x=\frac{8\pm 12}{8}$$

6. The solutions to the equation are: $x=\frac{5}{2}$, $x=-\frac{1}{2}$.

EXAMPLE 6.1b: Solve the equation $\frac{6x}{x-9}=4$.

SOLUTION:

1. $x\neq 9$ since the expression $\frac{6x}{x-9}$ is not defined for $x=9$.
2. $\frac{6x}{x-9}=4$
3. $\frac{6x}{x-9}-4=0$ Subtract 4 from both sides.
4. $\frac{6x}{x-9}-4\frac{x-9}{x-9}=0$ Write all expressions in an equivalent form with the same denominator.
5. $\frac{2x+36}{x-9}=0$ Add the rational expressions.
6. $2x+36=0$ A rational expression is zero if and only if the numerator is zero.
7. $2x=-36$ Solve
8. $x=-18$
9. The equation has one solution: $x=-18$.

In the next set of examples, you are expected to explain most of the steps on your own.

EXAMPLE 6.1c: Solve the equation $\frac{x^2+1-2x}{x-1}-x=0$.

SOLUTION: Condition: $x \neq 1$.

1. $\frac{x^2+1-2x}{x-1}-x=0$
2. $\frac{x^2+1-2x-x(x-1)}{x-1}=0$
3. $x^2+1-2x-x(x-1)=0$
4. $x^2+1-2x-x^2+x=0$
5. $1-x=0$
6. $x=1$.

Since $x=1$ was excluded as a solution, there is no solution to this equation. You can see how important it is to specify the domains of the expressions involved beforehand.

EXAMPLE 6.1d: Solve the equation $\frac{x+1}{x-5}=\frac{x-3}{x+6}$.

SOLUTION: Condition: $x \neq 5$ and $x \neq -6$.

1. $\frac{x+1}{x-5}=\frac{x-3}{x+6}$
2. $\frac{x+1}{x-5}-\frac{x-3}{x+6}=0$
3. A common denominator here is $(x-5)(x+6)$. Hence:
4. $\frac{(x+1)(x+6)}{(x-5)(x+6)}-\frac{(x-3)(x-5)}{(x+6)(x-5)}=0$
5. $\frac{(x+1)(x+6)-(x-3)(x-5)}{(x-5)(x+6)}=0$
6. $(x^2+7x+6)-(x^2-8x+15)=0$
7. $x^2+7x+6-x^2+8x-15=0$

8. $15x - 9 = 0$

9. $x = \frac{9}{15}$

10. The solution to the given equation is $x = \frac{9}{15}$.

EXAMPLE 6.1e: Solve the equation $\frac{2x-9}{1-x} + \frac{2x+1}{x+1} = \frac{1}{1-x^2}$.

SOLUTION:

1. $\frac{2x-9}{1-x} + \frac{2x+1}{x+1} - \frac{1}{1-x^2} = 0$
2. First, we factor $1 - x^2$.
3. $1 - x^2 = (1-x)(1+x)$
4. Thus, $(1-x)(1+x)$ is a common denominator for the three rational expressions in the given equation.
5. Condition: $x \neq 1$ and $x \neq -1$.
6. Changing rational expressions into equivalent forms that have the same common denominator, we get $\frac{(2x-9)(x+1)}{(1-x)(x+1)} + \frac{(2x+1)(1-x)}{(x+1)(1-x)} - \frac{1}{(1-x)(x+1)} = 0$
7. $\frac{(2x-9)(x+1) + (2x+1)(1-x) - 1}{(1-x)(x+1)} = 0$
8. $(2x^2 + 2x - 9x - 9) + (2x - 2x^2 + 1 - x) - 1 = 0$
9. $-6x - 9 = 0$
10. $x = -\frac{9}{6} = -1.5$
11. The given equation has only one solution: $x = -1.5$.

EXAMPLE 6.1f: Solve the equation $\frac{2x}{x^2-9x+14}-\frac{1}{x^2-3x+2}=\frac{1}{x-1}$.

SOLUTION: As before, we find a common denominator to the rational fractions in this equation. For this, as we did in the previous examples, we factor the given denominators into unfactorable expressions.

1. $x-1$ is unfactorable.

2. Next we factor the other two denominators:

3. $x^2-9x+14=(x-2)(x-7)$

4. $x^2-3x+2=(x-1)(x-2)$

5. Thus, our equation can be written as $\frac{2x}{(x-2)(x-7)}-\frac{1}{(x-1)(x-2)}-\frac{1}{x-1}=0$

6. $(x-1)(x-2)(x-7)$ is a common denominator of the three rational expressions in our equation, because each denominator of the rational expressions divides it.

7. Thus, we have the condition: $x\neq 1$, $x\neq 2$, $x\neq 7$.

8. Writing these rational expressions in equivalent forms that have this common denominator, we get $\frac{2x(x-1)}{(x-1)(x-2)(x-7)}-\frac{x-7}{(x-1)(x-2)(x-7)}-\frac{(x-2)(x-7)}{(x-1)(x-2)(x-7)}=0$

9. Adding, we get $\frac{x^2+6x-7}{(x-2)(x-7)(x-1)}=0$

10. Solving, we get $(x+7)(x-1)=0$

11. $x=-7$ or $x=1$

12. Since $x=1$ cannot be a solution, the only solution to the equation is $x=-7$

EXAMPLE 6.1g: Solve the equation $\frac{x-3}{x^2-5x+6}=0$.

SOLUTION 1: First we find the values of x for which the denominator x^2-5x+6 is 0, and we exclude these values.

1. Solving $x^2-5x+6=0$, we get $x=2$, $x=3$.
2. Condition: $x \neq 2$ and $x \neq 3$.
3. Now we proceed to solve the equation:
4. $\frac{x-3}{x^2-5x+6}=0$
5. $x-3=0$
6. $x=3$
7. Since $x=3$ was excluded as a solution, our equation has no solution.

SOLUTION 2: Factoring out the denominator, we get $\frac{x-3}{x^2-5x+6}=0$

1. $\frac{x-3}{(x-3)(x-2)}=0 \qquad x \neq 3$
2. $\frac{1}{(x-2)}=0$
3. Since the denominator is different from zero for all x, the equation has no solution.

Section 6.1.1
EXERCISES

1. Solve the following equations:

a. $\frac{3x-2}{x-3}=\frac{x+2}{x+3}$

b. $\frac{x+0.5}{9x+3}+\frac{8x^2+3}{9x^2-1}=\frac{x+2}{3x-1}$

c. $\frac{x^2-2x+1}{x-3}+\frac{x+1}{3-x}=4$

d. $\dfrac{1-2x}{6x^2+3x}+\dfrac{2x+1}{14x^2-7x}=\dfrac{8}{12x^2-3}$

e. $\dfrac{x+3}{4x^2-9}-\dfrac{3-x}{4x^2+12x+9}=\dfrac{2}{2x-3}$

f. $\dfrac{2x+7}{x^2+5x-6}+\dfrac{3}{x^2+9x+18}=\dfrac{1}{x+3}$

2. Find the domain of the following functions:

a. $y=\dfrac{2}{x^2+2}$

b. $y=\dfrac{x-3}{2x^3+x^2-13x+6}$

6.1.2 What Do the Graphs of Some Rational Functions Look Like?

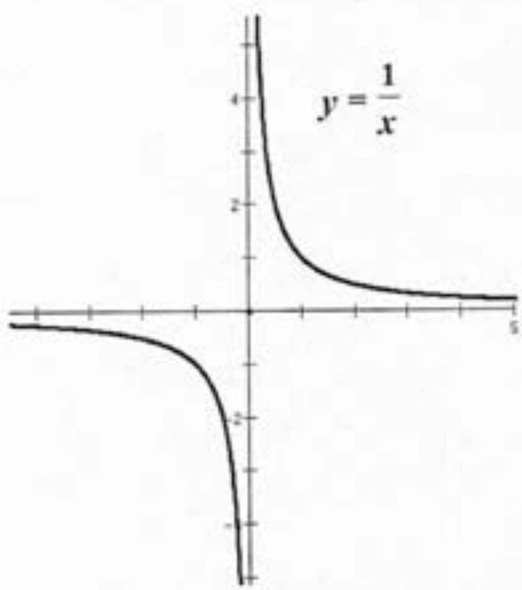


FIGURE 6–1

The graph of the function $f(x)=\dfrac{1}{x}$ is shown in FIGURE 6–1.

Note that the function is not defined for $x=0$. As x gets closer to zero along the positive x-axis, y gets larger; and as x gets closer to zero along the negative x-axis, y gets smaller, and in both cases, the graph approaches the y-axis.

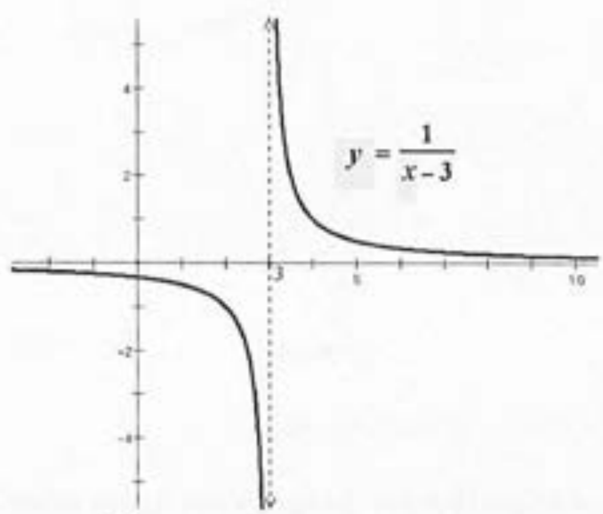
Also, as x gets farther from zero along the x-axis, y gets closer to zero, and the graph approaches the x-axis. The graph never intersects the x-axis or the y-axis. In general, rational functions can approach, without intersecting, certain lines. These lines are called ***asymptotes***. The lines $x=0$ and $y=0$ are asymptotes of the function $f(x)=\frac{1}{x}$.

EXAMPLE 6.1h: Graph the function $f(x)=\frac{x}{x-3}$.

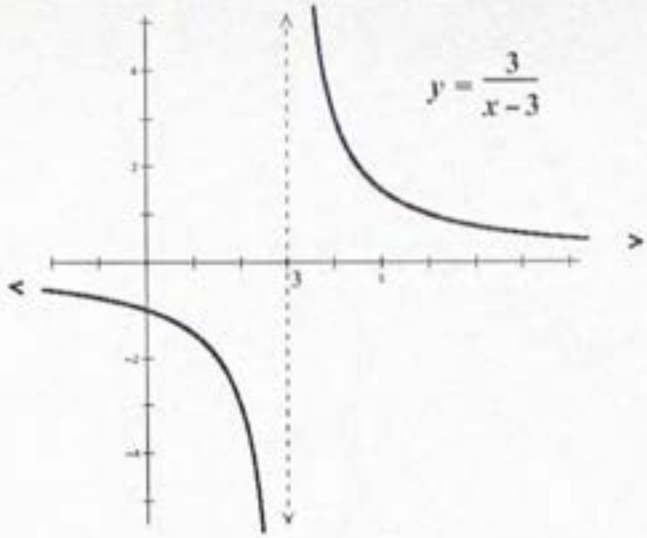
SOLUTION:

1. We first note that $f(x)=1+\frac{3}{x-3}$.
2. You can get this by the Division Algorithm or by noting that $f(x)=\frac{x}{x-3}=\frac{(x-3)+3}{x-3}=1+\frac{3}{x-3}$
3. From this, we see that $f(x)$ is of the form $f(x)=Ag(x)+C$ where $g(x)=\frac{1}{x-3}$, $A=3$, and $C=1$.
4. We can graph this function in three steps:

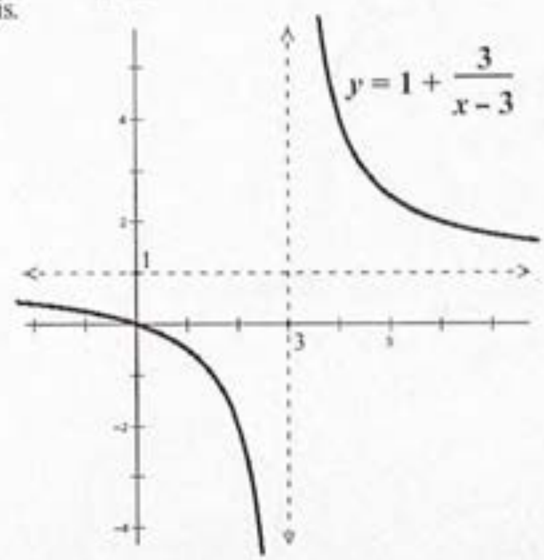
STEP 1: Graph the function $g(x)=\frac{1}{x-3}$. We can graph this function by shifting the graph of $\frac{1}{x}$ along the x-axis by 3 units.



STEP 2: Graph the function $\frac{3}{x-3}$. We can graph this function by expanding the graph of $\frac{1}{x-3}$ by a factor of 3 along the y-axis to the right.



STEP 3: Graph the function $1+\frac{3}{x-3}$. We graph this function by moving the graph of $\frac{3}{x-3}$ up by 1 unit along the y-axis.



EXAMPLE 6.1i: Graph the function $f(x)=\frac{2x+5}{x+4}$.

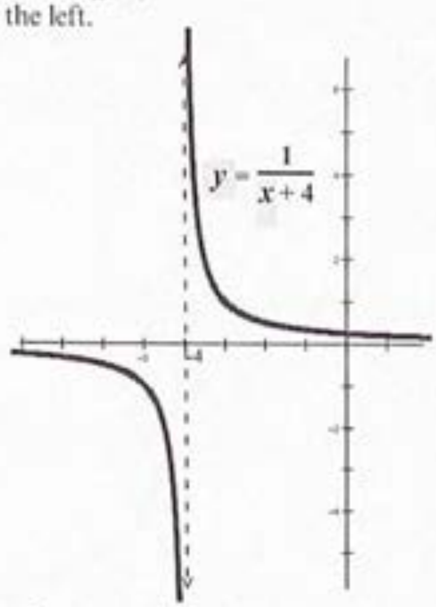
SOLUTION: We first note that $f(x)=2-\frac{3}{x+4}$.

1. You can get this by the Division Algorithm or by noting that $f(x)=\frac{2x+5}{x+4}=\frac{2(x+4)-3}{x+4}=2-\frac{3}{x+4}$

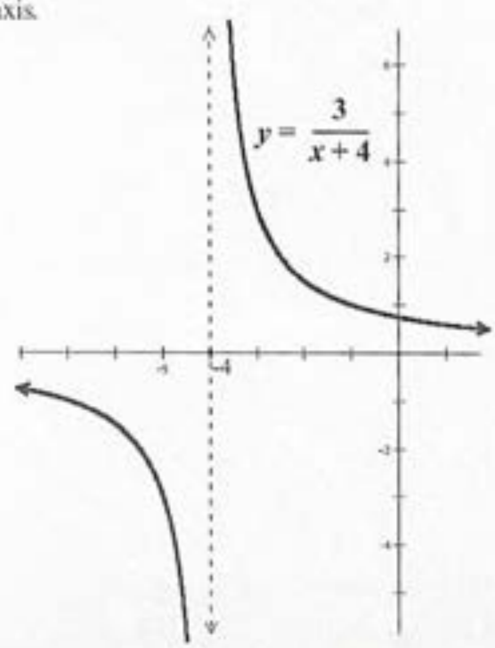
2. From here we see that $f(x)$ is of the form $f(x)=Ag(x)+C$ where $g(x)=\dfrac{1}{x+4}$, $A=-3$, and $C=2$.

We can graph this function in four steps:

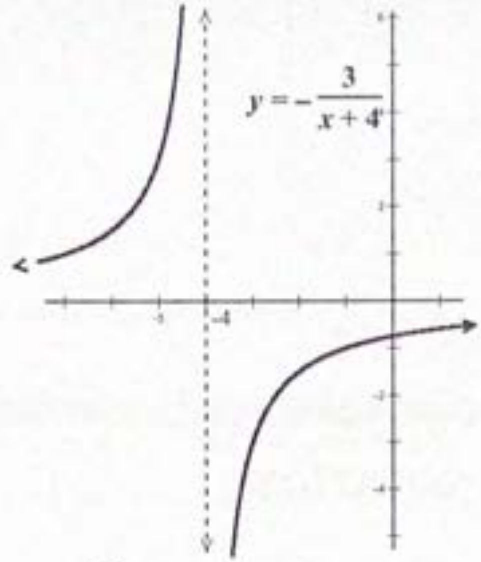
STEP 1: Graph the function $g(x)=\dfrac{1}{x+4}$. We graph this function by shifting the graph of $\dfrac{1}{x}$ along the x-axis by 4 units to the left.



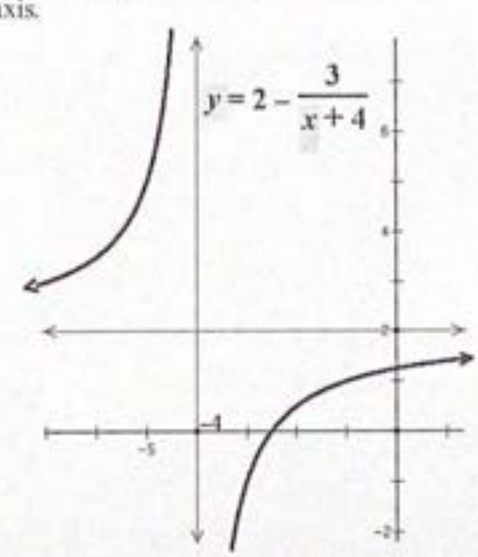
STEP 2: Graph the function $\dfrac{3}{x+4}$. We graph this function by expanding the graph of $\dfrac{1}{x+4}$ by a factor of 3 along the y-axis.



STEP 3: Graph the function $-\frac{3}{x+4}$. We graph this function by reflecting the graph of $\frac{3}{x+4}$ with respect to the y-axis.



STEP 4: Graph the function $2-\frac{3}{x+4}$. We graph this function by moving the graph of $-\frac{3}{x+4}$ up by 2 units along the y-axis.



Section 6.1.2
EXERCISES

1. Graph the following functions:

   a. $f(x) = -\dfrac{1}{x-2}$

   b. $f(x) = \dfrac{1}{2} - \dfrac{3}{x+1}$

   c. $f(x) = \dfrac{x+1}{x-3}$

6.2 EXPONENTIAL EQUATIONS
6.2.1 Basic Properties

Recall the meanings of expressions of the form a^n. For example:

$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

$9^{\frac{1}{2}} = \sqrt{9}$

$27^{\frac{1}{3}} = \sqrt[3]{27}$

$6^{-2} = \dfrac{1}{6^2}$

The expression a^n is called a ***power***. a is called the ***base*** of the power, and n is called the ***exponent*** of the power. Recall the following important properties of powers: For any real numbers $a > 0$, $b > 0$ $a \neq 0$, and real numbers m, and n, the following properties, 1 through 6, hold true:

1. $a^0 = 1$;
2. $a^m \cdot a^n = a^{m+n}$
3. $a^m \div a^n = a^{m-n}$
4. $(ab)^n = a^n \cdot b^n$
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

6. $(a^n)^m = a^{nm}$

When m and n are integers, the above properties, 1 through 6, hold true for all real numbers $a \neq 0, b \neq 0$.

And, for $a > 0$ and n a natural number,

7. $a^{\frac{1}{n}} = \sqrt[n]{a}$.

We note that $(-2)^x = -8$ has the solution $x = 3$.

6.2.2 How Do We Solve Exponential Equations?

The basic idea for solving exponential equations of the form $a^x = b$ is to write b as the power with the base a. Consider, for example, the equation $3^{x-4} = 81$. We write $81 = 3^4$, and accordingly our equation becomes: $3^{x-4} = 3^4$. Since $a^b = a^c$ if and only if $b = c$, we get $x - 4 = 4$. Hence, the solution to the equation is $x = 8$.

Let's now turn to solve more complicated exponential equations.

EXAMPLE 6.2a: Solve $3^{2x+1} - 2 \cdot 9^x = 5 - 3^{2x-1}$

SOLUTION:

All the powers involving the variable x can be written as powers with base 3:

$3^{2x+1} - 2 \cdot (3^2)^x = 5 - 3^{2x-1}$

$3^{2x+1} - 2 \cdot 3^{2x} = 5 - 3^{2x-1}$

Notice that each exponent in the last equation involves $2x$:

$3^{2x+1} = 3^{2x} \cdot 3^1 = 3 \cdot 3^{2x}$

$3^{2x-1} = 3^{2x} \cdot 3^{-1} = \dfrac{3^{2x}}{3}$

$3 \cdot 3^{2x} - 2 \cdot 3^{2x} = 5 - \dfrac{3^{2x}}{3}$

Writing the given equation in terms of 3^{2x}, we get

$3 \cdot 3^{2x} - 2 \cdot 3^{2x} = 5 - \dfrac{3^{2x}}{3}$

Let $3^{2x} = t$. Now, our equation turns into a linear equation:

$3\cdot t-2\cdot t=5-\frac{t}{3}$.

Solving for t, we get

$t=\frac{15}{4}=3.75$

Thus, $3^{2x}=3.75$.

Recall that $a^b=c$ if and only if $\log_a c=b$. Therefore: $3^{2x}=3.75$ if and only if $2x=\log_3 3.75$, if and only if $x=\frac{1}{2}\log_3 3.75$.

Thus: $x=\frac{1}{2}\log_3 3.75$ is the only solution to the given equation.

Using a calculator, we can find that solution x is approximately 0.60155.

EXAMPLE 6.2b: Solve the equation $4^{x+1}-5^{x-\frac{3}{2}}=5^{x+\frac{1}{2}}-2^{2x-4}$

SOLUTION: Notice $4^x=2^{2x}$

Hence, our equation can be written as $4^{x+1}-5^{x-\frac{3}{2}}=5^{x+\frac{1}{2}}-4^{x-2}$

We separate between the powers of 4 and the powers of 5: $4\cdot 4^x+4^{-2}\cdot 4^x=5^{\frac{1}{2}}\cdot 5^x+5^{-\frac{3}{2}}\cdot 5^x$

Factoring out 4^x, we get: $(4+4^{-2})4^x=(5^{\frac{1}{2}}+5^{-\frac{3}{2}})5^x$

$\frac{4^x}{5^x}=\frac{5^{\frac{1}{2}}+5^{-\frac{3}{2}}}{4+4^{-2}}$

We will compute the part on the right of the equation separately:

$$\frac{5^{\frac{1}{2}}+5^{-\frac{3}{2}}}{4+4^{-2}}=\frac{5^{\frac{1}{2}}+\frac{1}{5^{\frac{3}{2}}}}{4+\frac{1}{4^2}}=\frac{\left(5^{\frac{1}{2}}\cdot 5^{\frac{3}{2}}+1\right)4^2}{5^{\frac{3}{2}}\left(4\cdot 4^2+1\right)}=\frac{(25+1)4^2}{(64+1)5^{\frac{3}{2}}}=\frac{\cancel{26}\cdot 4^2}{\cancel{65}\cdot 5^{\frac{3}{2}}}=\frac{2\cdot 4^2}{5\cdot 5^{\frac{3}{2}}}=\frac{4^{\frac{1}{2}}\cdot 4^2}{5\cdot 5^{\frac{3}{2}}}=\frac{4^{2+\frac{1}{2}}}{5^{1+\frac{3}{2}}}=\frac{4^{\frac{5}{2}}}{5^{\frac{5}{2}}}=\left(\frac{4}{5}\right)^{\frac{5}{2}}$$

We have $\left(\frac{4}{5}\right)^x=\left(\frac{4}{5}\right)^{\frac{5}{2}}$

Hence: $x=\frac{5}{2}$.

The solution to the given equation is $x=\frac{5}{2}$.

Section 6.2
EXERCISES

1. Solve the following equations:

   a. $\left(\frac{3}{7}\right)^{x}=\left(\frac{7}{3}\right)^{-5}$

   b. $2^{x^2+x-0.5}=4\sqrt{2}$

   c. $10^{1+x^2}-10^{1-x^2}=99$

   d. $5^{\frac{1}{x^2-x}}\cdot 0.2^{\frac{1}{x}}=\sqrt[3]{25}$

   e. $4^{x}-3^{x-0.5}=3^{x+0.5}-2^{2x-1}$

   f. $27^{x}-13\cdot 9^{x}+13\cdot 3^{x+1}-27=0$

   g. $3\cdot 16^{x}+36^{x}=2\cdot 81^{x}$

6.3 LOGARITHMIC EQUATIONS
6.3.1 Basic Properties

The simplest logarithmic equation is of the form $\log_a x=b$. Its solution is $x=a^b$. For example, the solution to the equation $\log_2 x=4$ is $x=2^4=16$. Logarithmic equations can, however, be more complicated. For example, to solve the equation $\log_2 x+\log_4 x^2=17$ we need to apply some properties of logarithms.

Before we do so, recall that the function $\log_a x$ is meaningful only for $a>0$ and $a\neq 1$, and its domain is the positive real numbers. Also, before proceeding, you should make sure that you understand the meaning of $\log_a c=b$. Here are a few identities that will help you review this meaning:

$\log_2 32=5$ is the same as saying $2^5=32$.

$\log_4 \frac{1}{64}=-3$ is the same as saying $4^{-3}=\frac{1}{64}$.

$\log_8 2\sqrt{2}=\frac{1}{2}$ is the same as saying $\sqrt{8}=2\sqrt{2}$.

$\log_6(-36)$ is not meaningful because there is no real number b such that $6^b=-36$.

We will turn now to the basic properties of logarithms. In the following, 1) through 8), let a be a positive real number different from 1. Then,

1) $\log_a 1 = 0$

2) $\log_a a = 1$

Let b be a positive real number. Then

3) $a^{\log_a b} = b$

Let b be a positive real number and p be any real number. Then,

4) $\log_a b^p = p\log_a b$

Let b be a positive real number and q be any real number different from zero. Then,

5) $\log_{a^q} b = \dfrac{1}{q}\log_a b$

Let b and d be positive real numbers. Then,

6) $\log_a bd = \log_a b + \log_a d$

7) $\log_a \dfrac{b}{d} = \log_a b - \log_a d$

Let b be a positive real number different from 1, and let c be any positive real number. Then,

8) $\log_a c = \dfrac{\log_b c}{\log_b a}$

Property (8) is called the **Change of Base Formula**.

In the next section, you will be guided as to how to prove these properties.

6.3.2 How Do We Solve Logarithmic Equations?

EXAMPLE 6.3a: Solve the equation $\log_5 x = 3$.

SOLUTION: By the definition of a logarithm, $\log_5 x = 3$ if and only $5^3 = x$. Hence, $x = 5^3 = 125$ is the solution to this equation.

EXAMPLE 6.3b: Solve the equation $\log_{10}(x^2 - 9x + 10) = \log_{10}(x - 6)$.

SOLUTION: Recall that the domain of the logarithmic function is the set of the positive real numbers. Hence, we must state the following two conditions:

1. $x^2 - 9x + 10 > 0$

2. $x-6>0$

Since the logarithmic function is one-to-one, $\log_{10}(x^2-9x+10)=\log_{10}(x-6)$ if and only $x^2-9x+10=x-6$.

Solving the last equation, we get two solutions: $x=8$ or $x=2$.

Now, we check whether these values satisfy inequalities 1 and 2 above.

Substituting $x=8$ in these inequalities, we get:

$8^2-9\cdot8+10=2>0$

$8-6=2>0$

$x=8$ satisfies conditions 1 and 2, and hence it is a solution.

Substituting $x=2$ in inequality 1, we get $2^2-9\cdot2+10=-4<0$

Since $x=2$ does not satisfy at least one of conditions 1 and 2, it is not a solution.

EXAMPLE 6.3c: Solve the equation $\log_3 x+\log_3(x+8)=2$.

SOLUTION: $\log_3 x+\log_3(x+8)=2$

Condition:

1. $x>0$
2. $x+8>0$

$\log_3 x(x+8)=2$ By Property 6

$x(x+8)=3^2$

$x^2+8x=9$

$x^2+8x-9=0$

$$x=\frac{-8\pm\sqrt{64+36}}{2}$$

$$x=\frac{-8\pm10}{2}$$

$x=-9, x=1$

Since $x=-9$ violates condition 1, the only solution our equation has is $x=1$.

EXAMPLE 6.3d: Solve the equation $\log_3 x + 2\log_3(x+8) = 4$

SOLUTION:

Condition:

1. $x > 0$
2. $x + 8 > 0$

$\log_3 x + 2\log_3(x+8) = 4$

$\log_3 x + \log_3(x+8)^2 = 4$ By Property 4

$\log_3 x(x+8)^2 = 4$ By Property 6

$x(x+8)^2 = 4^3$

$x^3 + 16x^2 + 64x = 81$

$x^3 + 16x^2 + 64x - 81 = 0$

By the Rational Root Theorem, since $x = 1$ is a solution to the last equation, the polynomial $x^3 + 16x^2 + 64x - 81$ is divisible by $x - 1$.

When dividing $x^3 + 16x^2 + 64x - 81$ by $x - 1$, we get $x^2 + 17x + 81$.

Hence, the last equation can be written as $(x-1)(x^2 + 17x + 81) = 0$.

Equation $x^2 + 17x + 81 = 0$ has no real solution (verify).

Therefore $x = 1$ is the only solution to the last equation.

Since $x = 1$ satisfies conditions 1 and 2, it is the (only) solution to the given equation.

EXAMPLE 6.3e: Solve the equation $\log_3^2(x-1) - \log_3(x-1) - 2 = 0$

SOLUTION:

Condition: $x - 1 > 0$.

We note that $\log_3^2(x-1) = (\log_3(x-1))^2$.

Let $\log_3(x-1) = t$.

We have the quadratic equation $t^2 - t - 2 = 0$.

Solving for t we get $t = -1$ or $t = 2$.

Hence: $\log_3(x-1) = -1$ or $\log_3(x-1) = 2$.

Solving each of these two equations, we get: $3^{-1} = x-1$ or $3^2 = x-1$

$x = \frac{4}{3}$ or $x = 10$.

Since these two values satisfy the condition $x - 1 > 0$, they are both solutions to the given equation.

EXAMPLE 6.3f: Solve the equation $\ln x + \ln 2x = e^{\ln 1}$

SOLUTION:

Condition: $x > 0$.

First we note that $e^{\ln 1} = 1$ (why?)

$\ln x + \ln 2x = e^{\ln 1}$

$\ln x \cdot 2x = 1$ (why?)

$2x^2 = e$

$x^2 = \frac{e}{2}$

$x = \pm\sqrt{\frac{e}{2}}$

Since $x > 0$, the equation has only one solution: $x = \sqrt{\frac{e}{2}}$.

Section 6.3
EXERCISES

1. Simplify the following expressions:

   a. $\log_3 \sqrt[3]{\sqrt[3]{3}}$

   b. $\log_{\sqrt{3}} 25 - \log_3 7\frac{58}{81}$

   c. $\dfrac{\log_{10} 8 + \log_{10} 18}{2\log_{10} 2 + \log_{10} 3}$

   d. $\log_3 2 \cdot \log_4 3 \cdot \log_5 4 \cdot \log_6 5 \cdot \log_7 6 \cdot \log_8 7$ (Hint: Use the Change of Base Formula.)

2. Use the Change of Base Formula and a calculator to evaluate the following logarithms, correct to four decimal places:

   a. $\log_3 19$

   b. $\log_2 3$

3. Solve the following equations:

   a. $\log_{0.5} x = -1$

   b. $\log_2^2(x+1) - \log_{\frac{1}{4}}(x+1) = 5$

   c. $\log_2 x + \log_4 x + \log_8 x = 11$

   d. $\log_2 x + \dfrac{4}{\log_x 2} = 5$

   e. $2\log_{10} x - \log_{10} 4 = -\log_{10}(5 - x^2)$

   f. $\ln x + \log_x e = 2$

6.4 RADICAL EQUATIONS

The following are examples of radical equations:

- $\sqrt{x+5}=2$
- $\sqrt{x-4}+\sqrt{2-x}=1$
- $\sqrt{5x-1}-x=1$
- $x^{\frac{2}{3}}+\sqrt[3]{x}=7$

In all these equations, the variable x appears under the radical sign, and none of them is of the types we have learned about so far. In this section of the resource guide, we will learn how solve such equations.

The fact that an equation includes a radical does not necessarily make it a radical equation. For example, the following equations are not radical equations:

- $x-\sqrt{2}=\sqrt{5}$
- $x^2+3\sqrt{3}=5$
- $\sqrt{x^2}-2x=3$

The first two are not radical because the variable x does not appear under the radical sign. The third equation is not radical because it is equivalent to $|x|-2x=3$, which is not radical.

6.4.1 Method 1

One useful method for solving a radical equation of the form $\sqrt[n]{u}=\sqrt[n]{v}$ is to raise both sides of the equation to the power n. In this way, we get a simpler equation of the form $u=v$. We must be careful, however, for the two equations may not be equivalent. This is the case because not every solution to the second equation is necessarily a solution to the first equation. For example, consider the following equation: $\sqrt{x+2}=x$.

1. Squaring both sides, we get $x+2=x^2$
2. Solving this quadratic equation, we get two solutions: $x=-1$ and $x=2$.
3. Note that while $x=2$ is a solution to our equation, $x=-1$ is not.

Let's demonstrate this method further by working through several examples:

EXAMPLE 6.4a: Solve the equation $\sqrt{5x-1}-\sqrt{3x-2}=\sqrt{2x+1}$

SOLUTION: $\sqrt{5x-1}-\sqrt{3x-2}=\sqrt{2x+1}$.

$\left(\sqrt{5x-1}\right)^2-2\sqrt{5x-1}\cdot\sqrt{3x-2}+\left(\sqrt{3x-2}\right)^2=\left(\sqrt{2x+1}\right)^2$ Square both sides

$5x-1-2\sqrt{(5x-1)(3x-2)}+3x-2=2x+1$

$8x-3-2\sqrt{(5x-1)(3x-2)}=2x+1$

$-2\sqrt{(5x-1)(3x-2)}=-6x+4$

$2\sqrt{(5x-1)(3x-2)}=6x-4$

$\sqrt{15x^2-13x+2}=3x-2$

$15x^2-13x+2=9x^2-12x+4$ Square both sides

$6x^2-x-2=0$

$x=\frac{2}{3},\ x=-\frac{1}{2}$

$x=-\frac{1}{2}$ cannot be a solution to our original equation because some of the expressions in the original equation are not defined for this value. (For example, when $x=-\frac{1}{2}$ is substituted in the expression $\sqrt{5x-1}$, we get a square root of a negative number.) On the other hand, all the expressions in our original equation are defined for $x=\frac{2}{3}$.

Hence $x=\frac{2}{3}$ is the only solution to our original equation.

EXAMPLE 6.4b: Solve the equation $\sqrt{x-5}+\sqrt{3-x}=1$.

SOLUTION:

$\left(\sqrt{x-5}\right)^2+2\sqrt{x-5}\cdot\sqrt{3-x}+\left(\sqrt{3-x}\right)^2=1$ Square both sides

$x-5+2\sqrt{x-5}\cdot\sqrt{3-x}+3-x=1$

$2\sqrt{-x^2+8x-15}=3$

$$\sqrt{-x^2+8x-15}=\frac{3}{2}$$

$$-x^2+8x-15=\frac{9}{4}$$

$$4x^2-32x+69=0$$

This equation has no solution. Hence, our original equation too has no solution.

6.4.2 Method 2

Now we will learn a second method for solving radical equations. It is called ***solution by substitution.***

EXAMPLE 6.4c: Solve the equation $\sqrt[3]{(x-1)^2}-\sqrt[3]{x-1}=12$.

SOLUTION: Note that if we let $\sqrt[3]{x-1}=t$, we will get the quadratic equation: $t^2-t=12$.

Solving for t, we get that $t=4$ or $t=-3$.

Therefore, $\sqrt[3]{x-1}=4$ or $\sqrt[3]{x-1}=-3$.

Raising both sides of each equation to the third power, we get $x-1=64$ or $x-1=-27$

$x=65$ or $x=-26$.

By checking each of these values in the original equation, we can conclude that both $x=65$ and $x=-26$ are solutions to our equation.

Section 6.4
EXERCISES

1. Solve the following equations:

   a. $\sqrt{3-2x^2}=1$

   b. $\sqrt{5+2x}=10-3\sqrt[4]{5+2x}$

   c. $(x+1)\sqrt{x^2-x-6}=6x+6$

   d. $\sqrt{x-3}+\sqrt{6-x}=\sqrt{3}$

   e. $\sqrt{x+\sqrt{x+11}}+\sqrt{x-\sqrt{x+11}}=4$

   f. $\sqrt{3x-15}-2x=3-\sqrt{5-x}$

Section 7
Inequalities

7.1 LINEAR INEQUALITIES

In Section 1, we listed the important set of properties of the real numbers concerning inequalities. These are:

1. For any two real numbers x and y, one and only one of the following is true: $x < y$, $x = y$, or $x > y$.
2. For any three real numbers x, y, and z, if $x < y$, $y < z$, then $x < z$.
3. For any three real numbers x, y, and z, if $x < y$ then $x + z < y + z$.
4. For any three real numbers x, y, and z, if $x < y$ and $z > 0$, then $zx < zy$.
5. For any three real numbers x, y, and z, if $x < y$ and $z < 0$, then $zx > zy$.

These properties are needed to solve inequalities. Let's consider a few examples:

EXAMPLE 7.1a: Solve the inequality $3x - 4 > 5$.

We are asked to find the set of all numbers x that satisfy this inequality.

SOLUTION:

$3x - 4 > 5$ Add 4 to both sides of this inequality to get the following inequality:

$3x > 9$ Divide both sides of the inequality by 3 to get the following inequality:

$x > 3$

The solution set consists of all the numbers that are greater than 3. For example, $x = 4$ satisfies the inequality because $3 \cdot 4 - 4 = 8 > 5$. On the other hand, 2 does not satisfy the inequality because $3 \cdot 2 - 4 = 2 < 5$.

EXAMPLE 7.1b: Solve the inequality $8x-2>2x-3$.

SOLUTION:

$8x-2>2x-3$	Add 2 to both sides of this inequality to get the following inequality:
$8x>2x-1$	Add $-2x$ to both sides of this inequality to get the following inequality:
$6x>-1$	Divide both sides of this inequality by 6 to get the following inequality:
$x>-\frac{1}{6}$	

Thus, the solution set of the given inequality consists of all the numbers that are greater than $-\frac{1}{6}$.

EXAMPLE 7.1c: Solve the inequality $-3x+12>45$.

SOLUTION:

$-3x+12>45$	Add -12 to both sides of this inequality to get the following inequality:
$-3x>33$	Divide both sides of the inequality by -3 to get the following inequality:
$x<-11$	

If you feel you understand the process of solving an inequality, there is no need to explain each step. Nor is there a need to say explicitly the meaning of the last expression. The answer $x<-11$ is understood to mean "the solution set of the given inequality consists of all the numbers that are smaller than -11."

EXAMPLE 7.1d: Solve the inequality $4(x-12.4)\leq 8x-\frac{1}{8}$.

SOLUTION: You should justify each step in the following process on your own:

$4(x-12.5)\leq 8x-\frac{1}{8}$

$4x - 50 \le 8x - \frac{1}{8}$

$32x - 400 \le 64x - 1$

$-32x - 400 \le -1$

$-32x \le 399$

$x \ge -\frac{399}{32} = -12\frac{15}{32}$

EXAMPLE 7.1e: Solve the inequality $2(-x+3) > -2x+19$.

SOLUTION:

$2(-x+3) > -2x+19$

$-2x+6 > -2x+19$

Look carefully at this inequality. Clearly, there is no number that satisfies it because if there were such a number, then we would get $6 > 19$, which is not true. Thus, the given inequality has no solution.

EXAMPLE 7.1f: Solve the inequality $-22(3x-3) > -6(11x+1) - 35$.

SOLUTION:

$-22(3x-3) > -6(11x+1) - 35$

$-66x + 66 > -66x - 6 - 35$

$-66x + 66 > -66x - 41$

Since $66 > -41$, the given inequality has infinitely many solutions: any number x satisfies it.

EXAMPLE 7.1g: Solve the system of inequalities $\begin{cases} 5(x-3) > 3x-7 \\ 2x+3 > 36-x \end{cases}$

SOLUTION: This time, we are looking for the set of all the numbers that satisfy two inequalities. We maintain the two inequalities together while solving each separately:

$$\begin{cases} 5x-15 > 3x-7 \\ 2x+3 > 36-x \end{cases}$$

$$\begin{cases} 5x-3x > -7+15 \\ 2x+x > 36-3 \end{cases}$$

$$\begin{cases} 2x > 9 \\ 3x > 33 \end{cases}$$

$$\begin{cases} x > 4.5 \\ x > 11 \end{cases}$$

The solution set of the given system consists of all the numbers that are greater than 4.5 ***and*** greater than 11. Clearly, any number that is greater than 11 is also greater than 4.5. Thus, the solution set is the set all the numbers x that are greater than 11. Or we simply write: $x > 11$.

Some linear inequalities involve ***absolute values***. The absolute value of a number is the distance of the number from zero on the number line. For example, the absolute value of 5, is 5, and the absolute value of -5 is also 5. The absolute value of zero is obviously zero.

The absolute value of a number x is denoted by $|x|$. The absolute value of a number x depends on where x is located on the number line: if x is located to right of 0, then $|x| = x$; if x is located to left of 0, then $|x| = -x$; and if x is zero, then $|x| = 0$.

The above paragraph can be written in more compact form:

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

The outputs of the absolute-value function $y = |x|$ are always non-negative. Its graph consists of two linear functions: $y = x$ for $x \geq 0$, and $y = -x$ for $x < 0$. (See FIGURE 7-1.)

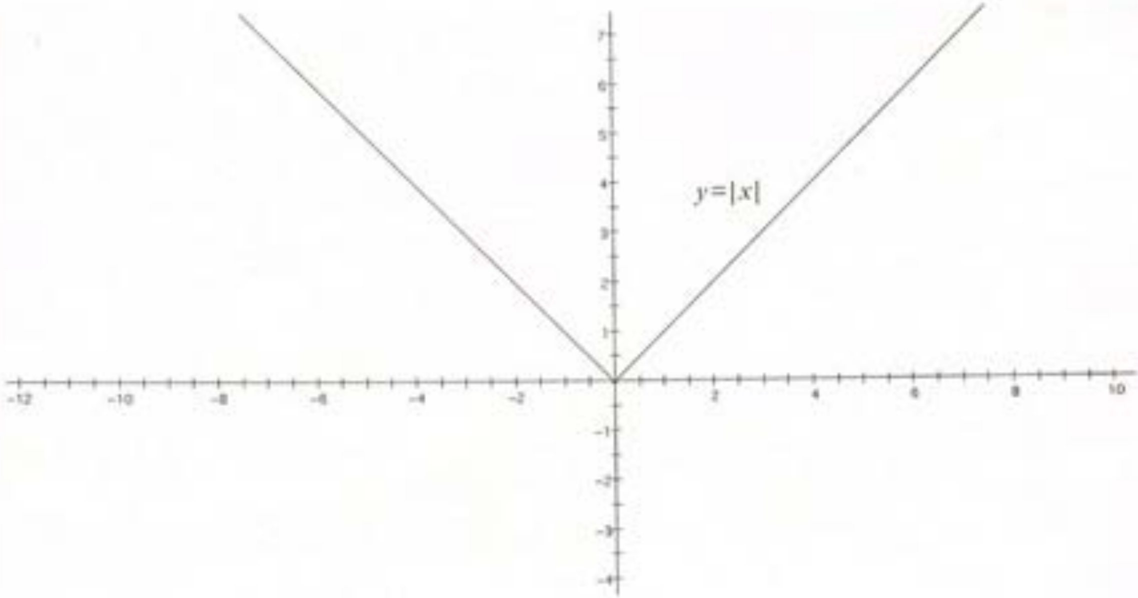


FIGURE 7–1

Let's now solve a few linear inequalities involving absolute values.

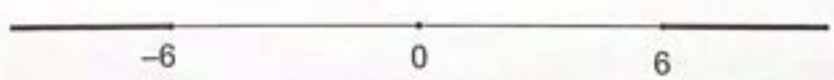
EXAMPLE 7.1h: Solve $|x| < 3$.

SOLUTION: We are looking for all the numbers x whose distance from zero is smaller than 3. The only numbers x that satisfy this inequality are $-3 < x < 3$.

EXAMPLE 7.1i: Solve $|x| \geq 6$.

SOLUTION:

We are looking for all the numbers x whose distance from zero is greater than or equal to 6. The only numbers x that satisfy this inequality are: $x \geq 6$ or $x \leq -6$.



EXAMPLE 7.1j: Solve $x+|x|<\frac{1}{2}$.

SOLUTION: This inequality is a bit harder. The algebraic definition of absolute value can be very helpful in solving this kind of inequality. We will deal with two cases:

CASE 1: $x \geq 0$.

In this case $|x| = x$, and so the given inequality is: $x+x<\frac{1}{2}$

Solving:

$2x<\frac{1}{2}$

$x<\frac{1}{4}$

So, when $x \geq 0$ the solution of the given inequality is $x<\frac{1}{4}$.

Hence, any x where $0 \leq x < \frac{1}{4}$ is a solution to the inequality.

CASE 2: $x<0$.

In this case, $|x| = -x$, and so the given inequality is: $x-x<\frac{1}{2}$.

Solving:

$0 \cdot x<\frac{1}{2}$

We are looking for all the numbers x for which $0 \cdot x<\frac{1}{2}$. Obviously, any multiple of zero is zero, which is less than $\frac{1}{2}$. Therefore, any number x satisfies the inequality $0 \cdot x<\frac{1}{2}$.

But, remember that we are dealing with the case $x<0$. Hence the solution in this case is $x<0$.

SUMMARIZING:

The solution to the given inequality consists of two sets of numbers:

The set of all the numbers x for which $0 \leq x < \frac{1}{4}$

The set of all the numbers x for which $x<0$.

Combining these two sets, we get:

The solution to the given inequality is $x<\frac{1}{4}$.

Section 7.1
EXERCISES

1. Solve the inequalities:

   a. $-3x+21>0$

   b. $2(x-2)-5(1-3x)<2$

   c. $\sqrt{6}(2-x)\geq 5-2x$

   d. $\frac{2x}{3}-1<3-2(1-2x)$

   e. $13-\frac{3-7x}{10}+\frac{x+1}{2}<14-\frac{7-8x}{2}$

   f. $\begin{cases}\frac{x}{2}-\frac{7}{4}>\frac{5x}{2}-\frac{7}{8}\\ \frac{2x+1}{4}<5-\frac{1-2x}{3}\end{cases}$

   g. $|x-2|\leq 5-x$

7.2 QUADRATIC INEQUALITIES

The following are examples of **quadratic inequalities**:

- $x^2-3x+6<0$
- $5x^2<125$
- $(x-3)(x+5)>0$
- $-x^2-3x\leq 0$
- $(-x+5)(3-x)\leq 6$

Any quadratic inequality can be brought by algebraic operations into one of two forms:

1. $ax^2+bx+c>0$ and $a>0$
2. $ax^2+bx+c<0$ and $a>0$

For example, the inequality $-2x^2+3x-9>0$ can be changed into the second form by multiplying both sides of the inequality by -1. It becomes: $2x^2-3x-9<0$. (Note that a, the coefficient of x^2, is positive).

7.2.1 Inequalities of the Form $ax^2 + bx + c > 0$ and $a > 0$

We first observe that when $a>0$, the function ax^2+bx+c has a minimum. Next we ask: Does the parabola intersect the x-axis? And, if it does, at how many points does it intersect the x-axis?

Recall that the answer to these questions depends on the value of the discriminant $\Delta=b^2-4ac$.

CASE 1: When $\Delta=b^2-4ac<0$ and $a>0$, the parabola does not intersect the x-axis, and it looks like this:

FIGURE 7–2

This implies that for any value of x, $ax^2+bx+c>0$. Hence: When $\Delta=b^2-4ac<0$ and $a>0$, the solution to the inequality $ax^2+bx+c>0$ is $-\infty<x<\infty$.

CASE 2: When $\Delta=b^2-4ac=0$, the parabola touches the x-axis at one point: $x=-\dfrac{b}{2a}$

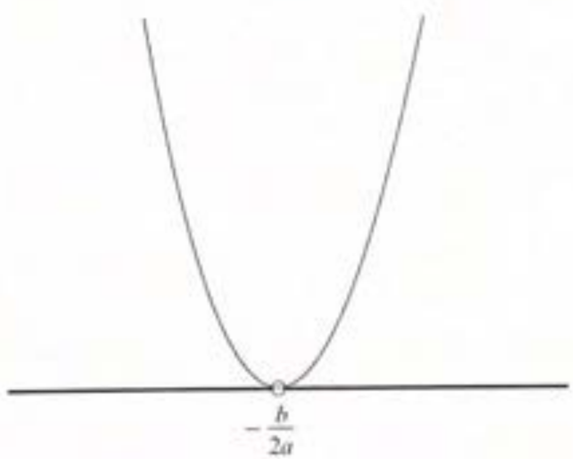


FIGURE 7–3

This implies that for any value of x that is different from $-\frac{b}{2a}$, $ax^2+bx+c>0$. Hence: When $\Delta=b^2-4ac=0$ and $a>0$, the solution to the inequality $ax^2+bx+c>0$ is any number $x\neq-\frac{b}{2a}$.

CASE 3: When $\Delta=b^2-4ac>0$ and $a>0$, the parabola intersects the x-axis at two points: $x_1=\frac{-b-\sqrt{b^2-4ac}}{2a}$ and $x_2=\frac{-b+\sqrt{b^2-4ac}}{2a}$.

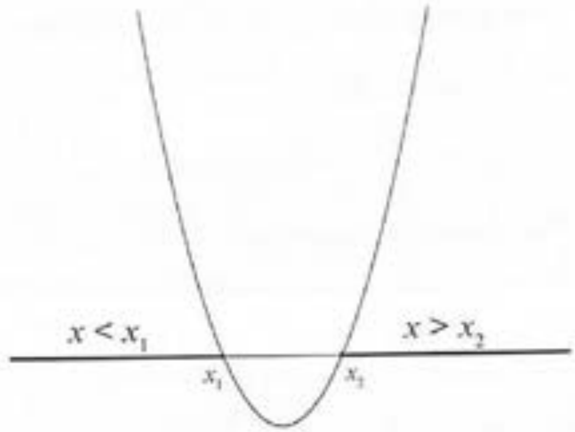


FIGURE 7-4

This implies that for any value of x for which $x>x_2$ or $x<x_1$, $ax^2+bx+c>0$.

Hence: When $\Delta=b^2-4ac>0$ and $a>0$, the solution to the inequality $ax^2+bx+c>0$ is $x<x_1$ or $x>x_2$.

EXAMPLE 7.2a: Solve the inequality $x^2+6x>-10$.

SOLUTION: First we bring this inequality into standard form $ax^2+bx+c>0$ by adding 10 to each side: $x^2+6x+10>0$

We find the value Δ for $x^2+6x+10$:

$\Delta=6^2-10\cdot4=36-40=-4<0$.

Since $a=1$, the quadratic function has a minimum, and its graph is above the x-axis. Therefore, any real number x satisfies the given inequality.

To show this, we write: $-\infty<x<\infty$.

EXAMPLE 7.2b: Solve $x^2+6x+9>0$.

SOLUTION: We find the value of Δ for x^2+6x+9

$\Delta=6^2-9\cdot4=36-36=0$.

Since $a=1$, the graph of the quadratic function touches the x-axis from above at one point.

To find this point, we solve: $x^2+6x+9=0$.

Solving, we get $x=-3$.

This implies that any $x\neq-3$ satisfies the given inequality.

EXAMPLE 7.2c: Solve $x^2+6x+5>0$.

SOLUTION: The value of Δ for x^2+6x+5 is $\Delta=6^2-5\cdot4=36-20=16>0$

Since $a=1$, the function has a minimum, and it intersects the x-axis at two points. These points are the solutions to the equation $x^2+6x+5=0$.

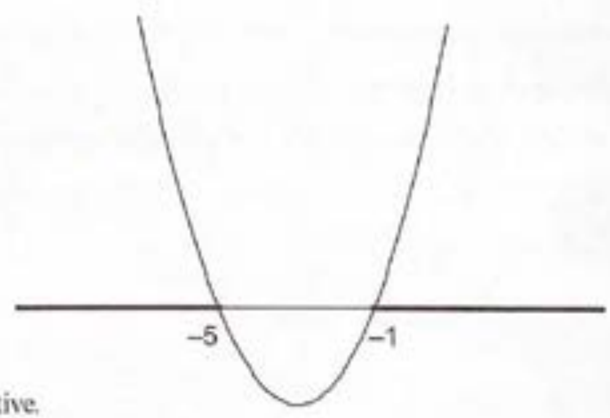
Solving, we get:

$$x=\frac{-6\pm4}{2}$$

$x=-5$

or

$x=-1$.



For $x<-5$ or $x>-1$, $y=x^2+6x+5$ is positive.

Thus, the solution to the given inequality is the set: $x<-5$ or $x>-1$.

7.2.2 Inequalities of the Form $ax^2 + bx + c < 0$ and $a > 0$

Recall that when $a > 0$ the function $ax^2 + bx + c$ has a minimum. As before we ask: Does the parabola intersect the x-axis, and if it does, at how many points does it intersect the x-axis?

The answer to this question depends on the value of the discriminant $\Delta = b^2 - 4ac$. As before, we have three cases:

CASE 1: When $\Delta = b^2 - 4ac < 0$, the parabola does not intersect the x-axis, as is shown in FIGURE 7–5.

FIGURE 7–5

This implies that there is no x for which $ax^2 + bx + c < 0$. Hence: When $\Delta = b^2 - 4ac < 0$ and $a > 0$, the inequality $ax^2 + bx + c < 0$ has no solution.

CASE 2: When $\Delta = b^2 - 4ac = 0$, the parabola touches the x-axis at one point: $x = -\dfrac{b}{2a}$

FIGURE 7–6

This implies that there is no x for which $ax^2 + bx + c < 0$. Hence: When $\Delta = b^2 - 4ac = 0$ and $a > 0$, the inequality $ax^2+bx+c<0$ has no solution.

CASE 3: When $\Delta=b^2-4ac>0$, the parabola intersects the x-axis at two points:

$x_1=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$ and $x_2=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$.

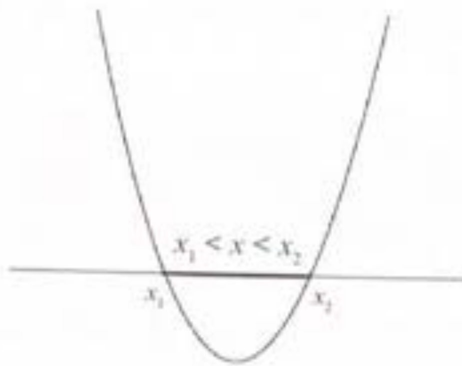


FIGURE 7–7

This implies that for any value of x between x_1 and x_2, $ax^2+bx+c<0$. Hence: When $\Delta=b^2-4ac>0$ and $a>0$, the solution to the inequality $ax^2+bx+c<0$ is $x_1<x<x_2$.

EXAMPLE 7.2d: Solve the inequality $x^2+8x+18<0$.

SOLUTION: $\Delta=8^2-18\cdot 4=64-72=-8<0$

Therefore, the graph of the function $y=x^2+8x+18$ does not intersect the x-axis. Since $a=1$, the graph is above the x-axis.

Therefore, the given inequality has no solution.

EXAMPLE 7.2e: Solve the inequality $x^2+8x+16<0$.

SOLUTION: $\Delta=8^2-16\cdot 4=64-64=0$

Therefore, the graph of the function $y=x^2+8x+16$ touches the x-axis at one point.

Since $a=1$, the graph is above the x-axis.

Hence, the given inequality has no solution.

EXAMPLE 7.2f: Solve the inequality $x^2+8x+7<0$.

SOLUTION: $\Delta=8^2-7\cdot 4=64-28=36>0$

Therefore, the graph of the function $y=x^2+8x+16$ intersects the x-axis at two points. These points are the solutions to the equation $x^2+8x+7=0$.

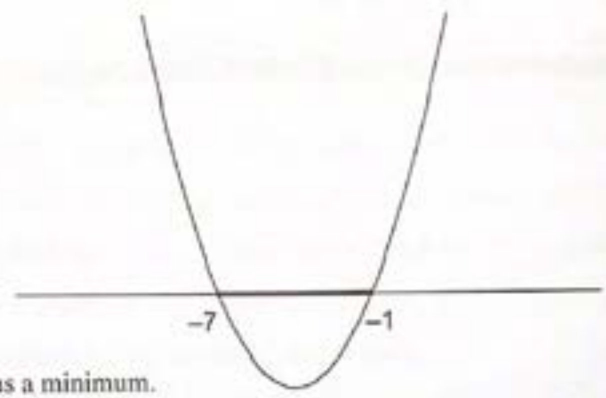
Solving, we get

$x=\frac{-8\pm 6}{2}$

$x=-7$

or

$x=-1$



Since $a=1$, the function $y=x^2+8x+16$ has a minimum.

Hence, for $-7<x<-1$ the inequality holds.

Section 7.2
EXERCISES

1. Solve the inequalities:

   a. $x^2-5x+6<0$

   b. $4x^2-4x+1\le 0$

   c. $81-(3+2x)^2<0$

   d. $3+2x\le x^2$

   e. $5-x^2\ge -4x$

   f. $\frac{3}{2x-1}>0$

g. $\dfrac{2x-9}{x-6}<1$

2. Find the domain of the following functions:

a. $y=\sqrt{(3x-2)(x-5)}$

b. $y=\dfrac{1}{\sqrt{112x+64+49x^2}}$

Section 8
Coordinate Geometry

The goal of **coordinate geometry** is to describe geometric objects, such as ***lines*** and ***circles***, algebraically through equations (and sometimes through inequalities). In this way, an investigation of an equation representing a geometric object can reveal important properties of the geometric object. The most basic geometric object is the ***point***. A basic idea of coordinate geometry is the creation of a correspondence between the set of all ordered pairs of real numbers and the set of all points in the plane: For any point in the plane, there corresponds exactly one ordered pair of real numbers (x, y), and, conversely, for any ordered pair of real numbers (x, y), there corresponds exactly one point in the plane.

8.1 THE PYTHAGOREAN THEOREM

Before we start with coordinate geometry, we will first prove an ancient theorem, called the Pythagorean Theorem. We will need this theorem to prove the distance formula.

Pythagorean Theorem: In a right triangle, the square of the hypotenuse is the sum of the squares of the other two sides of the triangle.

PROOF: Consider a right triangle with sides a and b and hypotenuse c as shown in FIGURE 8–1 below. We are to prove that $a^2 + b^2 = c^2$.

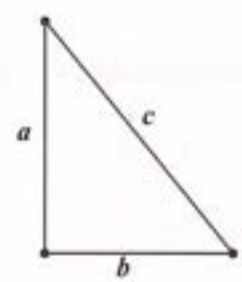


FIGURE 8–1

Four copies of the right triangle are assembled to form FIGURE 8–2. In FIGURE 8–2 we have two squares: the outer square with side $a + b$, and the inner square with side c.

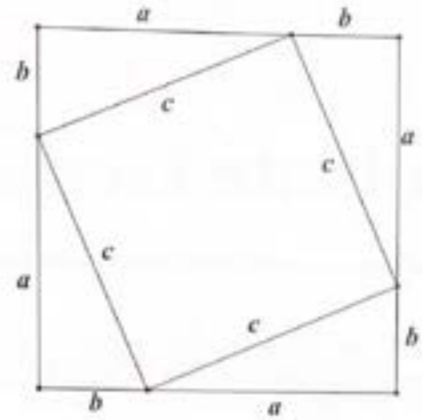


FIGURE 8–2

Let A be the area of the outer square: On the one hand, $A=(a+b)^2$. On the other hand, A is the sum of the areas of the four triangles and the area of the inner square. That is: $A=4\left(\frac{1}{2}ab\right)+c^2$

Therefore, $(a+b)^2=4\left(\frac{1}{2}ab\right)+c^2$.

This leads to: $a^2+2ab+b^2=2ab+c^2$

Hence: $a^2+b^2=c^2$, as was required.

The Converse of the Pythagorean Theorem: If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.

PROOF: Let ABC be a triangle such that
(1) $AC^2+BC^2=AB^2$.

We are to prove that $C=90°$.

Construct a triangle $AB'C$ such that $B'C$ is perpendicular to AC and $B'C=BC$, as is shown in Figure 8–3:

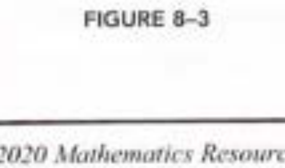


FIGURE 8–3

Since triangle $AB'C$ is a right triangle, according to the Pythagorean Theorem,

(2) $AC^2 + B'C^2 = AB'^2$

Since $B'C = BC$, we get

(3) $AC^2 + BC^2 = AB'^2$

From (1) and (3), we get $AB' = AB$.

The triangles ABC and $AB'C$ are congruent by the side-side-side triangle congruence theorem (SSS).

Hence $C' = C = 90^\circ$.

This completes the proof.

8.2 POINTS

Distance between two points: Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, find the distance d between them.

SOLUTION: In Figure 8–4, $QL \perp PR$

By the Pythagorean Theorem, $\overline{PQ} = \sqrt{\overline{QL}^2 + \overline{PL}^2}$

$\overline{QL} = x_1 - x_2$

$\overline{PL} = y_1 - y_2$

Therefore $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

This is called the ***distance formula***.

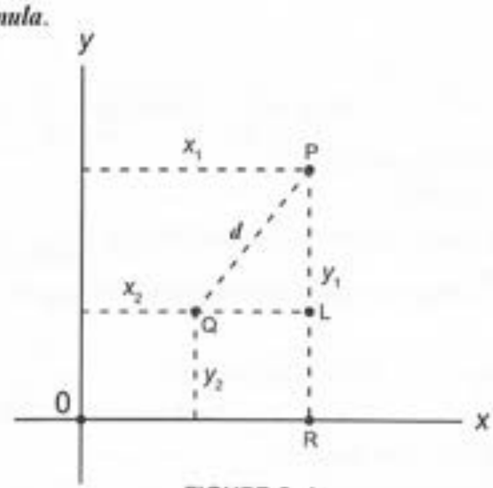


FIGURE 8–4

Midpoint: Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, find the midpoint $M(x, y)$ between them.

SOLUTION: We solve this problem for the case where the segment $\overline{PQ}$ is not parallel to the y-axis (see FIGURE 8–5). You can pursue the case where $\overline{PQ}$ is parallel to the y-axis on your own as an independent exercise.

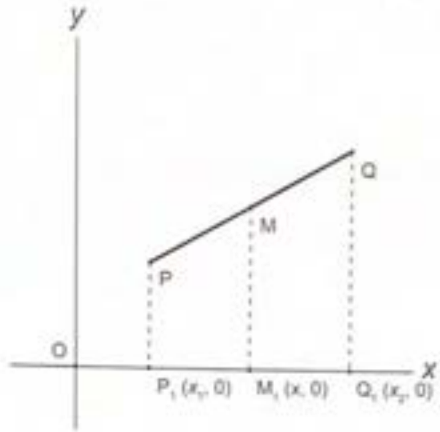


FIGURE 8–5

Since $\overline{PQ}$ is not parallel to the y-axis, $x_1 \neq x_2$.

From the points P, Q, and M, drop lines parallel to the y-axis, and let $P_1(x_1, 0)$, $Q_1(x_2, 0)$, and $M_1(x, 0)$ be the intersections of these lines with the x-axis, respectively.

From geometry, we know that the point M_1 is the midpoint of $\overline{P_1Q_1}$. This implies that $\overline{P_1M_1} = \overline{M_1Q_1}$.

Therefore, $|x - x_1| = |x - x_2|$.

Solving for x, we get either $x - x_1 = x - x_2$, which is not possible since $x_1 \neq x_2$, or $x - x_1 = -(x - x_2)$, which implies that $x = \dfrac{x_1 + x_2}{2}$.

In a similar way, we find that $y = \dfrac{y_1 + y_2}{2}$. Thus, $M\left(\dfrac{x_1 + x_2}{2}; \dfrac{y_1 + y_2}{2}\right)$ is the midpoint between $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Dividing a Segment into a Given Ratio: Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, find the point C on the segment $\overline{PQ}$ such that $\dfrac{\overline{PC}}{\overline{CQ}} = \dfrac{m}{n}$, where m and n are two positive integers.

SOLUTION: Let $C(x, y)$ be the point on the segment $\overline{PQ}$ such that $\dfrac{\overline{PC}}{\overline{CQ}} = \dfrac{m}{n}$ (see FIGURE 8–6).

From the points P, C, and Q, drop lines parallel to the y-axis, and let $P_1(x_1, 0)$, $Q_1(x_2, 0)$, $C_1(x, 0)$ be the intersections of these lines with the x-axis. From geometry, we know that $\frac{\overline{P_1C_1}}{\overline{C_1Q_1}} = \frac{\overline{PC}}{\overline{CQ}}$.

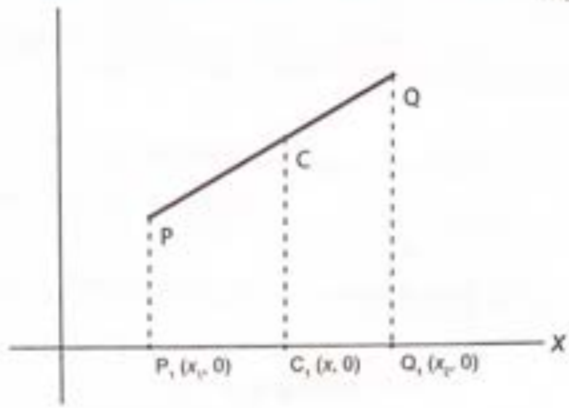


FIGURE 8–6

Therefore,

$$\frac{x-x_1}{x_2-x} = \frac{m}{n}$$

$$n(x-x_1) = m(x_2-x)$$

$$nx - nx_1 = mx_2 - mx$$

$$x(n+m) = nx_1 + mx_2$$

$$x = \frac{nx_1 + mx_2}{n+m}$$

In a similar way, we find that $y = \frac{ny_1 + my_2}{n+m}$.

Thus, $C\left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m}\right)$.

8.3 LINES

In the previous section, we dealt with points in the plane. In this section, we will deal with straight lines. We will begin with the following problem:

Perpendicular Bisector: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane, and let l be the perpendicular bisector of the segment PQ.[1] Find the equation of l.

[1] The perpendicular bisector of the segment PQ is the line that goes through the midpoint of PQ and is perpendicular to PQ.

Hence, γ is a line.

Take any two distinct points on γ. For simplicity's sake, we will take the points $A\left(0,-\frac{c}{b}\right)$ and $B\left(1,-\frac{a}{b}-\frac{c}{b}\right)$. (Check that these points are indeed on γ.)

Let l be the line between A and B. Using the Point-Point Form, the equation of l is

$$y=\frac{\left(-\frac{a}{b}-\frac{c}{b}\right)-\left(-\frac{c}{b}\right)}{1-0}x-\frac{c}{b}.$$

Simplifying, we get $y=-\frac{a}{b}x-\frac{c}{b}$.

So, the graph γ and the line l have exactly the same equation.

You can work through the case where $b=0$ as an exercise on your own.

Position of a Line. How does the position of the line whose equation is $ax+by+c=0$ depend on the coefficients a, b, and c?

SOLUTION: We will distinguish among several cases:

CASE 1: $a=0$ and $b\neq 0$.

In this case, the equation of the line is $y=-\frac{c}{b}$. (See FIGURE 8–8.) This line is parallel to the x-axis and passes through the point $(0,-\frac{c}{b})$ (Why?)

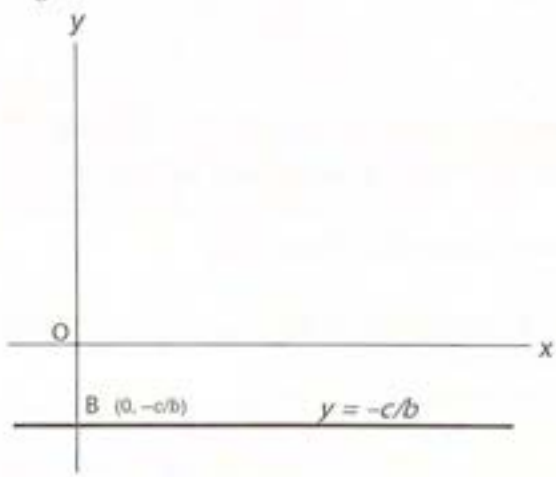


FIGURE 8–8

CASE 2: $a\neq 0$ and $b=0$.

In this case, the equation of the line is $x=-\frac{c}{a}$. (See FIGURE 8–9.) This line is parallel to the y-axis and passes through the point $(-\frac{c}{a},0)$ (Why?)

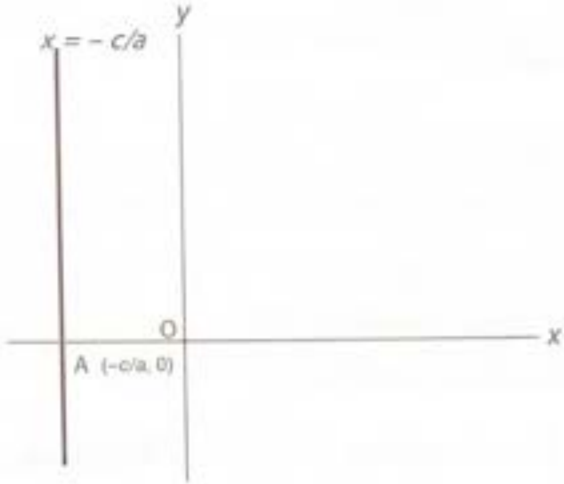


FIGURE 8–9

CASE 3: $c = 0$.

In this case, the equation of the line is $ax + by = 0$. (See FIGURE 8–10.) This line goes through the origin (Why?)

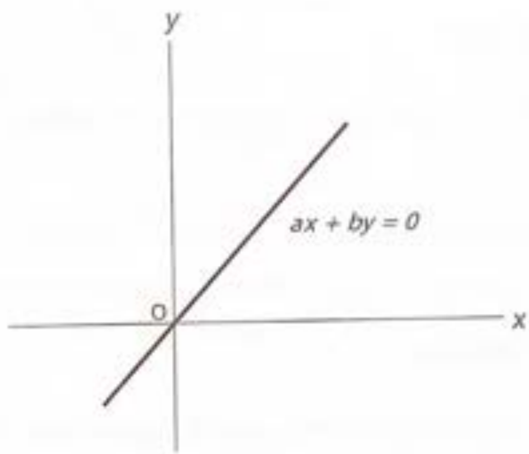


FIGURE 8–10

8.3.3 Slope–Point Form

Determining a line by slope and one point. The equation of a line with slope m that passes through the point $P_1(x_1, y_1)$ is $y - y_1 = m(x - x_1)$.

PROOF: Recall that the slope of a line is independent of the two distinct points we choose on the line to calculate it.

Let $P(x, y)$ be any point on the line other than $P_1(x_1, y_1)$. Then $\frac{y-y_1}{x-x_1} = m$.

Multiplying by $x - x_1$ we get: $y - y_1 = m(x - x_1)$.

8.3.4 Mutual Positions of Lines

If we are given two lines by their equations, can we determine whether the lines are parallel to each other? Can we determine whether they are perpendicular to each other? The following two theorems answer these questions.

THEOREM 4: Let l_1 and l_2 be two distinct lines, whose equations are $y = m_1x + d_1$ and $y = m_2x + d_2$, respectively. l_1 and l_2 are parallel if and only if $m_1 = m_2$.

PROOF: Let $m_1 = m_2 = m$. Then, the equations of l_1 and l_2 are, respectively, $y = mx + d_1$ and $y = mx + d_2$.

Since the two lines are different, $d_1 \neq d_2$.

Clearly, there is no ordered pair (x, y) that satisfies both equations (why?).

Hence, l_1 and l_2 have no point in common.

This proves that l_1 and l_2 are parallel.

Now let l_1 and l_2 be two distinct parallel lines. Therefore, the equations of these lines do not have any common solution.

Therefore, when subtracting the two equations, we get that the equation $0 = (m_2 - m_1)x + (d_2 - d_1)$ has no solution.

Since this equation has no solution, $m_1 = m_2$.

This is so because if $m_1 \neq m_2$, then this equation does have a solution. It is $x = \frac{d_1 - d_2}{m_2 - m_1}$.

THEOREM 5: Let l_1 and l_2 be two lines with non-zero slopes whose equations are $y = m_1x + d_1$ and $y = m_2x + d_2$, respectively. l_1 and l_2 are perpendicular to each other if and only if $m_2 = -\frac{1}{m_1}$.

PROOF: Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two distinct points on l_1. Then, the slope of l_1 is

$m_1 = \frac{y_2 - y_1}{x_2 - x_1}$ (Note that $x_2 - x_1 \neq 0$.)

We have seen that the perpendicular bisector l' of P_1P_2 has the equation $2(x_2 - x_1)x + 2(y_2 - y_1)y = x_2^2 + y_2^2 - x_1^2 - y_1^2$.

From here, the slope of l' is $m' = -\frac{2(x_2 - x_1)}{2(y_2 - y_1)} = -\frac{x_2 - x_1}{y_2 - y_1} = -\frac{1}{m_1}$.

From geometry, we can conclude that l_1 is perpendicular to l_2 if and only if l' is parallel to l_2.

On the other hand, l' is parallel to l_2 if and only if $m_2 = m_2' = -\frac{1}{m_1}$.

8.4 CIRCLES

A circle is the set of all points in the plane that are equidistant from a given point in the plane. What is the equation of the circle? Let's begin with a simple case where a circle with radius r is centered at the origin. By the Pythagorean Theorem (see FIGURE 8–11), for any point $A(x, y)$ on the circle, $x^2 + y^2 = r^2$.

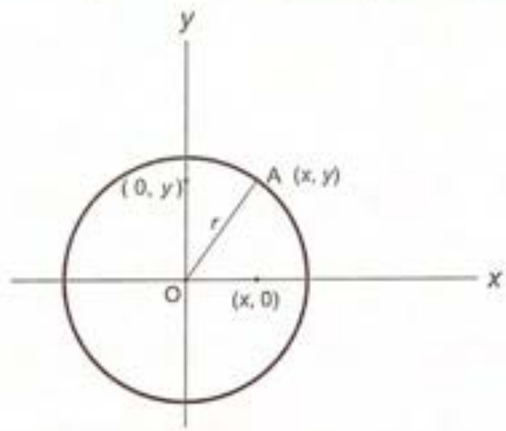


FIGURE 8–11

Thus, $x^2 + y^2 = r^2$ is the equation of a circle with radius r centered at the origin.

Now let's take a circle with a center at any arbitrary point $C(m, n)$ in the plane, and let r be the radius of the circle. (See FIGURE 8–12.) By the distance formula, $(x-m)^2 + (y-n)^2 = r^2$

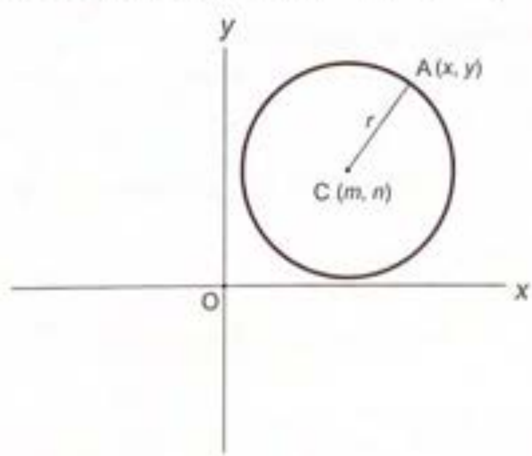


FIGURE 8–12

Thus, $(x-m)^2+(y-n)^2=r^2$ is the equation of a circle with radius r centered at $C(m,n)$.

8.5 SOLVING GEOMETRY PROBLEMS USING COORDINATE GEOMETRY

THEOREM: The diagonals of a parallelogram bisect each other.

PROOF: We are given a parallelogram $ABCD$. To prove this theorem using coordinate geometry, we must first introduce a coordinate system. We choose the coordinate *axes* so that AB is on the x-axis and the point A is at the origin (see FIGURE 8–13).

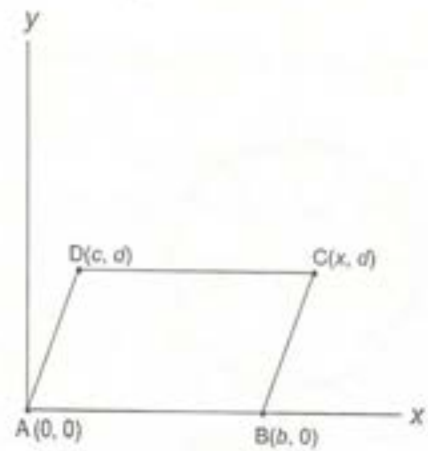


FIGURE 8–13

Therefore, $B=(b,0)$.

Let $D=(c,d)$.

Since CD is parallel to AB, the y-coordinate of C is d.

Let $C=(x,d)$.

To find x, we use the fact that BC is parallel to AD. We will consider two cases:

CASE 1. AD and BC are not parallel to the y-axis.

Since they are parallel to each other, their slopes must be equal. Hence, $\frac{d}{c}=\frac{d}{x-b}$.

From this we get $x=b+c$.

CASE 2. AD and BC are parallel to the y-axis.

In this case, $c = 0$ and $x = b = b + c$.

So, in both cases we get $C = (b + c, d)$.

To show that the diagonals bisect each other, it is sufficient to show that the midpoint of AC is the same as the midpoint of BD.

The midpoint M_1 of AC has the coordinates: $x = \frac{1}{2} \cdot (0 + (b + c)) = \frac{1}{2}(b + c)$, $y = \frac{1}{2}(0 + d) = \frac{1}{2}d$.

Therefore, $M_1 = \left(\frac{1}{2}(b + c), \frac{1}{2}d\right)$.

The midpoint M_2 of BD has the coordinates: $x = \frac{1}{2} \cdot (b + c) = \frac{1}{2}(b + c)$, $y = \frac{1}{2}(0 + d) = \frac{1}{2}d$.

Therefore, $M_2 = \left(\frac{1}{2}(b + c), \frac{1}{2}d\right)$.

Hence: $M_1 = M_2$.

This implies that the diagonals of a parallelogram bisect each other.

THEOREM: The three perpendicular bisectors of the sides of a triangle are concurrent.

PROOF: Let ABC be any triangle positioned in a coordinate system as shown in FIGURE 8–14:

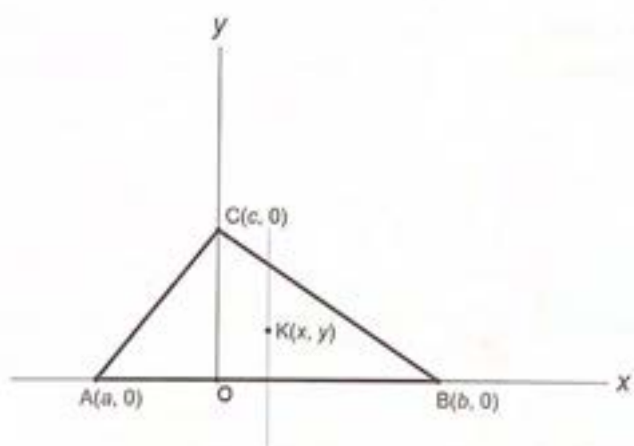


FIGURE 8–14

$A = (a, 0)$, $B = (b, 0)$, $C = (0, c)$.

Let us find the equation of the perpendicular bisector of AB.

Take any point $K = (x, y)$ on the perpendicular bisector of AB. Then, since it is equidistant from A and B, we have: $(x - a)^2 + (y - 0)^2 = (x - b)^2 + (y - 0)^2$

Opening parentheses and simplifying, we get:

(1) $x=\frac{1}{2}(a+b)$.

The equations of the other two perpendicular bisectors can be found in a similar way. They are $2ax-2cy=a^2-c^2$ and $2bx-2cy=b^2-c^2$.

To find the point of intersection of the last two bisectors, we solve their system of equations by subtracting the second from the first. We get:

$2ax-2bx=a^2-b^2$,

$2x(a-b)=a^2-b^2$,

(2) $x=\frac{1}{2}(a+b)$.

From (1) and (2) we get that the point of intersection lies on the first perpendicular bisector. Hence, the three perpendicular bisectors are concurrent.

THEOREM: The three medians of a triangle are concurrent, and the distance from any vertex to the intersection point is two-thirds the length of the median drawn from that vertex.

PROOF: Let $P_1P_2P_3$ be a triangle in a coordinate system as shown in Figure 8–15:
$P_1=(x_1,y_1)$, $P_2=(x_2,y_2)$, $P_3=(x_3,y_3)$.

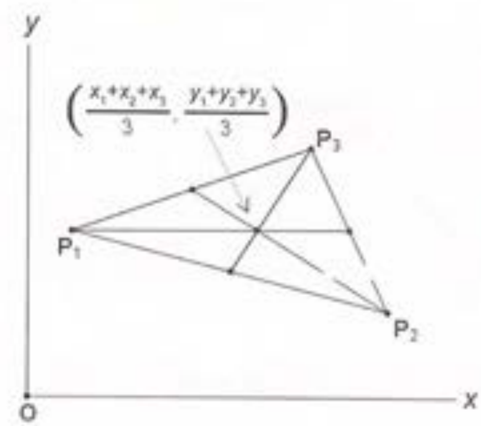


FIGURE 8–15

The midpoint of P_1P_2 has the coordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Thus, the point that is two-thirds the way down the median from P_3 has the coordinates

$$x=\frac{1\cdot x_3+2\cdot\frac{1}{2}(x_1+x_2)}{3}=\frac{x_1+x_2+x_3}{3},\ y=\frac{1\cdot y_3+2\cdot\frac{1}{2}(y_1+y_2)}{3}=\frac{y_1+y_2+y_3}{3}.$$

To find the corresponding point for P_1, we can simply interchange indices. (Replace 3 by 1, 1 by 2, and 2 by 3.) We do the same for P_2.

But the above expressions for x and y are invariant under a permutation of the indices. This proves the theorem.

Section 8
EXERCISES

1. Which of the given points $A(3,-4)$, $B(1,0)$, $C(0,5)$, $D(0,0)$, and $E(0,1)$ belong to the following circles?
   a. $(x-1)^2+(y+3)^2=9$
   b. $x^2+y^2=25$
2. Find which of the following equations represent a circle. If it is a circle, identify the radius and the center of the circle.
   a. $x^2+y^2+8x-4y+40=0$
   b. $x^2+y^2-4x-2y=-1$
3. The vertices of a triangle ABC are $A(4,6)$, $B(-4,0)$, $C(-1,-4)$. Find the equation of the median CM.
4. What are the values of a and b in the equation of the line $ax+by=1$ going through the points $(1,2)$ and $(2,1)$?
5. Three vertices of a parallelogram $ABCD$ are $A(1,0)$, $B(2,3)$, $C(3,2)$. Find the coordinates of the fourth vertex, and find the coordinates of the point of intersection of the diagonals.
6. The distance of the point A from the x-*axis* is q_1, and its distance from the y-axis is q_2, and its distance from the point $B(3,6)$ is q_3. Find the coordinates of A if $q_1=q_2=q_3$.

7. Find the point that is one-fourth of the distance from the point $A(-1, 2)$ to the point $B(6, 7)$ along the segment AB.

8. Find the point that is five-sevenths of the distance from the point $A(-\frac{3}{2}, 2)$ to the point $B(2, 2)$ along the segment AB.

Section 9

Trigonometric Functions

Trigonometry means triangle measurement. Trigonometry is used to compute the lengths of segments and measures of angles, and therefore it is an important tool in many branches of science, such as astronomy, geography, and navigation. To illustrate how trigonometry is used in measurement, let's consider the following example.

In FIGURE 9-1, AD is the height of a mountain sitting on a plane. There is no direct way to measure the height of the mountain. However, with a special instrument, we can measure the angle in which we see the summit A of the mountain from a point B on the plane. Let's denote the measure of this angle by β. Now, along the segment BD we move a meters toward the mountain, and again measure the angle in which we see the summit A. Let's denote the measure of this angle by γ. The numbers β, γ, and a are sufficient to compute the height AD. In this section of the resource guide, you will learn trigonometric tools for finding measurements of sides and angles of a triangle.

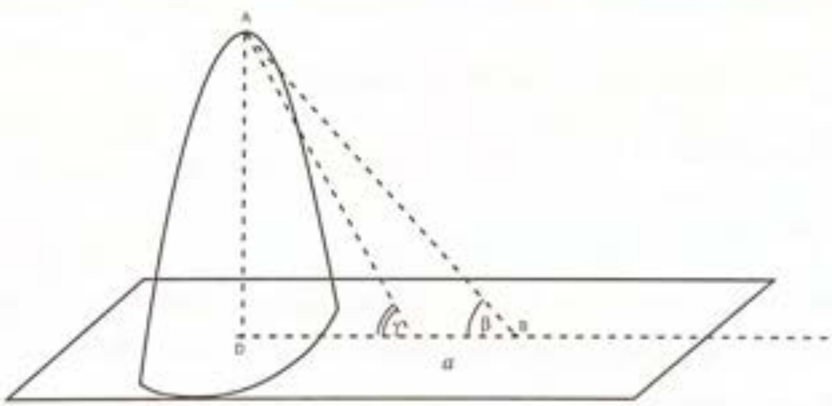


FIGURE 9–1

We will begin with the trigonometric functions of an acute angle.

9.1 THE SINE FUNCTION FOR ACUTE ANGLES

Take an acute angle α and position it in a coordinate system so that its vertex is at the origin O and one of its sides is on the positive x-axis. Let $P(x, y)$ be a point on the second side of the angle, and let r be the distance between P and the origin O. (See FIGURE 9–2.)

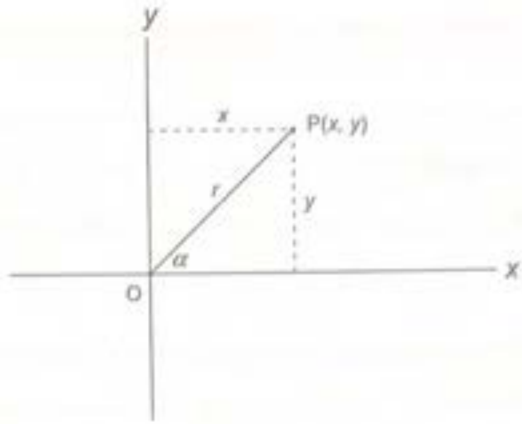


FIGURE 9–2

As α varies from $0°$ to $90°$, the ratio $\frac{y}{r}$ also varies. Moreover, as α gets larger, $\frac{y}{r}$ gets larger, and as α gets smaller, $\frac{y}{r}$ gets smaller. The ratio $\frac{y}{r}$ is called the ***sine of*** α and is written as $\sin\alpha$.

Since α is acute, and y is positive, the $\sin\alpha$ is therefore positive. And, since y is smaller than r, $\sin\alpha$ is smaller than 1.

We summarize this as follows: For $0° < \alpha < 90°$, $0 < \sin\alpha < 1$.

At $90°$, $y = r$. Hence: $\sin 90° = 1$.

At $0°$, $y = 0$. Hence: $\sin 0° = 0$.

Is $\sin\alpha$ a function? As defined, there is no guarantee that $\sin\alpha$ is a function, or in other words there is no guarantee that for any input α, there exists exactly one output $\sin\alpha$. Thus, to insure that $\sin\alpha$ is a function, we must prove the following theorem, which asserts that the value $\sin\alpha$ is independent of the choice of the point $P(x, y)$ on the second side of the angle.

THEOREM: Let angle α be an acute angle such that its vertex is at the origin and one of its sides is on the positive x-axis, as is shown in FIGURE 9–3. Let $P(x, y)$ and $P'(x', y')$ be two distinct points on the non-horizontal side of the angle. Call $OP = r$ and $OP' = r'$. Then, $\frac{y}{r} = \frac{y'}{r'}$.

PROOF: Drop two perpendiculars PC and PC' as shown in FIGURE 9–3.

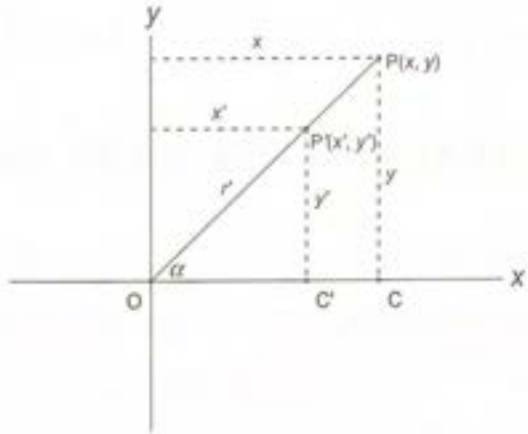


FIGURE 9–3

Clearly, $\triangle OPC$ and $\triangle OP'C'$ are similar. Hence, $\frac{y}{r} = \frac{y'}{r'}$.

One of the important applications of the sine function is expressed in the following theorem:

THEOREM: The area of any triangle is one-half the product of two sides of the triangle and the sine of the angle between them. That is, the area of $\triangle ABC$ is given by the formula: $area(\triangle ABC) = \frac{1}{2}\overline{AB} \cdot \overline{AC} \cdot \sin\alpha$.

PROOF: We will prove this theorem for the case where α is acute. The theorem, however, is also valid when α is not acute.

The area of any triangle ABC (see FIGURE 9–4) is half the product of a side of the triangle and the altitude to that side. Hence, $area(\triangle ABC) = \frac{1}{2}\overline{AB} \cdot \overline{CH}$.

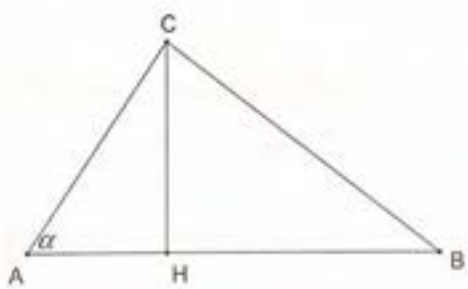


FIGURE 9–4

Point C is at distance $\overline{CH}$ from the line AB. This distance can be seen from the angle $\measuredangle CAB$ (let's call it α). Therefore, ratio $\frac{\overline{CH}}{\overline{AC}} = \sin\alpha$. From here, $\overline{CH} = \overline{AC} \cdot \sin\alpha$. Substituting the value of $\overline{CH}$ into the initial formula, we get $area(\triangle ABC) = \frac{1}{2}\overline{AB}\cdot\overline{CH} = \frac{1}{2}\overline{AB}\cdot\overline{AC}\cdot\sin\alpha$.

Therefore, $area(\triangle ABC) = \frac{1}{2}\overline{AB}\cdot\overline{AC}\cdot\sin\alpha$.

9.2 THE TANGENT FUNCTION FOR ACUTE ANGLES

As with the sine function, take an acute angle α and position it in a coordinate system so that its vertex is at the origin O and one of its sides is on the positive x-axis. Let $P(x, y)$ be a point on the second side of the angle. (See FIGURE 9–5.)

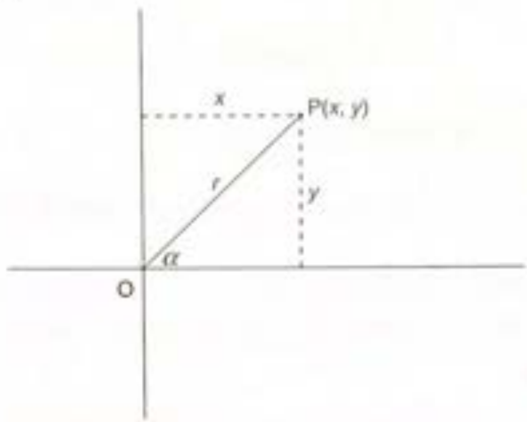


FIGURE 9–5

As α varies from $0°$ to $90°$, the ratio $\frac{y}{x}$ also varies but always remains positive. Moreover, as α gets larger, y gets larger and x gets smaller, and so $\frac{y}{x}$ gets larger. As α gets smaller, y gets smaller and x gets larger, and so $\frac{y}{x}$ gets smaller.

The ratio $\frac{y}{x}$ is called the ***tangent of*** α, and is written as $\tan\alpha$. Indeed $\tan\alpha$ is the slope of the line going through the origin O and the point P. For $0 < \alpha < 90$, $0 < \tan\alpha < \infty$.

As with the sine function, here too we must guarantee that $\tan\alpha$ is a function. That is, we must show that the value of $\tan\alpha$ is independent of the choice of the point $P(x, y)$ on the second side of the angle.

THEOREM: Let angle α be an acute angle such that its vertex is at the origin and one of its sides is on the positive x-axis, as is shown in FIGURE 9–6. Let $P(x, y)$ and $P'(x', y')$ be two distinct points on the non-horizontal side of the angle. Then $\frac{y}{x} = \frac{y'}{x'}$.

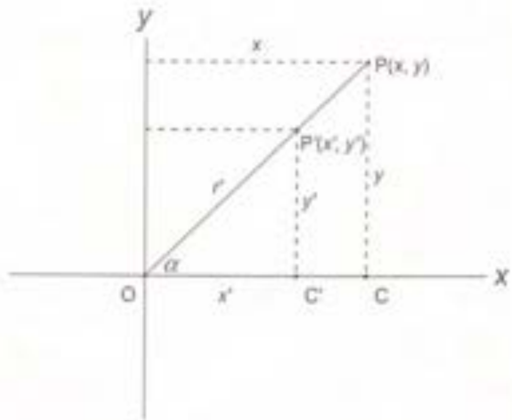


FIGURE 9–6

PROOF: Drop two perpendiculars PC and PC' as shown in FIGURE 9–6. Clearly, $\triangle OPC$ and $\triangle OP'C'$ are similar. Hence, $\frac{y}{x} = \frac{y'}{x'}$.

9.3 THE COSINE AND COTANGENT FUNCTIONS FOR ACUTE ANGLES

In addition to the functions sine and tangent, we will define two additional trigonometric functions. The four functions are needed to write trigonometric formulas more effectively.

The ***cosine*** of an angle $0° < \alpha < 90°$ is denoted by $\cos\alpha$ and is defined as follows: $\cos\alpha = \sin(90° - \alpha)$.

The ***cotangent*** of an angle $0° < \alpha < 90°$ is denoted by $\cot\alpha$ and is defined as follows: $\cot\alpha = \tan(90° - \alpha)$

From here, it is easy to see that $\sin\alpha = \cos(90° - \alpha)$ and that $\tan\alpha = \cot(90° - \alpha)$

Here is why:

By the definition of cosine, $\cos(90° - \alpha) = \sin(90° - (90° - \alpha)) = \sin\alpha$.

By the definition of cotangent, $\cot(90° - \alpha) = \tan(90° - (90° - \alpha)) = \tan\alpha$.

Thus, in a right triangle ABC (see FIGURE 9–7), we have the following relations:

1. $\sin A = \frac{a}{c} = \cos B$
2. $\cos A = \frac{b}{c} = \sin A$
3. $\tan A = \frac{a}{b} = \cot B$
4. $\cot A = \frac{b}{a} = \tan B$

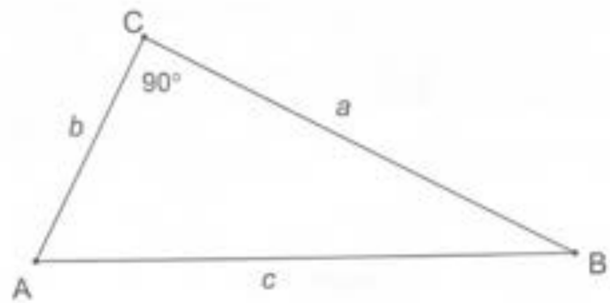


FIGURE 9–7

9.4 RELATIONS AMONG TRIGONOMETRIC FUNCTIONS

There are six basic relations among trigonometric functions. They are referred to as basic **identities**:

1. $\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$
2. $\cot \alpha = \frac{\cos \alpha}{\sin \alpha}$
3. $\tan \alpha \cdot \cot \alpha = 1$
4. $\sin^2 \alpha + \cos^2 \alpha = 1$
5. $\tan^2 \alpha + 1 = \frac{1}{\cos^2 \alpha}$, $\cos \alpha \neq 0$
6. $1 + \cot^2 \alpha = \frac{1}{\sin^2 \alpha}$, $\sin \alpha \neq 0$

Often $\frac{1}{\sin \alpha}$ is denoted by $\csc \alpha$ and $\frac{1}{\cos \alpha}$ by $\sec \alpha$.

Thus identities 5 and 6 can be written as:

7. $\tan^2\alpha + 1 = \sec^2\alpha$

8. $1 + \cot^2\alpha = \csc^2\alpha$.

To prove these formulas for an acute angle α, let $\triangle ABC$ be a right triangle in C with the acute angle $\measuredangle BAC = \alpha$. Let $AC = b$, $CB = a$, and $AB = c$, as shown in FIGURE 9–8.

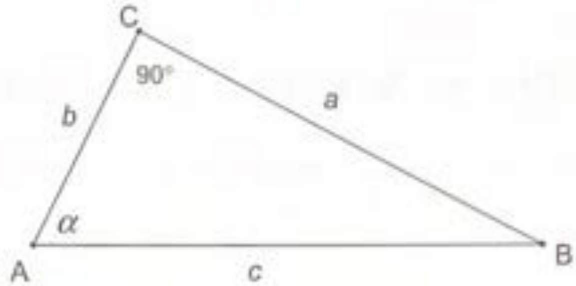


FIGURE 9–8

1. $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$

PROOF: Since $\sin\alpha = \dfrac{a}{c}$ and $\cos\alpha = \dfrac{b}{c}$, the ratio $\dfrac{\sin\alpha}{\cos\alpha} = \dfrac{a}{c} : \dfrac{b}{c} = \dfrac{a}{b}$. By definition, $\dfrac{a}{b} = \tan\alpha$. Therefore, $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$.

2. $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$

PROOF: Since $\cos\alpha = \dfrac{b}{c}$, we have: $\dfrac{\cos\alpha}{\sin\alpha} = \dfrac{b}{c} : \dfrac{a}{c} = \dfrac{b}{a}$

Since $\dfrac{b}{a} = \cot\alpha$, we get $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$.

3. $\tan\alpha \cdot \cot\alpha = 1$

PROOF: By definition, $\cot\alpha = \dfrac{b}{a}$ and $\tan\alpha = \dfrac{a}{b}$. Hence: $\tan\alpha \cdot \cot\alpha = \dfrac{a}{b} \cdot \dfrac{b}{a} = 1$.

4. $\sin^2\alpha + \cos^2\alpha = 1$

PROOF: By the Pythagorean Theorem, in a right triangle ABC, we have $a^2 + b^2 = c^2$.

Dividing both sides of the equality by the non-zero c^2, we get $\dfrac{a^2}{c^2} + \dfrac{b^2}{c^2} = \dfrac{c^2}{c^2} = 1$

Since $\dfrac{a}{c} = \sin\alpha$ and $\dfrac{b}{c} = \cos\alpha$, we get: $\sin^2\alpha + \cos^2\alpha = 1$.

5. $\tan^2\alpha + 1 = \dfrac{1}{\cos^2\alpha}$

PROOF: $\tan^2\alpha + 1 = \left(\dfrac{\sin\alpha}{\cos\alpha}\right)^2 + 1 = \dfrac{\sin^2\alpha}{\cos^2\alpha} + 1 = \dfrac{\sin^2\alpha + \cos^2\alpha}{\cos^2\alpha} = \dfrac{1}{\cos^2\alpha}$.

6. $\cot^2\alpha + 1 = \dfrac{1}{\sin^2\alpha}$

PROOF: $\cot^2\alpha + 1 = \left(\dfrac{\cos\alpha}{\sin\alpha}\right)^2 + 1 = \dfrac{\cos^2\alpha}{\sin^2\alpha} + 1 = \dfrac{\cos^2\alpha + \sin^2\alpha}{\sin^2\alpha} = \dfrac{1}{\sin^2\alpha}$.

9.5 TRIGONOMETRIC FUNCTIONS OF SPECIAL ANGLES

We will now show how the identities we just proved can be used to compute the trigonometric functions for the angles 30°, 45°, 60°, and 90°.

First, let's determine the values of all trigonometric functions for 30° and 60°. For this, consider a right triangle ABC ($AC = b$, $CB = a$, and $AB = c$) with acute angles $\angle A = \alpha = 30°$ and $\angle B = \beta = 60°$. From geometry, we know that the hypotenuse c is twice the side opposite the 30°. Hence, $\dfrac{a}{c} = \dfrac{1}{2}$, and so: $\sin 30° = \dfrac{1}{2}$.

Using the identity $\sin^2\alpha + \cos^2\alpha = 1$, it is easy to get $\cos 30° = \dfrac{\sqrt{3}}{2}$.

We have $\cos 30° = \sqrt{1 - \sin^2 30°} = \sqrt{1 - \dfrac{1}{4}} = \sqrt{\dfrac{3}{4}} = \dfrac{\sqrt{3}}{2}$.

Since $\sin 30° = \cos 60°$, $\cos 60° = \dfrac{1}{2}$. Also, $\sin 60° = \cos 30°$, $\cos 30° = \dfrac{\sqrt{3}}{2}$.

Using the identity $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$, we compute $\tan 30° = \dfrac{\sin 30°}{\cos 30°} = \dfrac{1}{2} : \dfrac{\sqrt{3}}{2} = \dfrac{1}{\sqrt{3}}$.

And

$\tan 60° = \dfrac{\sin 60°}{\cos 60°} = \dfrac{\sqrt{3}}{2} : \dfrac{1}{2} = \sqrt{3}$.

Using the identity $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$, we get $\cot 30° = \dfrac{\cos 30°}{\sin 30°} = \dfrac{\sqrt{3}}{2} : \dfrac{1}{2} = \sqrt{3}$.

And

$\cot 60° = \dfrac{\cos 60°}{\sin 60°} = \dfrac{1}{2} : \dfrac{\sqrt{3}}{2} = \dfrac{1}{\sqrt{3}}$.

Now let's compute the values of all trigonometric functions for the angle 45°. For this we take a right isosceles (i.e., having two equal sides) triangle ABC ($AC = a$, $CB = a$, and $AB = c$) with $\beta = \alpha = 45°$.

By the Pythagorean Theorem, $a^2 + a^2 = c^2$. Hence: $2a^2 = c^2$.

From this, we get $\frac{a}{c} = \frac{1}{\sqrt{2}}$.

Since $\frac{a}{c} = \sin 45° = \cos 45°$, we have: $\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}$.

And

$$\tan 45° = \cot 45° = \frac{1}{\sqrt{2}} : \frac{1}{\sqrt{2}} = 1.$$

We will now conclude with the values of the trigonometric functions for angle 90°. We have already seen that: $\sin 90° = 1$ and $\sin 0° = 0$.

From these two facts, we can calculate the rest:

$$\cos 90° = \sqrt{1 - \sin^2 90°} = \sqrt{1-1} = 0.$$

$$\tan 90° = \frac{\sin 90°}{\cos 90°} = \frac{1}{0} = \infty.$$

$$\cot 90° = \frac{\cos 90°}{\sin 90°} = \frac{0}{1} = 0.$$

The following table summarizes the values of the special angles:

α	30°	45°	60°	90°
$\sin\alpha$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\alpha$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\alpha$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–
$\cot\alpha$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

9.6 TRIGONOMETRIC FUNCTIONS OF ANGLES OF ANY MEASURE

9.6.1 Definitions and Properties

The definitions of the trigonometric functions for acute angles we learned in the previous sections are extendable to angles of any measure. As before, we take any angle α and position it in a coordinate system such that its vertex is at the origin O and one of its sides is on the positive x-axis. Let $P(x, y)$ be a point on the second side of the angle, and let $PO = r$. Here we think of the angle α as the angle generated by moving counterclockwise from the positive x-axis to the segment PO. See FIGURE 9–9.

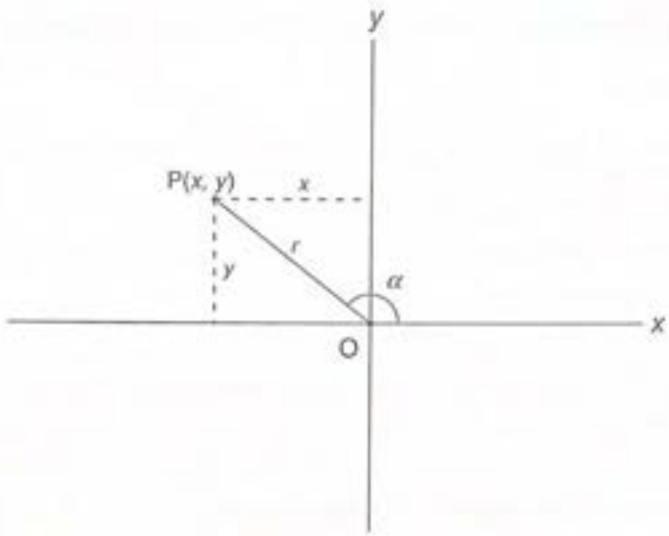


FIGURE 9–9

The definitions of the four trigonometric functions for such an angle α are as follows:

1. $\sin\alpha = \dfrac{y}{r}$
2. $\cos\alpha = \dfrac{x}{r}$
3. $\tan\alpha = \dfrac{y}{x},\ x \neq 0$
4. $\cot\alpha = \dfrac{x}{y},\ y \neq 0$

From these definitions, it is easy to derive the following identities:

- $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$

- $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$

Since r is positive, the sign of the sine function (that is, whether the function is positive or negative) is determined by the sign of y, whereas the sign of cosine is determined by the sign of x.

- For $0 < \alpha < 90°$, $y > 0$ and $x > 0$. Hence: $\sin\alpha > 0$ and $\cos\alpha > 0$
- For $90° < \alpha < 180°$, $y > 0$ and $x < 0$. Hence: $\sin\alpha > 0$ and $\cos\alpha < 0$
- For $180° < \alpha < 270°$, $y < 0$ and $x < 0$. Hence: $\sin\alpha < 0$ and $\cos\alpha < 0$
- For $270° < \alpha < 360°$, $y < 0$ and $x > 0$. Hence: $\sin\alpha < 0$ and $\cos\alpha > 0$

In addition, the following table gives the values of sine and cosine for $\alpha = 0°, 90°, 180°, 270°, 360°$:

α	0°	90°	180°	270°	360°
$\sin\alpha$	0	1	0	-1	0
$\cos\alpha$	1	0	-1	0	1

As for $\tan\alpha$ and $\cot\alpha$, these functions are not defined for all angles. $\tan\alpha$ is not defined for $\alpha = 90°$ and $\alpha = 270°$ because for these values $x = 0$; $\cot\alpha$ is not defined for $\alpha = 0°$, $\alpha = 180°$, and $\alpha = 360°$ because for these values $y = 0$.

From the definitions of tangent and cotangent we get:

- For $0° < \alpha < 90°$, $\tan\alpha > 0$ and $\cot\alpha > 0$
- For $90° < \alpha < 180°$, $\tan\alpha < 0$ and $\cot\alpha < 0$
- For $180° < \alpha < 270°$, $\tan\alpha > 0$ and $\cot\alpha > 0$
- For $270° < \alpha < 360°$, $\tan\alpha < 0$ and $\cot\alpha < 0$

In addition, the following table gives the values of tangent and cotangent for $\alpha = 0°, 90°, 180°, 270°, 360°$:

α	0°	90°	180°	270°	360°
$\tan\alpha$	0	–	0	–	0
$\cot\alpha$	–	0	–	0	–

9.6.2 Negative Angles

Using the above identities, we can determine the values of trigonometric functions for negative angles. As before, we take any angle α and position it in a coordinate system so that its vertex is at the origin O and one of its sides is on the positive x-axis. And, as before, we let $P(x, y)$ be a point on the second side of the angle, and $PO = r$. Here, however, we think of the angle α as the angle generated by moving clockwise from the positive x-axis to the segment PO. In the exercises at the end of this section, you will prove the following relations using identities we will learn in this section:

1. $\sin(-\alpha) = -\sin\alpha$
2. $\cos(-\alpha) = \cos\alpha$
3. $\tan(-\alpha) = -\tan\alpha$
4. $\cot(-\alpha) = -\cot\alpha$

9.7 TRIGONOMETRIC IDENTITIES

9.7.1 Sum and Difference Identities

The following identities are helpful in computing the measure of many angles:

1. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
2. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
3. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
4. $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Proofs of Identities 1 and 2:

These identities hold for α and β of any measure. Here we will prove them only for the case where $0 < \alpha + \beta < 90^\circ$.

We first construct a figure like that shown in FIGURE 9–10.

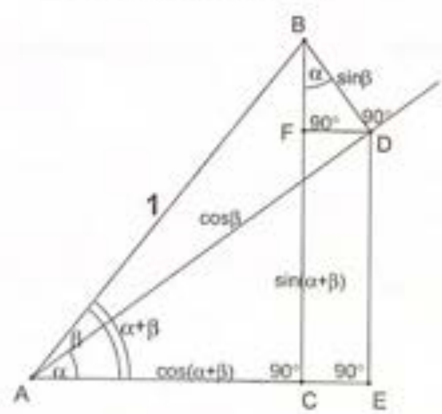


FIGURE 9–10

In this figure we have:

$BC = \sin(\alpha+\beta)$

$AC = \cos(\alpha+\beta)$

$BD = \sin\beta$

$AD = \cos\beta$

$\sin(\alpha+\beta) = \overline{BC} = \overline{BF} + \overline{DE}$

$\cos(\alpha+\beta) = \overline{AC} = \overline{AE} - \overline{FD}$

$\overline{BF} = \overline{BD}\cdot\cos\alpha = \sin\beta\cos\alpha$

$\overline{DE} = \overline{AD}\cdot\sin\alpha = \cos\beta\sin\alpha$

$\overline{AE} = \overline{AD}\cdot\cos\alpha = \cos\beta\cos\alpha$

$\overline{FD} = \overline{BD}\cdot\sin\alpha = \sin\beta\sin\alpha$

Therefore, $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

and

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

This completes the proofs for Identities 1 and 2.

Proofs of Identities 3 and 4:

We will now prove Identities 3 and 4:

Let $\alpha = (\alpha-\beta)+\beta$. Then,

$\sin\alpha = \sin((\alpha-\beta)+\beta) = \sin(\alpha-\beta)\cos\beta + \cos(\alpha-\beta)\sin\beta$

$\cos\alpha = \cos((\alpha-\beta)+\beta) = \cos(\alpha-\beta)\cos\beta - \sin(\alpha-\beta)\sin\beta$.

We treat these two equations as the system of equations with two unknowns $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$.

Multiplying the first equation by $\cos\beta$, and the second by $-\sin\beta$ and adding, we get:

$\sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)\cdot(\cos^2\beta+\sin^2\beta) = \sin(\alpha-\beta)\cdot 1 = \sin(\alpha-\beta)$.

This completes the proof for Identity 3.

To prove Identity 4, solve the system of equations 1 and 2 for $\cos(\alpha-\beta)$. To do this, multiply the first equation by $\sin\beta$, and the second equation by $\cos\beta$, and then add them. You will get: $\sin\alpha\sin\beta+\cos\alpha\cos\beta=\cos(\alpha-\beta)\cdot(\cos^2\beta+\sin^2\beta)=\cos(\alpha-\beta)\cdot 1=\cos(\alpha-\beta)$.

The following identities are also useful:

1. $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}$, where $\alpha+\beta\neq(1+2n)90°$, $\alpha\neq(1+2n)90°$, $\beta\neq(1+2n)90°$, n is an integer.

2. $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$ where $\alpha-\beta\neq(1+2n)90°$, $\alpha\neq(1+2n)90°$, $\beta\neq(1+2n)90°$, n is an integer.

PROOF:

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}=\frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}=$$

$$=\frac{(\tan\alpha)\cdot 1+1\cdot(\tan\beta)}{1-\tan\alpha\tan\beta}=\frac{(\tan\alpha)+(\tan\beta)}{1-\tan\alpha\tan\beta}.$$

The proof for the second identity is similar.

EXAMPLE 9.7a: Find $\cos 135°$

SOLUTION:

$$\cos 135°=\cos(90°+45°)=\cos 90°\cos 45°-\sin 90°\sin 45°=0\cdot\frac{1}{\sqrt{2}}-1\cdot\frac{1}{\sqrt{2}}=-\frac{1}{\sqrt{2}}.$$

EXAMPLE 9.7b: Find $\tan 210°$

SOLUTION:

$$\tan 210°=\tan(180°+30°)=\frac{\tan 180°+\tan 30°}{1-\tan 180°\cdot\tan 30°}=\frac{0+\dfrac{1}{\sqrt{3}}}{1-0\cdot\dfrac{1}{\sqrt{3}}}=\frac{1}{\sqrt{3}}.$$

EXAMPLE 9.7c: Find $\sin 300°$

SOLUTION:

$\sin 300° = \sin(270° + 30°) = \sin 270° \cos 30° + \cos 270° \sin 30° = (-1)\cdot\frac{\sqrt{3}}{2} + 0\cdot\frac{1}{2} = -\frac{\sqrt{3}}{2}$.

EXAMPLE 9.7d: Find $\cot 75°$

SOLUTION:

$\cot 75° = \cot(45° + 30°) = \frac{1 - \tan 45° \tan 30°}{\tan 45° + \tan 30°} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$.

Summarizing: we have the following **Sum and Difference Identities:**

1. $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
2. $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
3. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
4. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
5. $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$, where $\alpha + \beta \neq (1 + 2n)90°$, $\alpha \neq (1 + 2n)90°$, $\beta \neq (1 + 2n)90°$, n is an integer.
6. $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta}$, where $\alpha - \beta \neq (1 + 2n)90°$, $\alpha \neq (1 + 2n)90°$, $\beta \neq (1 + 2n)90°$, n is an integer.

9.7.2. Double-Angle Identities

From the above identities, it is easy to prove the following identities. You can complete the proofs of these identities as an exercise on your own:

1. $\sin 2\alpha = 2\sin\alpha\cos\alpha$
2. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

3. $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

EXAMPLE 9.7e: Given that $\sin\alpha = \dfrac{4}{5}$ where $0 < \alpha < 90°$, find $\sin 2\alpha$, $\cos 2\alpha$, $\tan 2\alpha$ and $\cot 2\alpha$.

SOLUTION:

$\cos\alpha = \pm\sqrt{1-\sin^2\alpha} = \pm\sqrt{1-\left(\dfrac{4}{5}\right)^2} = \pm\sqrt{\dfrac{9}{25}} = \pm\dfrac{3}{5}$.

Since $0° < \alpha < 90°$, we take $\cos\alpha$ positive, and so $\cos\alpha = \dfrac{3}{5}$.

From here, we have $\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\cdot\dfrac{4}{5}\cdot\dfrac{3}{5} = \dfrac{24}{25}$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \left(\dfrac{3}{5}\right)^2 - \left(\dfrac{4}{5}\right)^2 = -\dfrac{7}{25}$

To find $\tan 2\alpha$ and $\cot 2\alpha$, we first find $\tan\alpha$ and $\cot\alpha$:

$\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{\frac{4}{5}}{\frac{3}{5}} = \dfrac{4}{3}$.

$\cot\alpha = \dfrac{1}{\tan\alpha} = \dfrac{1}{\frac{4}{3}} = \dfrac{3}{4}$

$\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \dfrac{2\cdot\frac{4}{3}}{1-\left(\frac{4}{3}\right)^2} = \dfrac{\frac{8}{3}}{-\frac{7}{9}} = -\dfrac{24}{7}$

$\cot 2\alpha = \dfrac{1}{\tan 2\alpha} = -\dfrac{7}{24}$.

Thus, if $\sin\alpha = \dfrac{4}{5}$ where $0 < \alpha < 90°$, then $\sin 2\alpha = \dfrac{24}{25}$, $\cos 2\alpha = -\dfrac{7}{25}$, $\tan 2\alpha = -\dfrac{24}{7}$, $\cot 2\alpha = -\dfrac{7}{24}$.

EXAMPLE 9.7f: Given that $\cos 2\alpha = \frac{4}{5}$ and $90° < \alpha < 180°$, find $\tan\alpha + \cot\alpha$.

SOLUTION: Recall that $\tan^2 2\alpha + 1 = \frac{1}{\cos^2 2\alpha}$. Hence:

$$\tan 2\alpha = \pm\sqrt{\frac{1-\cos^2 2\alpha}{\cos^2 2\alpha}} = \pm\sqrt{\frac{1-\left(\frac{4}{5}\right)^2}{\left(\frac{4}{5}\right)^2}} = \pm\sqrt{\frac{9}{16}} = \pm\frac{3}{4}.$$

Since $90° < \alpha < 180°$, so $180° < 2\alpha < 360°$, but $\cos 2\alpha > 0$, therefore $270° < 2\alpha < 360°$ and $\tan 2\alpha = -\frac{3}{4}$.

To find $\tan\alpha$, we use the identity $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$.

We have: $-\frac{3}{4} = \frac{2\tan\alpha}{1-\tan^2\alpha}$, $1-\tan^2\alpha \neq 0$.

Solving for $\tan\alpha$, we get: $3\tan^2\alpha - 8\tan\alpha - 3 = 0$

$\tan\alpha = 3$ or $\tan\alpha = -\frac{1}{3}$.

Since $90° < \alpha < 180°$, $\tan\alpha = -\frac{1}{3}$ and, therefore, $\cot\alpha = -3$.

Now we are ready to compute $\tan\alpha + \cot\alpha$: $\tan\alpha + \cot\alpha = -\frac{1}{3} - 3 = -3\frac{1}{3}$

9.7.3 Half-Angle Identities

We can also obtain the following identities for half angles:

1. $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$
2. $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$
3. $\tan\frac{\alpha}{2} = \pm\frac{\sin\alpha}{1+\cos\alpha}$
4. $\cot\frac{\alpha}{2} = \pm\frac{1+\cos\alpha}{\sin\alpha}$

PROOF:

Proof of (1):

$$\cos\alpha = \cos\left(2\frac{\alpha}{2}\right) = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \left(1-\sin^2\frac{\alpha}{2}\right) - \sin^2\frac{\alpha}{2} = 1 - 2\sin^2\frac{\alpha}{2}$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

Proof of (2):

$$\cos\alpha = \cos\left(2\frac{\alpha}{2}\right) = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \cos^2\frac{\alpha}{2} - \left(1-\cos^2\frac{\alpha}{2}\right) = 2\cos^2\frac{\alpha}{2} - 1$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Proof of (3):

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}\cdot\frac{\sqrt{1+\cos\alpha}}{\sqrt{1+\cos\alpha}} = \pm\sqrt{\frac{1-\cos^2\alpha}{(1+\cos\alpha)^2}} = \pm\frac{\sin\alpha}{\pm(1+\cos\alpha)} =$$

$= \dfrac{\sin\alpha}{(1+\cos\alpha)}$, where $\alpha \neq 180°n$ and n is an integer.

Proof of (4): You should complete the proof of (4) on your own as an exercise.

EXAMPLE 9.7g: Find $\sin 15°$.

SOLUTION:

$$\sin 15° = \pm\sqrt{\frac{1-\cos 30°}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \pm\sqrt{\frac{2-\sqrt{3}}{4}} = \pm\frac{\sqrt{2-\sqrt{3}}}{2}.$$

Since $0° < \alpha < 90°$, we take the positive value: $\sin 15° = \dfrac{\sqrt{2-\sqrt{3}}}{2}$.

EXAMPLE 9.7h: Simplify the expression $A = 0.125\cos 4\alpha + \sin^2\alpha\cos^2\alpha$.

SOLUTION: (Justify each step using the identities you have learned thus far.)

$A = 0.125\cos 4\alpha + (\sin\alpha\cos\alpha)^2$

$A = 0.125\cos 4\alpha + \dfrac{(2\sin\alpha\cos\alpha)^2}{4}$

$A = 0.125\cos 4\alpha + \frac{(\sin 2\alpha)^2}{4}$

$A = 0.125\cos 4\alpha + \frac{1}{4}\sin^2 2\alpha$.

$A = 0.125\cos 4\alpha + \frac{1}{4}\sin^2 2\alpha = 0.125\cos 4\alpha + \frac{1}{4}\cdot\frac{1-\cos 4\alpha}{2} = 0.125\cos 4\alpha + 0.125(1-\cos 4\alpha) =$

$= \cancel{0.125\cos 4\alpha} + 0.125 - \cancel{0.125\cos 4\alpha} = 0.125.$

Thus, $A = 0.125$.

9.7.4 Sum–to–Product Identities

In the previous sections, we learned identities of trigonometric functions for the sum and difference of angles. In this section, we will learn identities that will help us deal with the sum and difference of trigonometric functions such as $\sin 18° + \cos 42°$.

Recall the following identities:

1. $\sin(x+y) = \sin x\cos y + \cos x\sin y$
2. $\sin(x-y) = \sin x\cos y - \cos x\sin y$

Adding Equations 1 and 2, we get $\sin(x+y)+\sin(x-y) = 2\sin x\cos y$

Subtracting Equations 1 and 2, we get $\sin(x+y)-\sin(x-y) = 2\cos x\sin y$

Let $x+y=\alpha$ and $x-y=\beta$

Adding these two equations, we get: $x = \frac{\alpha+\beta}{2}$

Subtracting these two equations, we get: $y = \frac{\alpha-\beta}{2}$

Substituting the values of x and y in $\sin(x+y)+\sin(x-y) = 2\sin x\cos y$, we get:
$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$.

Substituting the values of x and y in $\sin(x+y)-\sin(x-y) = 2\cos x\sin y$, we get:
$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

In a similar manner, using sum and difference identities for *cosine*, we can prove that:

1. $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

2. $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

You can complete this proof on your own as an exercise.

EXAMPLE 9.7i: Compute $\sin 18° + \cos 42°$.

SOLUTION:

$$\sin 18° + \sin 42° = 2\sin\frac{18°+42°}{2}\cos\frac{18°-42°}{2} = 2\sin 30°\cos(-12°) = 2\cdot\frac{1}{2}\cos 12° = \cos 12°.$$

We will now derive similar formulas for *tangent* and *cotangent* :

$$\tan\alpha + \tan\beta = \frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta} = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$$

and

$$\tan\alpha - \tan\beta = \frac{\sin\alpha}{\cos\alpha} - \frac{\sin\beta}{\cos\beta} = \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta} = \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta},$$ where $\alpha \neq 90°(1+2n)$, $\beta \neq 90°(1+2n)$, and n is an integer.

Similarly, $\cot\alpha + \cot\beta = \frac{\cos\alpha}{\sin\alpha} + \frac{\cos\beta}{\sin\beta} = \frac{\sin\beta\cos\alpha + \cos\beta\sin\alpha}{\sin\alpha\sin\beta} = \frac{\sin(\beta+\alpha)}{\sin\alpha\sin\beta}$

and

$$\cot\alpha - \cot\beta = \frac{\cos\alpha}{\sin\alpha} - \frac{\cos\beta}{\sin\beta} = \frac{\sin\beta\cos\alpha - \cos\beta\sin\alpha}{\sin\alpha\sin\beta} = \frac{\sin(\beta-\alpha)}{\sin\alpha\sin\beta},$$ $\alpha \neq 180°\cdot n$, $\beta \neq 180°\cdot n$, and n is an integer.

Summarizing: we have the following **Sum-to-Product Identities**:

1. $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
2. $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$
3. $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
4. $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$
5. $\tan\alpha \pm \tan\beta = \frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}$, $\alpha \neq 90°(1+2n)$, $\beta \neq 90°(1+2n)$, and n is an integer.
6. $\cot\alpha \pm \cot\beta = \frac{\sin(\beta\pm\alpha)}{\sin\alpha\sin\beta}$, $\alpha \neq 180°\cdot n$, $\beta \neq 180°\cdot n$, and n is an integer.

EXAMPLE 9.7j: Compute (without calculator) $\dfrac{2\sin^2 49^\circ - 1}{\cos 53^\circ - \cos 37^\circ}$.

SOLUTION: (Justify each step in the solution.)

$$\frac{2\sin^2 49^\circ - 1}{\cos 53^\circ - \cos 37^\circ} = \frac{2\frac{1-\cos 98^\circ}{2} - 1}{-2\sin 45^\circ \sin 8^\circ} = \frac{1-\cos 98^\circ - 1}{-\sqrt{2}\sin 8^\circ} = \frac{\cos 98^\circ}{\sqrt{2}\sin 8^\circ}$$

Notice, $\cos 98^\circ = \cos(90^\circ + 8^\circ) = -\sin 8^\circ$.

Thus, $\dfrac{-\sin 8^\circ}{\sqrt{2}\sin 8^\circ} = -\dfrac{1}{\sqrt{2}}$.

EXAMPLE 9.7k: Simplify the expression $\dfrac{\sin 7\beta + \sin 11\beta}{\cos 10\beta - \cos 8\beta}$ and find its value for $\beta = 30^\circ$.

SOLUTION:

$$\frac{\sin 7\beta + \sin 11\beta}{\cos 10\beta - \cos 8\beta} = \frac{2\sin 9\beta \cos(-2\beta)}{-2\sin 9\beta \sin \beta}.$$

Recall that $\cos(-2\beta) = \cos 2\beta$. Hence: $\dfrac{\sin 7\beta + \sin 11\beta}{\cos 10\beta - \cos 8\beta} = \dfrac{2\sin 9\beta \cos(-2\beta)}{-2\sin 9\beta \sin \beta} = -\dfrac{\cos 2\beta}{\sin \beta}$.

For $\beta = 30^\circ$, we have $-\dfrac{\cos 2\beta}{\sin \beta} = -\dfrac{\cos 60^\circ}{\sin 30^\circ} = -\dfrac{1}{2} : \dfrac{1}{2} = -1$

9.7.5 Product–to–Sum Identities

The following are three more useful identities; they allow us to transfer the product of trigonometric functions into the sum or difference of trigonometric functions.

1. $\sin\alpha \sin\beta = \frac{1}{2}\left(\cos(\alpha - \beta) - \cos(\alpha + \beta)\right)$
2. $\cos\alpha \cos\beta = \frac{1}{2}\left(\cos(\alpha + \beta) + \cos(\alpha - \beta)\right)$
3. $\sin\alpha \cos\beta = \frac{1}{2}\left(\sin(\alpha + \beta) + \sin(\alpha - \beta)\right)$

Use the sum and difference identities to prove these identities.

EXAMPLE 9.7l: Compute (without a calculator) $A = \sin 15° \cos 7° - \cos 11° \cos 79° - \sin 4° \sin 86°$.

SOLUTION:

$A = \frac{1}{2}\left(\sin 22° + \sin 8°\right) - \frac{1}{2}\left(\cos 90° + \cos 68°\right) - \frac{1}{2}\left(\cos(-82°) - \cos 90°\right)$.

Opening the parentheses, we get $A = \frac{1}{2}\left(\sin 22° + \sin 8° - \cos 68° - \cos 82°\right)$.

(Recall that $\cos(-82°) = \cos 82°$.) Note, $\cos 68° = \cos(90° - 22°) = \sin 22°$ and $\cos 82° = \cos(90° - 8°) = \sin 8°$.

We have: $A = \frac{1}{2}\left(\sin 22° + \sin 8° - \sin 22° - \sin 8°\right) = 0$.

EXAMPLE 9.7m: Show that $\tan 30° + \tan 40° + \tan 50° + \tan 60° = \frac{8\cos 20°}{\sqrt{3}}$.

SOLUTION: Recall that $\tan\alpha + \tan\beta = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$.

Hence: $\tan 30° + \tan 40° + \tan 50° + \tan 60° = (\tan 30° + \tan 60°) + (\tan 40° + \tan 50°) =$

$$\frac{\sin(30°+60°)}{\cos 30° \cos 60°} + \frac{\sin(40°+50°)}{\cos 40° \cos 50°} = \frac{\sin 90°}{\frac{\sqrt{3}}{2}\cdot\frac{1}{2}} + \frac{\sin 90°}{\cos 40° \cos 50°} = \frac{4}{\sqrt{3}} + \frac{1}{\cos 40° \cos 50°} =$$

$$= \frac{4\cos 40° \cos 50° + \sqrt{3}}{\sqrt{3}\cos 40° \cos 50°}.$$

To proceed, we use $\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$:

$$\frac{4\cos 40° \cos 50° + \sqrt{3}}{\sqrt{3}\cos 40° \cos 50°} = \frac{4\cdot\frac{1}{2}\left(\cos 90° + \cos(-10°)\right) + \sqrt{3}}{\sqrt{3}\cdot\frac{1}{2}\left(\cos 90° + \cos(-10°)\right)} = \frac{2(0+\cos 10°)+\sqrt{3}}{\frac{\sqrt{3}}{2}(0+\cos 10°)} = \frac{4\cos 10° + 2\sqrt{3}}{\sqrt{3}\cos 10°}$$

Recall that $\frac{\sqrt{3}}{2} = \cos 30°$. Hence $\frac{4\cos 10° + 2\sqrt{3}}{\sqrt{3}\cos 10°} = \frac{4\left(\cos 10° + \frac{\sqrt{3}}{2}\right)}{\sqrt{3}\cos 10°} = \frac{4(\cos 10° + \cos 30°)}{\sqrt{3}\cos 10°}$

Now we can use the identity $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$.

We get $\frac{4(\cos 10° + \cos 30°)}{\sqrt{3}\cos 10°} = \frac{4\cdot 2\cos 20° \cos(-10°)}{\sqrt{3}\cos 10°} = \frac{8\cos 20° \cancel{\cos 10°}}{\sqrt{3}\cancel{\cos 10°}} = \frac{8\cos 20°}{\sqrt{3}}$.

Sections 9.1 – 9.7
EXERCISES

1. Find $\sin 2\alpha$ and $\cos 2\alpha$ if $\cos\alpha = \frac{7}{25}$, $270° < \alpha < 360°$.

2. Find $\sin(\alpha + \beta)$ if $\sin\alpha = \frac{3}{4}$, $\cos\beta = -\frac{1}{5}$, $90° < \alpha < 180°$, $180° < \beta < 270°$.

3. An 18-ft ladder leans against a building so that the angle between the ground and the ladder is 70 degrees. At what height does the ladder reach the building?

4. In a triangle ABC, the length of BC is 1 inch, and angles A and B are $30°$ and $45°$, respectively. Find the other two sides of the triangle.

5. Simplify the following expressions:

   a. $(\cos^2 18° - \cos^2 72°) \cdot 2\cos 27°$

   b. $\cos^2\alpha + \cos^2\beta - \cos(\alpha+\beta)\cos(\alpha-\beta)$

   c. $\dfrac{1+\cos\alpha+\cos 2\alpha+\cos 3\alpha}{\cos\alpha + 2\cos^2\alpha - 1}$

   d. $\dfrac{\sin 58°\cos 52° + \sin 52°\cos 58°}{\cos 72°\cos 37° + \sin 72°\sin 37°}$

6. Prove:

   a. $\sin(-\alpha) = -\sin\alpha$

   b. $\cos(-\alpha) = \cos\alpha$

   c. $\tan(-\alpha) = -\tan\alpha$

   d. $\cot(-\alpha) = -\cot\alpha$

9.8 GRAPHS OF TRIGONOMETRIC FUNCTIONS

You can use a calculator to verify that the graphs of the six trigonometric functions, *sine*, *cosine*, *tangent*, *cotangent*, *secant*, and *cosecant* look like the graphs shown here for each:

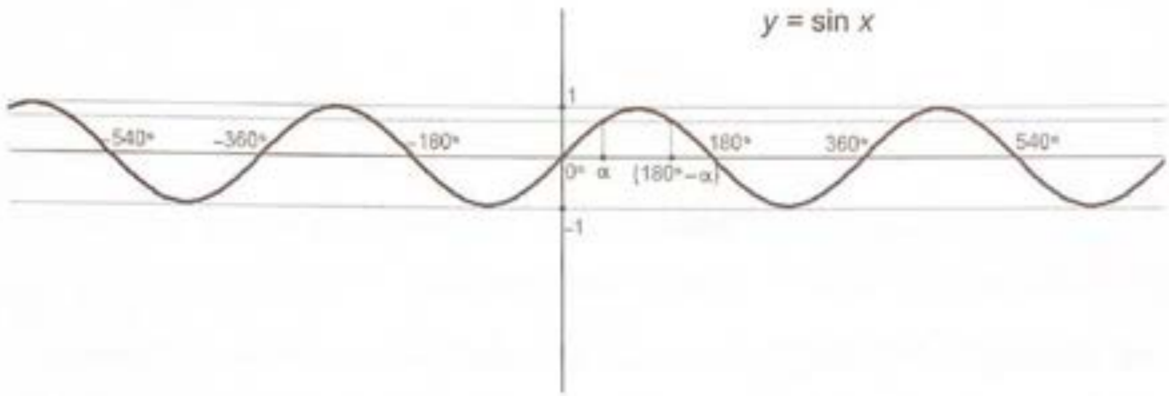


FIGURE 9–11

It is important to note in this graph (FIGURE 9–11) that for any integer k,

- $\sin(\alpha + 360°k) = \sin\alpha$
- $\sin(180°k - \alpha) = \sin\alpha$
- $\sin(-\alpha) = -\sin\alpha$

You can use the identities you learned in the previous sections to verify these identities.

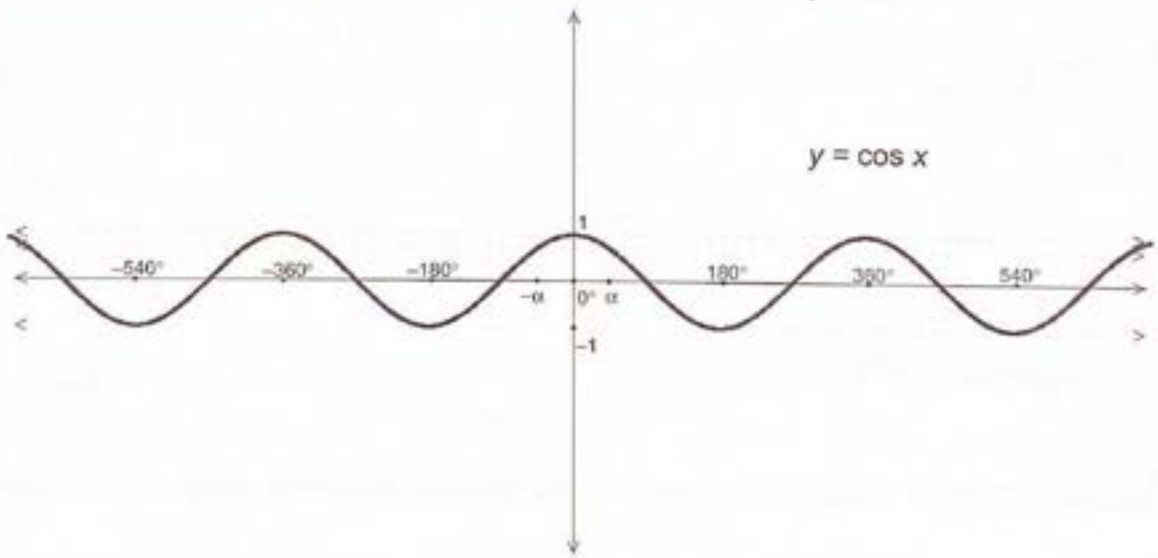


FIGURE 9–12

It is important to note that in this graph (Figure 9–12), for any integer k,

- $\cos(\alpha + 360°k) = \cos\alpha$
- $\cos(-\alpha) = \cos\alpha$

You can use the identities you learned in the previous sections to verify these identities.

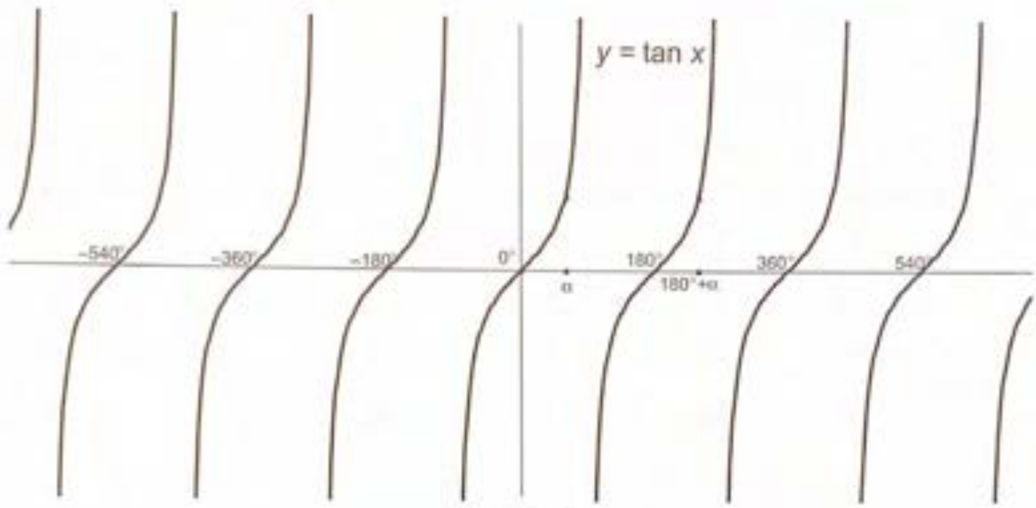


FIGURE 9–13

It is important to note in this graph (Figure 9–13) that for any integer k,

- $\tan(\alpha + 180°k) = \tan\alpha$
- $\tan(-\alpha) = -\tan\alpha$

You can use the identities you learned in the previous sections to verify these identities.

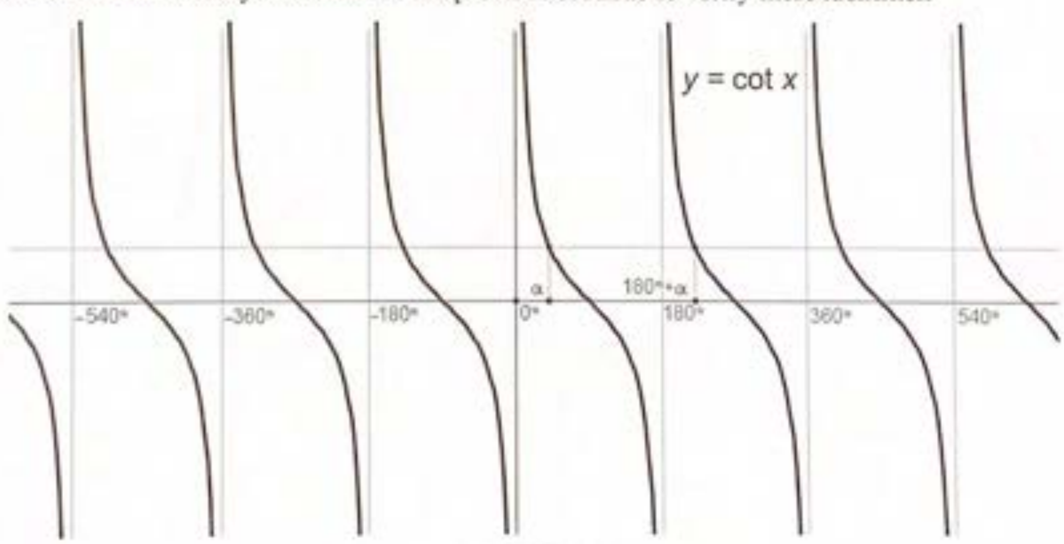


FIGURE 9–14

It is important to note in this graph (Figure 9–14) that for any integer k,

- $\cot(\alpha + 180°k) = \cot \alpha$
- $\cot(-\alpha) = -\cot \alpha$

You can use the identities you learned in the previous sections to verify these identities.

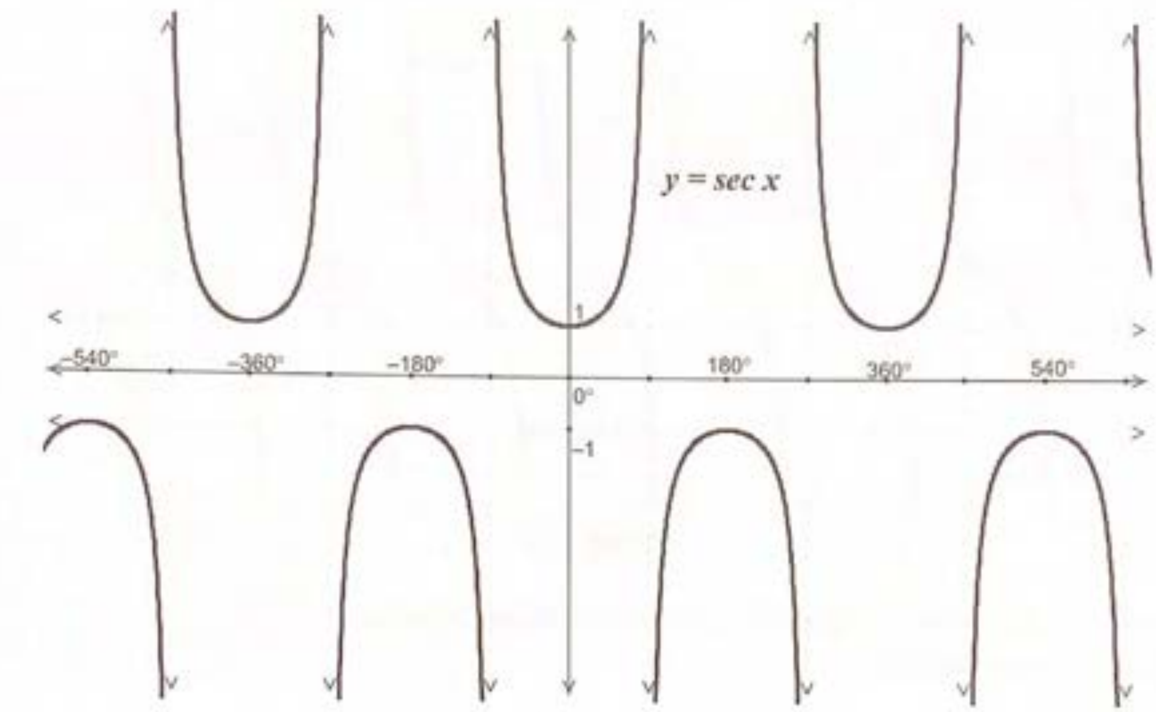


FIGURE 9–15

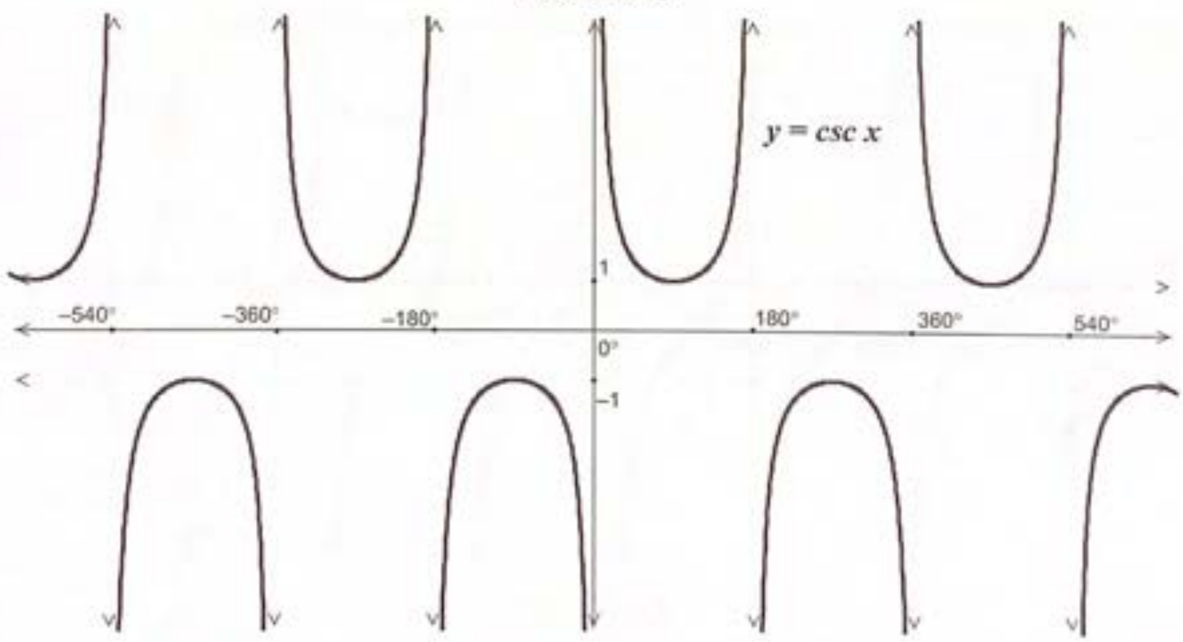


FIGURE 9–16

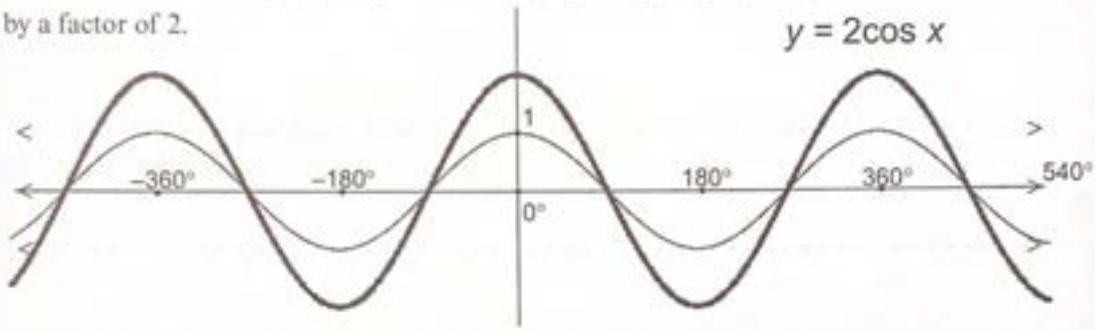
Before proceeding, it is recommended that you revisit SECTION 5.6.

EXAMPLE 9.8a: Graph the function $y = 4 - 2\cos x$.

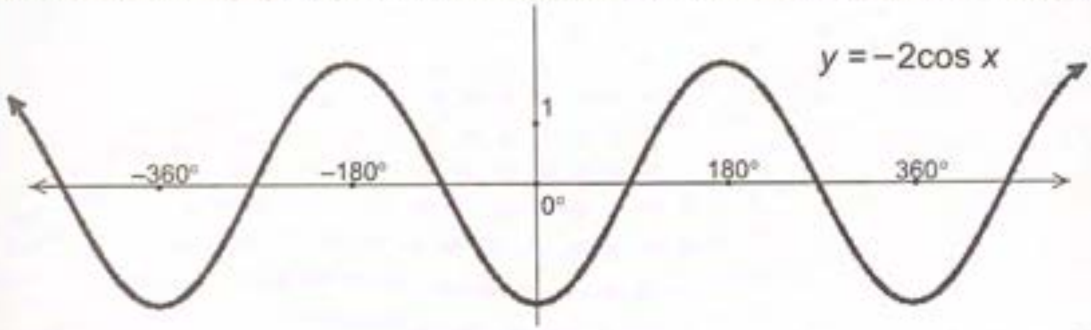
SOLUTION: This function is of the form $y = Af(x) + C$, where $f(x) = \cos x$, $A = -2$, $C = 4$.

The graph of this function can be obtained from the graph of $y = \cos x$ in the following steps:

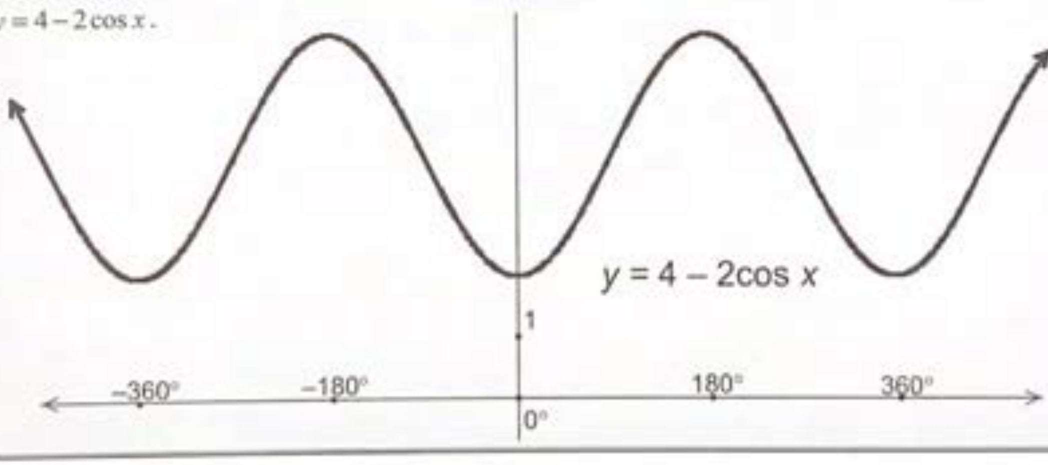
STEP 1: Construct the graph of $y = 2\cos x$ by expanding the graph of $y = \cos x$ along the y-axis by a factor of 2.



STEP 2: Reflect the graph of $y = 2\cos x$ with respect to the x-axis to get the graph of $y = -2\cos x$.



STEP 3: Move the graph of $y = -2\cos x$ along the y-axis by 4 units to get the graph of $y = 4 - 2\cos x$.

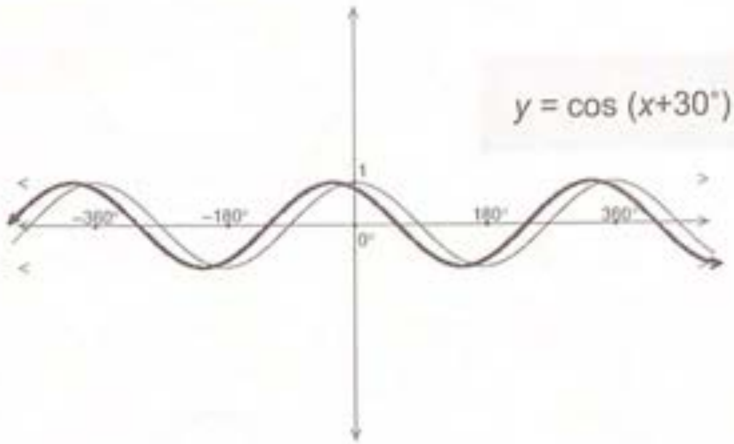


EXAMPLE 9.8b: Graph the function $y=-5\cos(2x-60^\circ)+1$.

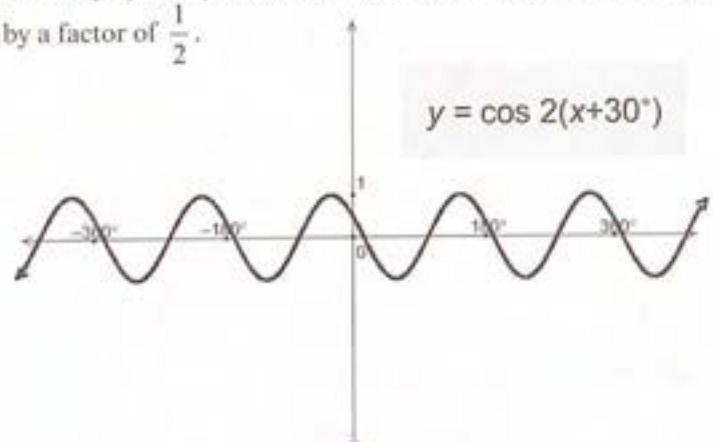
SOLUTION: First we rewrite this function as $y=-5\cos 2(x-30^\circ)+1$. This function is of the form $y=Af(ax-c)+C$, where $f(x)=\cos x$, $A=-5$, $C=1$, $a=2$, $c=30^\circ$.

The graph of this function can be obtained from the graph of $y=\cos x$ in the following steps:

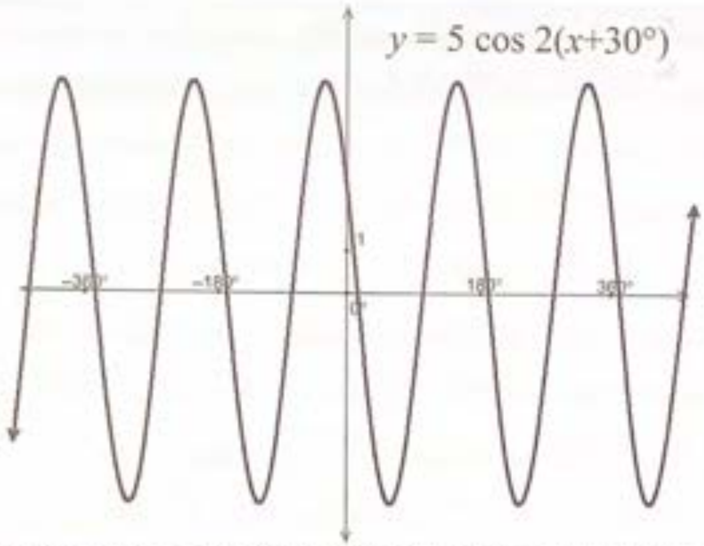
STEP 1: Move the graph of $y=\cos x$ along the x-axis by 30° to the left to get the graph of $y=\cos(x+30^\circ)$.



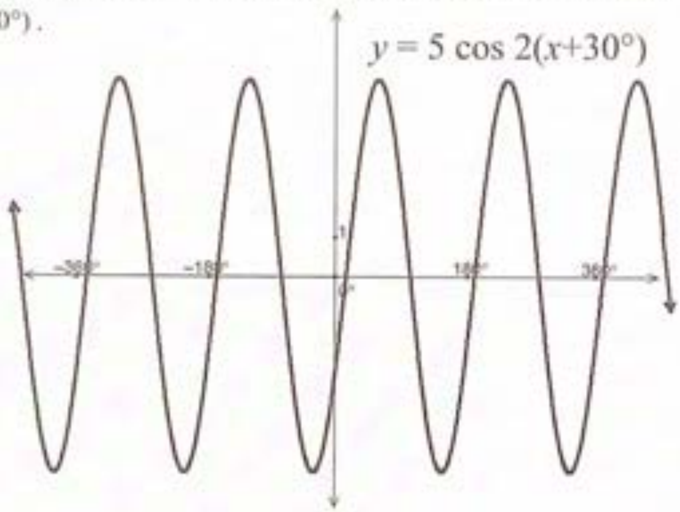
STEP 2: Construct the graph of $y=\cos 2(x+30^\circ)$ by shrinking the graph of $y=\cos(x+30^\circ)$ along the x-axis by a factor of $\frac{1}{2}$.



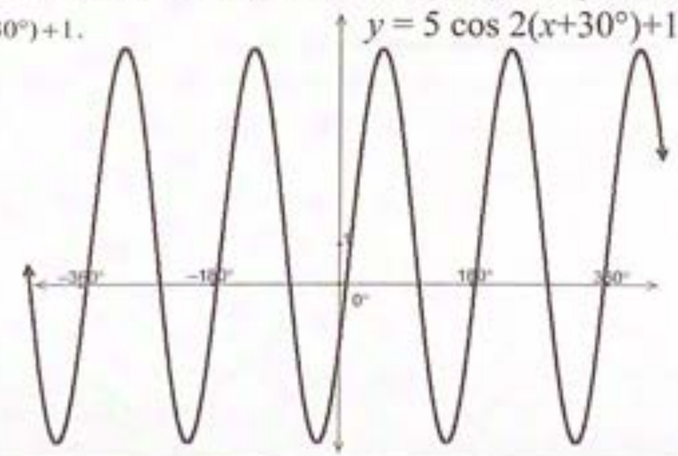
STEP 3: Construct the graph of $y=5\cos 2(x+30^\circ)$ by expanding the graph of $y=\cos 2(x+30^\circ)$ along the y-axis by a factor of 5.



STEP 4: Reflect the graph of $y = 5\cos 2(x+30°)$ with respect to the y-axis to get the graph of $y = -5\cos 2(x+30°)$.



STEP 5: Move the graph of $y = -5\cos 2(x+30°)$ along the y-axis by 1 unit to get the graph of $y = -5\cos 2(x+30°)+1$.



Graphs of the trigonometric functions are ***periodic***. This means that there exists a number T for which $f(x+T)=f(x)$ for any x in the domain of the function. T is called the ***period*** of the function. For the *cosine* and *sine* functions, the period is $360°$ because $\cos(\alpha+360°k)=\cos\alpha$ and $\sin(\alpha+360°k)=\sin\alpha$. For the *tangent* and *cotangent* functions, the period is $180°$ because $\tan(\alpha+180°k)=\tan\alpha$ and $\cot(\alpha+180°k)=\cot\alpha$.

The period of a function can change if the function is of the form $y=f(ax)$ $a\neq 0$. Recall from Section 5.6 that the graph of the function $y=f(ax)$ shrinks or expands along the x-axis by a factor of $\frac{1}{a}$. From here we conclude that the period of $y=f(ax)$ is determined from the period of $y=f(x)$ by the following formula: $T_1=\frac{T}{a}$, where T_1 is the period of $y=f(ax)$ and T is the period of $y=f(x)$.

EXAMPLE 9.8c: Find the period of the function $y=-5\cos(2x+60°)+1$.

SOLUTION: By the formula just discussed, the period is: $T_1=\frac{360°}{2}=180°$.

When a periodic function is of the form $y=Af(x)$, its graph expands or shrinks along the y-axis by the factor A. The absolute value $|A|$ is called the ***amplitude*** of the function.

- The amplitude of the function $y=-5\cos(2x+60°)+1$ is $|-5|=5$.
- The amplitude of the function $y=4-2\cos x$ is $|-2|=2$.

9.9 INVERSE TRIGONOMETRIC FUNCTIONS

Clearly, the sine function is a many-to-one function because for a value of y between -1 and 1, there are infinitely many values of x such that $\sin x=y$, as is shown in Figure 9–17:

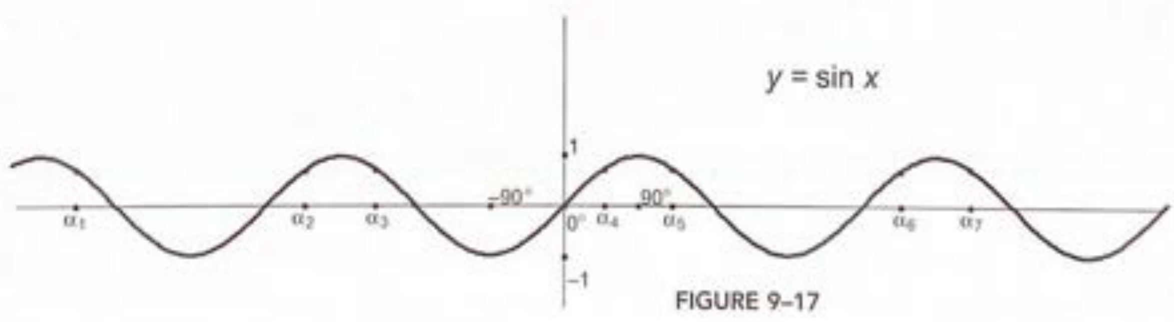


FIGURE 9–17

Recall from SECTION 4 that this implies that the function $y = \sin x$ is not invertible. However, if we restrict the domain of the function to $-90° \le x \le 90°$, the function $y = \sin x$ over this interval is one-to-one, and, hence, it is invertible.

The inverse of the function $y = \sin x$ over the interval $-90° \le x \le 90°$ is called ***arcsine*** and is denoted by $y = \arcsin x$. Recall, that the domain of function f will be the range of function f^{-1} (see SECTION 4). Since we restrict the domain of $y = \sin x$ to be $-90° \le x \le 90°$, the range of $y = \arcsin x$ is $-90° \le y \le 90°$. Likewise, the range of $y = \sin x$ is the domain of $y = \arcsin x$. Hence the domain of $y = \arcsin x$ is: $-1 \le x \le 1$. In summary: $y = \arcsin x$ if only if $\sin y = x$ and $-90° \le y \le 90°$.
High school mathematics books often use $\sin^{-1}$ instead of arcsin. Consider the following example: $\sin^{-1} = 90$, so we can use this easy (though semi invalid) step: $\frac{1}{\sin} = 90$, and thus $1 = \sin 90$. The graph of $y = \arcsin x$ is shown in FIGURE 9–18.

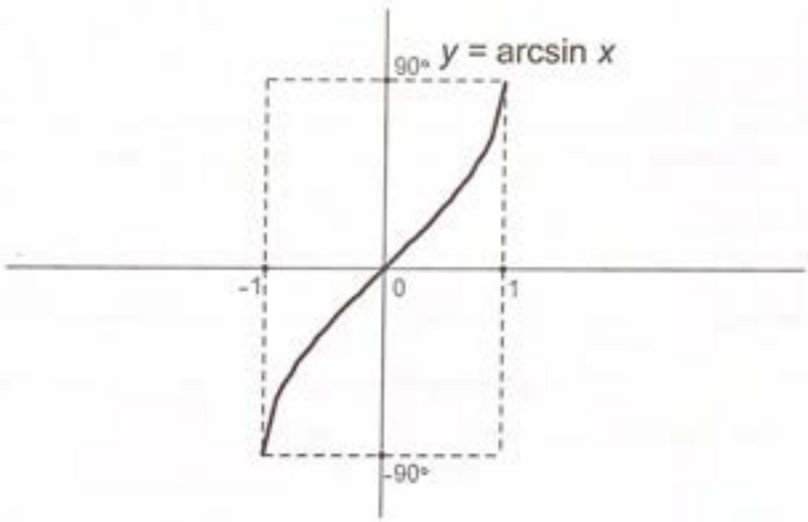


FIGURE 9–18

EXAMPLES:

- $\arcsin \frac{1}{2} = 30°$ because $\sin 30° = \frac{1}{2}$
- $\arcsin 0 = 90°$ because $\sin 90° = 0$
- $\arcsin 2$ does not exist since there is no angle α for which $\sin \alpha = 2$.
- $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -60°$ because $\sin(-60°) = -\frac{\sqrt{3}}{2}$

EXAMPLE 9.9a: For any $|a| \le 1$, $\arcsin(-a) = -\arcsin a$.

PROOF: Let:

(1) $\arcsin(-a) = \alpha$, $-90° \le \alpha \le 90°$.

Therefore, $\sin \alpha = -a$.

Dividing both sides of this equation by -1, we get $-\sin \alpha = a$.

Recall that $\sin(-\alpha) = -\sin \alpha$. So, $\sin(-\alpha) = a$.

Note that $-90° \le -\alpha \le 90°$. Thus, $-\alpha = \arcsin a$.

Therefore:

(2) $\alpha = -\arcsin a$

From (1) and (2), we get $\arcsin(-a) = -\arcsin a$.

EXAMPLE 9.9b: $\arcsin\left(-\frac{1}{2}\right) = -\arcsin\left(\frac{1}{2}\right) = -30°$.

The cosine function, too, is not one-to-one because for a value of y between -1 and 1, there are infinitely many values of x such that $\cos x = y$, as is shown in Figure 9–19:

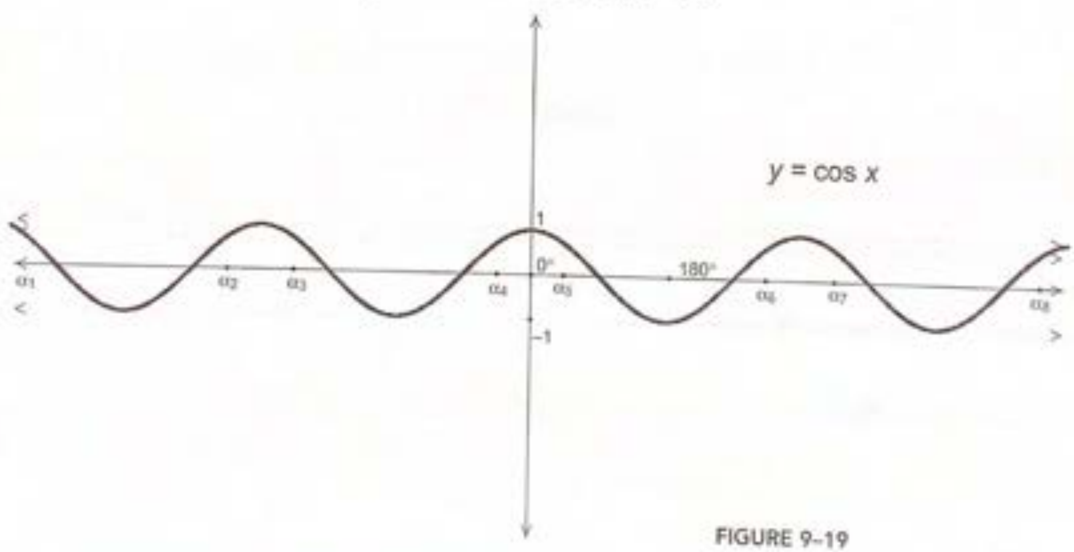


FIGURE 9–19

This implies that the function $y=\cos x$ is not invertible. However, if we restrict the domain of the function to $0° \le x \le 180°$, the function $y=\cos x$ over this interval is one-to-one, and, hence, it is invertible.

The inverse of the function $y=\cos x$ over the interval $0° \le x \le 180°$ is called ***arccosine*** and is denoted by $y=\arccos x$. Again, since we take the interval $0° \le x \le 180°$ as the domain of the function $y=\cos x$, the range of the function $y=\arccos x$ is $0° \le y \le 180°$. Likewise, the range of $y=\cos x$ is the domain of $y=\arccos x$. Hence the domain of $y=\arccos x$ is: $-1 \le x \le 1$. In summary: $y=\arccos x$ if and only if $\cos y = x$ and $0° \le y \le 180°$. The graph of $y=\arcsin x$ is shown in FIGURE 9–20.

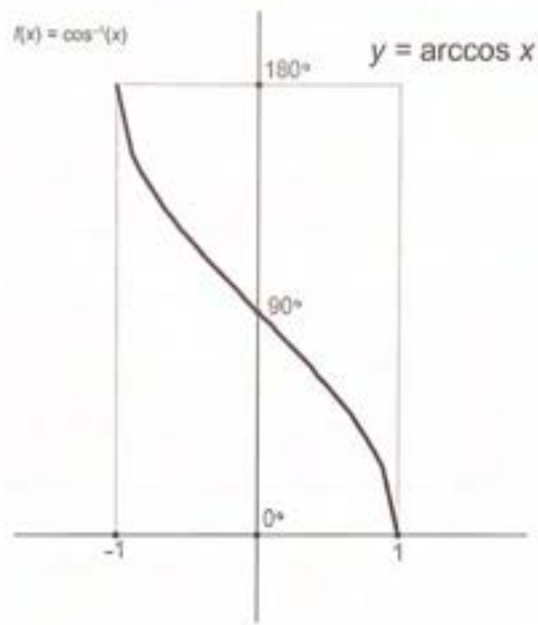


FIGURE 9–20

EXAMPLES:

- $\arccos\frac{\sqrt{2}}{2}=45°$ because $\cos 45°=\frac{\sqrt{2}}{2}$
- $\arccos 1=0°$ as $\cos 0°=1$
- $\arccos(-3)$ does not exist because there is no angle α for which $\cos\alpha=-3$.
- $\arccos\left(-\frac{1}{2}\right)=-120°$ because $\cos 120°=-\frac{1}{2}$.

EXAMPLE 9.9c: For any $|a| \le 1$, $\arccos(-a) = 180° - \arccos a$.

PROOF: Let:

(1) $\arccos(-a) = \alpha$, $0° \le \alpha \le 180°$.

Therefore, $\cos \alpha = -a$.

Using the fact that $\cos \alpha = -\cos(180° - \alpha)$, we get $-a = \cos \alpha = -\cos(180° - \alpha)$

or just $a = \cos(180° - \alpha)$.

Note that $0° \le 180° - \alpha \le 180°$. Thus, $180° - \alpha = \arccos a$.

Therefore, (2) $\alpha = 180° - \arccos a$.

From (1) and (2), we get $\arccos(-a) = 180° - \arccos a$.

EXAMPLE 9.9d: $\arccos\left(-\dfrac{\sqrt{2}}{2}\right) = 180° - \arccos\left(\dfrac{\sqrt{2}}{2}\right) = 180° - 45° = 135°$.

As with the sine and cosine functions, the tangent and cotangent functions too are not one-to-one because for any value of y there are infinitely many values of x such that $\tan x = y$ and $\cot x = y$, as is shown in the following figures:

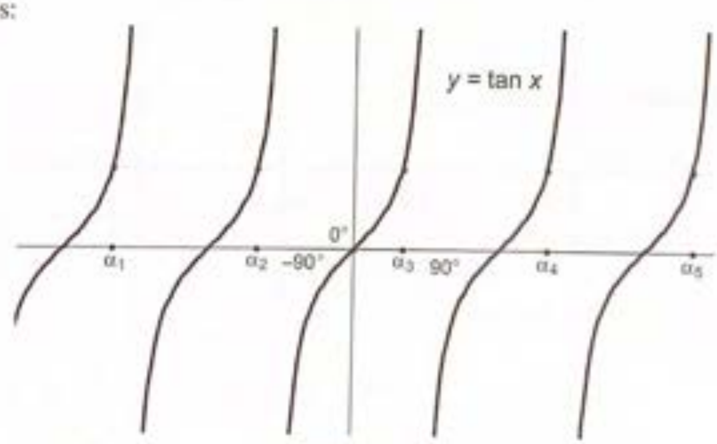


FIGURE 9–21

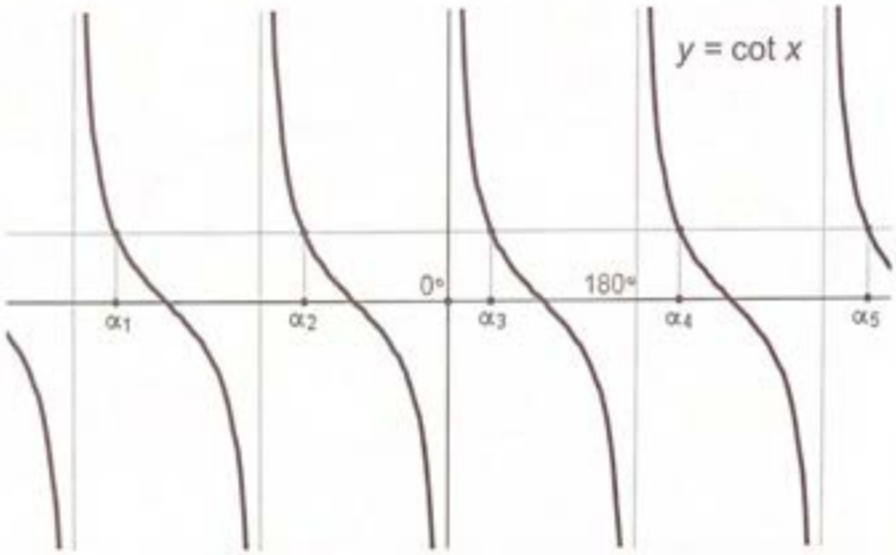


FIGURE 9–22

This implies that the functions $y = \tan x$ and $y = \cot x$ are not invertible. However, if we appropriately restrict their domains, we can make them invertible functions. For the tangent function, we only consider the interval $-90° < x < 90°$, and for the cotangent function, we only consider the interval $0° < x < 180°$.

The inverse of the function $y = \tan x$ over the interval $-90° < x < 90°$ is called the ***arctangent***, and is denoted by $y = \arctan x$. The domain of this function is $-\infty < x < \infty$ because the range of $y = \tan x$ is any number. The range of $y = \arctan x$ is $-90° < y < 90°$. Note that since $y = \tan x$ is undefined for all $\alpha = \pm 90°(1+2k),\ k \in \mathbb{Z}$, the angles 90° and –90° will be excluded from the range of $y = \arctan x$.

The inverse of the function $y = \cot x$ over the interval $-90° < x < 90°$ is called the ***arccotangent***, and is denoted by $y = \operatorname{arccot} x$. The domain of this function is also $-\infty < x < \infty$ and the range is $0° < y < 180°$. Note, angles 0° and 180° will be excluded from the range of $y = \operatorname{arccot} x$ for $y = \cot x$ is undefined for all $\alpha = 180°k, k \in \mathbb{Z}$.

Thus: $y = \arctan x$ if only if $\tan y = x$ and $-90° < y < 90°$. $y = \operatorname{arccot} x$ if and only if $\cot y = x$ and $0° < x < 180°$.

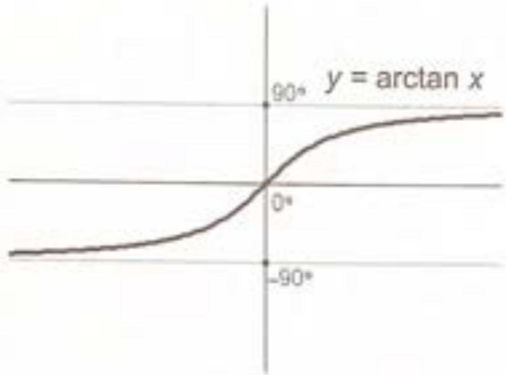


FIGURE 9–23

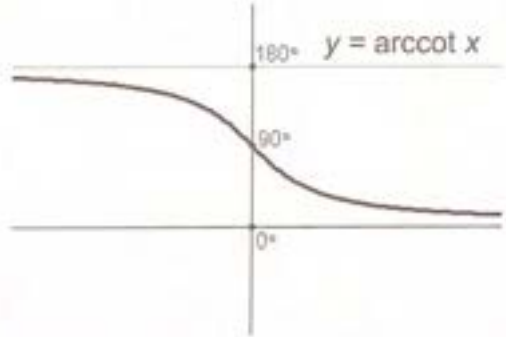


FIGURE 9–24

EXAMPLES:

- $\arctan 1 = 45°$ because $\tan 45° = 1$
- $\operatorname{arccot}\left(\dfrac{\sqrt{3}}{3}\right) = 60°$ because $\cot 60° = \dfrac{\sqrt{3}}{3}$
- $\arctan\left(-\sqrt{3}\right) = -60°$ because $\tan(-60°) = -\sqrt{3}$
- $\operatorname{arccot}(-1) = 135°$ because $\cot 135° = -1$

EXAMPLE 9.9e: Prove that for all $-1 \le y \le 1$ $\cos(\arccos y) = y$.

PROOF: For any y and x, $-1 \le y \le 1$ and $0° \le x \le 180°$, $\arccos y = x$ if and only if $\cos x = y$. Substituting x in the last equation, we get $\cos x = \cos(\arccos y) = y$.

In the same way, we can prove that for any y, $-1 \le y \le 1$, $\sin(\arcsin y) = y$; and for any $y \in \mathbb{R}$

- $\tan(\arctan y) = y$
- $\cot(\operatorname{arccot} y) = y$ (You can complete the proof of this on your own as an exercise.)

EXAMPLE 9.9f: Prove that for all $-90° \le x \le 90°$ $\arcsin(\sin x) = x$.

PROOF: For any y, $-1 \le y \le 1$ and any x, $-90° \le x \le 90°$, $\arcsin y = x$ if and only if $\sin x = y$. Substituting y in the first equation, we get $\arcsin y = \arcsin(\sin x) = x$.

In the same way, we can prove that for any x, $0° \le x \le 180°$, $\arccos(\cos x) = x$; And for any x, $-90° < x < 90°$, $\arctan(\tan x) = x$; And for any x, $0° < x < 180°$, $\text{arc}\cot(\cot x) = x$. (You can complete the proof of this on your own as an exercise.)

EXAMPLES:

- $\cos(\arccos\frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{2}$
- $\arcsin(\sin 30°) = 30°$
- $\arctan(\tan(-60°)) = -60°$
- $\text{arc}\cot(\cot 270°)$ is not defined (why?)

EXAMPLE 9.9g: Compute $\cos(\arcsin\frac{2}{5})$.

SOLUTION: Let $\arcsin\frac{2}{5} = \alpha$ where $-90° \le \alpha \le 90°$.

Then, $\sin\alpha = \frac{2}{5}$, $0° \le \alpha \le 90°$ (why?)

Therefore, $\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{2}{5}\right)^2} = \pm\frac{\sqrt{21}}{5}$.

Since $0° \le \alpha \le 90°$, $\cos\alpha = \frac{\sqrt{21}}{5}$.

From here we have $\arccos\frac{\sqrt{21}}{5} = \alpha$.

Therefore, $\arccos\frac{\sqrt{21}}{5} = \arcsin\frac{2}{5}$.

Hence, $\cos(\arcsin\frac{2}{5}) = \cos(\arccos\frac{\sqrt{21}}{5}) = \frac{\sqrt{21}}{5}$.

EXAMPLE 9.9h: Compute $\sin(\arctan(-\frac{4}{5}))$.

SOLUTION: Let $\arctan(-\frac{4}{5}) = \alpha$, where $-90° < \alpha < 90°$.

Then $\tan\alpha = -\frac{4}{5}$, $-90° < \alpha < 0°$ (why?)

To find $\sin\alpha$, we first find $\cot\alpha$ using the identity $\cot\alpha\cdot\tan\alpha=1$ and then we find $\sin\alpha$ using the identity $\cot^2\alpha+1=\frac{1}{\sin^2\alpha}$.

$\cot\alpha=\frac{1}{\tan\alpha}=-\frac{5}{4}$.

$\sin\alpha=\pm\sqrt{\frac{1}{\cot^2\alpha+1}}=\pm\sqrt{\frac{1}{\frac{25}{16}+1}}=\pm\sqrt{\frac{16}{41}}=\pm\frac{4}{\sqrt{41}}$.

Since $-90°<\alpha<0°$, $\sin\alpha=-\frac{4}{\sqrt{41}}$.

From here, $\arcsin(-\frac{4}{\sqrt{41}})=\alpha$.

Therefore, $\arctan(-\frac{4}{5})=\arcsin(-\frac{4}{\sqrt{41}})$.

Hence, $\sin(\arctan(-\frac{4}{5}))=\sin(\arcsin(-\frac{4}{\sqrt{41}}))=-\frac{4}{\sqrt{41}}$.

9.10 TRIGONOMETRIC EQUATIONS

The simplest trigonometric equations are equations such as $\sin x=a$, $\cos x=a$, $\tan x=a$ and $\cot x=a$. Since the range of the function $y=\sin x$ is $-1\le y\le 1$, the equation $\sin x=a$ has a solution if and only if $-1\le a\le 1$.

EXAMPLE 9.10a: Solve the equation $\sin 2x=-\frac{1}{2}$.

SOLUTION: $\sin 2x=-\frac{1}{2}$

$2x=\arcsin\left(-\frac{1}{2}\right)+360°k$

or

$2x=180°-\arcsin\left(-\frac{1}{2}\right)+360°k$

Since $\arcsin\left(-\frac{1}{2}\right)=-\arcsin\frac{1}{2}=-30°$, we get $2x=-30°+360°k$

or $2x=(180°-(-30°))+360°k=210°+360°k$.

Thus: $x = -15° + 90°k$

or $x = 105° + 180°k$

EXAMPLE 9.10b: Solve the equation $\frac{1}{2}\sin 2x = \sqrt{3}$.

SOLUTION: $\frac{1}{2}\sin 2x = \sqrt{3}$

$\sin 2x = 2\sqrt{3}$.

Since $2\sqrt{3} > 1$, the equation has no solution.

EXAMPLE 9.10c: Solve the equation $\cos(x - 45°) = \frac{\sqrt{3}}{2}$.

SOLUTION: $\cos(x - 45°) = \frac{\sqrt{3}}{2}$

$x - 45° = \arccos\frac{\sqrt{3}}{2} + 360°k$

or

$x - 45° = -\arccos\frac{\sqrt{3}}{2} + 360°k$

Since $\arccos\frac{\sqrt{3}}{2} = 30°$, we get $x - 45° = 30° + 360°k$

or

$x - 45° = -30° + 360°k$

Thus: $x = 75° + 180°k$

or

$x = 15° + 180°k$

EXAMPLE 9.10d: Solve the equation $\cos(-2x)=1$.

SOLUTION: $\cos(-2x)=1$

$-2x=\arccos 1+360°k$

or $-2x=-\arccos 1+360°k$

Since $\arccos 1=0°$, we get $-2x=0+360°k$

$x=-180°k$

Since k varies over all the integers, we can write the solution as: $x=180°k$.

EXAMPLE 9.10e: Solve the equation $\tan\left(\frac{2}{3}x-30°\right)=\sqrt{3}$.

SOLUTION: $\tan\left(\frac{2}{3}x-30°\right)=\sqrt{3}$

$\frac{2}{3}x-30°=\arctan\sqrt{3}+180°k$

$\frac{2}{3}x-30°=60°+180°k$

$x=135°+270°k$.

EXAMPLE 9.10f: Solve the equation $\cot 2x=-\frac{1}{\sqrt{3}}$.

SOLUTION: $\cot 2x=-\frac{1}{\sqrt{3}}$

$2x=\operatorname{arccot}\left(-\frac{1}{\sqrt{3}}\right)+180°k$

$2x=120°+180°k$

$x=60°+90°k$.

EXAMPLE 9.10g: Solve the equation $\sin^4\frac{x}{2}-\cos^4\frac{x}{2}=\frac{1}{2}$.

SOLUTION: $\sin^4\frac{x}{2}-\cos^4\frac{x}{2}=\frac{1}{2}$

$\left(\sin^2\frac{x}{2}-\cos^2\frac{x}{2}\right)\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)=\frac{1}{2}$

Since $\sin^2\frac{x}{2}+\cos^2\frac{x}{2}=1$ and $\sin^2\frac{x}{2}-\cos^2\frac{x}{2}=-\cos x$, we get $-\cos x=\frac{1}{2}$

Multiplying both sides of the equation by -1, we get $\cos x=-\frac{1}{2}$

$x=\pm\arccos\left(-\frac{1}{2}\right)+360°k$

Since $\arccos\left(-\frac{1}{2}\right)=180°-\arccos\frac{1}{2}=180°-60°=120°$, we get $x=\pm120°+360°k$.

EXAMPLE 9.10h: Solve the equation $\cos x+\sqrt{3}\sin x=2$.

SOLUTION: $\cos x+\sqrt{3}\sin x=2$

$\frac{1}{2}\cos x+\frac{\sqrt{3}}{2}\sin x=1$.

Notice that $\sin 30°=\frac{1}{2}$ and $\cos 30°=\frac{\sqrt{3}}{2}$.

Hence, the equation can be written as $\sin 30°\cos x+\cos 30°\sin x=1$.

Using one of the sum identities we learned earlier, we get $\sin(30°+x)=1$.

$30°+x=90°+360°k$

$x=60°+360°k$.

EXAMPLE 9.10i: Solve the equation $\cos 2x-\cos 6x=0$.

SOLUTION: $-2\sin\frac{2x-6x}{2}\sin\frac{2x+6x}{2}=0$

$-2\sin(-2x)\sin 4x = 0$

$2\sin 2x \sin 4x = 0$

In the last equation the product is equal to zero. Therefore, $\sin 2x = 0$

or $\sin 4x = 0$

Solving, we get $2x = 180°k$

or $4x = 180°k$

That is, $x = 90°k$

or $x = 45°k$

It is not difficult to see that the last set of solutions comprises the first one. Thus: $x = 45°k$.

EXAMPLE 9.10j: Solve the equation $6\sin^2 x - 5\sin x + 1 = 0$.

SOLUTION: Let $\sin x = t$. Then, we get $6t^2 - 5t + 1 = 0$.

$t = \frac{5 \pm 1}{12}$

$t = \frac{1}{2}$

or $t = \frac{1}{3}$.

Hence, $\sin x = \frac{1}{2}$ or $\sin x = \frac{1}{3}$.

From the first equation we get $x = 30° + 360°k$

or $x = 150° + 360°k$

From the second equation we get $x = \arcsin\frac{1}{3} + 360°k$

or $x = 180° - \arcsin\frac{1}{3} + 360°k$

EXAMPLE 9.10k: Solve the equation $\sin\frac{x}{2}+\cos x=2$.

SOLUTION: Since the functions $y=\sin x$ and $y=\cos x$ cannot exceed the value 1, $\sin\frac{x}{2}+\cos x=2$ if and only if each of the addends $\sin\frac{x}{2}$ and $\cos x$ is 1. Hence, we have the following system of equations:

$$\begin{cases}\sin\frac{x}{2}=1\\ \cos x=1\end{cases}$$

Solving the first equation, we get $\frac{x}{2}=90°+360°k$

$x=180°+720°k$.

Solving the second equation, we get $x=360°k$.

The question now is: Are there integers k_1 and k_2 such that $180°+720°k_1=360k_2$?

This is equivalent to $180°=360°(k_2-2k_1)$

The last equation, in turn, is equivalent to $\frac{1}{2}=k_2-2k_1$.

This obviously is not possible, since the difference between two integers cannot be a fraction. Hence, our equation has no solution.

Sections 9.8 – 9.10
EXERCISES

1. Which of the following expressions are meaningful?

   a. $\arccos(-1)$

   b. $\tan(\arctan(-3))$

   c. $\sin(\arcsin(-\frac{\sqrt{2}}{2}))$

2. Compute the following:

   a. $\tan(\arcsin\frac{24}{25})$

b. $\cos(\arctan(-\frac{1}{3}))$

3. Solve the following equations:

a. $\sin\left(-4x+30^\circ\right)=-\frac{1}{2}$

b. $1-2\sin^2 x=\frac{7}{3}\sin x$

c. $\sin\left(15^\circ+x\right)+\sin\left(45^\circ-x\right)=1$

d. $\sqrt{3}\sin x-\cos x=3$

e. $\cos^2 x-3\sin x\cos x=-1$

f. $1+\cos x+\cos 2x=0$

g. $\sin 2x=\cos^4\frac{x}{2}-\sin^4\frac{x}{2}$

h. $\cot^4 2x+\frac{1}{\sin^4 2x}=25$

9.11 THE LAW OF SINES AND COSINES

So far, we have learned how the trigonometric functions can help us solve geometric problems involving right triangles. In this section, you will learn two theorems that will help us solve geometric problems involving any triangle.

The Law of Sines: Let ABC be any triangle, with sides a, b, c, and angles α, β, and γ, as shown in Figure 9-25. Let r be the radius of the circle circumscribing the triangle. Then $\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}=2r$.

PROOF:

Let ABC be any triangle, with sides a, b, c, and angles α, β, and γ, as shown in Figure 9-25. We are to prove:

1) $\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$.

2) $\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}=2r$.

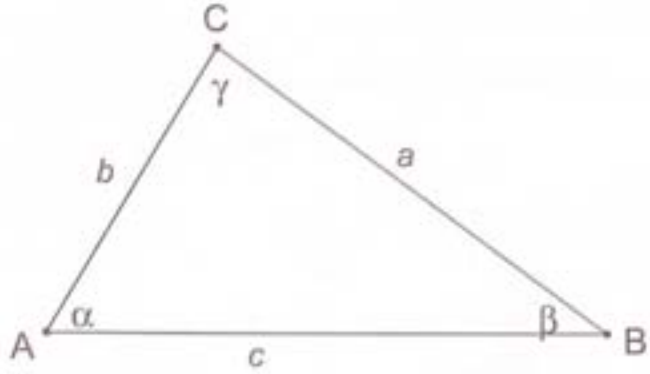


FIGURE 9–25

Proof of 1): We know that

- $area(\Delta ABC) = \frac{1}{2}ab\sin\gamma$
- $area(\Delta ABC) = \frac{1}{2}bc\sin\alpha$
- $area(\Delta ABC) = \frac{1}{2}ac\sin\beta$.

Hence:

(1) $\frac{1}{2}ab\sin\gamma = \frac{1}{2}bc\sin\alpha$

(2) $\frac{1}{2}bc\sin\alpha = \frac{1}{2}ac\sin\beta$

From (1), we get

(3) $\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}$.

From (2), we get

(4) $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$.

From (3) and (4), we get

(5) $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$.

Proof of 2): It is sufficient to prove that $\frac{a}{\sin\alpha} = 2r$.

Circumscribe the triangle ABC.

CASE 1: $\alpha = 90°$

The center of the circle is on the side BC as is shown in FIGURE 9–26.

In this case, $BC = a = 2r$.

Hence: $\dfrac{a}{\sin\alpha} = \dfrac{2r}{\sin 90°} = \dfrac{2r}{1} = 2r$.

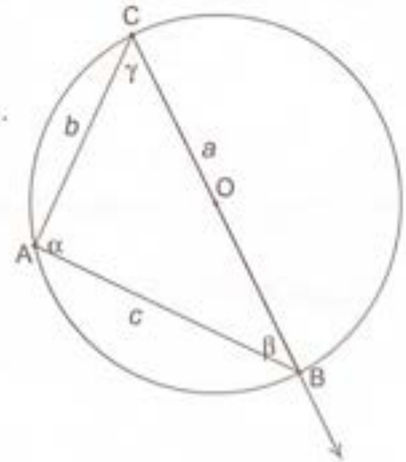


FIGURE 9–26

CASE 2 $\alpha \neq 90°$

Let CA_1 be a diameter. Clearly triangle A_1BC is a right triangle (why?) with angle $\measuredangle CBA_1 = 90°$.

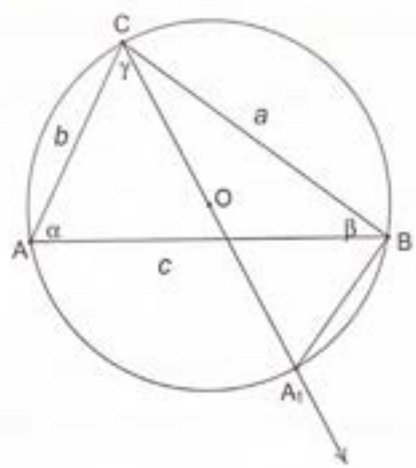


FIGURE 9–27

We have: $\sin A_1 = \dfrac{a}{A_1C} = \dfrac{a}{2r}$.

Therefore, $\dfrac{a}{\sin A_1} = 2r$.

Since angle A_1 and angle α are inscribed within the same arc, they are equal, and, hence, $\sin\alpha = \sin A_1$.

This shows that $\dfrac{a}{\sin\alpha} = 2r$, as was required.

Note, however, that the points A and B can be on the same side of the diameter CA_1, as is shown in FIGURE 9–28.

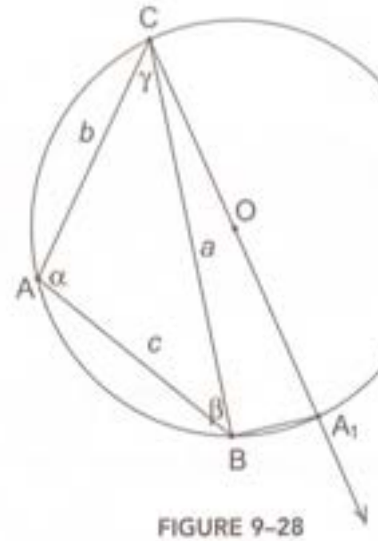


FIGURE 9–28

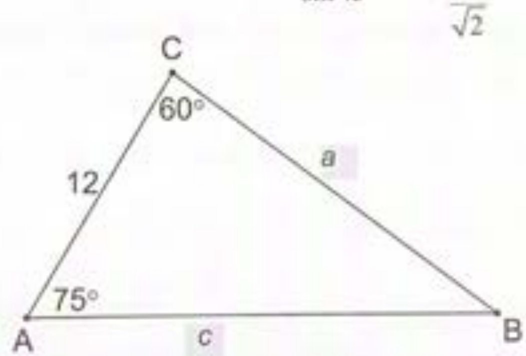
In this case, $\alpha + A_1 = 180$, or $A_1 = 180 - \alpha$.

Still, $\sin A_1 = \sin(180^{\circ} - \alpha) = \sin\alpha$, and so the relation $\frac{a}{\sin\alpha} = 2r$ still holds.

EXAMPLE 9.11a: In the triangle ABC $AC = 12$, $\measuredangle A = 75^{\circ}$, $\measuredangle C = 60^{\circ}$. Find side AB.

SOLUTION: To find side $AB = c$ we will use the Law of Sines: $\frac{12}{\sin B} = \frac{c}{\sin 60^{\circ}}$.

Since $\measuredangle B = 180^{\circ} - (75^{\circ} + 60^{\circ}) = 45^{\circ}$, we have $c = \frac{12 \cdot \sin 60^{\circ}}{\sin 45^{\circ}} = \frac{12 \cdot \frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}}} = 6\sqrt{3} \cdot \sqrt{2} = 6\sqrt{6}$.



The Law of Cosines: Let ABC be any triangle, with sides a, b, c, and angles α, β, and γ, as shown in FIGURE 9–29. Then:

$a^2 = b^2 + c^2 - 2bc\cos\alpha$
$b^2 = a^2 + c^2 - 2ac\cos\beta$
$c^2 = a^2 + b^2 - 2ab\cos\gamma$

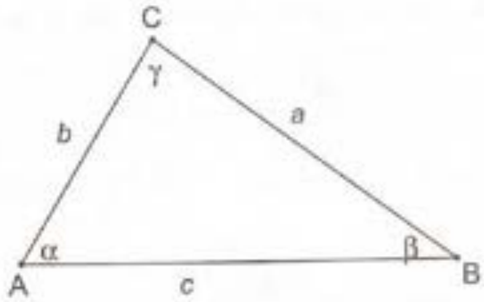


FIGURE 9–29

Proof: We will prove the first equality $a^2 = b^2 + c^2 - 2bc\cos\alpha$. The proof for the other two are analogous.

Situate the triangle in a coordinate system as is shown in FIGURE 9–30.

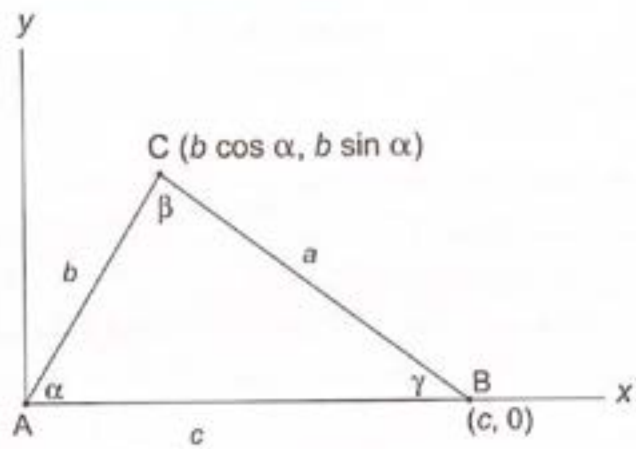


FIGURE 9–30

We have $A = (0,0)$, $B = (c,0)$ and $C = (b\cos\alpha, b\sin\alpha)$ (Why? Hint: drop the altitude CH from the vertex C to the segment AB and consider the right triangle ACH).

By the distance formula: $BC^2 = a^2 = (b\cos\alpha - c)^2 + (b\sin\alpha - 0)^2 = b^2\cos^2\alpha + b^2\sin^2\alpha - 2bc\cos\alpha + c^2 =$
$b^2(\sin^2\alpha + \cos^2\alpha) - 2bc\cos\alpha + c^2 = b^2 + c^2 - 2bc\cos\alpha$.

Thus, $a^2 = b^2 + c^2 - 2bc\cos\alpha$.

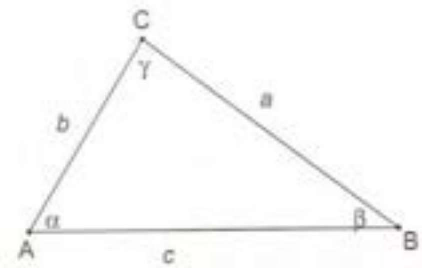
EXAMPLE 9.11b: In the triangle ABC $\measuredangle A = 60°$ and $AB = 6\sqrt{6}$, $AC = 12$. Find side BC.

SOLUTION: By the Law of Cosines Theorem, we have:

$BC^2 = \left(6\sqrt{6}\right)^2 + 12^2 - 2 \cdot 6\sqrt{6} \cdot 12 \cdot \cos 60^\circ = 216 + 144 - 144\sqrt{6} \cdot \frac{1}{2} = 360 - 72\sqrt{6}$.

Then, $BC = \sqrt{360 - 72\sqrt{6}} = \sqrt{36(10 - 2\sqrt{6})} = 6\sqrt{10 - 2\sqrt{6}}$.

Section 9.11
EXERCISES



1. Find all the sides and angles of the triangle ABC if
   a. $\measuredangle A = 60^\circ$, $\measuredangle B = 40^\circ$, $c = 14$
   b. $\measuredangle A = 80^\circ$, $a = 16$, $b = 10$
   c. $a = 14$, $b = 18$, $c = 20$.

2. Determine the angles of a triangle ABC if its sides are
   a. 5, 4, and 4
   b. 17, 8, and 15
   c. 9, 5, and 6.

3. Find the bisectors in the triangle with sides 5, 6, and 7.

4. A group of students was assigned to determine the width of a river. They marked two trees on one bank of the river, A and B, and measured the distance between them. The distance was 70 yards. On the other bank of the river, adjacent to the water, they marked another tree C. They also found that $\measuredangle CAB = 12^\circ 30'$ and $\measuredangle ABC = 72^\circ 42'$. The students determined the width of the river from this information. What is the river's width?

5. Use the Law of Cosines to prove the Pythagorean Theorem.

6. State the converse of the Pythagorean Theorem and prove it using the Law of Cosines.

9.12 RADIANS

So far we have measured angles by degree units. In this section, we will learn how to measure angles by length units, called ***radians***. To define the radian measure of an angle α, take any circle with radius r

centered at the vertex of α, as shown in FIGURE 9–31, and let l be the length of the arc intercepted by α. l is considered to be positive if α is a positive angle and negative if α is a negative angle.

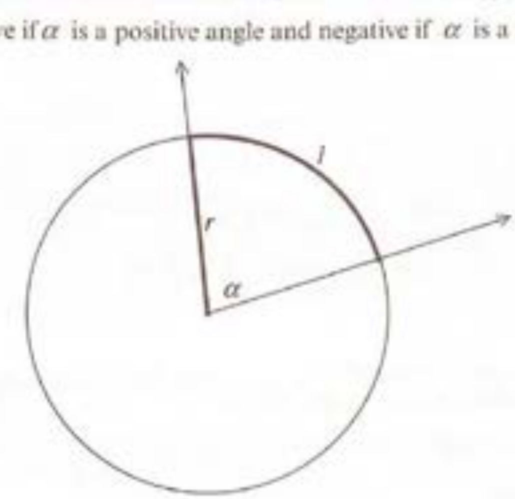


FIGURE 9–31

The ***radian measure*** of the angle α is defined to be the ratio $\frac{l}{r}$.

For this definition to be meaningful, we must show that the radian measure of the angle α is independent of the choice of the radius r. In the exercises at the end of this section, you will be guided to justify this fact.

Since the ratio $\frac{l}{r}$ is independent of the radius r, we may choose $r = 1$. So, the measure of α in radians is the length of the arc intercepted by α in a circle of radius 1. When we say the radian measure of an angle α is, for example, 1.5, we mean that the length of the arc intercepted by α in a circle of radius 1 is 1.5.

Thus, it easy to see that:

1. If the degree measure of an angle α is $360°$, then the radian measure of α is 2π.

2. If the degree measure of an angle α is $180°$, then the radian measure of α is π.

3. If the degree measure of an angle α is $1°$, then the radian measure of α is $\frac{\pi}{180}$.

And in general:

4. If the degree measure of an angle α is $n°$, then the radian measure of α is $\frac{\pi}{180} \cdot n$.

We can also convert radians to degrees:

5. If the radian measure of an angle α is 1, then the degree measure of α is $\left(\frac{180}{\pi}\right)^{\circ}$.

And in general:

6. If the radian measure of an angle α is s, then the degree measure of α is $\left(\frac{\pi}{180}\cdot s\right)^{\circ}$.

When we write $135^{\circ}=\frac{3\pi}{4}$, we mean that the radian measure of the angle whose degree measure is 135° is $\frac{3\pi}{4}$.

EXAMPLES:

- $360^{\circ}=\frac{\pi}{180^{\circ}}\cdot 360^{\circ}=2\pi$
- $135^{\circ}=\frac{\pi}{180^{\circ}}\cdot 135^{\circ}=\frac{3\pi}{4}$
- $2.5=\frac{180^{\circ}}{\pi}\cdot 2.5\approx 143^{\circ}$
- $\frac{\pi}{6}=\frac{180^{\circ}}{\pi}\cdot\frac{\pi}{6}=30^{\circ}$.

Section 9.12
EXERCISES

1. Find the radian measure for each angle.

°	15°	30°	45°	60°	90°	120°	150°	180°	225°	270°	315°	360°
radians												

2. Determine to which quarter each of the following angles belongs:
   $\frac{2\pi}{3}, \frac{3\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{18}, \frac{7\pi}{12}, \frac{2\pi}{5}, 35\pi, 146\pi, \frac{15}{2}\pi, 1, 8, \frac{2}{\pi}$.

3. Find the radian measures of the angles whose degree measures are:
   a. $45^{\circ}30'$
   b. $12'$

4. Solve the following equations:
   a. $\tan(3\sin\pi x)=\sqrt{3}$
   b. $\sin(\cos 3x)=1$