THE 1991 ASIAN PACIFIC MATHEMATICAL OLYMPIAD SOLUTION

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1 Solution

Problem 1. As $XY||BC$, by Ceva's theorem, AM, BY and CX are concurrent. By sine law, $\dfrac{BP}{sin\angle BCP} = \dfrac{BC}{sin\angle BPC}$ and

$\dfrac{PY}{sin\angle ACP} = \dfrac{CY}{sin\angle CPY} = \dfrac{\frac{1}{3}AC}{sin\angle BPC}$. Hence, $\dfrac{BP}{PY} = \dfrac{BCsin\angle BCP}{\frac{1}{3}sin\angle ACP}$. Let M be the midpoint of AB. Using the similar arguments, we have

$\dfrac{BM}{AM} = 1 = \dfrac{BCsin\angle BCP}{ACsin\angle ACP}$. Hence, $\dfrac{BP}{PY} = 3$. From Ceva's theorem, it follows that $QP||BC||XY$. Hence, $\dfrac{BQ}{QC} = 3$. Let N be the midpoint of AC. It follows that Q is the midpoint of BN. Hence, $QM||AC$. Using similar arguments $PM||AB$, it follows that ΔABC and ΔMPQ have parallel sides. Therefore, they are similar. ■

Problem 2. Consider a rectangular coordinate system on the set of points such that all the points have distinct y-coordinates. Denote these points $y_1, y_2, \cdots, y_{997}$ in increasing order. Let A_i be the point with y-coordinate y_i.
Now, consider the midpoints of $\overline{A_iA_{i+1}}$ for $1 \leq i \leq 996$. There are 996 such midpoints. By order all of them have distinct y-coordinate. Hence, they do not coincide. Consider the midpoints of $\overline{A_iA_{i+2}}$ for $1 \leq i \leq 995$. These points cannot coincide with any of the former midpoints as all y_i are distinct. This yeids $996 + 995 = 1991$ distinct midpoints.

Problem 3. Solution 1. By Cauchy-Schwarz Inequality, we have

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} (\frac{a_i}{\sqrt{a_i + b_i}})^2 \sum_{i=1}^{n} (\sqrt{a_i + b_i})^2$$
$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} \frac{a_i^2}{a_i + b_i} \sum_{i=1}^{n} (a_i + b_i)$$

By dividing each side by $\sum_{i=1}^{n} (a_i + b_i)$ each side and
$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$, it follows that

$$\frac{a_1^2}{a_1+b_1}+\frac{a_2^2}{a_2+b_2}+\cdots+\frac{a_n^2}{a_n+b_n}\geq\frac{a_1+a_2+\cdots+a_n}{2}$$
which completes the proof. ■

Solution 2. By Titu's Lemma, we have
$$\frac{a_1^2}{a_1+b_1}+\frac{a_2^2}{a_2+b_2}+\cdots+\frac{a_n^2}{a_n+b_n}\geq\frac{(a_1+a_2+\cdots+a_n)^2}{a_1+a_2+\cdots+a_n+b_1+b_2+\cdots+b_n}$$
Since $a_1+a_2+\cdots+a_n=b_1+b_2+\cdots+b_n$, we get
$$\frac{a_1^2}{a_1+b_1}+\frac{a_2^2}{a_2+b_2}+\cdots+\frac{a_n^2}{a_n+b_n}\geq\frac{a_1+a_2+\cdots+a_n}{2}$$
which completes the proof. ■

Problem 4. Denote each child the number of the set $0,1,2,\cdots,n-1$ in closewise direction. The first child which receive a candy is 1. The k-th is the remainder when $\dfrac{k(k+1)}{2}$ is divided by n.
If n is an odd number, then $\dfrac{(n+1)(n+2)}{2}\equiv 1\ (mod\ n)$. This implies that $(n+1)$-th child is the first one so that the teacher will take the same steps. It is obvious that the $(n-1)$-th and n-th children are the same child. Therefore, there is one child who didn't take a candy as there is one who took two on the first round.
Consider the case that n is an even number. Let $C_{(1,i)}=\{i,i+\dfrac{n}{2}\}$, for each $i\in\{1,2,\ldots,\dfrac{n}{2}\}$. Consider a circle with $C_{(1,i)}$ written in clockwise direction. Note that the steps taken on the circle with n children will be seen as if the teacher had $\dfrac{n}{2}$ children on the latter circle.. Hence, $\dfrac{n}{2}$ must be an even number. Otherwise, there exist a set $C_{(1,i)}$ which wasn't awarded with a candy.
If $\dfrac{n}{2}$ is an even number, call $C_{(2,i)}=\{C_{(1,i)},C_{(1,i+\frac{n}{4})}\}$. The same argument can be taken. Following this, the only possibility for each child to get a candy is $n=2^k$ where $k\in\mathbb{Z}$ and $k\geq 0$.

Problem 5. Let C be the intersection of two circles. Let l be one of the common external tangents. The circle we are searching for is the inverse of the line l with respect to circle (P,CP). We can obtain one more circle if we use the other external common tangent.