THE 2004 ASIAN PACIFIC MATHEMATICAL OLYMPIAD SOLUTION

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1 Solution

Problem 1. Let $k \in S$. Then $\dfrac{k+k}{(k,k)} = 2 \in S$. Let M be the largest odd element of S. $\dfrac{M+2}{(M,2)} = M+2 \in S$, which leads to a contradiction. Let $m = 2n$ be the smallest element of S greater than 2. Then $\dfrac{m+2}{2} = n+1 \in S$. But n must be larger than 1 or $m = 2$. Therefore, $2n > n+1$. We get $2n = 2$ by minimality of m. Hence, $n = 1$ which leads to a contradiction. Therefore, the only non-empty finite set S is $\{2\}$.

Problem 2. Let M be the midpoint of BC. Note that $AH = 2OM$. Let A, B, C be the angle of ΔABC. Also
$\angle HAO = \angle CAH - \angle CAO = |(90^\circ - C) - (90^\circ - B)| = |B - C|$. Also, let a, b and c be the sides of ΔABC and R be its circumradius. Then,
$AH = 2OM = 2\sqrt{BO^2 - BM^2} = 2\sqrt{R^2 - (\frac{a}{2})^2} = 2\sqrt{R^2 - (Rsin\ A)^2} = |2Rcos\ A|$ which implies that
$[AOH] = \frac{1}{2}(AH)(AO)sin\angle HAO = |R^2 cos\ Asin(B-C)|$
$= \frac{1}{2}R^2|sin(A+B-C) - sin(A-B+C)| = \frac{1}{2}R^2|sin\ 2C - sin\ 2B|$
Similarly, we get $[BOH] = \frac{1}{2}|sin\ 2A - sin\ 2C|$ and $[COH] = \frac{1}{2}|sin\ 2B - sin\ 2A|$.
WLOG, assume $sin\ 2A \geq sin\ 2B \geq sin\ 2C$. Then we get
$[AOH] + [COH] = [BOH]$. ■

Problem 3. Call a pair of vertices good if they have the same color with odd number of lines separating them or they have different colors with even number of lines separating them.
First, color P_1 red. Then color each other P_i red or blue such that (P_1, P_i) is good.
Lemma. If (P, Q) and (P, R) are good, then (Q, R) is good.
Proof: Let $d(PQ)$ be the number of lines separating P and Q i.e. cutting through the segment $\overline{PQ}$.

Claim: $d(PQ)+d(PR)+d(QR)$ is an odd number.
Proof of Claim: Consider two types of lines.
Type 1: Line joining two points that are not P, Q, R.
If it cuts through ΔPQR, it must cut at exactly two points. Therefore, in total they cut through ΔPQR for time of even number.
Type 2: Three lines joining an other point X with P, Q and R.
If X lies outside ΔPQR, then exactly one of these three lines cut ΔPQR at exactly one point. If X lies inside ΔPQR, then each of these three lines cut ΔPQR at exactly one point so in total at three points. Since there are 2001 other points X, in total they cut through ΔPQR for times of odd number.
Therefore, $d(PQ)+d(PR)+d(QR)$ is an odd number.
Back to the lemma, WLOG assume P is red.
Case 1: Q and R are red: $d(PQ)$ and $d(PR)$ are odd numbers so $d(QR)$ is an odd number. Hence, (Q,R) is good.
Case 2: Q and R are blue: $d(PQ)$ and $d(PR)$ are even numbers so $d(QR)$ is an odd number. Hence, (Q,R) is good.
Case 3: Q is red and R is blue: $d(PQ)$ is an odd number and $d(PR)$ is even numbers so $d(QR)$ is an even number. Hence, (Q,R) is good.
Case 4: Q is blue and R is red: This is similar to case 3.
From the lemma, it follows that every pair of vertices is good and the proof is done. ■

Problem 4. By Legendre's Formula, we have $\upsilon_2((n-1)!\,) = n-1-s_2(n-1)$. By the bound $s_2(n-1) \le log_2(n-1)+1$, we have $n-1-s_2(n-1) \ge n-log_2(n-1)$. However, $\upsilon_2(n(n+1)) < log_2(n)+log_2(n+1)$. Therefore, it remains to show that $n-log_2(n-1) \ge log_2(n)+log_2(n+1)+1$. Note that
$n-log_2(n-1) \ge log_2(n)+log_2(n+1)+1$
$\Leftrightarrow n \ge log_2(n)+log_2(n+1)+log_2(n-1)+1$
$\Leftrightarrow 2^{n-1} \ge n(n+1)(n-2)$ which is clearly true for $n \ge 12$. Therefore, $\lfloor \dfrac{(n-1)!}{n(n+1)} \rfloor$ must be an even number for $n \ge 12$. We just need to check the inequality is also true for $n \le 11$ which is easy to check. Then the proof is done. ■

Problem 5. Let $x=a+b+c, y=ab+bc+ac, z=abc$. Then the inequality is equivalent to $z^2+2y^2-4xz+4x^2-17y+8 \ge 0$. Note that $z^2+2y^2-4xz+4x^2-17y+8 \ge 0 \Leftrightarrow (z-\frac{x}{3})^2+\frac{8}{9}(y-3)^2+\frac{10}{9}(y^2-3xz)+\frac{35}{9}(x^2-3y) \ge 0$. It is easy to prove that $y^2-3xz \ge 0$ and $x^2-3y \ge 0$. The proof is done then. ■