Mathematical Methods Cheatsheet

Calculus of Single and Multiple Variables

- **Basic Differentiation**: $\frac{d}{dx}[x^n] = nx^{n-1}$
- **Basic Integration**: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$
- **Partial Derivatives**: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$
- **Exact vs. Inexact Differentials**: df is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for $Mdx + Ndy = 0$
- **Jacobian Determinant**: $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$
- **Taylor Expansion**: $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$
- **Fourier Series (Periodic Function $f(x)$)**:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

  where $a_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(x)dx$, $a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \cos \frac{n\pi x}{L} dx$, and $b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \sin \frac{n\pi x}{L} dx$.
- **Tips and Tricks**:
  - **Chain Rule Shortcut**: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$. Example: $\frac{d}{dx}[\sin(x^2)] = \cos(x^2) \cdot 2x$.
  - **Small Angle Approximations**: For small x, $\sin(x) \approx x$, $\cos(x) \approx 1 - \frac{x^2}{2}$, and $e^x \approx 1 + x$.

Vector Algebra and Calculus

- **Vector Dot Product**: $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$
- **Vector Cross Product**: $\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin\theta \hat{n}$
- **Gradient**: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$
- **Divergence**: $\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
- **Curl**: $\nabla \times \mathbf{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$
- **Green's Theorem**: $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \hat{n} \; dA$
- **Stokes' Theorem**: $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS$
- **Divergence Theorem**: $\iiint_V \nabla \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS$
- **Multiple Integrals**: Evaluate volume integrals as nested integrals, e.g., $\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x,y,z) dz \, dy \, dx$.
- **Tips and Tricks**:
  - **Gradient Insight**: ∇f points in the direction of steepest ascent. Level curves are orthogonal to it.
  - **Divergence Theorem Symmetry**: For symmetric fields, use spherical or cylindrical coordinates to simplify volume integrals.

Differential Equations

- **First Order Linear ODE**: $\frac{dy}{dx} + P(x)y = Q(x)$

$$\text{Solution: } y \cdot \mu(x) = \int Q(x) \cdot \mu(x)dx, \text{ where } \mu(x) = e^{\int P(x)dx}$$

- **Second Order Linear ODE (Constant Coefficients)**:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Solution depends on roots of characteristic equation $ar^2 + br + c = 0$

  - Distinct Roots: $y = c_1e^{r_1x} + c_2e^{r_2x}$
  - Repeated Roots: $y = (c_1 + c_2x)e^{r_1x}$
  - Complex Roots: $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

- **Homogeneous Equations**: $Mdx + Ndy = 0$ is solved using substitution $y = vx$.
- **Tips and Tricks**:
  - **Variable Separation**: For $\frac{dy}{dx} = g(y)/h(x)$, separate variables and integrate.
  - **Particular Solution Guessing**: Match the form of the non-homogeneous term (e.g., exponential, polynomial, or sine/cosine).

Matrices and Determinants

- **Matrix Multiplication**: $(AB)_{ij} = \sum_k A_{ik}B_{kj}$
- **Determinant of 2x2 Matrix**: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- **Inverse Matrix (2x2)**: $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- **Eigenvalues**: Solve $\det(A - \lambda I) = 0$ for eigenvalues λ.
- **Diagonalization**: For $A = PDP^{-1}$, where D is diagonal and columns of P are eigenvectors.
- **Trace and Determinant**: $\text{Tr}(A) = \sum \text{eigenvalues}, \det(A) = \prod \text{eigenvalues}.$
- **Tips and Tricks**:
  - **Row Reduction for Determinants**: Use elementary row operations to simplify calculation.
  - **Symmetric Matrices**: Eigenvalues are real, and eigenvectors are orthogonal.
  - **Complex Matrices**: For Hermitian matrices, eigenvalues are real.