STATS 406 F15: Lab 05

1 More on rejection sampling

- **Recap:**(What is rejection sampling?)
  Goal: sample from PDF $f(x)$.
  Method: use g that satisfies
  $$f(x) \leq Mg(x)$$
  for all x, where:
  - g is a distribution we know how to sample from.
  - M is a proper constant.

  Sample X from PDF $g(x)$, and accept with probability $f(X)/(Mg(X))$.

- **Remarks:**
  - **We only need to know f up to a constant factor.** (Theoretical proof omitted.) This is helpful when the normalizing constant for $f(x)$ is expensive to compute.
    Examples:
    $$f_1(x) \propto \frac{1}{(1+|x|^\alpha)^\beta} \text{ , for } \alpha\beta > 1$$
    $$f_2(x) \propto e^{-x^2} \cdot \mathbb{1}[x > C] \text{ , for } C = \text{constant}$$
    $$f_3(x) \propto \Phi(x) \text{ , where } \Phi = \text{CDF}(N(0,1)) \text{ and } x \leq 0$$
  - Meaningfulness: rejection sampling enables sampling from many "peculiar" distributions.

- **Another scenario where rejection sampling is meaningful.**

  Example:(Truncated standard normal distribution)
  $$f(x) \propto e^{-\frac{x^2}{2}} \cdot \mathbb{1}[x \geq C] \text{ , for } C = \text{constant}$$

  * **A nerdy approach:**(Essentially a naive rejection sampling)
    1. Sample X from $N(0,1)$.
    2. Reject X if $X < C$.

    This works fine when C is negative or a small positive number. **But what if $C \geq 2$?** (Recall the 1-, 2- and 3- sigma rule you learned in a preliminary course like STATS 250.)
    If $C = 3$, about 99% of the candidate sampled X will be rejected!
  * **A refined rejection sampling:**(only consider the challenging $C > 0$ case)
    1. Choose a proper M such that $f(x) \leq Mg(x)$ but $f(x)$ and $Mg(x)$ as close as possible.

* Set

$$M = \sup_{x \geq C} \frac{f(x)}{g(x)}$$

where the RHS equals $f(C)/g(C)$ in this example, since both $f(x)$ and $g(x)$ are decreasing, and $f(x)$ decreases at a faster rate than $g(x)$.

2. Sample X from $g(x) := e^{-(x-C)} \mathbb{1}[x \geq C]$.
   * This is an exponential distribution with rate $\lambda = 1$ and location shift C.
3. Given X, sample Y from Unif$[0, Mg(X)]$. Accept X if $Y \leq f(X)$ and reject X otherwise.
   * Equivalent to sampling Y from Unif$[0, 1]$ and accept X if $Y \leq f(X)/(Mg(X))$, as in textbooks, but we did differently to facilitate graphical illustration.

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# Implementation: see Lab_5.r
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* **NOTICE:** The code example used two ways to empirically check that the rejection method did produce the desired distribution:
  1. Fact: sampling from a distribution $f(x) \Leftrightarrow$ uniformly sample from the region between x-axis and $f(x)$.
     * In the example, accepted(dark green) points uniformly spread over this region.
  2. For some fixed a, b in the domain of x, compare $\hat{F}(b) - \hat{F}(a)$(recall what $\hat{F}(x)$ is – we saw it last week) and $\int_a^b f(x)\mathrm{d}x$.
     * In the example, the proportion of accepted points falling in $\{(x, y) : a \leq x \leq b\}$(which is $\hat{F}(b) - \hat{F}(a)$, check this!) is close to $\int_a^b f(x)\mathrm{d}x$.

2 Monte Carlo integration

- **Goal:** Compute $I := \int_a^b f(x)\mathrm{d}x$, where $a < b$, by Monte Carlo methods.
- **Method:**
  1. Choose a proper distribution $\pi(x)$, such that $\pi(x)>0$ on $[a, b]$.
  2. Rewrite the integral as

  $$I = \int_a^b \frac{f(x)}{\pi(x)} \cdot \pi(x)\mathrm{d}x = \mathbb{E}\left[\frac{f(x)}{\pi(x)}\right]$$

  and the integral I can be estimated by

  $$\hat{I} = \hat{\mathbb{E}}\left[\frac{f(x)}{\pi(x)}\right] = \frac{1}{n}\left\{\frac{f(X_1)}{\pi(X_1)} + \ldots + \frac{f(X_n)}{\pi(X_n)}\right\}$$

  for $X_1, \ldots, X_n$ sampled from PDF $\pi(x)$.
- **Easy case:** when a and b are finite, usually choosing $\pi(x) = 1/(b - a)$(uniform distribution on $[a, b]$) is good enough.
- **More challenging case:** with a and/or b being infinity, we need to carefully choose $\pi(x)$.

  Example: Let

  $$f(x) = \sin\left\{\frac{\cos(x)}{x^3}\right\}$$

  Compute the following Lebesgue integral:

  $$I = \int_{-\infty}^{+\infty} f(x)\mathrm{d}x = \int_{-\infty}^{+\infty} \sin\left\{\frac{\cos(x)}{x^3}\right\}\mathrm{d}x$$

* This is indeed a valid integral, because $\sin\left|\cos(x)/x^3\right|$ is integrable.
* By negative symmetry of the integrand, we immediately know $I = 0$.
* Now following the steps above, we should choose a proper $\pi(x)$, and rewrite

$$I = \int_{-\infty}^{+\infty} \frac{\sin\left\{\frac{\cos(x)}{x^3}\right\}}{\pi(x)} \cdot \pi(x)\mathrm{d}x = \mathbb{E}\left[\sin\left\{\frac{\cos(X)}{X^3}\right\}\Big/\pi(X)\right]$$

for $X \overset{\text{PDF}}{\sim} \pi(x)$. Then estimate I by

$$\hat{I} = \hat{\mathbb{E}}\left[\sin\left\{\frac{\cos(X)}{X^3}\right\}\Big/\pi(X)\right] = \frac{1}{n}\left[\sin\left\{\frac{\cos(X_1)}{X_1^3}\right\}\Big/\pi(X_1) + \ldots + \sin\left\{\frac{\cos(X_n)}{X_n^3}\right\}\Big/\pi(X_n)\right]$$

with $X_1, \ldots, X_n$ drawn from PDF $\pi(x)$.
* Consider two choices of $\pi(x)$.
  1. Cauchy: $\pi(x) \propto 1/(1+x^2)$.
     (+) Here $\lim_{x\to+\infty} f(x)/\pi(x) = 0$. The estimator $\hat{I}$ is "stable" in the sense that

     $$\mathrm{Var}\left[\sin\left\{\frac{\cos(X)}{X^3}\right\}\Big/\pi(X)\right] < +\infty$$

     Cauchy PDF is a fine choice for $\pi(x)$.
  2. Normal: $\pi(x) \propto e^{-x^2/2}$.
     (-) Here, however, we notice that $\lim_{x\to+\infty} f(x)/\pi(x) = 0$, and worse

     $$\begin{aligned}
     &\mathrm{Var}\left[\sin\left\{\frac{\cos(X)}{X^3}\right\}\Big/\pi(X)\right] = \mathbb{E}\left[\sin\left\{\frac{\cos(X)}{X^3}\right\}\Big/\pi(X)\right]^2 \\
     = &\int_{-\infty}^{+\infty} \sin^2\left\{\frac{\cos(x)}{x^3}\right\}\Big/\pi(x)\mathrm{d}x \\
     \sim &\int_{-\infty}^{+\infty} \cos^2(x)\frac{x^{-6}}{e^{-\frac{x^2}{2}}}\mathrm{d}x = +\infty \quad \text{(Why?)}
     \end{aligned}$$

     (-) As a consequence, asymptotically, we have a consistent estimation for I, but cannot consistently estimate a confidence interval using standard error(whose expectation is infinity). In finite sample, we cannot sense how close our estimator $\hat{I}$ is to the true I.

     Normal PDF is a bad choice for $\pi(x)$.

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# Implementation: see Lab_5.r
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