STAT 406: HW9

- All computer code should be written using the language R. Type ALL your code into one PLAIN Text format file. Plain text format is available by default in R. Please do not use Microsoft Word .doc format or .rtf format of .pdf format. Inside your plain text file, make sure you identify each problem in a comment placed at the beginning of the problem. The file name should match your name as in 'JohnDoe.R'.

- Submit your R code file online (under Assignments) at or before the due date, and hand in a hard copy of the code as well as a printed copy of your answers to the questions. The hard copy is due at the beginning of your respective lab sessions.

- I recommend that before submitting your homework, you also create a new directory and run your R code, to make sure that it is self-contained and runs as you intended.

1. Consider estimating the integral $I = \int_0^1 e^{-x^2} dx$. Implement the following two methods. In both cases, use a Monte Carlo sample size $N = 10^4$, and return the estimate and its Monte Carlo error.

   (a) Write the integral as an expectation against the Beta density $\mathcal{B}(2, \beta)$. Use this to implement an importance sampling method to approximate I. Try different values of β between 0.1 and 5, and use the value of β with the smaller Monte Carlo error. Use this value of β to report your estimate of I and its Monte Carlo error.

   (b) Let $g(x) = e^{-x} + e^{-1} - 1$. If $X_1, \ldots, X_N$ are i.i.d. random variables from the uniform distribution $\mathcal{U}(0, 1)$, show that for any $\alpha \in \mathbb{R}$,

$$\hat{I}_N = \frac{1}{N} \sum_{i=1}^{N} \left(e^{-X_i^2} + \alpha g(X_i) \right),$$

is a valid estimate of I (it converges to I as $N \to \infty$). What value of α gives the smallest Monte Carlo error (approximately). Use this value of α to report your estimate of I and its Monte Carlo error.

2. Consider the vector

$$m = (2.16,\ 0.74,\ 1.87,\ 3.03,\ 3.11,\ 2.74,\ 1.23,\ 3.64,\ 1.57,\ 2.12)\,,$$

and the density f studied in HW5:

$$f(x) = \frac{x^{\alpha-1}\exp\left(-\beta x - \frac{1}{2}\sum_{i=1}^{10}(x-m_i)^2\right)}{\int_0^\infty x^{\alpha-1}\exp\left(-\beta x - \frac{1}{2}\sum_{i=1}^{10}(x-m_i)^2\right)dx},\quad x \geq 0,$$

where $\alpha = 2$, and $\beta = 0.5$. Again we wish estimate the integral $I = \int_0^\infty x f(x)dx$ by Monte Carlo, but this time using importance sampling. As instrument density g we use the Gamma density $\mathcal{G}(a, b)$ with density $g(x) = b^a\Gamma(a)^{-1}x^{a-1}e^{-bx}$, $x \geq 0$, with $a = \alpha = 2$ and for some positive parameter b that needs to be chosen.

(a) Estimate the integral I by importance sampling without computing the normlizing constant of f. Use a Monte Carlo size $N = 10^4$. Choose the value b for which the coefficient of variation (CV) of the importance sampling weights is the smallest (approximately). Report your estimate of I and its Monte Carlo error.