(1) Find the least squares solution (i.e., $(A^T A)\hat{x} = A^T b$) and the projection ($p = A\hat{x}$) of the following

(a)

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}; \quad b = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}; \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}; \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(2) Find the least squares regression, and sketch the points and curve on the same graph.

(a) line for data points

(i) $(1,1)$, $(2,3)$, $(4,5)$.

(ii) $(-2,4)$, $(-1,3)$, $(0,1)$, $(2,0)$

(iii) $(-1,2)$, $(0,0)$, $(1,-3)$, $(2,-5)$

(b) parabola (quadratic polynomial) for data points $(-2,0)$, $(-1,0)$, $(0,1)$, $(1,2)$, $(2,5)$.

(3) Project the vector b onto he line through a, and check that $e = b - p$ is perpendicular to a:

(a) $b = (1,2,2)$ and $a = (1,1,1)$

(b) $b = (1,3,1)$ and $a = (-1,-3,-1)$

(4) Draw the projection of b onto a and also compute it from $p = \hat{x}a$:

(a) $(\cos\theta, \sin\theta)$ and $a = (1,0)$ for an arbitrary constant θ

(b) $b = (1,1)$ and $a = (1,-1)$

(5) Apply Gram-Schmidt to

(a) $a = (0,0,1)$, $b = (0,1,1)$, $c = (1,1,1)$.

(b) $a = (1,1,0)$, $b = (1,0,1)$, $c = (0,1,1)$.

(c) $a = (1,2,2)$, $b = (1,3,1)$.

(6) Find the eigenvalues and eigenvectors of the following matrices

(a)

$$\begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$$

(b)

$$\begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

(d)
$$\begin{bmatrix} 3 & 2 & -3 \\ -3 & -4 & 9 \\ -1 & -2 & 5 \end{bmatrix}$$
(e)
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$
(f)
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$
(g)
$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$
(h)
$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$
and A^T. Are the eigenvalues the same? Are the eigenvectors the same?

(i)
$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$
and A^T. Are the eigenvalues the same? Are the eigenvectors the same?

(j) Consider
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \cdots & \cdots & \cdots \end{bmatrix}$$
What should the third row of A be in order to give us the characteristic polynomial $-\lambda^3 + 4\lambda^2 + 5\lambda + 6$.

(k) Compute the eigenvalues of
$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$
and $A + I$. How are they related?

(l) Compute the eigenvalues of
$$A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$$
and A^{-1}. How are they related?

(m) Compute the eigenvalues of
$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$
and A^2. How are they related?