5.3 Gram-Schmidt

By now we are used to finding bases, but recall that orthogonal, or even better, orthonormal bases are preferred.

Definition 1. The vectors $q_1, \dots, q_n$ are orthonormal if

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \quad \text{(giving orthogonality)}, \\ 1 & \text{if } i = j \quad \text{(giving the normalization)}; \end{cases} \tag{1}$$

We can also create matrices out of these bases. Notice that the standard basis for an Euclidean space is in the columns of the identity matrix. However, if we want a generic orthonormal basis we need to apply the Gram-Schmidt orthogonalization procedure.

Theorem 1. *If Q (square or rectangular) has orthonormal columns, then $Q^T Q = 1$.*

Definition 2. An orthogonal matrix is a square matrix with orthonormal columns.

Theorem 2. *For orthogonal matrices, the transpose is the inverse.*

Ex: Consider

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Rightarrow Q^T = Q^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

which we can verify by multiplying.

Ex: Any permutation matrix P (consisting of only row exchanges) is an orthogonal matrix. The criteria of orthonormal columns and square are trivially satisfied. Then we check $P^{-1} = P^T$ by checking $PP^T = I$.

Theorem 3. *Multiplication by any Q preserves lengths: $||Qx|| = ||x||$ for all x. It also preserves inner products and angles: $(Qx)^T(Qy) = x^T Q^T Q y = x^T y$.*

Consider $Qx = b$ where q_i are the columns of Q. Then we can write

$$b = x_1 q_1 + x_2 q_2 + \cdots + x_i q_i + \cdots + x_{n-1} q_{n-1} + x_n q_n$$

If we multiply both sides by q_i^T we get

$$q_i^T = 0 + \cdots + x_i q_i^T q_i + \cdots + 0 = x_i \Rightarrow x = Q^T b.$$

So if your A is an orthogonal matrix, you don't have to do Gaussian Elimination.

The Gram-Schmidt Process

Suppose you are given three independent vectors $\vec{a}, \vec{b}, \vec{c}$. If they are orthonormal we can project a vector $\vec{v}$ onto $\vec{a}$ by doing $(\vec{a}^T\vec{v})\vec{a}$. To project onto the $\vec{a} - \vec{b}$ plane we do $(\vec{a}^T\vec{v})a + (\vec{b}^T\vec{v})b$, etc.

Process: We are given $\vec{a}, \vec{b}, \vec{c}$ and we want $\vec{q}_1, \vec{q}_2, \vec{q}_3$. No problem with q_1; i.e., $q_1 = a/||a||$ (we don't have to change its direction, just normalize.) The problem begins with q_2, which has to be orthogonal to q_1. If the vector b has any component in the direction of q_1 (i.e., direction of a) it has to be subtracted: $B = b - (q_1^T b)q_1$, then $q_2 = B/||B||$, and this continues for q_3: $C = c - (q_1^T c)q_1 - (q_2^T c)q_2$, then $q_3 = C/||C||$, so on and so forth.

Ex: $a = (1,0,1)$, $b = (1,0,0)$, and $c = (2,1,0)$ for $A = [a \quad b \quad c]$.

Solution:

Step 1: Make the first vector into a unit vector: $\boxed{q_1 = a/\sqrt{2} = (1/\sqrt{2}, 0, 1/\sqrt{2})}$.

Step 2a: Subtract from the second vector its component in the direction of the first: $\boxed{B = b - (q_1^T b)q_1 = (1/2, 0, -1/2)}$.

Step 2b: Divide B by its magnitude: $\boxed{q_2 = B/||B|| = (1/\sqrt{2}, 0, -1/\sqrt{2})}$.

Step 3a: Subtract from the third vector its component in the firs and second directions: $\boxed{C = c - (q_1^T c)q_1 - (q_2^T c)q_2 = (0,1,0)}$.

Step 3b: We normalize C, but C is already a unit vector so $\boxed{q_3 = (0,1,0)}$

Then we can write Q as the matrix

$$Q = \begin{pmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}$$

From the matrix Q we can get a $A = QR$ factorization. This means that $A = QR \Rightarrow Q^T A = R$, then

$$R = \begin{pmatrix} --- & q_1^T & --- \\ --- & q_2^T & --- \\ --- & q_3^T & --- \end{pmatrix} \begin{pmatrix} | & | & | \\ a & b & c \\ | & | & | \end{pmatrix} = \begin{pmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{pmatrix} \tag{2}$$

Now lets do a bunch of examples. I skip some of the computation steps in the notes, but they are done in detail in the lectures.

Ex: Are the vectors orthonormal?

$$\left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

They are orthogonal but not normal, so they are not orthonormal.

Ex: Are the vectors orthonormal?

$$\left\{ \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ -17 \\ -1 \end{bmatrix} \right\}$$

They are orthogonal but not normal, so they are not orthonormal.

Ex: Orthonormalize the following set of vectors

$$\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Solution: We can get $\boxed{q_1 = (3,4)/5}$ immediately. Then

$$B = b - (q_1^T b)q_1 = (1,0) - \frac{3}{5}(3,4)/5 = \boxed{(16/25, -12/25)} \Rightarrow q_2 = \frac{(4^2/5^2, -12/5^2)}{\sqrt{(4^4/5^4) + (3^2 \cdot 4^2)/5^4}} = \boxed{(4/5, -3/5)}.$$

Ex: Orthonormalize the following set of vectors

$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$$

Solution: $\boxed{q_1 = (0,1)}$. Then

$$B = b - (q_1^T b)q_1 = (2,5) - 5(0,1) = \boxed{(2,0)} \Rightarrow \boxed{q_2 = (1,0)}.$$

Ex: Orthonormalize the following set of vectors

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Solution: The vectors are already orthogonal, so just divide by the magnitude.

Ex: Orthonormalize the following set of vectors

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Solution: $\boxed{q_1 = (0,1,1)/\sqrt{2}}$. Then

$$B = b - (q_1^T b)q_1 = (1,1,0) - \frac{1}{\sqrt{2}}(0, 1/\sqrt{2}, 1/\sqrt{2}) = (1, 1/2, -1/2) \Rightarrow q_2 = (1, 1/2, -1/2)/\sqrt{3/2} = (\sqrt{2/3}, \sqrt{2/3}/2, -\sqrt{2/3}/2).$$

And

$$C = c - (q_1^T c)q_1 - (q_2^T c)q_2 = (1,0,1) - \frac{1}{\sqrt{2}}(0, 1/\sqrt{2}, 1/\sqrt{2}) - \frac{\sqrt{2/3}}{2}(\sqrt{2/3}, \sqrt{2/3}/2, -\sqrt{2/3}/2)$$
$$= (1,0,1) - (0, 1/2, 1/2) - (1/3, 1/6, -1/6) = (2/3, -2/3, 2/3).$$

So,

$$q_3 = (2/3, -2/3, 2/3)/(2\sqrt{2/3}) = (1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}).$$