(1) Given the system

$$\begin{aligned} 3x_1 + 3x_2 + 3x_3 + 9x_4 &= b_1 \\ 2x_1 - x_2 + 4x_3 + 7x_4 &= b_2 \\ 3x_1 - 5x_2 - x_3 + 7x_4 &= b_3 \end{aligned}$$

(a) Write the system in matrix form $Ax = b$.
(b) Will this have solutions for all $b \in \mathbb{R}^3$? If not, which vectors b give no solution?
(c) Solve the system of equations for $b = (b_1, b_2, b_3) = (3, 2, 3)$.

(2) Are there any vectors x such that

$$\begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 3 & 5 \end{bmatrix} x = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

(3) Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

(a) Find the inverse of A.
(b) Are the columns of A linearly independent (You don't have to do any work, just explain why or why not)?

(4) Find an LU factorization of

$$A = \begin{bmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ 2 & -4 & 7 & 5 \end{bmatrix}$$

(5) Suppose we know the LU factorization of some matrix A to be

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{bmatrix}; \qquad U = \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve for $Ax = b$ where

$$b = \begin{bmatrix} 1 \\ 2 \\ 9 \\ -6 \end{bmatrix}$$

(6) Let A be and $m \times n$ matrix where r is the number of its pivot columns. What are the conditions on m, n, and r (other than $r \leq m$ and $r \leq n$, which is always true) such that $Ax = b$
(a) has infinitely many solutions for each b.
(b) has exactly one solution for each b.

(7) Suppose A has the row echelon form R; i.e.,

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 6 & 4 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 7 & 10 & -3 & 12 \end{bmatrix}; \qquad R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the bases for the row space, column space, and null space of A.

(8) Let

$$A = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix}$$

be a 3×3 matrix with rows $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and let $\det(A) = 2$.
(a) Find an elementary matrix E such that

$$EA = B = \begin{bmatrix} \mathbf{c} + 3\mathbf{b} \\ 2\mathbf{b} \\ \mathbf{a} \end{bmatrix}$$

(b) Compute the determinant of B.
(c) Compute the determinant of $2BA^2(B^T)^{-1}$.

(9) Find the least squares solution (i.e., $(A^T A)\hat{x} = A^T b$) and the projection ($p = A\hat{x}$) of the following
(a)

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}; \quad b = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}; \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}; \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(10) Find the least squares regression, and sketch the points and curve on the same graph.
  (a) line for data points
    (i) $(1,1)$, $(2,3)$, $(4,5)$.
    (ii) $(-2,4)$, $(-1,3)$, $(0,1)$, $(2,0)$
    (iii) $(-1,2)$, $(0,0)$, $(1,-3)$, $(2,-5)$
  (b) parabola (quadratic polynomial) for data points $(-2,0)$, $(-1,0)$, $(0,1)$, $(1,2)$, $(2,5)$.

(11) Project the vector b onto he line through a, and check that $e = b - p$ is perpendicular to a:
  (a) $b = (1,2,2)$ and $a = (1,1,1)$
  (b) $b = (1,3,1)$ and $a = (-1,-3,-1)$

(12) Draw the projection of b onto a and also compute it from $p = \hat{x}a$:
  (a) $(\cos\theta, \sin\theta)$ and $a = (1,0)$ for an arbitrary constant θ
  (b) $b = (1,1)$ and $a = (1,-1)$

(13) Apply Gram-Schmidt to
  (a) $a = (0,0,1)$, $b = (0,1,1)$, $c = (1,1,1)$.
  (b) $a = (1,1,0)$, $b = (1,0,1)$, $c = (0,1,1)$.
  (c) $a = (1,2,2)$, $b = (1,3,1)$.

(14) Find the eigenvalues and eigenvectors of the following matrices, then state whether they are diagonalizable, and if they are find the diagonalization $A = S\Lambda S^{-1}$.
  (a)

$$\begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$$

  (b)

$$\begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}$$

  (c)

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

  (d)

$$\begin{bmatrix} 3 & 2 & -3 \\ -3 & -4 & 9 \\ -1 & -2 & 5 \end{bmatrix}$$

  (e)

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

  (f)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

(g)

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

(h)

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

and A^T. Are the eigenvalues the same? Are the eigenvectors the same?

(i)

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

and A^T. Are the eigenvalues the same? Are the eigenvectors the same?

(j) Consider

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \cdots & \cdots & \cdots \end{bmatrix}$$

What should the third row of A be in order to give us the characteristic polynomial $-\lambda^3 + 4\lambda^2 + 5\lambda + 6$.

(k) Compute the eigenvalues of

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

and $A + I$. How are they related?

(l) Compute the eigenvalues of

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$$

and A^{-1}. How are they related?

(m) Compute the eigenvalues of

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$

and A^2. How are they related?

(15) Consider the following ODE,

$$\frac{d}{dt}\vec{x} = \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} \vec{x};$$

Find the eigenvalues and eigenvectors of the corresponding matrix.

(16) Consider the following ODE,

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & -3 \end{pmatrix} \vec{x};$$

Find the eigenvalues and eigenvectors of the corresponding matrix.

(17) Consider the following ODE,

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \vec{x};$$

Find the eigenvalues and eigenvectors of the corresponding matrix.