(1) Given the system

$$\begin{aligned} 3x_1 + 3x_2 + 3x_3 + 9x_4 &= b_1 \\ 2x_1 - x_2 + 4x_3 + 7x_4 &= b_2 \\ 3x_1 - 5x_2 - x_3 + 7x_4 &= b_3 \end{aligned}$$

(a) Write the system in matrix form $Ax = b$.
(b) Will this have solutions for all $b \in \mathbb{R}^3$? If not, which vectors b give no solution?
(c) Solve the system of equations for $b = (b_1, b_2, b_3) = (3, 2, 3)$.

(2) Are there any vectors x such that

$$\begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 3 & 5 \end{bmatrix} x = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

(3) Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

(a) Find the inverse of A.
(b) Are the columns of A linearly independent (You don't have to do any work, just explain why or why not)?

(4) Find an LU factorization of

$$A = \begin{bmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ 2 & -4 & 7 & 5 \end{bmatrix}$$

(5) Suppose we know the LU factorization of some matrix A to be

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{bmatrix}; \qquad U = \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve for $Ax = b$ where

$$b = \begin{bmatrix} 1 \\ 2 \\ 9 \\ -6 \end{bmatrix}$$

(6) Let A be and $m \times n$ matrix where r is the number of its pivot columns. What are the conditions on m, n, and r (other than $r \leq m$ and $r \leq n$, which is always true) such that $Ax = b$

(a) has infinitely many solutions for each b.

(b) has exactly one solution for each b.

(7) Suppose A has the row echelon form R; i.e.,

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 6 & 4 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 7 & 10 & -3 & 12 \end{bmatrix}; \qquad R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the bases for the row space, column space, and null space of A.

(8) Let

$$A = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix}$$

be a 3×3 matrix with rows $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and let $\det(A) = 2$.

(a) Find an elementary matrix E such that

$$EA = B = \begin{bmatrix} \mathbf{c} + 3\mathbf{b} \\ 2\mathbf{b} \\ \mathbf{a} \end{bmatrix}$$

(b) Compute the determinant of B.

(c) Compute the determinant of $2BA^2(B^T)^{-1}$.