

(1) Given the system

$$3x_1 + 3x_2 + 3x_3 + 9x_4 = b_1$$

$$2x_1 - x_2 + 4x_3 + 7x_4 = b_2$$

$$3x_1 - 5x_2 - x_3 + 7x_4 = b_3$$

- (a) Write the system in matrix form  $Ax = b$ .  
 (b) Will this have solutions for all  $b \in \mathbb{R}^3$ ? If not, which vectors  $b$  give no solution?  
 (c) Solve the system of equations for  $b = (b_1, b_2, b_3) = (3, 2, 3)$ .

**Solution:**

$$\begin{aligned} \frac{1}{3} \begin{bmatrix} 3 & 3 & 3 & 9 & | & 3 \\ 2 & -1 & 4 & 7 & | & 2 \\ 3 & -5 & -1 & 7 & | & 3 \end{bmatrix} &= \frac{2}{3} \begin{bmatrix} 1 & 1 & 1 & 3 & | & 1 \\ 2 & -1 & 4 & 7 & | & 2 \\ 3 & -5 & -1 & 7 & | & 3 \end{bmatrix} = \frac{8}{3} \begin{bmatrix} 1 & 1 & 1 & 3 & | & 1 \\ 0 & -3 & 2 & 1 & | & 0 \\ 0 & -8 & -4 & -2 & | & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 & 3 & | & 1 \\ 0 & -3 & 2 & 1 & | & 0 \\ 0 & 0 & -28/3 & -14/3 & | & 0 \end{bmatrix} \end{aligned}$$

$$\text{Then } x_4 = -2x_3 \Rightarrow x_2 = 0 \Rightarrow x_1 = 5x_3.$$

(2) Are there any vectors  $x$  such that

$$\begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 3 & 5 \end{bmatrix} x = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\textbf{Solution:} \quad \text{Choose } x_4 = 0 \Rightarrow x_3 = 1 \Rightarrow x_2 = -1 \Rightarrow x_1 = 3$$

(3) Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

(a) Find the inverse of  $A$ .

$$\begin{aligned} \begin{bmatrix} 0 & 1 & -1 & | & 1 & 0 & 0 \\ 2 & -2 & -1 & | & 0 & 1 & 0 \\ -1 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & -1 & -1 & | & 0 & 0 & -1 \\ 0 & 1 & -1 & | & 1 & 0 & 0 \\ 2 & -2 & -1 & | & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & | & 0 & 0 & -1 \\ 0 & 1 & -1 & | & 1 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 0 & | & 0 & 1 & 1 \\ 0 & 1 & 0 & | & 1 & 1 & 2 \\ 0 & 0 & 1 & | & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & | & 1 & 2 & 3 \\ 0 & 1 & 0 & | & 1 & 1 & 2 \\ 0 & 0 & 1 & | & 0 & 1 & 2 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \end{aligned}$$

(b) Are the columns of  $A$  linearly independent (You don't have to do any work, just explain why or why not)?

**Solution:** As Severus Snape would say, "Obviously". We were able to take the inverse, which means the matrix is nonsingular, and therefore the columns of  $A$  must be linearly independent.

(4) Find the row echelon form of

$$A = \begin{bmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ 2 & -4 & 7 & 5 \end{bmatrix}$$

**Solution:**

$$\begin{array}{c} -1 \\ 4 \\ 2 \end{array} \begin{bmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ 2 & -4 & 7 & 5 \end{bmatrix} = \begin{array}{c} \\ 5 \\ 5 \end{array} \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & -10 & 15 & 5 \\ 0 & -10 & 17 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

(5) Suppose we know the  $LU$  factorization of some matrix  $A$  to be

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{bmatrix}; \quad U = \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve for  $Ax = b$  where

$$b = \begin{bmatrix} 1 \\ 2 \\ 9 \\ -6 \end{bmatrix}$$

**Solution:** Since we know the  $LU$  factorization, we recall that  $Ax = b \Leftrightarrow LUx = b \Leftrightarrow Ux = L^{-1}b$ , and

$$L^{-1}b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 9 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

Then,

$$Ux = \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 4 \\ 4 \\ -2 \end{bmatrix} \Rightarrow \boxed{x_4 = 0} \Rightarrow \boxed{x_3 = 0} \Rightarrow \boxed{x_2 = -2} \Rightarrow \boxed{x_1 = 7}.$$

- (6) Let  $A$  be an  $m \times n$  matrix where  $r$  is the number of its pivot columns. What are the conditions on  $m$ ,  $n$ , and  $r$  (other than  $r \leq m$  and  $r \leq n$ , which is always true) such that  $Ax = b$
- (a) has infinitely many solutions for each  $b$ .
  - (b) has exactly one solution for each  $b$ .

(7) Suppose  $A$  has the row echelon form  $R$ ; i.e.,

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 6 & 4 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 7 & 10 & -3 & 12 \end{bmatrix}; \quad R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the bases for the row space, column space, and null space of  $A$ .

**Solution:** Row space:  $\{(1, 0, 1, 0), (0, 1, -1, 0), (0, 0, 0, 1)\}$ .

Column space:  $\{(3, 6, 1, 0, 7), (2, 4, 0, 1, 10), (0, 0, 0, 3, 12)\}$ .

Nullspace:  $\{(-1, 1, 1, 0)\}$ .

(8) Let

$$A = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix}$$

be a  $3 \times 3$  matrix with rows  $\mathbf{a}, \mathbf{b}, \mathbf{c}$ , and let  $\det(A) = 2$ .

(a) Find an elementary matrix  $E$  such that

$$EA = B = \begin{bmatrix} \mathbf{c} + 3\mathbf{b} \\ 2\mathbf{b} \\ \mathbf{a} \end{bmatrix}$$

**Solution:** Here we need to fill out the row operations in  $E$ ,

$$E = \begin{bmatrix} 0 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) Compute the determinant of  $B$ .

**Solution:**  $|B| = |E||A| = 2|E|$ , then

$$|B| = 2 \begin{vmatrix} 0 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 2 \cdot (-1 \cdot 2 \cdot 1) = -4.$$

(c) Compute the determinant of  $2BA^2(B^T)^{-1}$ .

**Solution:**  $|2BA^2(B^T)^{-1}| = 2^3|B||A|^2|(B^T)^{-1}| = 2^3|B||A|^2/|B| = 2^3|A|^2 = 32.$