These will not appear on the Final, but some similar problems might.

(1) Find the least squares solution (i.e., $(A^TA)\hat{x} = A^Tb$) and the projection ($p = A\hat{x}$) of the following
(a)

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}; \quad b = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix}; \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}; \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(2) Find the least squares regression, and sketch the points and curve on the same graph.
(a) line for data points
(i) $(1, 1)$, $(2, 3)$, $(4, 5)$.
(ii) $(-2, 4)$, $(-1, 3)$, $(0, 1)$, $(2, 0)$
(iii) $(-1, 2)$, $(0, 0)$, $(1, -3)$, $(2, -5)$
(b) parabola (quadratic polynomial) for data points $(-2, 0)$, $(-1, 0)$, $(0, 1)$, $(1, 2)$, $(2, 5)$.

(3) Project the vector $b$ onto he line through $a$, and check that $e = b - p$ is perpendicular to $a$:
(a) $b = (1, 2, 2)$ and $a = (1, 1, 1)$
(b) $b = (1, 3, 1)$ and $a = (-1, -3, -1)$

(4) Draw the projection of $b$ onto $a$ and also compute it from $p = \hat{x}a$:
(a) $(\cos\theta, \sin\theta)$ and $a = (1, 0)$ for an arbitrary constant $\theta$
(b) $b = (1, 1)$ and $a = (1, -1)$

(5) Apply Gram-Schmidt to
(a) $a = (0, 0, 1)$, $b = (0, 1, 1)$, $c = (1, 1, 1)$.
(b) $a = (1, 1, 0)$, $b = (1, 0, 1)$, $c = (0, 1, 1)$.
(c) $a = (1, 2, 2)$, $b = (1, 3, 1)$.

(6) Find the eigenvalues and eigenvectors of the following matrices

(a)
$$\begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$$

(b)
$$\begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

(d)
$$\begin{bmatrix} 3 & 2 & -3 \\ -3 & -4 & 9 \\ -1 & -2 & 5 \end{bmatrix}$$

(e)
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

(f)
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

(g)
$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

(h)
$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

and $A^T$. Are the eigenvalues the same? Are the eigenvectors the same?

(i)
$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

and $A^T$. Are the eigenvalues the same? Are the eigenvectors the same?

(j) Consider
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \cdots & \cdots & \cdots \end{bmatrix}$$

What should the third row of $A$ be in order to give us the characteristic polynomial $-\lambda^3 + 4\lambda^2 + 5\lambda + 6$.

(k) Compute the eigenvalues of
$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

and $A + I$. How are they related?

(l) Compute the eigenvalues of

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$$

and $A^{-1}$. How are they related?

(m) Compute the eigenvalues of

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$

and $A^2$. How are they related?

(7) Given the system

$$\begin{aligned} 3x_1 + 3x_2 + 3x_3 + 9x_4 &= b_1 \\ 2x_1 - x_2 + 4x_3 + 7x_4 &= b_2 \\ 3x_1 - 5x_2 - x_3 + 7x_4 &= b_3 \end{aligned}$$

(a) Write the system in matrix form $Ax = b$.
(b) Will this have solutions for all $b \in \mathbb{R}^3$? If not, which vectors $b$ give no solution?
(c) Solve the system of equations for $b = (b_1, b_2, b_3) = (3, 2, 3)$.

(8) Are there any vectors $x$ such that

$$\begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 3 & 5 \end{bmatrix} x = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

(9) Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

(a) Find the inverse of $A$.
(b) Are the columns of $A$ linearly independent (You don't have to do any work, just explain why or why not)?