(1) Given the system

$$\begin{aligned} 3x_1 + 3x_2 + 3x_3 + 9x_4 &= b_1 \\ 2x_1 - x_2 + 4x_3 + 7x_4 &= b_2 \\ 3x_1 - 5x_2 - x_3 + 7x_4 &= b_3 \end{aligned}$$

(a) Write the system in matrix form $Ax = b$.
(b) Will this have solutions for all $b \in \mathbb{R}^3$? If not, which vectors b give no solution?
(c) Solve the system of equations for $b = (b_1, b_2, b_3) = (3, 2, 3)$.
Solution:

$$\begin{array}{c} 1/3 \\ \\ \\ \end{array}\begin{bmatrix} 3 & 3 & 3 & 9 & | & 3 \\ 2 & -1 & 4 & 7 & | & 2 \\ 3 & -5 & -1 & 7 & | & 3 \end{bmatrix} = \begin{array}{c} \\ 2 \\ 3 \end{array}\begin{bmatrix} 1 & 1 & 1 & 3 & | & 1 \\ 2 & -1 & 4 & 7 & | & 2 \\ 3 & -5 & -1 & 7 & | & 3 \end{bmatrix} = \begin{array}{c} \\ \\ 8/3 \end{array}\begin{bmatrix} 1 & 1 & 1 & 3 & | & 1 \\ 0 & -3 & 2 & 1 & | & 0 \\ 0 & -8 & -4 & -2 & | & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 3 & | & 1 \\ 0 & -3 & 2 & 1 & | & 0 \\ 0 & 0 & -28/3 & -14/3 & | & 0 \end{bmatrix}$$

Then $\boxed{x_4 = -2x_3} \Rightarrow \boxed{x_2 = 0} \Rightarrow \boxed{x_1 = 5x_3}$.

(2) Are there any vectors x such that

$$\begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 3 & 5 \end{bmatrix} x = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Solution: Choose $\boxed{x_4 = 0} \Rightarrow \boxed{x_3 = 1} \Rightarrow \boxed{x_2 = -1} \Rightarrow \boxed{x_1 = 3}$

(3) Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

(a) Find the inverse of A.

$$\begin{bmatrix} 0 & 1 & -1 & | & 1 & 0 & 0 \\ 2 & -2 & -1 & | & 0 & 1 & 0 \\ -1 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & | & 0 & 0 & -1 \\ 0 & 1 & -1 & | & 1 & 0 & 0 \\ 2 & -2 & -1 & | & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & | & 0 & 0 & -1 \\ 0 & 1 & -1 & | & 1 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 & 0 & | & 0 & 1 & 1 \\ 0 & 1 & 0 & | & 1 & 1 & 2 \\ 0 & 0 & 1 & | & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & | & 1 & 2 & 3 \\ 0 & 1 & 0 & | & 1 & 1 & 2 \\ 0 & 0 & 1 & | & 0 & 1 & 2 \end{bmatrix} \Rightarrow \boxed{A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}}$$

(b) Are the columns of A linearly independent (You don't have to do any work, just explain why or why not)?
Solution: As Severus Snape would say, "Obviously". We were able to take the inverse, which means the matrix is nonsingular, and therefore the columns of A must be linearly independent.

(4) Find an LU factorization of

$$A = \begin{bmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ 2 & -4 & 7 & 5 \end{bmatrix}$$

Solution:

$$\begin{array}{c} \\ -1 \\ 4 \\ 2 \end{array}\begin{bmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ 2 & -4 & 7 & 5 \end{bmatrix} = \begin{array}{c} \\ \\ 5 \\ 5 \end{array}\begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & -10 & 15 & 5 \\ 0 & -10 & 17 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U}$$

Then

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 4 & 5 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{bmatrix}}$$

(5) Suppose we know the LU factorization of some matrix A to be

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{bmatrix}; \qquad U = \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve for $Ax = b$ where

$$b = \begin{bmatrix} 1 \\ 2 \\ 9 \\ -6 \end{bmatrix}$$

Solution: Since we know the LU factorization, we recall that $Ax = b \Leftrightarrow LUx = b \Leftrightarrow Ux = L^{-1}b$, and

$$L^{-1}b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 9 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 4 \\ -2 \end{bmatrix}$$

Then,

$$Ux = \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 4 \\ 4 \\ -2 \end{bmatrix} \Rightarrow \boxed{x_4 = -2} \Rightarrow \boxed{x_3 = 6} \Rightarrow \boxed{x_2 = 0} \Rightarrow \boxed{x_1 = 25}.$$

(6) Let A be and $m \times n$ matrix where r is the number of its pivot columns. What are the conditions on m, n, and r (other than $r \leq m$ and $r \leq n$, which is always true) such that $Ax = b$

(a) has infinitely many solutions for each b.

(b) has exactly one solution for each b.

(7) Suppose A has the row echelon form R; i.e.,

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 6 & 4 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 7 & 10 & -3 & 12 \end{bmatrix}; \qquad R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the bases for the row space, column space, and null space of A.

Solution: Row space: $\boxed{\{(1,0,1,0),\ (0,1,-1,0),\ (0,0,0,1)\}}$.

Column space: $\boxed{\{(3,6,1,0,7),\ (2,4,0,1,10),\ (0,0,0,3,12)\}}$.

Nullspace: $\boxed{\{(-1,1,1,0)\}}$.

(8) Let

$$A = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix}$$

be a 3×3 matrix with rows $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and let $\det(A) = 2$.

(a) Find an elementary matrix E such that

$$EA = B = \begin{bmatrix} \mathbf{c} + 3\mathbf{b} \\ 2\mathbf{b} \\ \mathbf{a} \end{bmatrix}$$

Solution: Here we need to fill out the row operations in E,

$$\boxed{E = \begin{bmatrix} 0 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$$

(b) Compute the determinant of B.

Solution: $|B| = |E||A| = 2|E|$, then

$$|B| = 2\begin{vmatrix} 0 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 2 \cdot (-1 \cdot 2 \cdot 1) = \boxed{-4}.$$

(c) Compute the determinant of $2BA^2(B^T)^{-1}$.

Solution: $|2BA^2(B^T)^{-1}| = 2|B||A|^2|(B^T)^{-1}| = 2|B||A|^2/|B| = 2|A|^2 = \boxed{8}$.