Math 112 - 009 Rahman Exam 3 Practice Problems

One word of advice: Keep It Simple Stupid! - no, I'm not calling *you* stupid, don't tell on me. We learned a lot of stuff and did a lot of confusing examples. There are some weird - convoluted ways of solving certain problems, but you will not see complicated examples on the exam. You have to know what form will go with what test, and this comes through practice.

Spring 2011 solutions

(1) (a) Notice that $\left|\frac{\sin^2 n}{2^n+1}\right| \leq \frac{1}{2^n}$. Since $\sum_{n=1}^{\infty} 1/2^n$ converges by geometric series because $|1/2| < 1$, $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n+1}$ converges absolutely by the direct comparison test, and hence converges.

(b) This is a typical root test example,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1+\ln(n)}{n} = 0 < 1.$$

Therefore the series converges by root test.

(2) (a) For this we simply take the limit because we can tell it's going to diverge,

$$\lim_{n\to\infty} \frac{e^n}{1+3e^n} = \lim_{n\to\infty} \frac{1}{e^{-n}+3} = \frac{1}{3} \neq 0.$$

This shows that the series indeed diverges.

(b) This is a typical integral test example because we have a nice u-sub. Since we are experts on u-sub I'll skip some steps. Of course, if you are having trouble with the integration please see me ASAP.

$$\int_2^{\infty} \frac{\mathrm{d}x}{x(\ln(x))^{3/2}} = \int_{\ln 2}^{\infty} \frac{\mathrm{d}u}{u^{3/2}} = -\frac{2}{\sqrt{u}}\bigg|_{\ln 2}^{\infty} = \frac{2}{\sqrt{\ln 2}}.$$

Since the integral converges, so does the series by integral test.

(3) (a) Notice $\left|\frac{\tan^{-1} n}{n^2}\right| \leq \frac{\pi/2}{n^2}$. Since $\pi/2\sum_{n=1}^{\infty} 1/n^2$ converges by p-series because $p > 1$, the series converges absolutely by the direct comparison test.

(b) We have a feeling the series wont converge absolutely because it's a lot like the series for $1/\sqrt{n}$. Lets try this using limit comparison,

$$\lim_{n\to\infty} \frac{1/(1+\sqrt{n})}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1+1/\sqrt{n}} = 1.$$

This proves that $1/\sqrt{n}$ is a valid comparison. Since $\sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges by p-series because p¡1, $\sum_{n=1}^{\infty}(-1)^{n+1}1/(1+\sqrt{n})$ <u>does not</u> converge absolutely. Now, $\lim_{n\to\infty} 1/(1+\sqrt{n}) = 0$ and $1/(1+\sqrt{n}) > 1/(1+\sqrt{n+1})$, so by the alternating series test the series converges conditionally.

(4) (a) This is a typical ratio test problem,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^2 10^{n+1}} \cdot \frac{n^2 10^n}{n!}\right| = \lim_{n\to\infty}\left|\frac{n^2}{n+1}\right| = \lim_{n\to\infty}\frac{n}{1+1/n} = \infty > 1.$$

Therefore, by the ratio test, the series diverges.

(b) This is a typical limit comparison problem (note: direct comparison wont work), Lets compare with $1/n^{3/2}$.

$$\lim_{n\to\infty} \frac{n/\sqrt{2n^5-1}}{1/n^{3/2}} = \lim_{n\to\infty} \frac{n^{5/2}}{\sqrt{2n^5-1}} = \lim_{n\to\infty} \frac{1}{\sqrt{2-1/n^5}} = \frac{1}{\sqrt{2}}.$$

Since $\sum_{n=1}^{\infty} 1/n^{3/2}$ converges by p-series because $p > 1$, $\sum_{n=1}^{\infty} n/\sqrt{2n^5-1}$ converges by the limit comparison test.

(5) As with most radius of convergence problems we shall employ ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{10^{n+1}(x-1)^{n-1}}{(n+1)!} \cdot \frac{n!}{10^n(x-1)^n}\right| = \lim_{n\to\infty} \frac{10}{n+1}|x-1| = 0.$$

Therefore, the radius of convergence is $R = \infty$, hence it converges for $x \in (-\infty, \infty)$.

(6) For this Taylor series I will use our theorems to arrive at the solution swiftly, however if you are not 100% confident about remembering the series, DO NOT use this method - just take the derivatives and compute the first few terms.

We use the series for $\sin x$ and go from there,

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \Rightarrow \sin(2x) = \sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n+1}}{(2n+1)!}$$

$$\Rightarrow x^2\sin(2x) = \sum_{n=0}^{\infty}(-1)^n 2^{2n+1} \frac{x^{2n+3}}{(2n+1)!} = 2x^3 - \frac{4}{3}x^5 + \frac{2^5}{5!}x^7 + \cdots.$$

(7) As per usual we use ratio test to find the radius of convergence,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{(n+1)4^{n+1}}\cdot\frac{n4^n}{(x+1)^n}\right| = \lim_{n\to\infty}\frac{n}{4(n+1)}|x+1|$$
$$= \lim_{n\to\infty}\frac{1}{4(1+1/n)}|x+1| = \frac{1}{4}|x+1|.$$

Hence, the series converges absolutely for $|x+1| < 4$, i.e. $x \in (-5,3)$. Now we test the end points: $x = 3$: $\sum_{n=1}^{\infty}\frac{(x+1)^n}{n4^n} = \sum_{n=1}^{\infty} 1/n$ diverges by p-series because $p = 1$. $x = -5$: $\sum_{n=1}^{\infty}\frac{(x+1)^n}{n4^n} = \sum_{n=1}^{\infty}(-1)^n/n$ converges by alternating series test because $\lim_{n\to\infty} 1/n = 0$ and $1/n > 1/(n+1)$. Therefore, the interval of convergence is $[-5,3)$.

(8) (a) For this we have to start from scratch. Recall the definition of Taylor series, $f(x) = \sum_{n=0}^{\infty} f^{(n)}(a)(x-a)^n/n!$. Now, $f^{(n)}(x) = e^x$, hence $f^{(n)}(1) = e$. This makes life easy and we only have to plug it into the formula,

$$e^x \approx e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3.$$

(b) We recall the formula to bound the remainder of a Taylor series, $R_n(x) \le \frac{M}{(n+1)!}(x-a)^{n+1}$ where $f^{(n+1)}(x) \le M$ on our domain. Since $f^{(n)}(2) = e^2$, $M = e^2$, so $|R_3(x)| \le |M(x-1)^4/4!| < e^2/24$ because $|x-1| \le 1$ in our domain.

Fall 2011 solutions

(1) (a) This is a typical root test problem,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{n} - \frac{1}{n^2} = 0 < 1.$$

Therefore, the series converges by root test.

(b) Here we simply take the limit because we can tell that it will diverge,

$$\lim_{n\to\infty} ne^{-1/n} = \lim_{n\to\infty} \frac{n}{e^{1/n}} = \infty > 1.$$

This shows that the series does diverge.

(c) This is a typical limit comparison test problem. The largest power in the numerator is 4^n and the largest power in the denominator is 5^n, so lets try $4^n/5^n$,

$$\lim_{n\to\infty} \frac{\frac{3^n+4^n}{4^n+5^n}}{4^n/5^n} = \lim_{n\to\infty} \frac{15^n+20^n}{16^n+20^n} = \lim_{n\to\infty} \frac{(3/4)^n+1}{(4/5)^n+1} = 1.$$

This shows that our comparison is valid. Since $\sum_{n=1}^{\infty}(4/5)^n$ converges by geometric series because $4/5 < 1$, $\sum_{n=1}^{\infty} \frac{3^n+4^n}{4^n+5^n}$ also converges by the limit comparison test.

(2) (a) This is another limit comparison example. We compare this to $1/n^2$.

$$\lim_{n\to\infty} \frac{\frac{n^2+3n}{n^4+\sqrt{n}}}{1/n^2} = \lim_{n\to\infty} \frac{n^4+3n^3}{n^4+\sqrt{n}} = \lim_{n\to\infty} \frac{1+3/n}{1+1/n^{7/2}} = 1.$$

Since $\sum_{n=1}^{\infty} 1/n^2$ converges by p-series because $p > 1$, $\sum_{n=1}^{\infty} \frac{n^2+3n}{n^4+\sqrt{n}}$ also converges by the limit comparison test.

(b) This has been done already in the Spring 2011 section.

(3) Again for radius of convergence problems we use ratio test,

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(x+2)^{n+1}}{\sqrt{n+1}3^{n+1}} \cdot \frac{\sqrt{n}3^n}{(x+2)^n}\right| = \lim_{n\to\infty} \frac{\sqrt{n}}{3\sqrt{n+1}}|x+2| \\ &= \lim_{n\to\infty} \frac{1}{3\sqrt{1+1/n}}|x+2| = \frac{1}{3}|x+2|.\end{aligned}$$

Hence, the series is absolutely convergent for $|x+2| < 3$, i.e. $(-5, 1)$. Now we test the endpoints: $x = 1$: $\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}3^n} = \sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges by p-series because $p < 1$. $x = -5$: $\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}3^n} = \sum_{n=1}^{\infty}(-1)^n/n$ converges by alternating series test because $\lim_{n\to\infty} 1/\sqrt{n} = 0$ and $1/\sqrt{n} > 1/\sqrt{n+1}$.

(4) Here we have no choice, we must use Taylor's formula and it'll be slightly annoying. Recall $f(x) = \sum_{n=0}^{\infty} f^{(n)}(a)(x-a)^n/n!$. Now, $f(\pi/6) = \sqrt{3}/2$, $f'(\pi/6) = -\sin x|_{x=\pi/6} = -1/2$, $f''(\pi/6) = -\cos x|_{x=\pi/6} = -\sqrt{3}/2$, and $f'''(\pi/6) = \sin x|_{x=\pi/6} = 1/2$. Therefore,

$$\cos x \approx \frac{\sqrt{3}}{2} - \frac{1}{2}(x-\pi/6) - \frac{\sqrt{3}}{4}(x-\pi/6)^2 + \frac{1}{12}(x-\pi/6)^3.$$

(5) As per usual we use ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x-3)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n!(x-3)^n}\right| = \lim_{n\to\infty}\frac{n+1}{2}|x-3| = \infty.$$

Therefore, the series only converges for $x = 3$.

(6) (a) This is an alternating series, and it seems like the series of the absolute values wont converge, so lets try to show that first. Notice $e/n \le e^{1/n}/n \le 1/n$, so since $e\sum_{n=1}^{\infty} 1/n$ diverges by p-series because $p = 1$, $\sum_{n=1}^{\infty}(-1)^n e^{1/n}/n$ does not converge absolutely. However, $\lim_{n\to\infty} e^{1/n}/n = 0$ and it's an alternating series. In order to show conditional convergence all we have to do is show the a_n are decreasing. For this problem we have to take the derivative,

$$\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{e^{1/n}}{n}\right) = -\frac{e^{1/n}(1+n)}{n^3} < 0 \quad \text{for } n > -1,$$

hence $e^{1/n}/n$ is decreasing, therefore by the alternating series test, the series converges conditionally.

(b) This is a typical ratio test problem,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!}\right| = \lim_{n\to\infty}\frac{1}{4n+2} = 0 < 1.$$

Therefore, the series converges absolutely by the ratio test.

(7) (a) Notice that $\frac{\sqrt{n}}{n^2+9} \le \frac{1}{n^{3/2}}$ and since $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges by p-series because $p > 1$, $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n^2+9}$ converges absolutely by the direct comparison test.

(b) For the remainder we shall use the bound for alternating series because for an alternating series that will be the most accurate, however we are free to use other types of remainders, but it is not advisable!

Recall, to bound the remainder for alternating series test we simply go to the next term, not including the $(-1)^n$.

$$|R_8| \le \frac{\sqrt{9}}{9^2+9} = \frac{3}{90} = \frac{1}{30}.$$