(1) Please use the method of undetermined coefficients to find the form of the particular solution (**WITHOUT SOLVING FOR CONSTANTS**) of the following ODEs.

(a)

$$y'' + 5y' + 6y = -t + e^{-3t} + te^{-2t} + e^{-3t}\cos t$$

Solution: The characteristic solution is $y_c = c_1e^{-2t} + c_2e^{-3t}$. We take a guess at the particular solution using our forcing function

$$y_p \stackrel{?}{=} a_0 + a_1t + ke^{-3t} + (b_0 + b_1t)e^{-2t} + e^{-3t}[A_1\cos t + A_2\sin t].$$

Notice that there is a repeat with e^{-3t} and e^{-2t}, so our particular solution becomes

$$y_p = a_0 + a_1t + kte^{-3t} + t(b_0 + b_1t)e^{-2t} + e^{-3t}[A_1\cos t + A_2\sin t].$$

(b)

$$y'' + 3y' + 2y = e^t(t^2+1)\sin(2t) + 3e^{-t}\cos t + 4e^t.$$

Solution: The characteristic solution is $y_c = c_1e^{-2t} + c_2e^{-t}$. We take a guess at our particular solution, but as we'll see, we won't have any repeats,

$$y_p = (a_0 + a_1t + a_2t^2)e^t[A\cos 2t + A_2\sin 2t] + e^{-t}[B_1\cos t + B_2\sin t] + b_0e^t.$$

(2) Please find the general solution of the ODE: $y'' + 4y' + 4y = t^{-2}e^{-2t};\ t > 0$

Solution: The characteristic solution is $y_c = (c_1 + c_2t)e^{-2t}$, so our two solutions are $\boxed{y_1 = e^{-2t}}$ and $\boxed{y_2 = te^{-2t}}$. The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = \boxed{e^{-4t}}.$$

Then we use our formula for variation of parameters

$$y = -y_1\int \frac{y_2f(t)}{W(y_1,y_2)}dt + y_2\int \frac{y_1f(t)}{W(y_1,y_2)}dt = -e^{-2t}\int \frac{te^{-2t}\cdot t^{-2}e^{-2t}}{e^{-4t}}dt + te^{-2t}\int \frac{e^{-2t}\cdot t^{-2}e^{-2t}}{e^{-4t}}dt$$
$$= \boxed{-e^{-2t}[\ln t + c_3] + te^{-2t}\left[-t^{-1} + c_4\right]}.$$

Since they ask for the general solution, there's no need to simplify it any further.

(3) Consider the ODE $y'' + 2y' + 2y = \cos t$.

(a) Please find the general solution.

Solution: The characteristic polynomial is $r = \frac{1}{2}(-2 \pm \sqrt{4-8}) = \boxed{-1 \pm i}$, then $y_c = e^{-t}[A_1\cos t + A_2\sin t]$. We'll see that there won't be any repeats, so the particular solution is

$$y_p = B_1\cos t + B_2\sin t. \Rightarrow y_p' = -B_1\sin t + B_2\cos t \Rightarrow y_p'' = -B_1\cos t - B_2\sin t.$$

Plugging it into the ODE gives

$$-B_1\cos t - B_2\sin t - 2B_1\sin t + 2B_2\cos t + 2B_1\cos t + 2B_2\sin t = \cos t$$

$$\Rightarrow (B_1 + 2B_2)\cos t + (B_2 - 2B_1)\sin t = \cos t \Rightarrow B_2 = 2B_1 \Rightarrow \boxed{B_1 = \frac{1}{5}}, \boxed{B_2 = \frac{2}{5}}$$

$$\Rightarrow \boxed{y = e^{-t}[A_1\cos t + A_2\sin t] + \frac{1}{5}\cos t + \frac{2}{5}\sin t}.$$

(b) What happens to the solution as $t \to \infty$?

Solution: As $t \to \infty$, $y \to \frac{1}{5}\cos t + \frac{2}{5}\sin t$

(4) Please solve the IVP: $y'' + 4y = 6\sin(4t);\ y(0) = y'(0) = 0$.

Solution: The characteristic solution is $y_c = A_1 \cos 2t + A_2 \sin 2t$, and we'll see that there are no repeats so the particular solution is

$$y_p = B_1 \cos 4t + B_2 \sin 4t \Rightarrow y_p' = -4B_1 \sin 4t + 4B_2 \cos 4t \Rightarrow y_p'' = -16B_1 \cos 4t - 16B_2 \sin 4t$$

Plugging it into the ODE gives

$$\begin{aligned} &-16B_1 \cos 4t - 16B_2 \sin 4t + 4B_1 \cos 4t + 4B_2 \sin 4t = -12B_1 \cos 4t - 12B_2 \sin 4t = 6 \sin 4t \\ &\Rightarrow \boxed{B_1 = 0, B_2 = -\frac{1}{2}} \Rightarrow y = A_1 \cos 2t + A_2 \sin 2t - \frac{1}{2} \sin 4t. \end{aligned}$$

The first initial condition gives $\boxed{y(0) = A_1 = 0}$, and the second initial condition gives $y'(0) = 2A_2 - 2 = 0 \Rightarrow \boxed{A_2 = 1}$, then our solution is

$$\boxed{y = \sin 2t - \frac{1}{2}\sin 4t}$$

(5) Consider the IVP $y'' - 3y' - 4y = t + 2;\ y(0) = 3,\ y'(0) = 0$.

(a) Please find the solution to the IVP.

Solution: The characteristic solution is $y_c = c_1 e^{-4t} + c_2 e^{-t}$, and we'll see that there are no repeats so our particular solution is $y_p = a_1 t + a_0 \Rightarrow y_p' = a_1 \Rightarrow y_p'' = 0$. Plugging into the ODE gives

$$-3a_1 - 4a_1 t - 4a_0 = t + 2 \Rightarrow \boxed{a_1 = -\frac{1}{4}} \Rightarrow \boxed{a_0 = -\frac{5}{16}} \Rightarrow y = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{4}t - \frac{5}{16}.$$

The first initial condition gives us $y(0) = c_1 + c_2 - 5/16 = 3 \Rightarrow c_1 + c_2 = 53/16$. The derivative of the solution is $y' = 4c_1 e^{4t} - c_2 e^{-t} - 1/4$, then the second initial condition gives us $y'(0) = 4c_1 - c_2 - 1/4 = 0 \Rightarrow c_1 - c_2/4 = 1/16$. Then we get $\boxed{c_2 = 13/5}$ and $\boxed{c_1 = 57/80}$ and our solution becomes

$$\boxed{y = \frac{57}{80}e^{4t} + \frac{13}{5}e^{-t} - \frac{1}{4}t - \frac{5}{16}}.$$

(b) What happens to the solution as $t \to \infty$?

Solution: As $t \to \infty$, $y \to \infty$.

(6) Consider the ODE $2t^2 y'' - ty' + y = t\sqrt{t}$.

(a) Verify the solutions to the homogeneous ODE are $y_1 = t$ and $y_2 = \sqrt{t}$

Solution: Verify solutions by plugging into the ODE.

(b) Use the characteristic solution $y_c = c_1 y_2 + c_2 y_2$ to find the general solution to the full ODE.

Solution: The forcing function is $\boxed{f(t) = \frac{1}{2}t^{-1/2}}$ (i.e. standard form) and Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} t & t^{1/2} \\ 1 & \frac{1}{2}t^{-1/2} \end{vmatrix} = \boxed{-\frac{1}{2}t^{1/2}}$$

Then we plug into our formula to get

$$\begin{aligned} y &= -y_1 \int \frac{y_2 f(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 f(t)}{W(y_1, y_2)} dt = -t \int \frac{t^{1/2} \cdot (1/2)t^{-1/2}}{-\frac{1}{2}t^{1/2}} dt + t^{1/2} \int \frac{t \cdot (1/2)t^{-1/2}}{-\frac{1}{2}t^{1/2}} dt \\ &= \boxed{-t\left[-2t^{1/2} + c_3\right] + t^{1/2}[-t + c_4]} \end{aligned}$$

Again, since only the general solution was required there is no need to simplify any further.

(7) A mass weighing 1/2 lb (i.e. mass $= 1/64 lb \cdot s^2/ft$) stretches a spring 1/2 ft.

(a) Suppose the system has no damping. The mass is initially pulled down 1/2 ft and released.

(i) Write down the IVP for this system.

Solution: $\boxed{k = F/x = \frac{1/2}{1/2} = 1}$, so our IVP is

$$\frac{1}{64}x'' + x = 0;\ x(0) = \frac{1}{2},\ x'(0) = 0.$$

(ii) Solve the IVP.

Solution: The general solution will be $\boxed{x = A\cos 8t + B\sin 8t}$. The initial conditions give us $x(0) = \boxed{A = 1/2}$ and $x'(0) = \boxed{8B = 0}$. Then the solution is

$$\boxed{x = \frac{1}{2}\cos 8t}.$$

(iii) When does the mass return to the equilibrium position (i.e. $x = 0$).

Solution: $x = 0 \Rightarrow \boxed{t = \pi/16}$ for the first time.

(b) Now suppose the system has a damping constant of $2lb \cdot s/ft$. The mass is initially pushed up 1/2 ft and released with a downward velocity of 1/2 ft/s.

(i) Write down the IVP for this system.

Solution: The damping adds a $2x'$ term, so

$$\frac{1}{64}x'' + 2x' + x = 0;\ x(0) = -\frac{1}{2},\ x'(0) = \frac{1}{2}.$$

(ii) Solve the IVP.

Solution: This is where the problem starts to be a pain, but basically, the roots are

$$r^2 + 128r + 64 = 0 \Rightarrow r = \frac{1}{2}\left(-128 \pm \frac{1}{2}\sqrt{128^2 - 4 \cdot 64}\right) \Rightarrow \boxed{r_{1,2} = -64 \pm 8\sqrt{63}}$$

From this point lets just write down things in the general form because it doesn't make sense to carry all those ridiculous number around.

$$x = c_1e^{r_1t} + c_2e^{r_2t}.$$

The first initial condition gives us $x(0) = c_1 + c_2 = -1/2$ and the second initial condition gives us $x'(0) = r_1c_1 + r_2c_2 = 1/2$, then we get

$$c_2 = \frac{r_1 + 1}{2(r_2 - r_1)}, \qquad c_1 = \frac{r_2 + 1}{2(r_1 - r_2)}.$$

Moving on...

(8) Please solve the following IVP

$$y'' + 4y = 3\sin 2t;\; y(0) = 2,\; y'(0) = -1.$$

Solution: The characteristic solution is $y_c = A_1 \cos 2t + A_2 \sin 2t$ and our guess for the particular solution is $y_p \stackrel{?}{=} B_1 \cos 2t + B_2 \sin 2t$, but look at that, we have a repeat, so our particular solution actually is

$$y_p = B_1 t \cos 2t + B_2 t \sin 2t \Rightarrow y_p' = B_1 \cos 2t - 2B_1 t \sin 2t + B_2 \sin 2t + 2B_2 t \cos 2t$$
$$\Rightarrow y_p'' = -4B_1 \sin 2t - 4B_1 t \cos 2t + 4B_2 \cos 2t - 4B_2 t \sin 2t.$$

Plugging this into the ODE gives

$$\begin{aligned} &-4B_1 \sin 2t - 4B_1 t \cos 2t + 4B_2 \cos 2t - 4B_2 t \sin 2t + 4B_1 t \cos 2t + 4B_2 t \sin 2t \\ &= -4B_1 \sin 2t + 4B_2 \cos 2t = 3\sin 2t \Rightarrow \boxed{B_2 = 0,\; B_1 = -\frac{3}{4}}. \end{aligned}$$

Then the general solution is

$$y = A_1 \cos 2t + A_2 \sin 2t - \frac{3}{4} t \cos 2t \Rightarrow y(0) = \boxed{A_1 = 2}$$

The derivative of this is

$$y' = -2A_1 \sin 2t + 2A_2 \cos 2t - \frac{3}{4}\cos 2t + \frac{3}{2} t \sin 2t \Rightarrow y'(0) = 2A_2 - \frac{3}{4} = -1 \Rightarrow \boxed{A_2 = -\frac{1}{8}}.$$

Then our solution is

$$\boxed{y = 2\cos 2t - \frac{1}{8}\sin 2t - \frac{3}{4} t \cos 2t.}$$