Spring 2011 solutions

(1) (a) We already did this problem in the previous practice exam.
(b) Notice $\Delta x = (b-a)/n = \pi/3$, and $x_0 = 0$, $x_1 = \pi/3$, $x_2 = 2\pi/3$, $x_3 = \pi$, then we plug this into our formula,

$$I \approx \frac{\pi}{6}\left[\frac{2\pi}{3}\cos\frac{\pi}{3} + \frac{4\pi}{3}\cos\frac{2\pi}{3} - \pi\right] = \frac{\pi}{6}\left[\frac{\pi}{3} - \frac{2\pi}{3} - \pi\right] = -\frac{2}{9}\pi^2$$

(2) (a) We already did this problem in the previous practice exam.
(b) We integrate this by partial fractions. The denominator is already factored, so we go straight to the partial fractions,

$$\frac{2x+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2}$$

We simplify the numerator of the RHS, $A(x-1)^2+Bx(x-1)+Cx = A(x^2-2x+1)+B(x^2-x)+Cx = (A+B)x^2 + (C-B-2A)x + A = 2x+1$. Matching the coefficients gives, $A = 1$ straight away, then $B = -1$, then $C = 3$. Plugging these back in and integrating gives,

$$I = \int \frac{\mathrm{d}x}{x} - \int \frac{\mathrm{d}x}{x-1} + \int \frac{3\mathrm{d}x}{(x-1)^2} = \ln|x| - \ln|x-1| - \frac{3}{x-1}.$$

(3) Both of these are improper integrals, so let's make that our focus. For these types of problems **do <u>not</u> use a test**, just evaluate the integral, then make your conclusion.
(a)

$$\int_1^\infty \frac{x^2\mathrm{d}x}{(2x^3+1)^{3/2}} = \lim_{t\to\infty}\int_1^t \frac{x^2\mathrm{d}x}{(2x^3+1)^{3/2}}$$

This is solved via u-sub where $u = 2x^3+1 \Rightarrow \mathrm{d}u = 6x^2\mathrm{d}x$,

$$I = \frac{1}{6}\lim_{t\to\infty}\int_3^{2t^2+1} u^{-3/2}\mathrm{d}u = \frac{1}{6}\lim_{t\to\infty} -2u^{-1/2} = \frac{1}{3}\lim_{t\to\infty}\frac{-1}{\sqrt{2t^2+1}} + \frac{1}{\sqrt{3}} = \frac{1}{3\sqrt{3}}.$$

Hence, the integral is convergent because the limit exists.
(b)

$$\int_0^8 \frac{\mathrm{d}x}{(x-8)^{2/3}} = \lim_{t\to 8^-}\int_0^t \frac{\mathrm{d}x}{(x-8)^{2/3}}.$$

We solve this via u-sub where $u = x-8 \Rightarrow \mathrm{d}u = \mathrm{d}x$,

$$I = \lim_{t\to 8^-}\int_{-8}^{t-8} u^{-2/3}\mathrm{d}u = \lim +t \to 8^- 3u^{1/3}\Big|_{-8}^{t-8} = \lim_{t\to 8^-} 3(t-8)^{1/3} + 6 = 6.$$

Hence, the integral is convergent because the limit exists.

(4) We already did this problem in the previous practice exam.

(5) Here are some more improper integral questions, but these allow you to use a test. When you are allowed to use a test, use a test and make your conclusion, and **do <u>not</u> evaluate the integral**.
(a) This is your typical limit comparison test problem because the numerator gives us difficulties, so lets divide through by the highest power of the numerator,

$$\frac{\frac{1}{x}\sqrt{x^2+1}}{\frac{1}{x}x^3} = \frac{\sqrt{1+\frac{1}{x^2}}}{x^2} \sim \frac{1}{x^2}.$$

Now, we take the limit to make sure we are allowed to make this comparison,

$$\lim_{x\to\infty}\frac{\frac{\sqrt{x^2+1}}{x^3}}{\frac{1}{x^2}} = \lim_{x\to\infty}\frac{\sqrt{x^2+1}}{x} = \lim_{x\to\infty}\sqrt{1+\frac{1}{x^2}} = 1.$$

So, we compare this to $1/x^2$. Since $p > 1$, $\int_1^\infty \mathrm{d}x/x^2$ converges, therefore by the limit comparison test $\int_1^\infty \frac{\sqrt{x^2+1}}{x^3}\mathrm{d}x$ converges.

(b) This is a typical direct comparison test problem. Notice that $-1 \leq \sin(2x) \leq 1$, then

$$\frac{1}{x} \leq \frac{2+\sin(2x)}{x} \leq \frac{3}{x}.$$

Decisions, decisions, which do we choose? If we are having trouble choosing we can go straight to limit comparison test. However, notice that we have to integrate $1/x$, and we know for the limits provided, this will diverge. So we pick the smaller of the two, i.e. $1/x$ instead of $3/x$. Now, since $\int_\pi^\infty \mathrm{d}x/x$ diverges, then by the direct comparison test $\int_\pi^\infty (2+\sin(2x)/x)$ also diverges.

(6) Notice that the highest power of the numerator is equal to the highest power of the denominator, so we must use long division first, from which we get

$$\frac{x^3+8}{x^3+4x} = 1 + \frac{-4x+8}{x^3+4x} = 1 + \frac{-4x+8}{x(x^2+4)}.$$

Now, we can break this up into partial fractions,

$$\frac{-4x+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} = \frac{A(x^2+4)+(Bx+C)x}{x(x^2+4)} = \frac{(A+B)x^2+Cx+4A}{x(x^2+4)}.$$

Now, we equate the coefficients to get $C=-4$ and $A=2$ straight away, then $B=-2$. Putting this back into the integral we get,

$$\int \frac{x^3+8}{x^3+4x}\mathrm{d}x = \int \left(1+\frac{2}{x}-\frac{2x+4}{x^2+4}\right)\mathrm{d}x = x + 2\ln|x| - \int \frac{2x\mathrm{d}x}{x^2+4} - \int \frac{4\mathrm{d}x}{x^2+4}$$

We solve the penultimate by u-sub with $u = x^2+4 \Rightarrow \mathrm{d}u = 2x\mathrm{d}x$,

$$\int \frac{2x\mathrm{d}x}{x^2+4} = \int \frac{\mathrm{d}u}{u} = \ln|u| = \ln|x^2+4|.$$

And for the last integral we divide through by 4 in the numerator and denominator and use $u = x/2 \Rightarrow \mathrm{d}u = \mathrm{d}x/2$, then

$$\int \frac{4\mathrm{d}x}{x^2+4} = \int \frac{\mathrm{d}x}{(x/2)^2+1} = 2\int \frac{\mathrm{d}u}{u^2+1} = 2\tan^{-1}u = 2\tan^{-1}\frac{x}{2}$$

Then our final answer is,

$$I = x + 2\ln|x| - \ln|x^2+4| - 2\tan^{-1}\frac{x}{2} + C.$$

(7) (a) We take the limit,

$$\lim_{n\to\infty}\left(\frac{n^3+5n^4}{2n^4+2n-1}\right)^{1/3} = \lim_{n\to\infty}\left(\frac{1/n+5}{2+2/n^3-1/n^4}\right)^{1/3} = \left(\frac{5}{2}\right)^{1/3}.$$

Since the limit exists, the sequence converges.

(b) Taking the limit gives,

$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\sin(1/n)}{1/n} = \lim_{n\to\infty}\frac{\frac{-1}{n^2}\cos(1/n)}{\frac{-1}{n^2}} = 1,$$

hence the sequence converges.

Fall 2011 solutions

(1) (a) Taking the limit gives,

$$\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{x}} = \lim_{n\to\infty} \frac{1/n}{1/2\sqrt{n}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0,$$

hence the sequence converges.

(b) Taking the limit gives,

$$\lim_{n\to\infty} \ln(2n) - \ln(n+1) = \lim_{n\to\infty} \ln \frac{2n}{n+1} = \lim_{n\to\infty} \ln \frac{2}{1+1/n} = \ln 2,$$

hence the sequence converges.

(2) (a) First we break up the fraction into partial fractions. Now for this problem, we may be able to see what the partial fractions are right away, in which case we don't have to carry out the operations, but if we choose to do the partial fractions in our head we must make sure it works!

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{Ax + A + Bx}{x(x+1)} = \frac{(A+B)x + A}{x^2+x}.$$

We get $A = 1$ straight away, then $B = -1$, so

$$I = \int \frac{\mathrm{d}x}{x} - \int \frac{\mathrm{d}x}{x+1} = \ln|x| - \ln|x+1|\Big|_1^5 = \ln|5| - \ln|6| + \ln|2| = \ln|5/3|$$

(b) Notice $\Delta x = (b-a)/n = 4/4 = 1$, then $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 5$. Plugging this into the trapezoid rule formula give,

$$I \approx \frac{\Delta x}{2}[f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)] = \frac{1}{2}\left[\frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{30}\right] = \frac{17}{30}.$$

(c) Using the above values and plugging them into the Simpson's rule formula gives,

$$I \approx \frac{\Delta x}{3}[f(1) + 4f(2) + 2f(3) + 4f(4) + f(5)] = \frac{1}{3}\left[\frac{1}{2} + \frac{2}{3} + \frac{1}{6} + \frac{1}{5} + \frac{1}{30}\right] = \frac{47}{90}.$$

(3) (a) We already did this problem in the previous practice exam.

(b) We first separate the fraction into partial fractions,

$$\begin{aligned}\frac{x^2}{(x-1)^3} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3} \\ &= \frac{Ax^2 - 2Ax + A + Bx - B + C}{(x-1)^3} = \frac{Ax^2 + (B-2A)x + A - B + C}{(x-1)^3}.\end{aligned}$$

Matching the coefficients gives us $A = 1$ straight away, then $B = 2$ and $C = 1$. Now we put this back in and integrate,

$$I = \int \frac{\mathrm{d}x}{x-1} + 2\int \frac{\mathrm{d}x}{(x-1)^2} + \int \frac{\mathrm{d}x}{(x-1)^3}$$

We use $u = x - 1 \Rightarrow \mathrm{d}u = \mathrm{d}x$, then

$$I = \ln|x-1| - \frac{2}{x-1} - \frac{1/2}{(x-1)^2}.$$

(4) We already did this problem in the previous practice exam.

(5) (a) We break this fraction into partial fractions,

$$\frac{4x-8}{x^3+4x} = \frac{4x-8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} = \frac{A(x^2+4)+(Bx+C)x}{x(x^2+4)} = \frac{(A+B)x^2+Cx+4A}{x^3+4x}.$$

Then we get $C = 4$ and $A = -2$ straight away, then $B = 2$.
Plugging this the integral gives,

$$\int \frac{4x-8}{x^3+4x} = -2\int \frac{\mathrm{d}x}{x} + \int \frac{2x+4}{x^2+4} = -2\ln|x| + \int \frac{2x\mathrm{d}x}{x^2+4} + \int \frac{4\mathrm{d}x}{x^2+4}$$
$$= -2\ln|x| + \ln|x^2+4| + 2\tan^{-1}\left(\frac{x}{2}\right) + C.$$

I didn't go through the details of the second and third integrals because these integrals were done a few pages back.

(b) We already did this problem in the previous practice exam.

(6) (a) We solve this via u-sub with $u = -x^2 \Rightarrow \mathrm{d}u = -2x\mathrm{d}x$, but this is also an improper integral,

$$\int_0^\infty xe^{-x^2}\mathrm{d}x = \lim_{t\to\infty}\int_0^t xe^{-x^2} = \lim_{t\to\infty}\frac{-1}{2}\int_0^{-t^2} e^u\mathrm{d}u = -\frac{1}{2}\lim_{t\to\infty} e^u\Big|_0^{-t^2} = -\frac{1}{2}\lim_{t\to\infty} e^{-t^2} - 1 = \frac{1}{2}.$$

(b) Again we do a u-sub with $u = \tan x \Rightarrow \mathrm{d}u = \sec^2 x\mathrm{d}x$, but we keep in mind that it is also an improper integral,

$$\int_0^{\pi/2} \tan x\sec^2 x\mathrm{d}x = \lim_{t\to\frac{\pi}{2}^-}\int_0^t \tan x\sec^2 x\mathrm{d}x = \lim_{t\to\frac{\pi}{2}^-}\int_0^{\tan t} u\mathrm{d}u$$
$$= \frac{1}{2}\lim_{t\to\frac{\pi}{2}^-} u^2\Big|_0^{\tan t} = \frac{1}{2}\lim_{t\to\frac{\pi}{2}^-}\tan^2 t = \infty$$

So, the integral diverges.

(7) For these problems just use the comparison tests without evaluating the integral.

(a) Notice $e^{1/x} \leq e$ for $x \in [1,\infty)$, so $\frac{e^{1/x}}{x^3} \leq \frac{e}{x^3}$ on $[1,\infty)$. Since $p > 1$, $\int_1^\infty \frac{\mathrm{d}x}{x^3}$ converges, hence any constant multiple of that integral will also converge, so $\int_1^\infty \frac{e\mathrm{d}x}{x^3}$ converges. Therefore, by the direct comparison test $\int_1^\infty \frac{e^{1/x}}{x^3}\mathrm{d}x$ converges.

(b) Now, this is a problem that would be difficult to find a direct comparison for, but we must find some sort of comparison nonetheless. We will experience difficulty due to the denominator, so lets divide through by the highest power of the denominator to see what we get. Now, we may be able to tell right away what the comparison is then we don't have to go through this process. But this is something that might help in finding the comparison.

$$\frac{x}{\sqrt{x^3+2}} = \frac{x/\sqrt{x^3}}{\sqrt{1+2/x^3}} = \frac{1/\sqrt{x}}{\sqrt{1+2/x^3}} \sim \frac{1}{\sqrt{x}}.$$

So, we compare our kernel to $1/\sqrt{x}$. We prove this is a valid comparison by taking the limit of the ratios,

$$\lim_{x\to\infty}\frac{x/\sqrt{x^3+2}}{1/\sqrt{x}} = \lim_{x\to\infty}\frac{\sqrt{x^3}}{\sqrt{x^3+2}} = \lim_{x\to\infty}\frac{1}{\sqrt{1+2/x^3}} = 1.$$

Since the limit exists, this is a valid comparison. Now, we know that $\int_1^\infty \mathrm{d}x/\sqrt{x}$ diverges because $p < 1$. Therefore, by the limit comparison test, $\int_1^\infty \frac{x\mathrm{d}x}{\sqrt{x^3+2}}$ also diverges.