Case study
Union-Find

Please note that these slides are based in part on material originally developed by Prof. Kevin Wayne of the CS Dept at Princeton University.

♠KT97♥8♦AK4♣AT982

Who knows what this is?

Steps to developing a usable algorithm

- Steps to developing a usable algorithm
    - Model the problem
    - Find an algorithm to solve it
    - Fast enough? Fits in memory?
    - If not, figure out why not
    - Find a way to address the problem
    - Iterate until satisfied
- The scientific method
- Mathematical analysis

Let’s look at an example

- Union-Find is a solution to a real problem:
  - It is a special case from graph theory (which we will look at in more detail later on)
  - Here, all we care about is whether two nodes are “connected” (directly, or indirectly).
  - Said connections have no attributes (as they probably would in a true graph).

Dynamic connectivity problem

Given a set of N objects, we support two operations:

- Connect two objects. (mutating)
- Is there a path connecting the two objects? (non-mutating)

connect 4 and 3

connect 3 and 8

connect 6 and 5

connect 9 and 4

connect 2 and 1

are 0 and 7 connected? ✗

are 8 and 9 connected? ✔

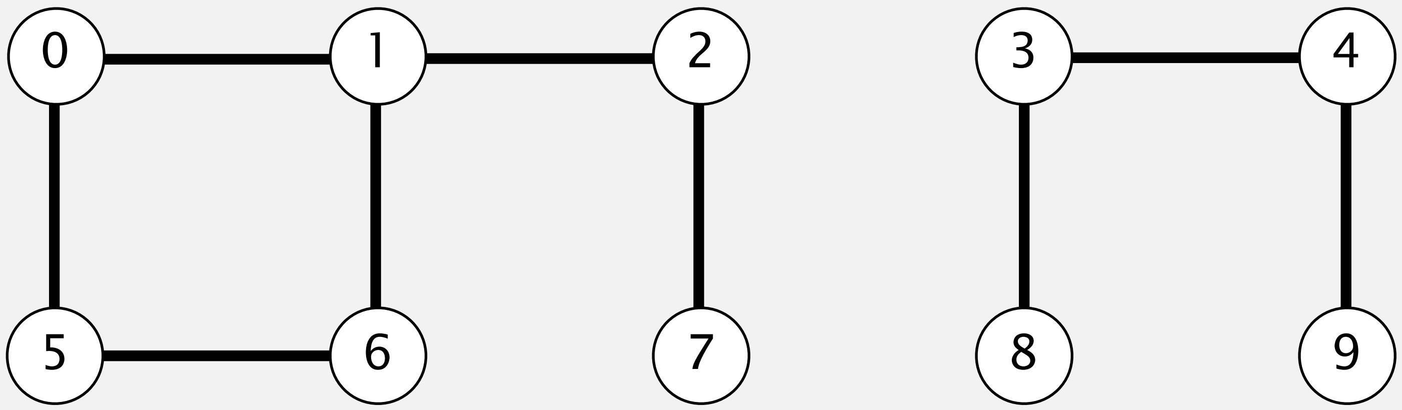
connect 5 and 0

connect 7 and 2

connect 6 and 1

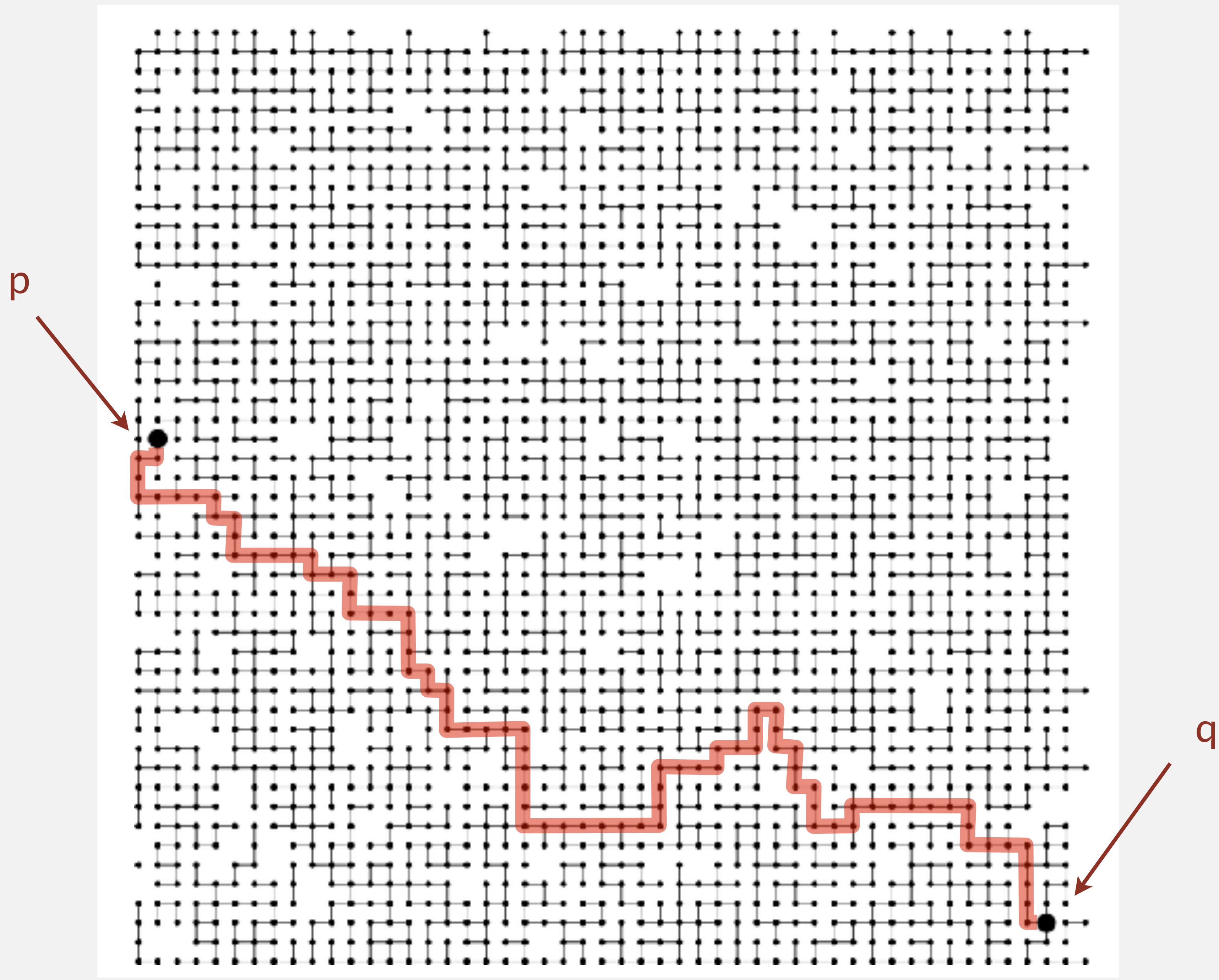
connect 1 and 0

are 0 and 7 connected? ✔



A larger connectivity example

Q. Is there a path connecting p and q ?



A. Yes.

Modeling the objects

Applications involve manipulating objects of all types.

- Pixels in a digital photo.
- Computers in a network.
- Friends in a social network.
- Transistors in a computer chip.
- Elements in a mathematical set.
- Variable names in a Fortran program.
- Metallic sites in a composite system.

When programming, convenient to name objects 0 to $N - 1$.

- Use integers as array index.
- Suppress details not relevant to union-find.

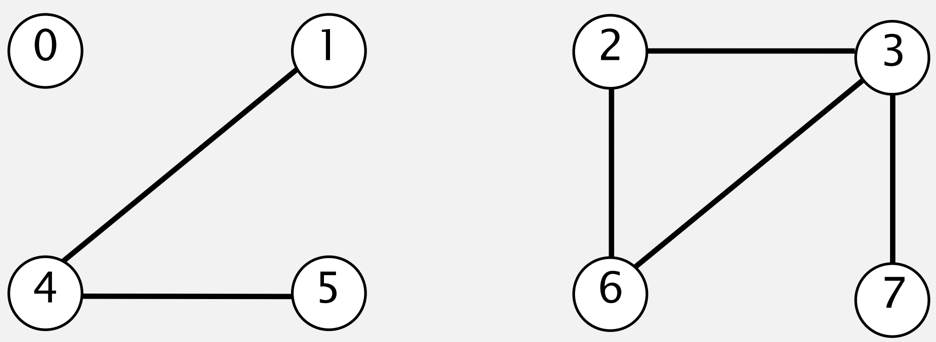
can use symbol table to translate from site names to integers: stay tuned (Chapter 3)

Modeling the connections

We assume "is connected to" is an equivalence relation:

- Reflexive: p is connected to p.
- Symmetric: if p is connected to q, then q is connected to p.
- Transitive: if p is connected to q and q is connected to r, then p is connected to r.

Connected component. Maximal set of objects that are mutually connected.



{ 0 } { 1 4 5 } { 2 3 6 7 }

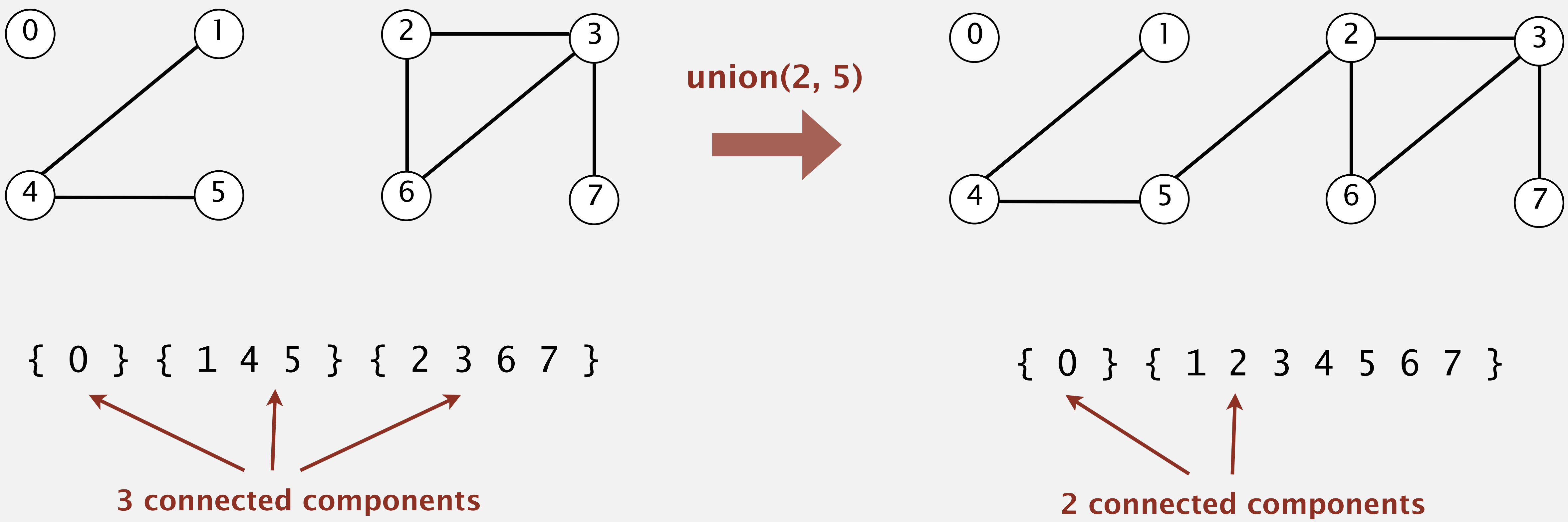
3 connected components

Implementing the operations

Find. In which component is object p ?

Connected. Are objects p and q in the same component?

Union. Replace components containing objects p and q with their union.



What just happened?

- We transformed the problem that we had, i.e. to implement for a no-attribute graph of vertices and edges:
  - connect(p,q); // connect object p to object q
  - isPath(p,q); // is there a path from p to q?

We call this “Reduction” and will learn about it later

- Into a slightly different problem, i.e. for a set of connected components:
  - find(p); // which component does object p belong to?
  - connected(p,q); // is p connected to q? i.e. find(p)==find(q)
  - union(p,q). // replace the components p and q with their union.

Union-find data type (API)

Goal. Design efficient data structure for union-find.

- Number of objects N can be huge.
- Number of operations M can be huge.
- Union and find operations may be intermixed.

public class UF		
	`UF(int N)`	*initialize union-find data structure with N singleton objects (0 to N – 1)*
`void`	`union(int p, int q)`	*add connection between p and q*
`private int`	`find(int p)`	*component identifier for p (0 to N – 1)*
`boolean`	`connected(int p, int q)`	*are p and q in the same component?*

```
public boolean connected(int p, int q)
{  return find(p) == find(q);  }
```

1-line implementation of connected()

Dynamic-connectivity client

- Read in number of objects N from standard input.
- Repeat:
  - read in pair of integers from standard input
  - if they are not yet connected, connect them and print out pair

```
public static void main(String[] args)
{
   int N = StdIn.readInt();
   UF uf = new UF(N);
   while (!StdIn.isEmpty())
   {
      int p = StdIn.readInt();
      int q = StdIn.readInt();
      if (!uf.connected(p, q))
      {
         uf.union(p, q);
         StdOut.println(p + " " + q);
      }
   }
}
```

```
% more tinyUF.txt
10
4 3
3 8
6 5
9 4
2 1
8 9
5 0
7 2
6 1
1 0
6 7
```

already connected

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1.5 Union-Find

- *dynamic connectivity*
- ***quick find***
- *quick union*
- *improvements*
- *applications*

Quick-find [eager approach]

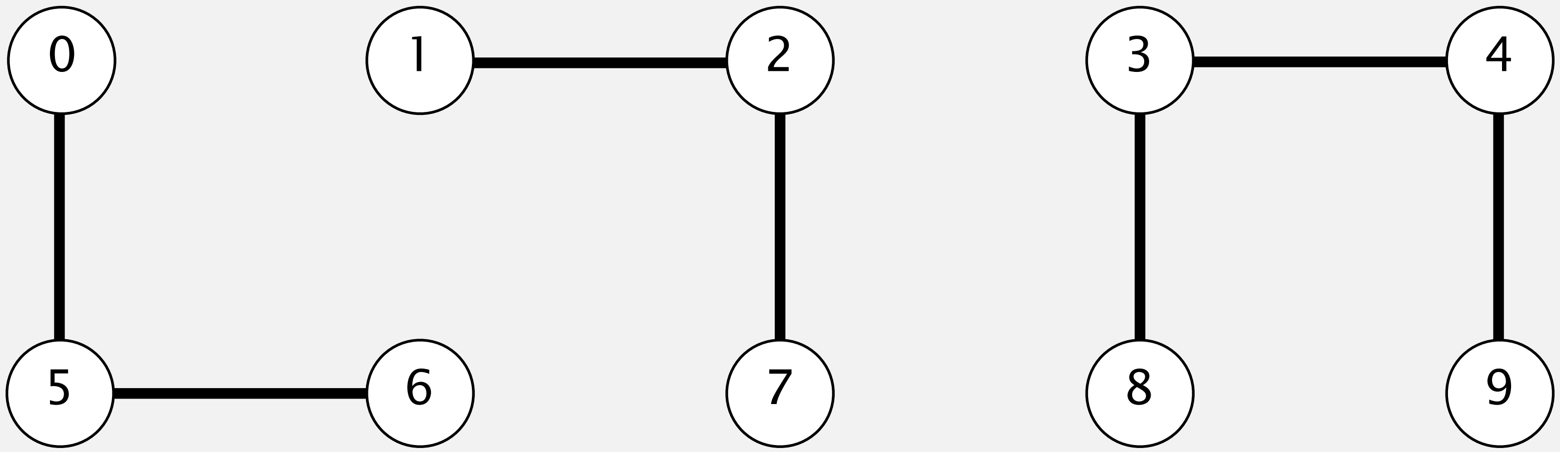
Data structure.

- Integer array `id[]` of length `N`.
- Interpretation: `id[p]` is the id of the component containing `p`.

if and only if

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8

0, 5 and 6 are connected
1, 2, and 7 are connected
3, 4, 8, and 9 are connected



Quick-find [eager approach]

Data structure.

- Integer array `id[]` of length `N`.
- Interpretation: `id[p]` is the id of the component containing `p`.

	0	1	2	3	4	5	6	7	8	9
`id[]`	0	1	1	8	8	0	0	1	8	8

Find. What is the id of `p`?

Connected. Do `p` and `q` have the same id?

`id[6] = 0; id[1] = 1`
6 and 1 are not connected

Union. To merge components containing `p` and `q`, change all entries whose id equals `id[p]` to `id[q]`.

Textbook: 從單獨的objects 開始 (N-1)(N-2)…(3)(2) -> O(N^2)

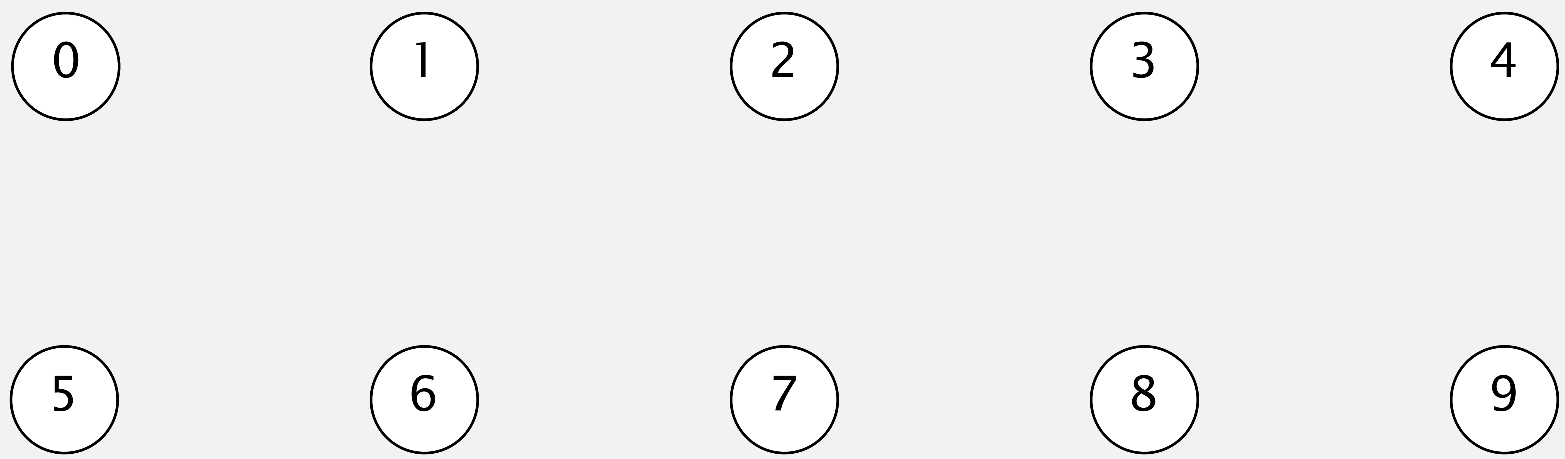
	0	1	2	3	4	5	6	7	8	9
`id[]`	1	1	1	8	8	1	1	1	8	8

after union of 6 and 1

Union -> k1*M*N = k1 * k2 * N^2
M = k2*N, M: # unions, N: # objects

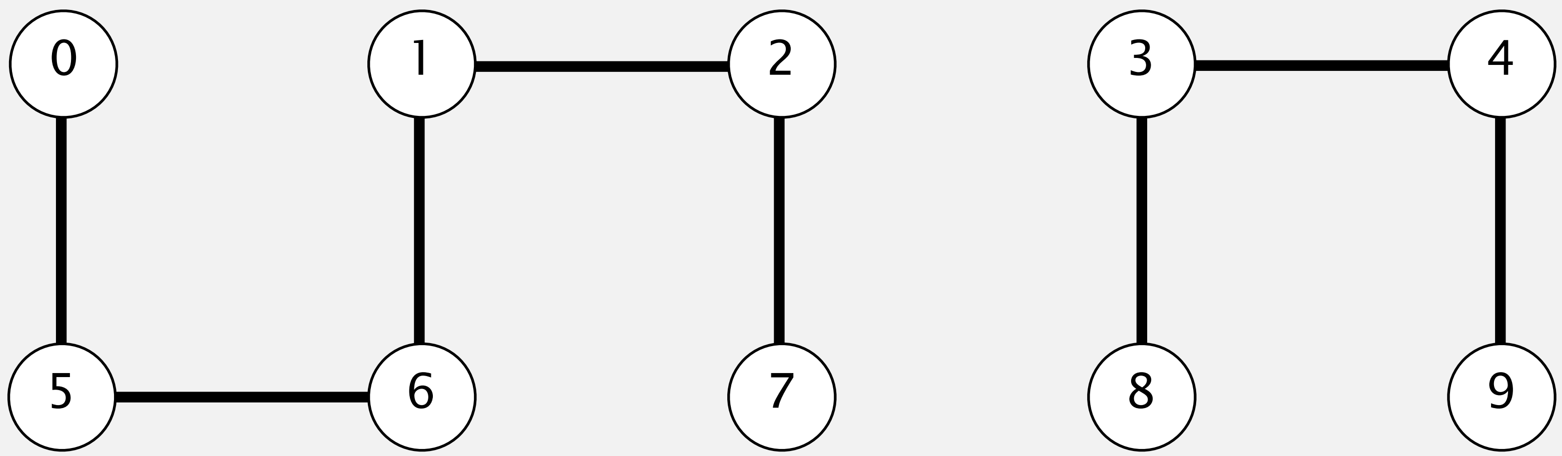
problem: many values can change

Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

Quick-find demo



	0	1	2	3	4	5	6	7	8	9
id[]	1	1	1	8	8	1	1	1	8	8

Quick-find: Java implementation

```
public class QuickFindUF
{
   private int[] id;

   public QuickFindUF(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++)
      id[i] = i;
   }

   public boolean find(int p)
   {  return id[p];  }

   public void union(int p, int q)
   {
      int pid = id[p];
      int qid = id[q];
      for (int i = 0; i < id.length; i++)
         if (id[i] == pid) id[i] = qid;
   }
}
```

set id of each object to itself
(N array accesses)

return the id of p
(1 array access)

change all entries with id[p] to id[q]
(at most 2N + 2 array accesses)

Quick-find is too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find	connected
quick-find	N	N	1	1

order of growth of number of array accesses

quadratic

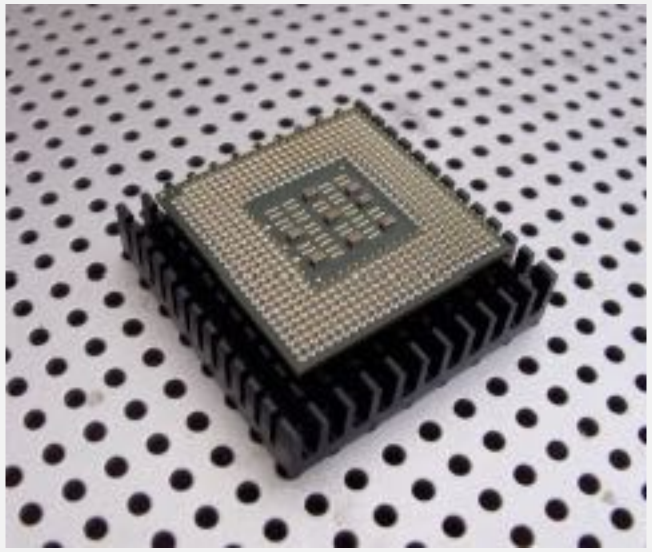
Union is too expensive. It takes N^2 array accesses to process a sequence of N union operations on N objects.

Quadratic algorithms do not scale

Rough standard (for now).

- 10^9 operations per second.
- 10^9 words of main memory.
- Touch all words in approximately 1 second.

a truism (roughly) since 1950!

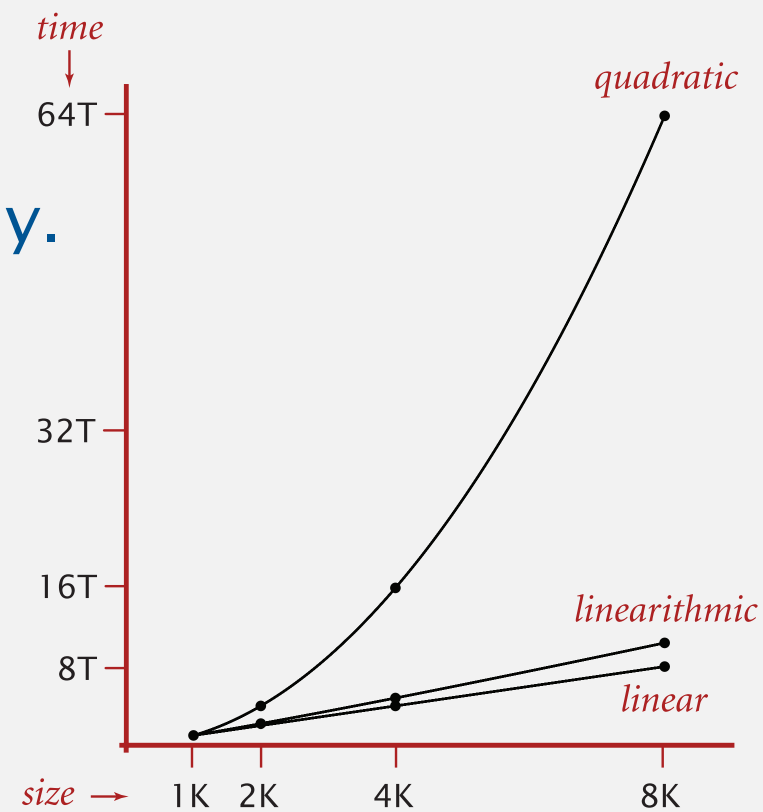


Ex. Huge problem for quick-find.

- 10^9 union commands on 10^9 objects.
- Quick-find takes more than 10^{18} operations.
- 30+ years of computer time!

Quadratic algorithms don't scale with technology.

- New computer may be 10x as fast.
- But, has 10x as much memory $\Rightarrow$ want to solve a problem that is 10x as big.
- With quadratic algorithm, takes 10x as long!



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1.5 Union-Find

- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

Quick-union [lazy approach]

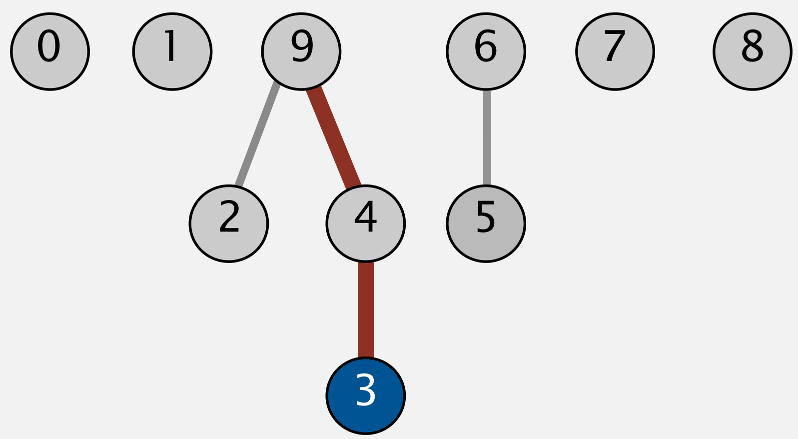
was `id[p]` is the id of the component containing p

Data structure.

- Integer array `id[]` of length `N`.
- Interpretation: `id[i]` is *parent* of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

keep going until it doesn't change
(algorithm ensures no cycles)

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



parent of 3 is 4

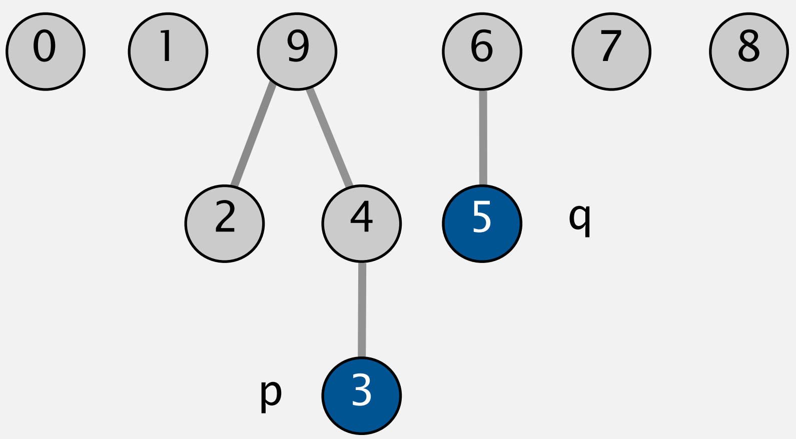
root of 3 is 9

Quick-union [lazy approach]

Data structure.

- Integer array `id[]` of length `N`.
- Interpretation: `id[i]` is parent of `i`.
- Root of `i` is `id[id[id[...id[i]...]]]`.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	9



root of 3 is 9
root of 5 is 6
3 and 5 are not connected

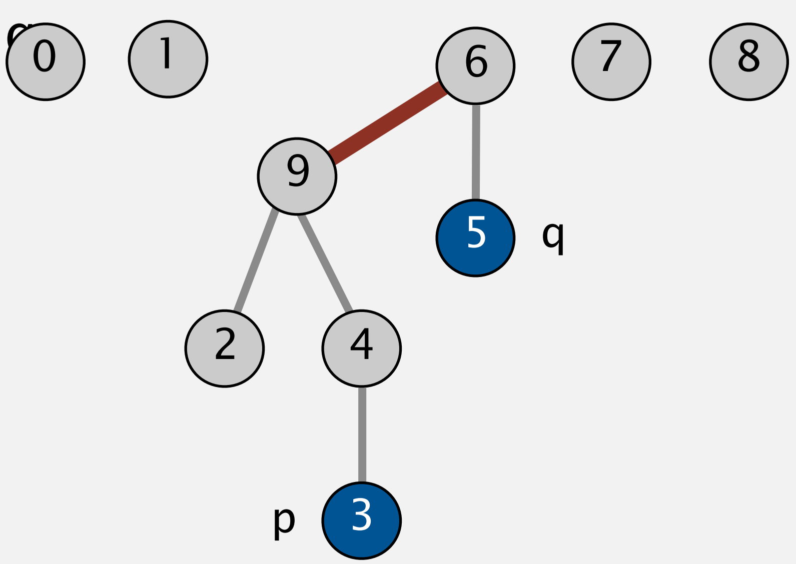
Find. What is the root of `p`?

Connected. Do `p` and `q` have the same root?

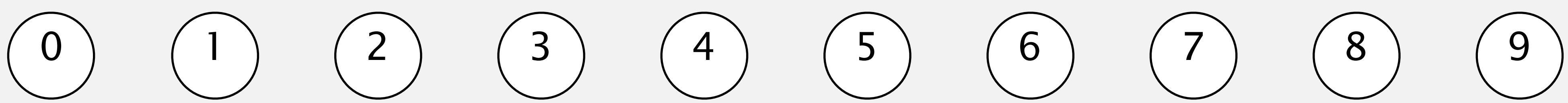
Union. To merge components containing `p` and `q`, set the id of `p`'s root to the id of `q`'s root.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	9	4	9	6	6	7	8	6

only one value changes



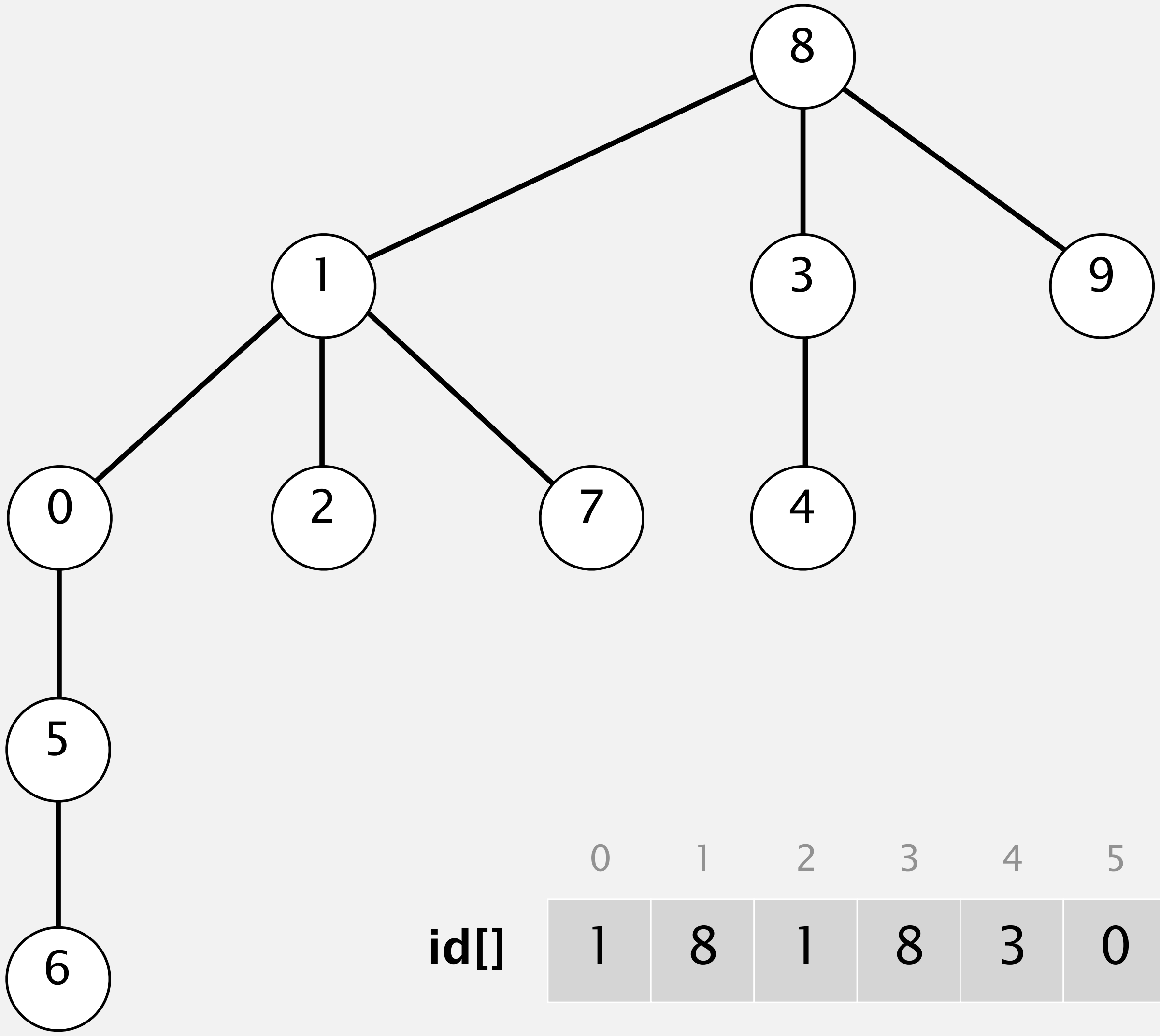
Quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

Quick-union demo

It’s OK, but not balance



	0	1	2	3	4	5	6	7	8	9
id[]	1	8	1	8	3	0	5	1	8	8

Quick-union: Java implementation

```
public class QuickUnionUF
{
   private int[] id;

   public QuickUnionUF(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++) id[i] = i;
   }

   public int find(int i)
   {
      while (i != id[i]) i = id[i];
      return i;
   }


   public void union(int p, int q)
   {
      int i = find(p);
      int j = find(q);
      id[i] = j;
   }
}
```

Need to fix builds

set id of each object to itself
(N array accesses)

chase parent pointers until reach root
(depth of i array accesses)

change root of p to point to root of q
(depth of p and q array accesses)

Quick-union is also too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find	connected
quick-find	N	N	1	1
quick-union	N	N †	N	N

← worst case

† includes cost of finding roots

Quick-find defect.

- Union too expensive (N array accesses).
- Trees are flat, but too expensive to keep them flat.

Quick-union defect.

- Trees can get tall.
- Find/connected too expensive (could be N array accesses).

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1.5 Union-Find

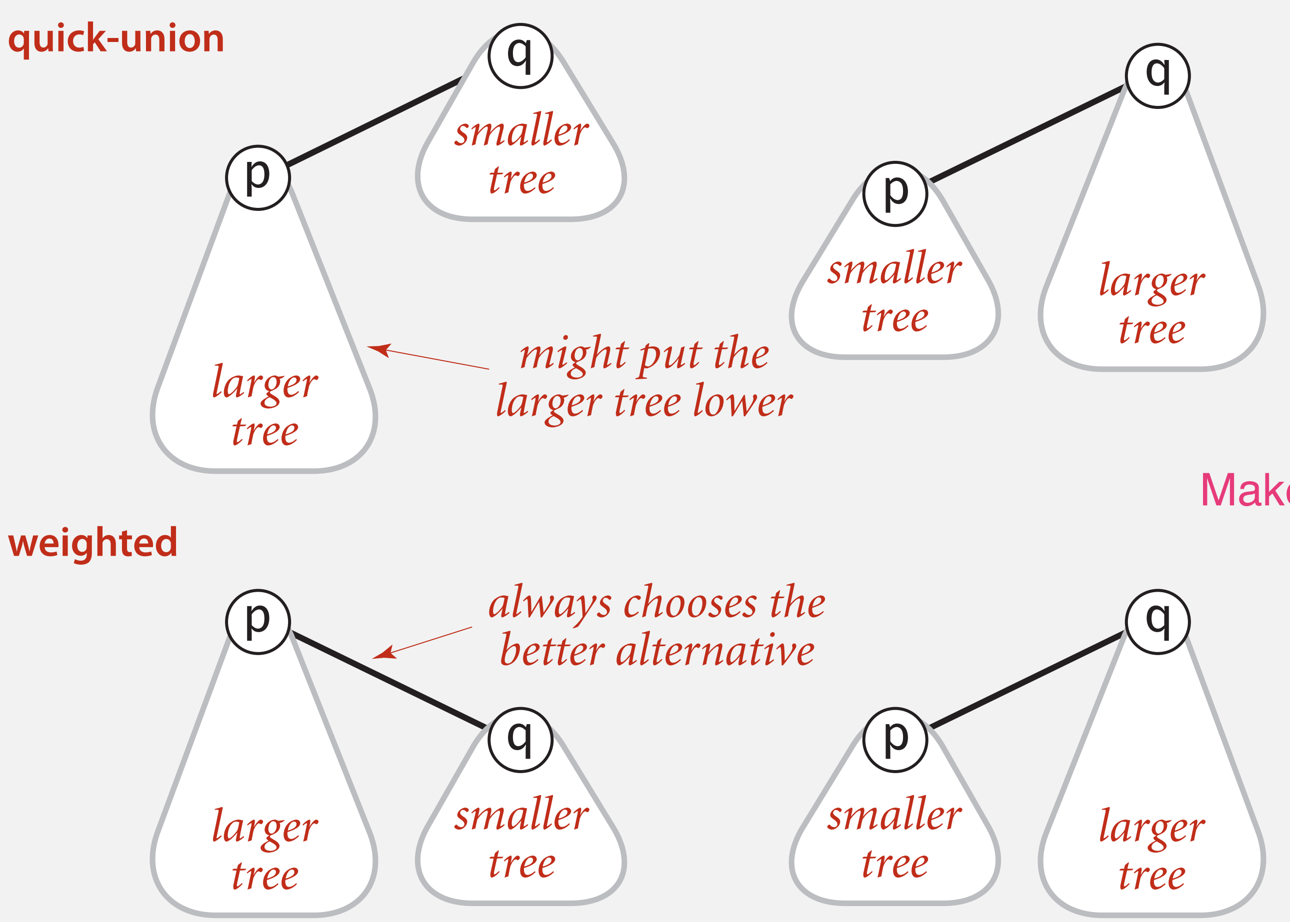
- *dynamic connectivity*
- *quick find*
- *quick union*
- *improvements*
- *applications*

Improvement 1: weighting

Weighted quick-union.

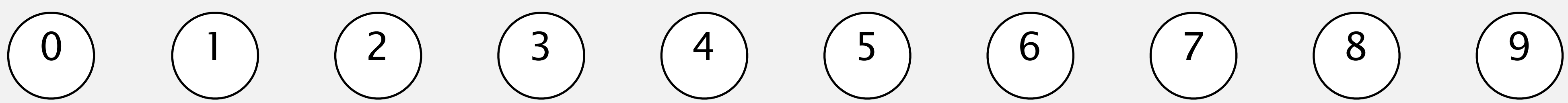
- Modify quick-union to avoid tall trees.
- Keep track of size of each tree (number of objects).
- Balance by linking root of smaller tree to root of larger tree.

reasonable alternatives:
union by height or "rank"



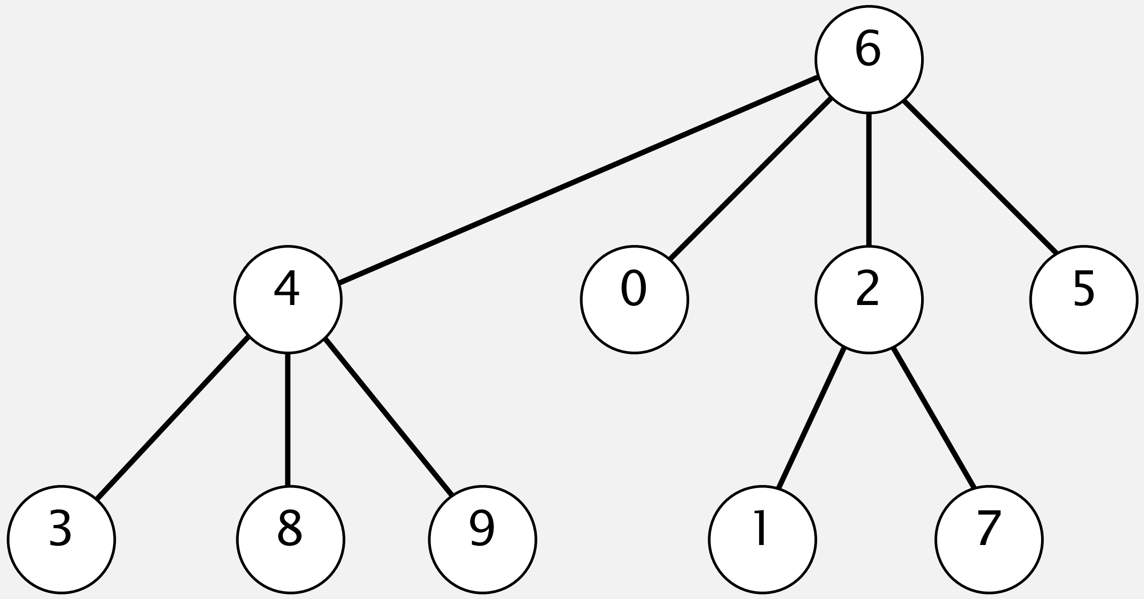
Make sure smaller tree to bigger tree

Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	0	1	2	3	4	5	6	7	8	9

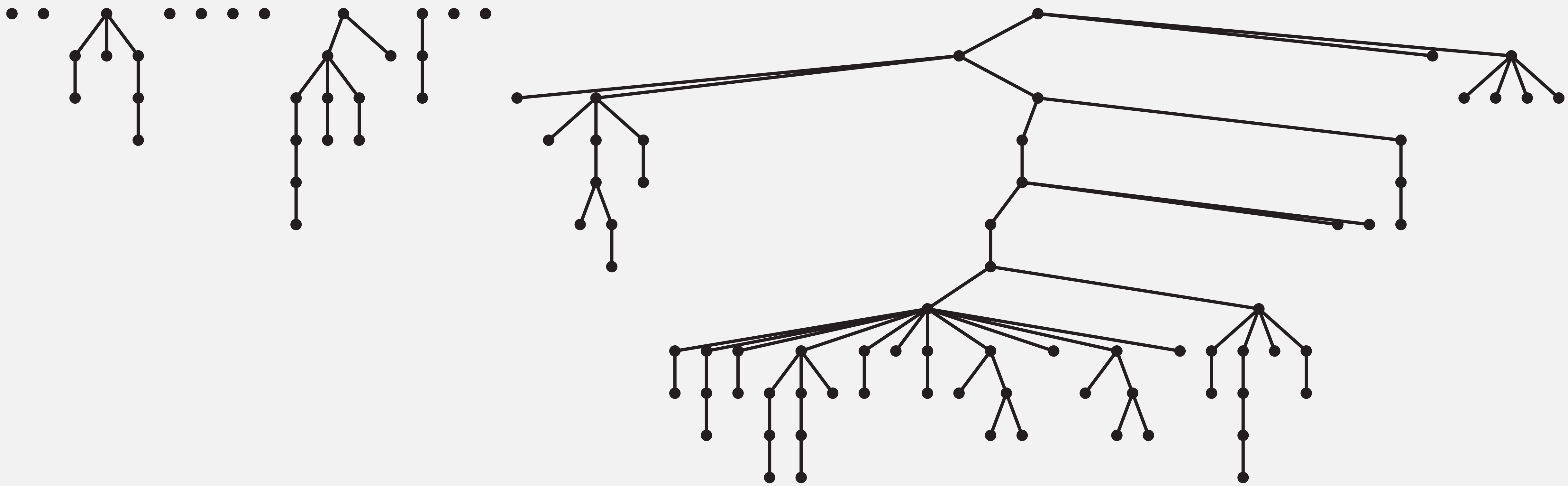
Weighted quick-union demo



	0	1	2	3	4	5	6	7	8	9
id[]	6	2	6	4	6	6	6	2	4	4

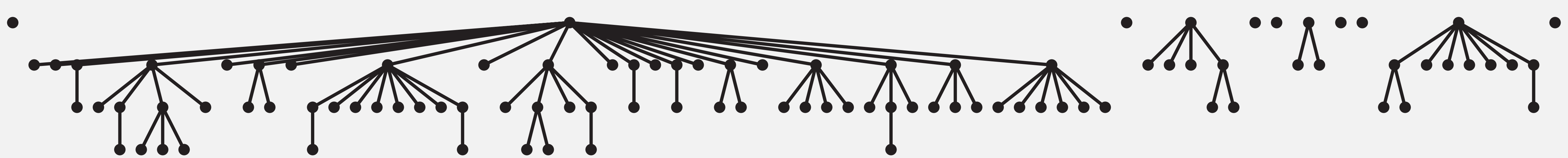
Quick-union and weighted quick-union example

quick-union



average distance to root: 5.11

weighted



average distance to root: 1.52

Quick-union and weighted quick-union (100 sites, 88 `union()` operations)

Weighted quick-union: Java implementation

Data structure. Same as quick-union, but maintain extra array `sz[i]` to count number of objects in the tree rooted at `i`.

Not height of subtree

Find/connected. Identical to quick-union.

Union. Modify quick-union to:

- Link root of smaller tree to root of larger tree.
- Update the `sz[]` array.

```
int i = find(p);
int j = find(q);
if (i == j) return;
if  (sz[i] < sz[j]) { id[i] = j; sz[j] += sz[i]; }
else                { id[j] = i; sz[i] += sz[j]; }
```

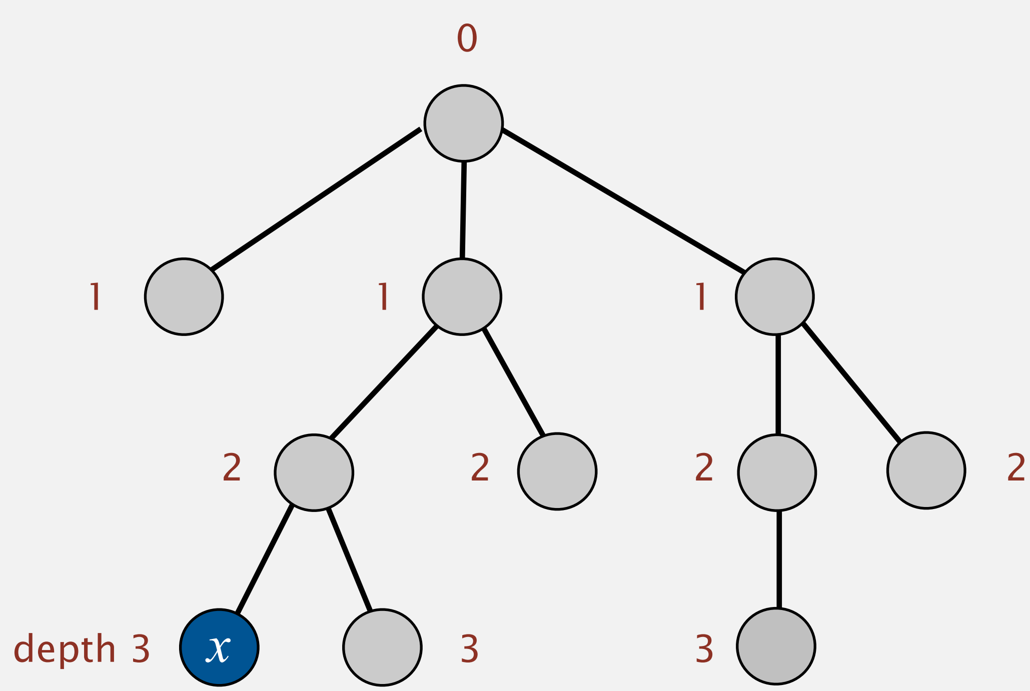
Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

lg = base-2 logarithm

Proposition. Depth of any node x is at most $\lg N$.



N = 11
depth(x) = 3 ≤ lg N

Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

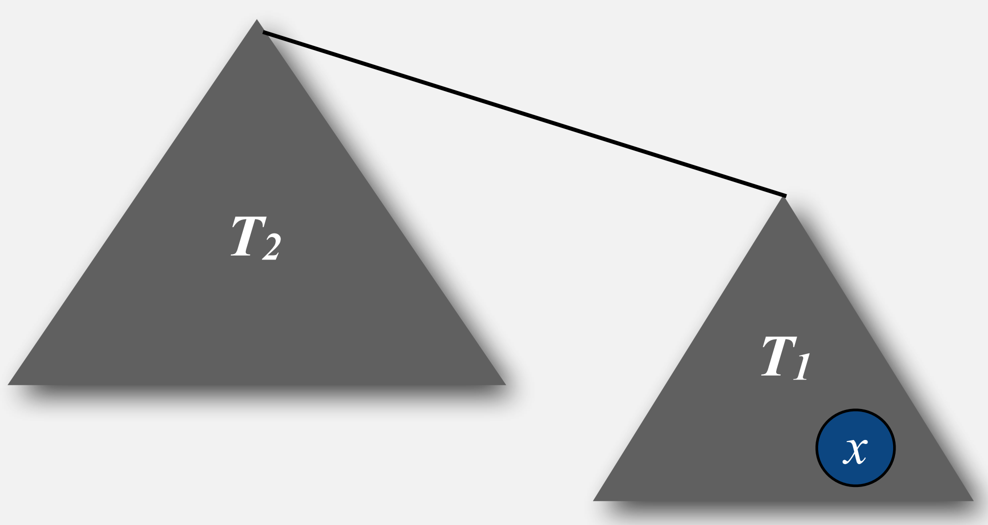
lg = base-2 logarithm

Proposition. Depth of any node x is at most $\lg N$.

Pf. What causes the depth of object x to increase?

Increases by 1 when tree T_1 containing x is merged into another tree T_2.

- The size of the tree containing x at least doubles since $|T_2| \geq |T_1|$.
- Size of tree containing x can double at most $\lg N$ times. Why?



$$\left.\begin{array}{c} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ \vdots \\ N \end{array}\right\} \lg N$$

Proposition H

- Proposition: the depth *d* of any forest built by weighted quick-union for *n* sites is at most *lg n*
- Prove: for every tree of size *s* in forest, $d <= lg\ s$
- Proof by induction:
  - Base case: when $n = 1$, $d = 0$ (d <= lg n)
  - Assume proposition is true for any tree *i* of size s_i. When we combine tree *i* of size s_i with tree *j* of size s_j, where $s_i <= s_j$, then
    - the *a priori* depths are: $d_i <= lg\ s_i <= lg\ s_j$ and $d_j <= lg\ s_j$
    - the *a posteriori* depths are: $d_i <= 1 + lg\ s_i$ and $d_j <= lg\ s_j$
    - but $d_i <= lg\ (s_i + s_i) <= lg\ (s_i + s_j)$
    - therefore *all* depths $d_k <= lg\ s_k$ where $s_k = s_i + s_j$

Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given roots.

Proposition. Depth of any node x is at most $\lg N$.

algorithm	initialize	union	find	connected
quick-find	N	N	1	1
quick-union	N	N †	N	N
weighted QU	N	lg N †	lg N	lg N

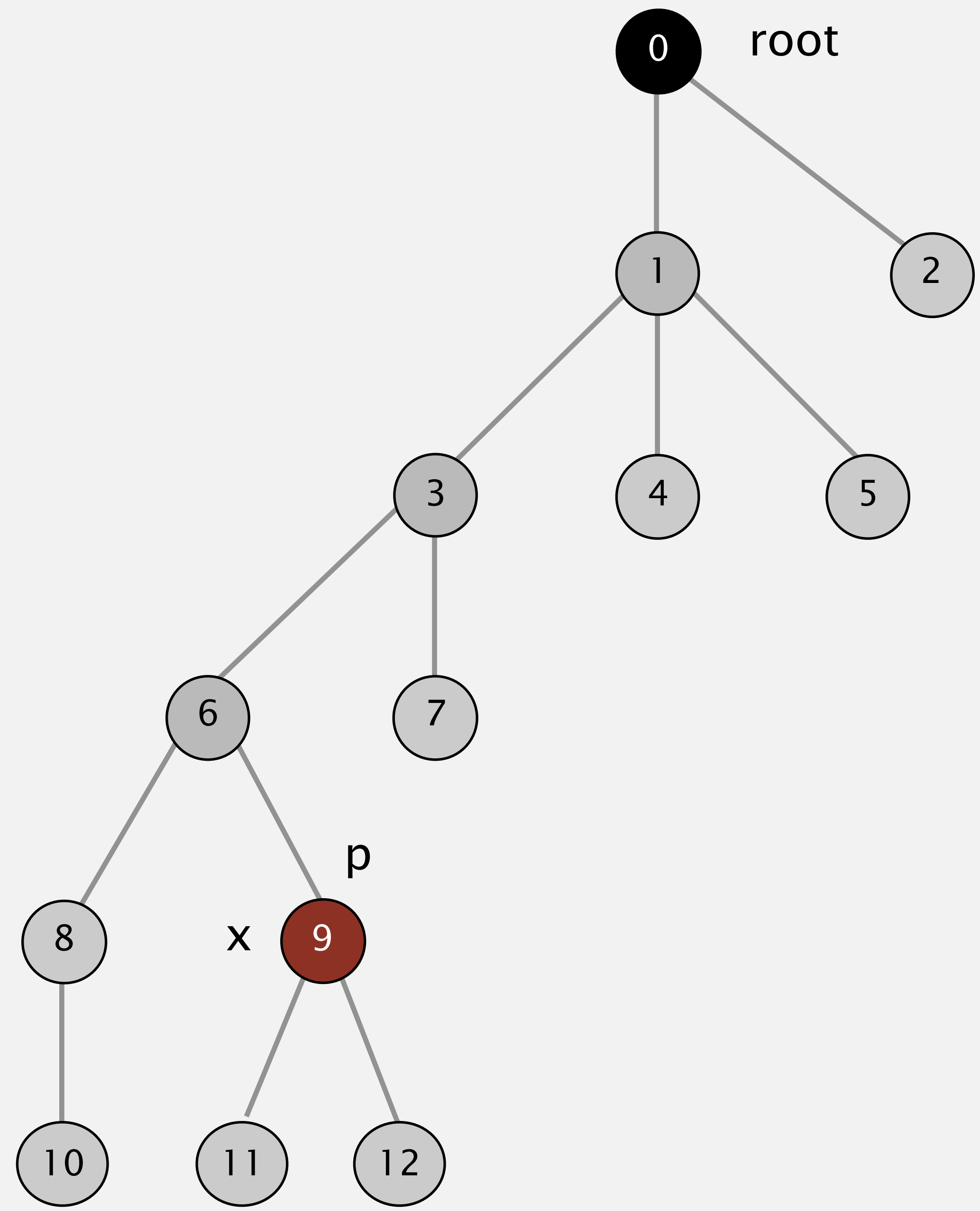
† includes cost of finding roots

Q. Stop at guaranteed acceptable performance?

A. No, easy to improve further.

Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the `id[]` of each examined node to point to that root.



Improvement 2: path compression

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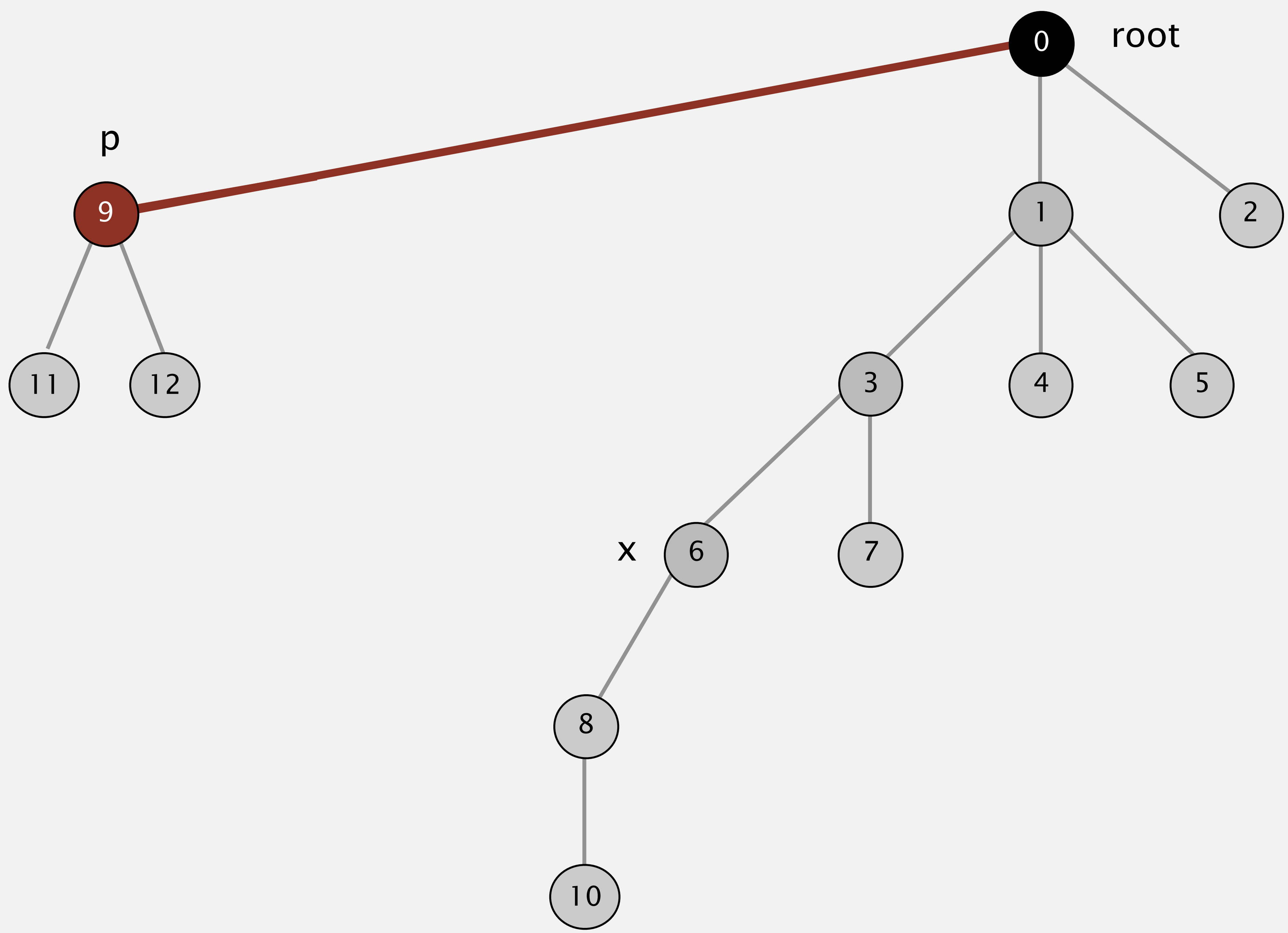
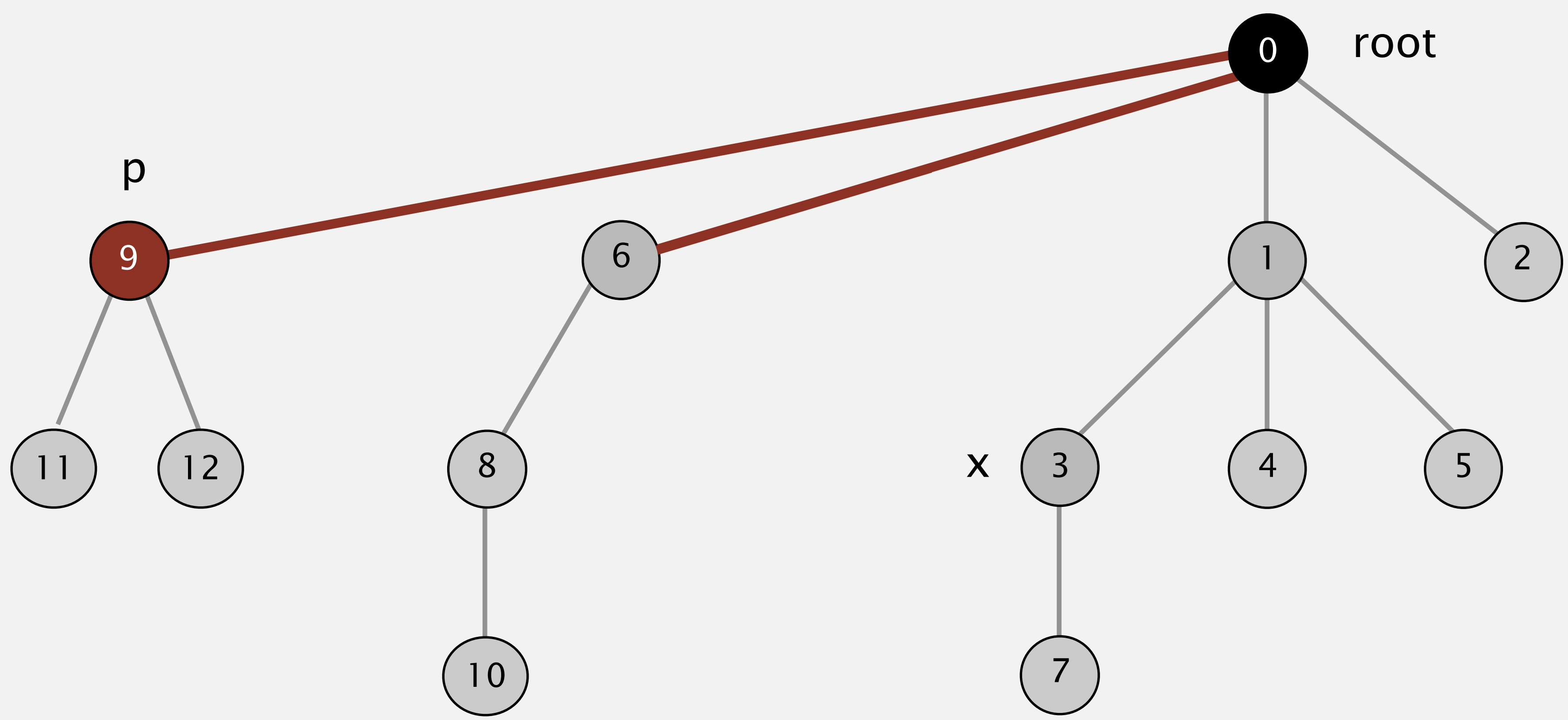


Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the `id[]` of each examined node to point to that root.



Improvement 2: path compression

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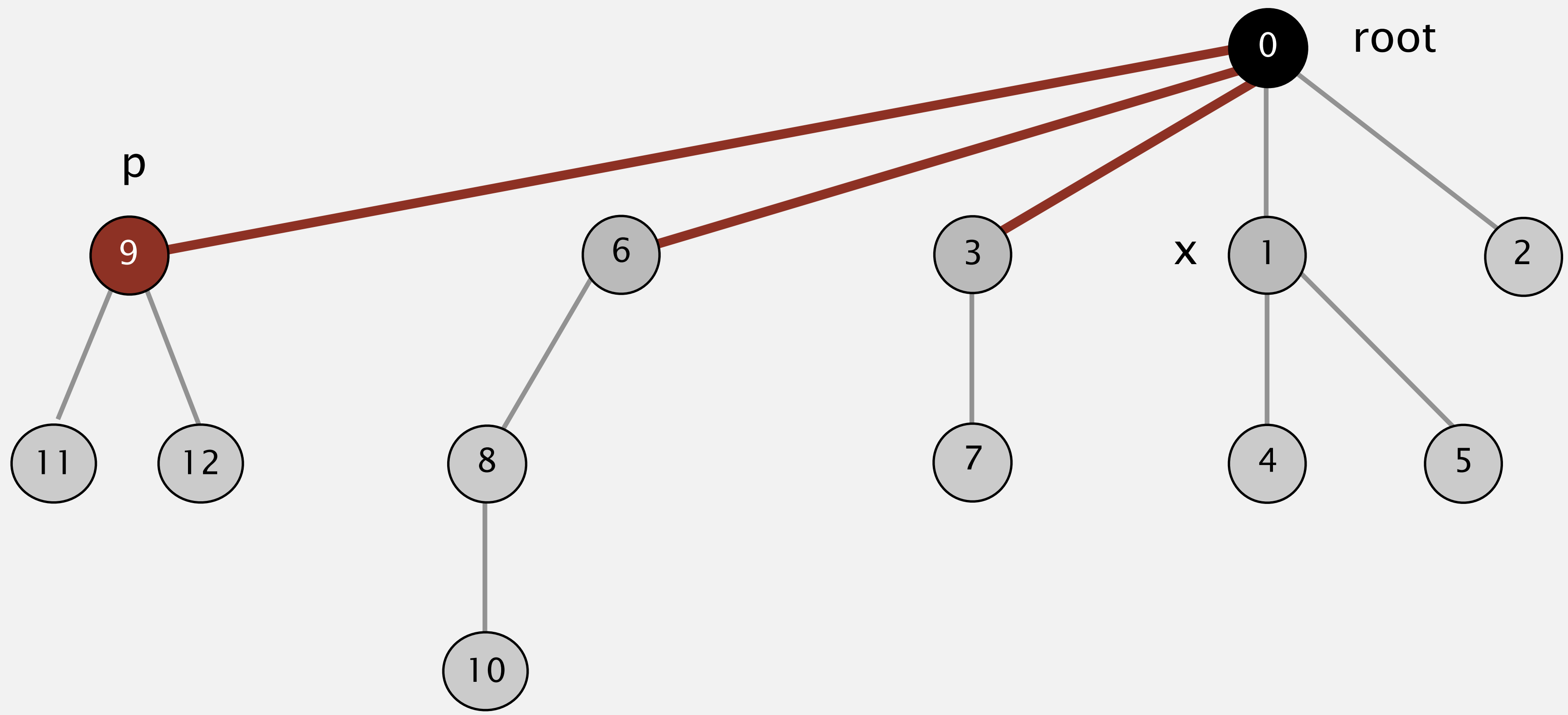
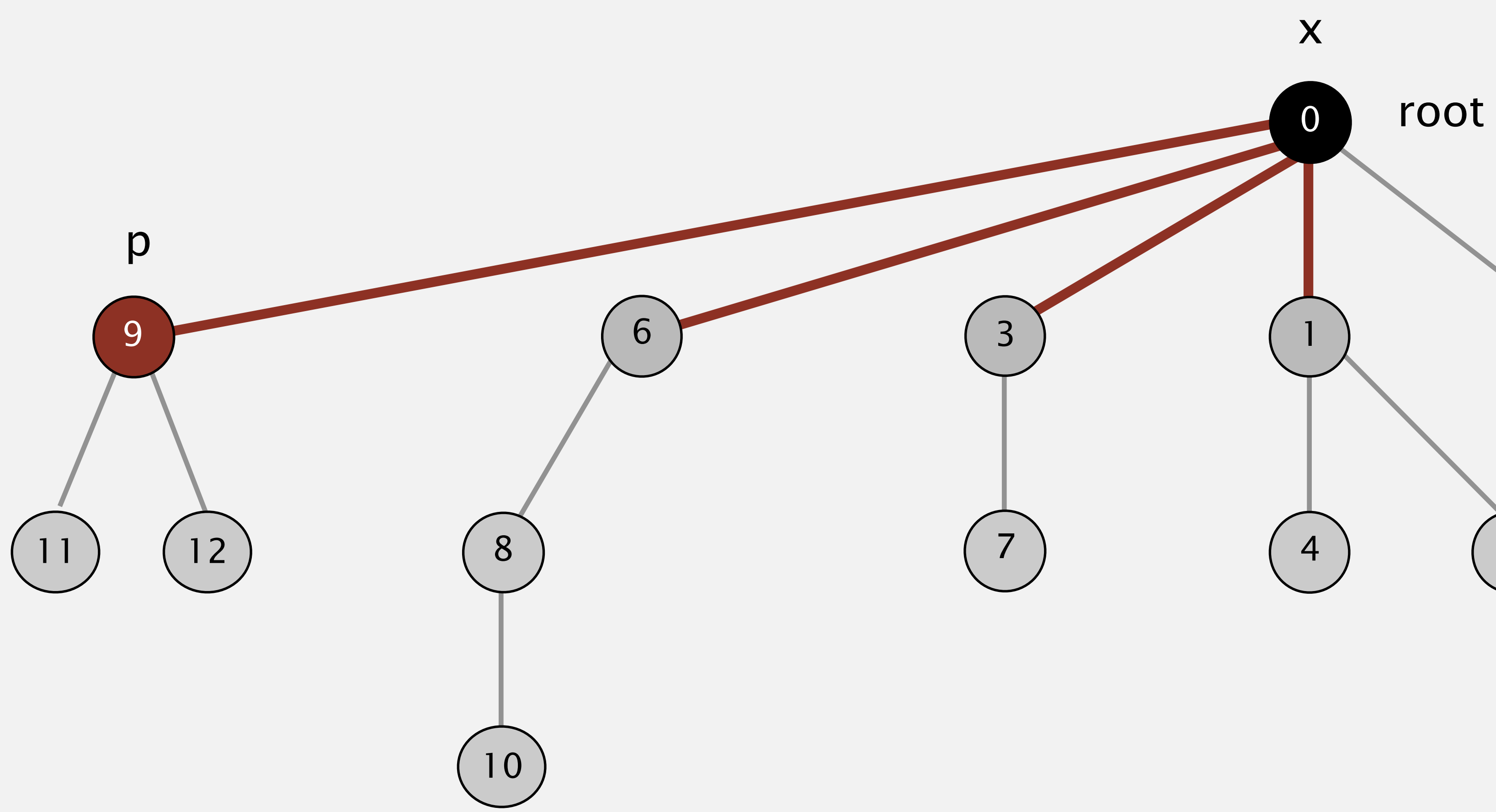


Improvement 2: path compression

Quick union with path compression. Just after computing the root of p, set the `id[]` of each examined node to point to that root.



Bottom line. Now, `find()` has the side effect of compressing the tree.

Path compression: Java implementation

Two-pass implementation: add second loop to `find()` to set the `id[]` of each examined node to the root.

Simpler one-pass variant (path halving): Make every other node in path point to its grandparent.

```
public int find(int i)
{
   while (i != id[i])
   {
      id[i] = id[id[i]];
      i = id[i];
   }
   return i;
}
```

only one extra line of code !

In practice. No reason not to! Keeps tree almost completely flat.

Iterated log function

- lg* x is defined recursively:

$$\lg^* n = \begin{cases} 1 & \text{if } n \leq 1 \\ 1 + \lg^*(\lg n) & \text{otherwise} \end{cases}$$

Weighted quick-union with path compression: amortized analysis

Proposition. [Hopcroft-Ulman, Tarjan] Starting from an empty data structure, any sequence of M union–find ops on N objects makes $\leq \ c\,(\,N + M \lg^*\, N\,)$ array accesses.

- Analysis can be improved to $N + M\,\alpha(M, N)$.
- Simple algorithm with fascinating mathematics.

lg* is the iterated log function

N	lg* N
1	0
2	1
4	2
16	3
65536	4
2^{65536}	5

iterated lg function

Linear-time algorithm for M union-find ops on N objects?

- Cost within constant factor of reading in the data.
- In theory, WQUPC is not quite linear.
- In practice, WQUPC is linear.

Amazing fact. [Fredman-Saks] No linear-time algorithm exists.

in "cell-probe" model of computation

Summary

Key point. Weighted quick union (and/or path compression) makes it possible to solve problems that could not otherwise be addressed.

algorithm	worst-case time
quick-find	M N
quick-union	M N
weighted QU	N + M log N
QU + path compression	N + M log N
weighted QU + path compression	N + M lg* N -> Almost linear

order of growth for M union-find operations on a set of N objects

Ex. [10^9 unions and finds with 10^9 objects]

- WQUPC reduces time from 30 years to 6 seconds.
- Supercomputer won't help much; good algorithm enables solution.

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

The scientific method.

Mathematical analysis.